NONCOMMUTATIVE GEOMETRY
AND QUANTUM GROUPS
BANACH CENTER PUBLICATIONS, VOLUME 61
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2003

QUANTUM-CLASSICAL INTERACTIONS AND GALOIS TYPE EXTENSIONS

WŁADYSŁAW MARCINEK
Institute of Theoretical Physics, University of Wrocław
Pl. Maxa Borna 9, 50-204 Wrocław, Poland
E-mail: wmar@ift.uni.wroc.pl

Abstract. An algebraic model for the relation between a certain classical particle system and the quantum environment is proposed. The quantum environment is described by the category of possible quantum states. The initial particle system is represented by an associative algebra in the category of states. The key new observation is that particle interactions with the quantum environment can be described in terms of Hopf-Galois theory. This opens up a possibility to use quantum groups in our model of particle interactions.

1. Introduction. The study of highly organized structures of matter leads to the investigation of some non-standard physical particle systems and effects. The fractional quantum Hall effect provides an example of a system with a well-defined internal order [8, 7, 10, 27]. Other interesting structures appear in the so-called $\frac{1}{2}$-electronic magneto-transport anomaly [11, 12, 13, 6], high temperature super-conductors or laser excitations of electrons. In these cases, anomalous behaviour of electrons occurs. An example is also given by the concept of statistical-spin liquids (see [1] and references therein).

It seems interesting to develop an algebraic approach to the unified description of all these new structures and effects. To this purpose, it is natural to assume that the whole world is divided into two parts: a classical particle system and its quantum environment. The classical system represents the observed reality, particles that really exist. The quantum environment represents all quantum possibilities that can become part of reality in the future [9]. The goal of this paper is to sketch a proposal of an algebraic model to describe interactions responsible for the appearance of the aforementioned highly organized structures. This model is based on a general algebraic formalism of Hopf algebras and Galois extensions of rings [24].

2000 *Mathematics Subject Classification*: 81R50, 17W30.

This work was partially sponsored by the Polish Committee for Scientific Research (KBN) under Grant No. 5P03B05620.

The paper is in final form and no version of it will be published elsewhere.

Our construction can be described in two steps. The first step concerns the transformation of the initial particles under interaction into composite systems consisting of quasi-particles and quanta. Such systems represent possible results of interactions [17, 20, 22, 21]. In the second step, we describe the algebra of realizations of quantum possibilities. This step is connected with the construction of an algebra extension and with a 'decision' which possibility can be realized and which one cannot. The problem how such 'decisions' are made was solved in [19] with the help of quantum commutativity and generalized Pauli exclusion principle. Our approach is based on the previously developed concept of particle systems with generalized statistics and quantum symmetries [15, 17, 20, 22, 21, 23].

The paper is organized as follows. In Section 2, we propose our general model within the framework of Hopf algebras and Galois extensions of rings. Then, in the subsequent section, we review Hopf-algebraic generalities. This recalls mathematical concepts employed in our proposal and allows us to specialize in the final section to the setting already appearing in some physical models. Since there are many interesting finite quantum groups related to spin coverings (e.g., [3, 5]) and the appearance of quantum symmetry in physics is more and more pronounced (e.g., [4]), our hope is that our general model will help us to understand some physical phenomena that cannot be adequately described by earlier methods.

2. The main idea. Let us consider a system of charged particles interacting with an external quantum environment. We assume that every charge is equipped with the ability to absorb and emit quanta of a certain nature. A system that contains a charge and a certain number of quanta as a result of interaction with the quantum environment is said to be a *dressed particle* [16, 18]. A particle dressed with a single quantum is a fictitious particle called a *quasi-particle*. Our model is based on the assumption that every charged particle transforms under interaction into a composite system consisting of quasi-particles and quanta [20]. This system represents possible results of interactions. Note that the process of absorption of quanta by a charged particle should be described as the creation of quasi-particles, whereas the emission as the annihilation of quasi-particles.

