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EQUIVARIANT SPECTRAL TRIPLES

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Abstract. We present the review of noncommutative symmetries applied to Connes' formulation of spectral triples. We introduce the notion of equivariant spectral triples with Hopf algebras as isometries of noncommutative manifolds, relate it to other elements of theory (equivariant K-theory, homology, equivariant differential algebras) and provide several examples of spectral triples with their isometries: isospectral (twisted) deformations (including noncommutative torus) and finite spectral triples.

1. Introduction. Spectral triples were proposed (in their present form) by Alain Connes [13] as the noncommutative generalisation of spin manifolds. The theory, formulated in an axiomatic way, uses the representation of an algebra on a Hilbert space together with an unbounded Dirac operator. The latter determines both the differential calculus as well as the metric. The natural symmetries, which appear in this approach, are related to the automorphism group of the algebra. Although this seems to be sufficient in the classical case, one may conjecture that for noncommutative algebras there is still place for symmetries in the form of Hopf algebras.

This paper is a self-contained review of a proposition for the definition of a noncommutative version of an *isometry* of a spectral triple, in the form of a Hopf algebra H. It is organised as follows: in the next section we review the basic notions and definitions used in the paper, in particular, these of equivariant modules and bimodules. We explicitly present the example of the tautological line bundle over the quantum sphere. We mention also the relation with the equivariant Hochschild (co)homology and K-theory. Further, we define equivariant Fredholm modules and cycles, then equivariant spectral triples and

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equivariant real spectral triples. In the last section we discuss the basic examples: the noncommutative torus and more general isospectral deformations and the spectral triples over discrete spaces.

2. Hopf algebras as symmetries. We briefly recall here the most important notions used further. H will always denote a Hopf algebra. We use Sweedler's notation for the coproduct: $\Delta h = h_{(1)} \otimes h_{(2)}$. Omitted proofs can be found in the textbooks by Shahn Majid [31] or Chari and Pressley [10].

2.1. *Action on algebras, modules and bimodules*

DEFINITION 2.1. An *H-module* is a pair (V, ρ), where V is a linear space and ρ is a complex linear representation of H on V. To simplify the notation we shall write $h \triangleright v$, $h \in H, v \in V$ instead of $\rho(h)v$.

DEFINITION 2.2. An algebra $\mathcal{A}$ is a left *H-module algebra* if $\mathcal{A}$ is a left H-module and the representation respects the algebra structure in $\mathcal{A}$:

$$h \triangleright (a_1 a_2) = (h_{(1)} \triangleright a_1)(h_{(2)} \triangleright a_2), \tag{1}$$

for all $h \in H, a_1, a_2 \in \mathcal{A}$. In the case of unital $\mathcal{A}$:

$$h \triangleright 1 = \epsilon(h), \qquad \forall h \in H, \tag{2}$$

whereas the action of $1 \in H$ is:

$$1 \triangleright a = a, \qquad \forall a \in \mathcal{A}. \tag{3}$$

The canonical examples of Hopf algebra actions are given through the adjoint action of the Hopf algebra on itself and the action of its dual:

EXAMPLE 2.3. Every Hopf algebra is an H-module algebra through the left adjoint action on itself:

$$h \triangleright_{\mathrm{Ad}} g = h_{(1)} g (S h_{(2)}), \tag{4}$$

for $h, g \in H$.

EXAMPLE 2.4. If there exist the dual Hopf algebra H^*, then there is a canonical action of the Hopf algebra H on its dual H^*:

$$h \triangleright \phi = \phi_{(1)} \langle h, \phi_{(2)} \rangle, \tag{5}$$

where $h \in H, \phi \in H^*$.

The adjoint action of the group algebra is the linear extension of the adjoint action of the group. In particular, for any algebra $\mathcal{A}$, if $\mathcal{A}^+$ is a group of its invertible elements, then the group algebra $\mathbb{C}\mathcal{A}^+$ acts on $\mathcal{A}$ by the adjoint action:

$$\Big(\sum_i c_i g_i \Big) \triangleright a = \sum_i c_i g_i a g_i^{-1}, \tag{6}$$

for $g_i \in \mathcal{A}^+$ and $c_i \in \mathbb{C}$, $i = 1, \ldots, n$.

There exists always a trivial action of H on any algebra $\mathcal{A}$, given through:

$$h \triangleright a = \epsilon(h) a. \tag{7}$$

Definition 2.5. For a pair $(H, \mathcal{A})$, a Hopf algebra H and an H-module algebra $\mathcal{A}$ the left *cross product algebra* $\mathcal{A} \rtimes H$[1] is the linear space $\mathcal{A} \otimes H$ equipped with the product:

$$(a \otimes h)(b \otimes g) = (a(h_{(1)} \triangleright b) \otimes h_{(2)} g). \tag{8}$$

One defines similarly right H-module algebras (and constructs right cross products). For right and left H-module algebras one can introduce the notion of an H-bimodule algebra:

Definition 2.6. We say that $\mathcal{A}$ is an *H-bimodule algebra* if $\mathcal{A}$ is an H-left and H-right module algebra and the left and right actions commute with each other:

$$(h \triangleright a) \triangleleft g = h \triangleright (a \triangleleft g), \tag{9}$$

for all $h, g \in H$ and $a \in \mathcal{A}$.

One can extend the notion of the action of Hopf algebras to modules over algebras.

Definition 2.7. Let M be an $\mathcal{A}$-left module and let $\mathcal{A}$ be an H-module algebra (left). Then M is an *H-equivariant $\mathcal{A}$-module* if M itself is an H-module and:

$$h \triangleright (am) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright m), \tag{10}$$

for all $h \in H$, $a \in \mathcal{A}$, $m \in M$.

Clearly, an equivalent definition (but with right $\mathcal{A}$-module structure) gives us right H-equivariant $\mathcal{A}$-modules. Putting these two conditions together we define H-equivariant $\mathcal{A}$-bimodules. In such a case we have:

$$h \triangleright (amb) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright m)(h_{(3)} \triangleright b), \tag{11}$$

for all $h \in H$, $a, b \in \mathcal{A}$, $m \in M$.

Note the following:

Proposition 2.8. *If $\mathcal{A}$ is a left H-module algebra and M is an H-equivariant left $\mathcal{A}$-module then M is a left module over the cross product algebra $\mathcal{A} \rtimes H$, with the module multiplication:*

$$(a \otimes h)m = a(h \triangleright m).$$

Since the converse is trivially true, we see the equivalence of H-equivariant modules with modules over the cross product algebras.

Before we proceed with the properties of equivariance, let us define invariant elements. We say that an element $m \in M$ of an H-equivariant module is *invariant under the action of H* if $h \triangleright m = \epsilon(h)m$ for all $h \in H$. A submodule $N \subset M$ is invariant if $(h \triangleright N) \subset N$ for all $h \in H$. It is clear that a submodule generated by invariant elements is invariant. One can state:

Proposition 2.9. *If $r : M \to N$ is a module epimorphism between left $\mathcal{A}$-modules, M is H-equivariant and $\ker r \subset M$ is an H-invariant submodule then N is also H-equivariant.*

[1] Note that in the literature this is also called a *smash product algebra* while the name cross product is used to denote a larger class of products, where the action of the Hopf algebra is twisted by a cocycle.

Proof. It is sufficient to define the action of $h \in H$ on $n \in N$ through:

$$h \triangleright n = r\left(h \triangleright r^{-1}(n)\right). \tag{12}$$

As an immediate corollary we have:

COROLLARY 2.10. *If M is an H-equivariant $\mathcal{A}$-module and N is its H-invariant submodule, then the quotient module M/N is also H-equivariant.*

It is natural to define equivariant module morphisms:

DEFINITION 2.11. Let M, N be equivariant (left-, right- or bi-) modules over an H-module algebra $\mathcal{A}$. A linear map $\psi : M \to N$ is an *equivariant* (left-, right-, bi-) *module morphism* if ψ is a (left-, right-, bi-) module morphism and for every $h \in H$, $m \in M$

$$h \triangleright \psi(m) = \psi(h \triangleright m).$$

It is easy to see that the projection r from Proposition 2.9 is equivariant. On the other hand, we may easily observe:

PROPOSITION 2.12. *If M, N are equivariant (left-, right- or bi-) modules over an H-module algebra $\mathcal{A}$ and ψ an equivariant (left- right-, bi-) module morphism between them, then the kernel of ψ is H-invariant.*

Of course, any H-module algebra is H-equivariant as a left module, right module, and a bimodule over $\mathcal{A}$. This extends also to any free left (or right) module over an H-module algebra $\mathcal{A}$. However, it is interesting to observe that there are many possible actions of H on $\mathcal{A}^n$:

PROPOSITION 2.13. *For every linear map $H \ni h \mapsto a^i_j(h) \in M_n(\mathcal{A})$, which satisfies:*

$$\sum_{j=1}^{n} \left(h_{(1)} \triangleright a^i_j(g)\right) a^j_k(h_{(2)}) = a^i_k(hg), \qquad h, g \in H, i, k = 1, \ldots, n, \tag{13}$$

there is an action of the Hopf algebra H on the free module $\mathcal{A}^n$, given on the canonical basis of $\mathcal{A}^n$, $e^i, i = 1, \ldots, n$ as:

$$h \triangleright e^i = \sum_j a^i_j(h) e^j. \tag{14}$$

In particular, every finite-dimensional representation of H, $a^j_i(h) \in M_n(\mathbb{C})$ satisfies the condition (13).