In our model the quantum environment is represented by a tensor category $\mathcal{C} = \mathcal{C}(\otimes, \mathbf{k})$ with duals [15]. All possible physical processes are represented as arrows of the category $\mathcal{C}$. If $f : \mathcal{U} \to \mathcal{V}$ is an arrow from $\mathcal{U}$ to $\mathcal{V}$, then the object $\mathcal{U}$ represents physical objects before interactions and $\mathcal{V}$ represents possible results of interactions. Different objects of the category $\mathcal{C}$ describe physical objects of different nature, charged particles, quasi-particles or different species of quanta of an external field, etc. If $\mathcal{U}$ is an object of $\mathcal{C}$ representing particles, quasi-particles or quanta, then the object $\mathcal{U}^*$ corresponds to antiparticles, or quasi-holes or dual fields, respectively. In the same fashion, if $\mathcal{U}$ represents charged particles and $\mathcal{V}$ describes certain quanta, then the product $\mathcal{U} \otimes \mathcal{V}$ encodes a composite system containing both particles and quanta. An arrow $\mathcal{U} \to \mathcal{U} \otimes \mathcal{V}$ means an interaction causing the passage from a single particle state to a composite quantum system. Thus the arrow $\mathcal{U} \to \mathcal{U} \otimes \mathcal{V}$ describes a process of absorption. Much in the same way, we conclude that the arrow $\mathcal{U} \otimes \mathcal{V} \to \mathcal{U}$ describes a process of emission.

In our approach a unital and associative algebra $\mathcal{A}$ in the category $\mathcal{C}$ represents the classical states of a system. The multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a morphism in this category representing the creation of a single object of reality from a composite system of objects of the same species. Quanta are encoded in a finitely generated coquasitriangular Hopf algebra $\mathcal{H}$. Quasi-particles are described by a new algebra $\mathcal{A}^{ext}$, which is an extension of $\mathcal{A}$. Interactions are described by a right action and coaction of $\mathcal{H}$ on the algebra $\mathcal{A}^{ext}$. It is natural to assume that the algebra $\mathcal{A}$ is invariant and coinvariant with respect to the action and coaction of $\mathcal{H}$, respectively, i.e., $\mathcal{A} = (\mathcal{A}^{ext})^{\mathcal{H}} = (\mathcal{A}^{ext})^{co\mathcal{H}}$. (Here $(\mathcal{A}^{ext})^{\mathcal{H}}$ is the set of $\mathcal{H}$-invariants and $(\mathcal{A}^{ext})^{co\mathcal{H}}$ the set of $\mathcal{H}$-coinvariants.)

We would like to represent the interaction of a charged particle with external quanta as a process of creation or annihilation of quasi-particles. A composite system of quasi-particles and quanta is described by a tensor product $\mathcal{A}^{ext} \otimes \mathcal{H}$ representing all possible quantum configurations coming as a result of the quantum absorption process. On the other hand, a composite system of two quasi-particles (related to the same particle) is described by a tensor product $\mathcal{A}^{ext}\,{}_{\mathcal{A}}\!\otimes \mathcal{A}^{ext}$. When $\mathcal{H}$ is a group-ring Hopf algebra $\mathbf{k}G$, our model assumes that an element of G can be understood as a specific charge characterizing an internal degree of freedom of a quasiparticle. We also assume that these charges are additive. Mathematically, this means that the algebra $\mathcal{A}^{ext}$ is G-graded, and that this grading is strong. As explained in the subsequent sections, the strongness of the G-grading of $\mathcal{A}^{ext}$ is known to be equivalent to the bijectivity of the canonical map

$$\beta : \mathcal{A}^{ext}\,{}_{\mathcal{A}}\!\otimes \mathcal{A}^{ext} \to \mathcal{A}^{ext} \otimes \mathcal{H}. \tag{1}$$

The bijectivity of this map means that the coaction $\mathcal{A}^{ext} \to \mathcal{A}^{ext} \otimes \mathcal{H}$ is Galois. ($\mathcal{A}^{ext}$ is a Hopf-Galois $\mathcal{H}$-extension of $\mathcal{A}$.)

An advantage of the above Galois condition is that it makes sense for an arbitrary Hopf algebra $\mathcal{H}$ and does not force $\mathcal{A}^{ext}$ to be a crossed-product algebra $\mathcal{A}^{ext} \rtimes \mathcal{H}$ [24, p. 101]. Thus, if we think of $\mathcal{H}$ as 'the group algebra of a quantum group', we have a rather general mathematical formalism capable of describing quasi-particles with the possible charges that are labeled by 'the elements of a quantum group' and additive according to the multiplication of $\mathcal{H}$. This way the Galois condition corresponds to the additivity of charges.