The equivariant left (right- and bi-) modules behave well under direct sums. Consider now their tensor products. Notice that for a left $\mathcal{A}$-module M and a vector space N, $M \otimes N$ has a natural left-module structure given by $a\,(m \otimes n) = am \otimes n$. We have:

PROPOSITION 2.14. *If M is an H-equivariant left $\mathcal{A}$-module, and N is an H-module, then $M \otimes N$ is an H-equivariant left $\mathcal{A}$-module with the action of H:*

$$h \triangleright (m \otimes n) = (h_{(1)} \triangleright m) \otimes (h_{(2)} \triangleright n). \tag{15}$$

This result can be extended to the situation where N is an H-equivariant right $\mathcal{A}$-module:

PROPOSITION 2.15. *If M, N are respectively left and right $\mathcal{A}$-modules which are H-equivariant then $M \otimes N$ is an H-equivariant bimodule over $\mathcal{A}$ with the module structure given by:*

$$a\,(m \otimes n)b = am \otimes nb,$$

and the action (15).

Finally, if both M and N are H-equivariant bimodules, we have:

PROPOSITION 2.16. *If M, N are H-equivariant $\mathcal{A}$-bimodules then their tensor product over $\mathcal{A}$ is an H-equivariant bimodule.*

Proof. Due to Proposition 2.15, $M \otimes N$ is already an H-equivariant bimodule. Therefore, it is sufficient to demonstrate that the subbimodule $\mathcal{I}$ generated by the elements $ma \otimes n - m \otimes an$ is H-invariant:

$$\begin{aligned} &h \triangleright (m \otimes an - ma \otimes n) \\ &\quad = h_{(1)} \triangleright m \otimes h_{(2)} \triangleright (an) - h_{(1)} \triangleright (ma) \otimes (h_{(2)} \triangleright n) \\ &\quad = (h_{(1)} \triangleright m) \otimes (h_{(2)} \triangleright a)(h_{(3)} \triangleright n) - (h_{(1)} \triangleright m)(h_{(2)} \triangleright a) \otimes (h_{(3)} \triangleright n), \end{aligned}$$

and the last expression is still in $\mathcal{I}$, so $\mathcal{I}$ is indeed H-invariant. For this reason the quotient $(M \otimes N)/\mathcal{I}$ is H-equivariant. ∎

2.2. *Equivariant bimodules and Yetter-Drinfeld H-algebras.* A particularly interesting example of equivariant bimodules is given by the Yetter-Drinfeld H-algebra structure (see [28] for details). Recall that an H-module algebra is a Yetter-Drinfeld module algebra $\mathcal{A}$ if it is simultaneously a right H^{op}-comodule algebra with the following compatibility condition between the coaction $\rho : \mathcal{A} \to \mathcal{A} \otimes H^{op}$, $\rho(a) =: a_{(0)} \otimes a_{(1)}$ (summation understood) and the H-module structure:

$$\rho(h \triangleright a) = \big(h_{(2)} \triangleright a_{(0)}\big) \otimes h_{(3)} a_{(1)} (S^{-1} h_{(1)}), \tag{16}$$

for all $a \in \mathcal{A}, h \in H$.

We assume here that H has an invertible antipode. We use here the left-module and right-comodule version of the definition but, clearly, all four versions (left-left, right-left left-right and right-right) are completely equivalent by replacing H with its opposite algebra, or coopposite coalgebra structure.

PROPOSITION 2.17. [2] *Let $\mathcal{A}$ be a left H-module algebra and a right H-comodule. Assume that $\mathcal{A} \otimes H$ is a left H-module via the diagonal action: $h \triangleright (a \otimes g) := (h_{(1)} \triangleright a) \otimes h_{(2)} g$. Then $\mathcal{A}$ is a Yetter-Drinfeld module algebra if and only if $\mathcal{A} \otimes H$ is an H-equivariant $\mathcal{A}$-bimodule with the natural left $\mathcal{A}$-multiplication and a right $\mathcal{A}$-multiplication such that $((a \otimes g)b)h = (a \otimes gh)b$, $\forall\, a, b \in \mathcal{A},\ g, h \in H$ (right H-linearity of the right $\mathcal{A}$-multiplication).*

Note that the aforementioned H-linearity of the right $\mathcal{A}$-module structure could be rephrased as the H-equivariance with respect to the trivial right H-action on $\mathcal{A}$.

[2] I am grateful to P. M. Hajac for a discussion that led me to this result.