3. Quantum commutativity and Hopf-Galois extensions. Let $\mathcal{H}$ be a Hopf algebra over a ground field $\mathbf{k}$, and let $m, \eta, \triangle, \varepsilon, S$ denote its multiplication, unit, comultiplication, counit and antipode, respectively. We use the following notation for the coproduct in $\mathcal{H}$. If $h \in \mathcal{H}$, then $\triangle(h) := \Sigma h_{(1)} \otimes h_{(2)} \in \mathcal{H} \otimes \mathcal{H}$. We assume that $\mathcal{H}$ is a coquasitriangular Hopf algebra (CQTHA) (e.g., see [24, p. 184]). This means that $\mathcal{H}$ is equipped with a convolution invertible homomorphism $b \in Hom(\mathcal{H} \otimes \mathcal{H}, \mathbf{k})$ such that

$$\Sigma b(h_{(1)}, k_{(1)}) k_{(2)} h_{(2)} = \Sigma h_{(1)} k_{(1)} b(h_{(2)}, k_{(2)}), \tag{2}$$

$$b(h, kl) = \Sigma b(h_{(1)}, k) b(h_{(2)}, l), \tag{3}$$

$$b(hk, l) = \Sigma b(h, l_{(2)}) b(k, l_{(1)}), \tag{4}$$

for every $h, k, l \in \mathcal{H}$. We call such a bilinear form b a *coquasitriangular structure* on $\mathcal{H}$.

Another ingredient of our model is the concept of quantum commutativity [2, 19]. Let $\mathcal{A}$ be a unital and associative algebra and $\mathcal{H}$ be a Hopf algebra. If $\mathcal{A}$ is a right $\mathcal{H}$-comodule such that the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit map $\eta : \mathbf{k} \to \mathcal{A}$ are $\mathcal{H}$-comodule maps, then we say that it is a right $\mathcal{H}$-comodule algebra. The algebra $\mathcal{A}$ is said to be *quantum commutative* with respect to the coaction of $\mathcal{H}$ and its coquasitriangular structure b if an only if we have the relation

$$a\ b = \Sigma\ b(a_{(1)}, b_{(1)})\ b_{(0)}\ a_{(0)}. \tag{5}$$

Here $\rho(a) = \Sigma a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes \mathcal{H}$, and $\rho(b) = \Sigma b_{(0)} \otimes b_{(1)} \in \mathcal{A} \otimes \mathcal{H}$ for every $a, b \in \mathcal{A}$. The Hopf algebra $\mathcal{H}$ is called a *quantum symmetry* of $\mathcal{A}$.

Finally, let us recall the definition of a Hopf-Galois extension. An algebra extension $\mathcal{A}^{ext}$ of $\mathcal{A}$ such that it is a right $\mathcal{H}$-comodule algebra and $\mathcal{A}$ is its coinvariant subalgebra

$$\mathcal{A} \equiv (\mathcal{A}^{ext})^{co\mathcal{H}} := \{a \in \mathcal{A}^{ext} : \delta(a) = a \otimes 1\} \tag{6}$$

is said to be an $\mathcal{H}$-extension. If in addition the map $\beta : \mathcal{A}^{ext}\ {}_{\mathcal{A}}\!\otimes \mathcal{A}^{ext} \to \mathcal{A}^{ext} \otimes \mathcal{H}$ defined by

$$\beta(a\ {}_{\mathcal{A}}\!\otimes b) := (a \otimes 1)\delta(b) \tag{7}$$

is bijective, then the $\mathcal{H}$-extension is called Hopf-Galois. If $\mathcal{A}^{ext}$ is a Hopf-Galois $\mathcal{H}$-extension, then there is also a bijection

$$\beta^n : \underbrace{\mathcal{A}^{ext}\ {}_{\mathcal{A}}\!\otimes \cdots\ {}_{\mathcal{A}}\!\otimes \mathcal{A}^{ext}}_{n+1} \leftrightarrow \mathcal{A}^{ext} \otimes \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n} \tag{8}$$

given by

$$\beta^n := (\beta \otimes id) \circ \cdots \circ (id\ {}_{\mathcal{A}}\!\otimes \beta \otimes id) \circ (id\ {}_{\mathcal{A}}\!\otimes \beta). \tag{9}$$

In our physical interpretation, the one-to-one correspondence β^n means that the n-th ${}_{\mathcal{A}}\!\otimes$-tensor product representing a composite system of n quasi-particles also corresponds to a system of a single quasi-particle and n quanta.

4. Strongly G-graded quantum-commutative algebras. Recall first that the group algebra $\mathbf{k}G$ is a Hopf algebra for which the comultiplication, the counit, and the antipode are given by the formulae

$$\triangle(g) := g \otimes g, \quad \varepsilon(g) := 1, \quad S(g) := g^{-1},$$

respectively. The coquasitriangular structure on $\mathbf{k}G$ is given by a commutation factor $b : G \times G \to \mathbf{k} \setminus \{0\}$ [24, 26, 14, 25], and the category of right $\mathcal{H}$-comodules is equivalent to the category of G-graded vector spaces.