Proof. Assume that $\mathcal{A}$ is a Yetter-Drinfeld module algebra. First, using its coaction ρ, we define a right $\mathcal{A}$-module structure on $\mathcal{A} \otimes H$:

$$(a \otimes h)b = ab_{(0)} \otimes b_{(1)}h, \quad \forall a, b \in \mathcal{A}, h \in H.$$

The left $\mathcal{A}$-module structure is the obvious one. Clearly, $\mathcal{A} \otimes H$ is an H-equivariant left module. Let us now check, using the condition (16), whether the action of H is compatible with the right multiplication:

$$\begin{aligned}\big(h_{(1)} \triangleright (a \otimes g)\big)\,(h_{(2)} \triangleright b) &= \big((h_{(1)} \triangleright a) \otimes h_{(2)}g\big)\,(h_{(3)} \triangleright b)\\ &= (h_{(1)} \triangleright a)(h_{(3)} \triangleright b)_{(0)} \otimes (h_{(3)} \triangleright b)_{(1)}h_{(2)}g\\ &= (h_{(1)} \triangleright a)\,\big(h_{(4)} \triangleright b_{(0)}\big) \otimes h_{(5)}b_{(1)}(S^{-1}h_{(3)})h_{(2)}g\\ &= (h_{(1)} \triangleright a)\,\big(h_{(4)} \triangleright b_{(0)}\big) \otimes h_{(5)}b_{(1)}S^{-1}\big((Sh_{(2)})h_{(3)}\big)\,g\\ &= (h_{(1)} \triangleright a)\,\big(h_{(2)} \triangleright b_{(0)}\big) \otimes h_{(3)}b_{(1)}g\\ &= h_{(1)} \triangleright (ab_{(0)}) \otimes h_{(2)}b_{(1)}g\\ &= h \triangleright ((a \otimes g)b)\,.\end{aligned}$$

The H-linearity of the right $\mathcal{A}$-module structure on $\mathcal{A} \otimes H$ is obvious.

Conversely, suppose that $\mathcal{A} \otimes H$ is an H-equivariant bimodule with a right H-linear right $\mathcal{A}$-module structure. Let us define a linear map $\rho : \mathcal{A} \to \mathcal{A} \otimes H^{op}$:

$$\rho(a) = (1 \otimes 1)a, \qquad \forall a \in \mathcal{A}.$$

Since the right $\mathcal{A}$-multiplication on $\mathcal{A} \otimes H$ is right H-linear, we clearly see that ρ is an algebra homomorphism:

$$\begin{aligned}\rho(ab) &= \big(a_{(0)} \otimes a_{(1)}\big)\,b = a_{(0)}\,((1 \otimes 1)b)\,a_{(1)}\\ &= a_{(0)}\,\big(b_{(0)} \otimes b_{(1)}\big)\,a_{(1)} = a_{(0)}b_{(0)} \otimes b_{(1)}a_{(1)}.\end{aligned}$$

The verification of the Yetter-Drinfeld compatibility condition (16) between the left H-module structure and the right H^{op}-comodule structure follows directly from the equivariance requirement and the right H-linearity:

$$\begin{aligned}\rho(h \triangleright a) &= (1 \otimes 1)(h \triangleright a)\\ &= \epsilon(h_{(2)}) \otimes h_{(3)}S^{-1}(h_{(1)}))(h_{(4)} \triangleright a)\\ &= (h_{(2)} \triangleright 1 \otimes h_{(3)}S^{-1}(h_{(1)}))(h_{(4)} \triangleright a)\\ &= (h_{(2)} \triangleright (1 \otimes S^{-1}(h_{(1)})))(h_{(3)} \triangleright a)\\ &= (h_{(2)} \triangleright ((1 \otimes S^{-1}(h_{(1)}))a))\\ &= (h_{(2)} \triangleright (((1 \otimes 1)a)S^{-1}(h_{(1)}))\\ &= (h_{(2)} \triangleright (a_{(0)} \otimes a_{(1)}S^{-1}(h_{(1)}))\\ &= h_{(2)} \triangleright a_{(0)} \otimes h_{(3)}a_{(1)}S^{-1}(h_{(1)}).\end{aligned}$$

This ends the proof. ∎

2.3. *Equivariant projective modules.* Having shown that free modules over an H-module algebra are equivariant, we shall now investigate the case of finitely generated projective modules. The example, which we shall demonstrate explicitly, is the tautological line bundle over the equatorial Podleś quantum sphere.

We take the description of finitely generated projective modules through the idempotents in $M_n(\mathcal{A})$. The problem is to define conditions for such idempotents, which determine whether they give equivariant left modules. For free modules the equivariant action is given by the relation (13). Let ρ be an n-dimensional right-representation of the Hopf algebra H. Then, for a free module $\mathcal{A}^n$, with its canonical basis e^i, we set:

$$h \triangleright e^i = \sum_{j=1}^{n} e^j \rho(h)^i_j, \tag{17}$$

and thus we obtain the structure of an H-equivariant left module.

Now, to say that a projection $P \in M_n(\mathcal{A})$ defines an equivariant H-module is equivalent to the statement that its kernel is H-invariant. We shall now derive the necessary conditions for this, restricting ourselves to the case of the action given by finite dimensional representations. Let us denote:

$$e^i P = \sum_{j=1}^{n} P^i_j e^j, \quad P^i_j \in \mathcal{A}, i = 1, \dots, n.$$

Then we have the following result:

PROPOSITION 2.18. *If there exists a linear map $M : H \to M_n(\mathcal{A})$ such that for every $h \in H$:*

$$\sum_{j=1}^{n} \rho^l_j(h) P^j_k = \sum_{i=1}^{n} (h_{(1)} \triangleright P^l_i) M^i_k(h_{(2)}), \qquad l, k = 1, \dots, n, \tag{18}$$

then the kernel of P is H-invariant.