Next, assume that an algebra $\mathcal{A}^{ext}$ is an object of this category. This means that it is a G-graded algebra. Now we come to the crucial theorem [24, p. 126] stating that, for an arbitrary G-graded algebra and $\mathbf{k}G$-coaction compatible with the grading ($\rho(a) = a \otimes g$ for $a \in \mathcal{A}^{ext}_g$), the coaction is *Galois* if and only if the algebra is *strongly* G-graded. The latter means that

$$\mathcal{A}^{ext} = \bigoplus\nolimits_{g \in G} \mathcal{A}^{ext}_g, \quad \mathcal{A}^{ext}_g \mathcal{A}^{ext}_h = \mathcal{A}^{ext}_{gh}, \quad \mathcal{A}^{ext}_e \equiv \mathcal{A}, \tag{10}$$

where e is the neutral element of G.

As an example, let us consider a G-graded b-commutative $\mathbb{C}$-algebra $\mathcal{A}^{ext}$ with the so-called standard gradation [14]. This means that we take as the *strongly* grading group

$Z^N := Z \oplus ... \oplus Z$ and assume

$$b(\xi^i, \xi^j) =: b^{ij} = (-1)^{\Sigma_{ij}} q^{\Omega_{ij}}. \tag{11}$$

Here $\xi^i := (0, \ldots, 1, \ldots, 0)$ (1 on the i-th place) is the set of generators of Z^N, $\Sigma := (\Sigma_{ij})$ and $\Omega := (\Omega_{ij})$ are integer-valued matrices such that $\Sigma_{ij} = \Sigma_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$, and $q \in \mathbb{C} \setminus \{0\}$ is a parameter [25]. Since our Hopf algebra is a group ring, the equation (2) for the bilinear form b is automatically satisfied, whereas the equations (3)-(4) uniquely determine b once we set its value on the generators. The convolution-invertibility of b follows from the fact that b^{ij}s are always non-zero. (Notice that $b(\xi^i, (\xi^j)^n) = (b^{ij})^n$, so that for $q = exp(\frac{2\pi i}{n})$ and n even, the grading group Z^N can be reduced to $Z_n \oplus ... \oplus Z_n$.) Combining (5) with (11), we obtain the following quantum commutativity relations:

$$a_{\xi^i}\, a_{\xi^j} = b^{ij}\, a_{\xi^j}\, a_{\xi^i}, \quad \text{where} \quad a_{\xi^i} \in \mathcal{A}^{ext}_{\xi^i}, \quad a_{\xi^j} \in \mathcal{A}^{ext}_{\xi^j}. \tag{12}$$

It is the behaviour of b^{ij} that determines whether we obtain a system with the q-statistics, or Fermi statistics and the Pauli exclusion principle, or whether we obtain bosons. On the other hand, the *strong* gradation ensures that the internal degrees of freedom of a quasi-particle are labeled by charges (N-tuples of integers), and that these charges are *additive*.

Acknowledgments. It is a pleasure to thank Cezary Juszczak for his help with typesetting this article.

References

[1] K. Byczuk and J. Spalek, *Universality classes, statistical exclusion principle, and properties of interacting fermions*, Phys. Rev. B51 (1995), 7934.

[2] M. Cohen and S. Westrich, *Quantum commutative algebras*, J. Algebra 168 (1994), 1.

[3] A. Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, Comm. Math. Phys. 182 (1996), 155–176.

[4] A. Connes and D. Kreimer, *From local perturbation theory to Hopf and Lie algebras of Feynman graphs*, Lett. Math. Phys. 56 (2001), 3–15.

[5] L. Dąbrowski and C. Reina, Quantum spin coverings and statistics, math.QA/0208088.

[6] R. R. Du, H. L. Stormer, D. C. Tsui, A. S. Yeh, L. N. Pfeiffer and K. W. West, *Drastic enhancement of composite fermion mass near Landau level filling* $v= \frac{1}{2}$, Phys. Rev. Lett. 73 (1994), 3274.

[7] Z. F. Ezawa and H. Hotta, *Field theory of anyons and the fractional quantum Hall effect*, Phys. Rev. B 46 (1992), 7765.

[8] A. C. Gossard, H. L. Stormer and D. C. Tsui, *Two-dimensional magnetotransport in the extreme quantum limit*, Phys. Rev. Lett. 48 (1982), 1559.

[9] R. Haag, *An evolutionary picture for quantum physics*, Comm. Math. Phys. 180 (1996), 733.

[10] B. I. Halperin, P. A. Lee and N. Read, *Theory of the half-filled Landau level*, Phys. Rev. B 46 (1993), 7312.