Proof. Let us assume that $\mathcal{A}^n \ni m = \sum_{i=1}^{n} v^i e^i$ is in the kernel of P. Then for every $j = 1, \dots n$, we have $\sum_{i=1}^{n} v^i P^i_j = 0$. Using (18) we calculate for a given $h \in H$:

$$\begin{aligned}
(h \triangleright m)P &= \sum_{i,j=1}^{n} (h_{(1)} \triangleright v^i) \rho^i_j(h_{(2)})(e^j P) \\
&= \sum_{i,j,k=1}^{n} (h_{(1)} \triangleright v^i) \rho^i_j(h_{(2)}) P^j_k e^k \\
&= \sum_{i,j,k=1}^{n} (h_{(1)} \triangleright v^i)(h_{(2)} \triangleright P^i_j) M^j_k(h_{(3)}) e^k \\
&= \sum_{j,k=1}^{n} \Big(h_{(1)} \triangleright \Big(\sum_{i=1}^{n} v^i P^i_j\Big)\Big) M^j_k(h_{(2)}) e^k = 0. \ \blacksquare
\end{aligned}$$

We shall illustrate the requirement (18) with an interesting example of the Podleś equatorial quantum sphere:

EXAMPLE 2.19. Let $\mathcal{A}(S^2_q)$ be the algebra defining the Podleś equatorial quantum sphere [37]. It is given by the relations:

$$\begin{aligned}
&ba = q^2 ab, && a^* b = q^2 b a^*, \\
&a^* a + b^2 = 1, && q^2 a a^* + \tfrac{1}{q^2} b^2 = q^2.
\end{aligned} \tag{19}$$

The action of $\mathcal{U}_q(su_2)$ (for the relations and coproduct relations of this quantum group, see for instance [31]) is expressed on the generators of $\mathcal{A}(S^2_q)$ through:

$$\begin{array}{lll} k \triangleright a = qa, & e \triangleright a = -(1+q^2)q^{-\frac{5}{2}}b, & f \triangleright a = 0 \\ k \triangleright a^* = \frac{1}{q}a^*, & e \triangleright a^* = 0, & f \triangleright a^* = (1+q^2)q^{-\frac{3}{2}}b, \\ k \triangleright b = b, & e \triangleright b = q^{\frac{1}{2}}a^*, & f \triangleright b = -q^{\frac{3}{2}}a. \end{array}$$

The tautological line bundles over S^2_q (see [7]) are defined as projective modules with the projectors:

$$P^{\pm}_q = \frac{1}{2}\begin{pmatrix} 1 \mp \frac{1}{q^2}b & \mp qa \\ \mp \frac{1}{q}a^* & 1 \pm b \end{pmatrix}. \tag{20}$$

Using the condition (18) and Proposition 2.18 we shall demonstrate that these are $\mathcal{U}_q(su_2)$-equivariant projective modules. The map $M^i_k(h)$ is in this particular case given by $M^i_k(h) = \rho^i_j(h)P^j_k$.

Let us show the calculation for $h = e$ only, it is done in a similar manner for other generators of $\mathcal{U}_q(su_2)$. We use the two-dimensional fundamental representation of $\mathcal{U}_q(su_2)$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad k = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix},$$

First, the left-hand side is:

$$\rho(e)P^{\pm}_q = \tfrac{1}{2}\begin{pmatrix} \mp\frac{1}{q}a^* & 1 \pm b \\ 0 & 0 \end{pmatrix},$$

and after some calculation we see that the right-hand side is the same:

$$\begin{aligned} &\frac{1}{4}\begin{pmatrix} \mp q^{-\frac{3}{2}}a^* & \pm q^{-\frac{3}{2}}(1+q^2)b \\ 0 & \pm q^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 \mp \frac{1}{q^2}b & \mp qa \\ \mp\frac{1}{q}a^* & 1 \pm b \end{pmatrix} + \\ &\qquad + \frac{1}{4}\begin{pmatrix} 1 \mp \frac{1}{q^2}b & \mp a \\ \mp a^* & 1 \pm b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \mp \frac{1}{q^2}b & \mp qa \\ \mp\frac{1}{q}a^* & 1 \pm b \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} \mp\frac{1}{q}a^* & 1 \pm b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \mp \frac{1}{q^2}b & \mp qa \\ \mp\frac{1}{q}a^* & 1 \pm b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mp\frac{1}{q}a^* & 1 \pm b \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