[11] J. K. Jain, *Composite-fermion approach for the fractional quantum Hall effect*, Phys. Rev. Lett. 63 (1989), 199.

[12] J. K. Jain, *Incompressible quantum Hall states*, Phys. Rev. B 40 (1989), 8079.

[13] J. K. Jain, *Theory of the fractional quantum Hall effect*, Phys. Rev. B 41 (1990), 7653.

[14] W. Marcinek, *On unital braidings and quantization*, Rep. Math. Phys. 34 (1994), 325.

[15] W. Marcinek, *Categories and quantum statistics*, in: Proc. Symposium: Quantum Groups and their Applications in Physics (Poznań, 1995), Rep. Math. Phys. 38 (1996), 149–179.

[16] W. Marcinek, *Topology and quantization*, in: Proc. IVth Internat. School on Theoretical Physics, Symmetry and Structural Properties (Zajączkowo, 1996), B. Lulek *et al.* (eds.), World Sci., Singapore, 1997, 415–424; hep-th/9705098.

[17] W. Marcinek, *Remarks on quantum statistics*, in: Proc. Conf. "Particles, Fields and Gravitation" (Łódź, 1998), J. Rembieliński (ed.), World Sci., Singapore, 1998; math.QA./9806158.

[18] W. Marcinek, *On commutation relations for quons*, Rep. Math. Phys. 41 (1998), 155.

[19] W. Marcinek, *Particles and quantum symmetries*, in: Proc. XVI Workshop on Geometric Methods in Physics (Białowieża, 1997), Rep. Math. Phys. 43 (1999), 239; math.QA/9805122.

[20] W. Marcinek, *On composite systems and quantum statistics*, in: Proc. Vth Internat. School on Theoretical Physics, Symmetry and Structural Properties (Zajączkowo, 1998), B. Lulek *et al.* (eds.), World Sci., Singapore, 1999; math.QA/9810060.

[21] W. Marcinek, *On generalized statistics and one dimensional systems*, in: Proc. III Internat. Seminar "Hidden Symmetry" (Rzeszów, 1998), Mol. Phys. Rep. 23 (1999), 170–173.

[22] W. Marcinek, *On generalized quantum statistics*, in: Proc. XII-th Max Born Symposium (Wrocław, 1998), A. Borowiec *et al.* (eds.), Theoretical Physics Fin de Siecle, Springer, 2000.

[23] W. Marcinek, *On generalized statistics and interactions*, in: Proc. XVII Workshop on Geometric Methods in Physics (Białowieża, 1998), Coherent States, Quantization and Gravity, M. Schlichenmaier *et al.* (eds.), Wyd. Uniw. Warszawskiego, Warszawa, 2001; math.QA/990029.

[24] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conf. Ser. Math. 82, AMS, 1993.

[25] Z. Oziewicz, *Lie algebras for arbitrary grading group*, in: Differential Geometry and Its Applications, J. Janyska and D. Krupka (eds.), World Sci., Singapore, 1990.

[26] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. 20 (1979), 712,.

[27] A. Zee, *Quantum Hall fluids*, in: Field Theory, Topology and Condensed Matter Physics, H. D. Geyer (ed.), Lecture Notes in Phys., Springer, 1995.

In memoriam

Shortly after the foregoing article had been completed, its author, Władysław Marcinek, suddenly passed away. This is, presumably, his last finished piece of work. It is therefore that we place this valediction of our friend and colleague herein.

Throughout his life, Władek struggled with a horrible muscle disease crippling his body ever since his youth. His life is a story of successful overcoming the tides of suffering, a story of a victory of the spirit. His choice not to resign himself to circumstances paid off. He was blessed with a happy marriage and published around 60 papers. Also, he was among those who set up the Society to Fight Muscle Diseases. He proved that even swimming against the currents one can reach the other shore.

This is an example of someone who did not feel excused by the circumstances from doing useful and important things. All the time we face some choices where we can choose to do more or less, to help, think, feel, act or to be excused by being too busy, too tired, too sick or too poor. A continuous and consistent heroism of micro-choices is a heroic deed and weighs as much. One should never think that an honest effort to do even a little bit of good here and there is too small to count. The Book of the Ecclesiastes says:

Whatsoever thy hand findeth to do, do it with thy might; for there is no work, nor device, nor knowledge, nor wisdom, in the grave, whither to thou goest. I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, neither yet bread to the wise, nor yet riches to men of understanding, nor yet favour to men of skill; but time and chance happeneth to them all.

We shall miss him…

The Editors