2.4. *Action and star structures*

Definition 2.20. If both $\mathcal{A}$ and H have a star structure and, additionally, H is a star Hopf algebra (see [31], p. 31, for instance) we say that the action is *compatible with the star structure* if:

$$(h \triangleright a)^* = (Sh)^* \triangleright a^*. \tag{21}$$

As a nice little exercise we demonstrate that the properties of the star as well as of the action hold and are compatible with the equation (21):

$$\begin{aligned} (h \triangleright (ab))^* &= (h_{(2)} \triangleright b)^*(h_{(1)} \triangleright a)^* = ((Sh_{(2)})^* \triangleright b^*)((Sh_{(1)})^* \triangleright a^*) \\ &= (Sh)^* \triangleright (b^*a^*) = (Sh)^* \triangleright (ab)^*, \end{aligned}$$

and:

$$(hg \triangleright (a))^* = ((S(hg))^* \triangleright a^*) = ((Sg)\,(Sh))^* \triangleright a^* = (Sh)^* \triangleright ((Sg)^* \triangleright a^*)$$
$$= (Sh)^* \triangleright (g \triangleright a)^* = (h \triangleright (g \triangleright a))^* .$$

We shall call such an algebra $\mathcal{A}$ a *star H-module algebra.* Having a star Hopf algebra H and a star H-module algebra $\mathcal{A}$ we define the star involution on the cross-product (smash-product) $\mathcal{A} \rtimes H$ through:

$$(a \otimes h)^* = (h^*_{(1)} \triangleright a^*) \otimes h^*_{(2)}. \tag{22}$$

It should be noted that this is a special case of a cleft Hopf-Galois extension with a star structure. Recall that a Hopf-Galois extension $\mathcal{A} \subset P$ is an algebra P with a left $\mathcal{A}$-module structure and a right H-comodule algebra structure such that $\mathcal{A}$ is the subalgebra of coinvariants of P satisfying certain conditions. For details on Hopf-Galois extensions we refer to [39]. We say that P is a star Hopf-Galois extension if all morphisms preserve the star structure.

An extension is cleft if there exists a cleaving map $j : H \to \mathcal{A} \otimes H$. A cleaving map is a (unital) homomorphism which is convolution invertible, i.e.,

$$j(h_{(1)})j^{-1}(h_{(2)}) = \epsilon(h) = j^{-1}(h_{(1)})j(h_{(2)}), \quad \forall h \in H,$$

and intertwines between the right coaction on P and the right coaction on H given by the coproduct. For cleft Hopf-Galois extensions we have an isomorphism i of $\mathcal{A} \otimes H$ with P as left $\mathcal{A}$-modules and right H-comodules:

$$i : (a \otimes h) \mapsto aj(h) \in P, \quad \forall a \in \mathcal{A}, h \in H.$$

Assuming that the cleaving map preserves the star structure, we can carry the star from P to $\mathcal{A} \otimes H$:

$$(a \otimes h)^* = i^{-1}(i(a \otimes h)^*) = i^{-1}((aj(h))^*) = i^{-1}(j(h)^* a^*)) =$$
$$= i^{-1}(j(h^*_{(1)})a^* j^{-1}(h^*_{(2)})j(h^*_{(3)}))$$
$$= j(h^*_{(1)})a^* j^{-1}(h^*_{(2)}) \otimes j(h^*_{(3)}),$$

where in the last line we used the properties of the cleaving map j and the isomorphism i (see [31], p. 268).

Note that in our particular case (smash product) when j is an algebra homomorphism, it defines an action of H on $\mathcal{A}$ in the following way:

$$h \triangleright a = j(h_{(1)})aj^{-1}(h_{(2)}).$$

Here we look at $\mathcal{A}$ as a subalgebra of P. This proves the formula (22).

2.5. *Equivariant differential bimodules.* Let $\mathcal{A}$ be a unital algebra and $\Omega^1(\mathcal{A})$ a differential bimodule over $\mathcal{A}$. Throughout this section we shall always assume that $\mathcal{A}$ is an H-module algebra.

Definition 2.21. $\Omega^1(\mathcal{A})$ is an *H-equivariant differential bimodule* over $\mathcal{A}$ if it is a differential, H-equivariant bimodule and the action of H intertwines with the $\Omega^1(\mathcal{A})$-valued derivation d:

$$h \triangleright (da) = d(h \triangleright a), \quad \forall a \in \mathcal{A}, h \in H. \tag{23}$$

The basic example of an H-equivariant differential bimodule is given by the universal differential bimodule over any H-module algebra. Indeed, $\Omega^1_U(\mathcal{A})$ is defined as a kernel of the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Therefore, since m is a bimodule morphism and, evidently, m is H-equivariant, from Proposition 2.12 we see that its kernel is H-invariant. It remains only to check the condition (23):

$$\begin{aligned} h \triangleright (da) &= h \triangleright (a \otimes 1 - 1 \otimes a) \\ &= (h \triangleright a) \otimes 1 - 1 \otimes (h \triangleright a) = d(h \triangleright a). \end{aligned} \tag{24}$$

Since, from the universality of $\Omega^1_U(\mathcal{A})$, every differential bimodule over an algebra $\mathcal{A}$ is obtained as a quotient of the universal differential bimodule, we get:

Corollary 2.22. *The differential bimodule $\Omega^1(\mathcal{A})$ is H-equivariant if and only if the submodule $\mathcal{I} \subset \Omega^1_U(\mathcal{A})$ such that $\Omega^1(\mathcal{A}) = \Omega^1_U(\mathcal{A})/\mathcal{I}$ is H-invariant, that is, $\forall h \in H$: $(h \triangleright \mathcal{I}) \subset \mathcal{I}$.*

The right equivariance is defined similarly and bicovariance (under the action of Hopf algebra) follows as the requirement that $\Omega^1(\mathcal{A})$ is an H-bimodule (like $\mathcal{A}$) and d intertwines both the left and right actions. Further, we extend the notion of equivariance to the full differential algebra:

Definition 2.23. Let $\Omega^*(\mathcal{A})$ be a graded differential algebra over $\mathcal{A}$. We say that it is *H-equivariant* if it is an H-module algebra and the action of H intertwines with the external derivative.

Again, it is an easy exercise to verify that the universal differential algebra over an H-module algebra is H-equivariant. Similarly like in the case of the differential bimodule, it follows from universality that every H-equivariant differential algebra is a quotient of the universal one by an H-invariant differential ideal.

2.6. *H-equivariant Hochschild homology.* The notion of equivariant Hochschild complex was studied by many persons. In particular, the equivariant version of Hochschild-Konstant-Rosenberg map was constructed by Block and Getzler [5]. Brylinski [6] showed that for any topological algebra and G a compact Lie group, which acts continuously on it, for the continuous Hochschild cohomology $HH^G_*(\mathcal{A}) = HH_*(\mathcal{A} \rtimes G)$.

In the group case the definition of a M-valued equivariant Hochschild complex, where M is a G-bimodule is straightforward, in fact the generalisation to the Hopf algebra case is also not difficult.

Let us recall the construction of the C^{bar}_* complex (for details, further definitions and properties of Hochschild (co)homology see, for instance, the book of Loday [30]).

Definition 2.24. Let $\mathcal{A}$ be an algebra. C^{bar}_* is the following complex:

$$\xrightarrow{b'} \cdots \mathcal{A}^{\otimes n} \xrightarrow{b'} \mathcal{A}^{\otimes (n-1)} \xrightarrow{b'} \cdots \xrightarrow{b'} \mathcal{A}^{\otimes 3} \xrightarrow{b'} \mathcal{A}^{\otimes 2},$$

where b' is a linear map on $\mathcal{A}^{\otimes n}$ defined as $b' = \sum_{i=0}^{n-1} (-1)^i \delta_i$, where:

$$\delta_i(a_0 \otimes a_1 \otimes \cdots a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) = a_0 \otimes a_1 \otimes \cdots a_i a_{i+1} \otimes \cdots \otimes a_n.$$

Proposition 2.25. *Let $\mathcal{A}$ be an H-module algebra and let us extend the action of H to the complex C^{bar}_* using (15). Then the boundary b' is an equivariant map.*

Proof. First, check that for each $0 \le i < n$, δ_i is an equivariant linear map:

$$\begin{aligned}
&\delta_i \left(h \triangleright (a_0 \otimes a_1 \otimes \cdots \otimes a_n)\right) \\
&\quad = \delta_i \left((h_{(1)} \triangleright a_0) \otimes (h_{(2)} \triangleright a_1) \otimes \cdots \otimes (h_{(n+1)} \triangleright a_n)\right) \\
&\quad = \left((h_{(1)} \triangleright a_0) \otimes \cdots \otimes (h_{(i+1)} \triangleright a_i)(h_{(i+2)} \triangleright a_{i+1}) \otimes \cdots \otimes (h_{(n+1)} \triangleright a_n)\right) \\
&\quad = \left((h_{(1)} \triangleright a_0) \otimes \cdots \otimes (h_{(i+1)} \triangleright (a_i a_{i+1}) \otimes \cdots \otimes (h_{(n)} \triangleright a_n)\right) \\
&\quad = h \triangleright (a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n) \\
&\quad = h \triangleright (\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n)).
\end{aligned}$$

So $b' = \sum_{i=0}^{n-1} (-1)^i \delta_i$ is also equivariant. ■

Note that the equivariant bar complex is still acyclic for a unital algebra $\mathcal{A}$. In fact, the operator s:

$$s(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n),$$

which gives the contracting homotopy, is again H-equivariant:

$$h \triangleright (s(a_0 \otimes a_1 \otimes \cdots \otimes a_n)) = s\left(h \triangleright (a_0 \otimes a_1 \otimes \cdots \otimes a_n)\right).$$

Therefore the equivariant C_*^{bar} complex is not an interesting object in itself. Note that a naive extension of the action onto the more interesting Hochschild complex does not work. Instead, we may turn to the Hochschild cohomology with values in an H-equivariant bimodule M.

PROPOSITION 2.26. *Take an H-equivariant bimodule M and the complex C_*^{bar} over an H-module algebra $\mathcal{A}$. Consider the linear space of all H-equivariant bimodule morphisms $\phi : C_*^{bar} \to M$, we denote by ϕ_n the restriction of ϕ to the subspace $\mathcal{A}^{\otimes(n+1)}$. Define the coboundary δ:*

$$(\delta\phi)_{n+1} = -(-1)^n \phi_n \circ b', \quad n = 1, 2, \ldots \tag{25}$$

Then the space $Hom_H(C_*^{bar}, M)$ with δ as a coboundary is a cochain complex, which we shall call an H-equivariant Hochschild cochain complex valued in M.

The subset of H-equivariant Hochschild cochains is a subcomplex of the complex of Hochschild cochains. Its cohomology is the H-equivariant Hochschild cohomology $HH_H^*(\mathcal{A}, M)$.

In practice, we shall use the equivalent description of Hochschild n-cochains as linear maps from $\mathcal{A}^{\otimes n} \to M$, with the standard formula for the coboundary. Clearly, we have some immediate follow-ups of the above proposition.

LEMMA 2.27. *$HH_H^0(\mathcal{A}, M)$ is an H-invariant subspace of central elements of M on which H acts trivially:*

$$HH_H^0(\mathcal{A}, M) = \{m \in M : h \triangleright m = \epsilon(h)m, \forall h \in H;\ am = ma, \forall a \in \mathcal{A}\}$$

LEMMA 2.28. *H-equivariant 1-cocycles are H-equivariant M-valued derivations on $\mathcal{A}$ and $HH_H^1(\mathcal{A}, M)$ are H-equivariant outer derivations.*

Proof. Indeed, the cocycle requirement for a map $D : \mathcal{A} \to M$ gives:

$$aD(b) - D(ab) + D(a)b = 0,$$

whereas the equivariance tells us that:

$$D(h \triangleright a) = h \triangleright (D(a)) \tag{26}$$

for all $a, b \in \mathcal{A}$, $h \in H$. ■

Note that (26) exactly matches the requirement for the H-equivariance of differential structures (23). Furthermore, the equivariant coboundaries are exactly inner derivations by an H-invariant element from M.

The presented definition and construction of an equivariant M-valued Hochschild cohomology is a special case. The Hochschild cohomology of an algebra $\mathcal{A}$ takes the opposite algebra $\mathcal{A}^{op}$ as the bimodule M. Since, in general, this is not an H-equivariant bimodule, we cannot define equivariant cochains.

There are several possible solutions to define equivariant Hochschild cohomology of $\mathcal{A}$. The trivial one is to use the trivial H-module structure on $\mathcal{A}^{op}$. One may also, for instance, take the bimodule of complex linear functionals on $\mathcal{A} \otimes H$, as M, with the bimodule structure:

$$(af)(b,h) = f(ba,h), \quad (fa)(b,h) = f((Sh_{(1)} \triangleright a)b, h_{(2)}). \tag{27}$$

This was first presented by Akbarpour and Khalkhali [1] and was used to extend the notion of cyclic cohomology on Hopf module algebras using the Connes-Moscovici cyclicity operator on Hopf algebras [15]. Further developments were presented recently by Khalkhali and Rangipour [27] and Hajac, Khalkhali, Rangipour and Sommerhäuser [24].

It is worth noting that the twisted version of Hochschild and cyclic cohomology (as developed by Kustermans, Murphy and Tuset [29]) is just a $\mathbb{Z}$-equivariant version of Hochschild (cyclic) cohomology, with the $\mathbb{Z}$ action given by a twisting automorphism related to the Haar measure.

2.7. *Equivariant K-theory.* In the K-theory of C^*-algebras, the notion of equivariance with respect to the action of a group is, in principle, well defined. Many results are, however, restricted to the case of finite or compact Lie groups. Basically, for commutative algebras of functions on a manifold the notion is that of a G vector bundle. This translates easily to an equivariant finitely generated projective module, or, equivalently, classes of G-invariant idempotents (see the book of Blackadar [4] for introduction).

In the case of not necessarily commutative C^*-algebras, we might consider their equivariant K-theory or the K-theory of cross product algebras. The latter is called *analytic equivariant K-theory.* In some cases they are known to be isomorphic, as shown in one of the crucial results of G-equivariant K-theory obtained by Julg [25]:

LEMMA 2.29. *For a C^*-algebra $\mathcal{A}$ and a continuous free action of a compact group G we have:*

$$K_0^G(\mathcal{A}) = K_0(\mathcal{A} \rtimes G).$$

From the algebraic point of view, K_0 of an algebra is the theory of projective modules over the algebra, hence, the step towards the definition of *H-equivariant* K-theory is, in principle, a simple one. The definition given here is similar to presented by Akbarpour and Khalkhali [1] as well as Neshveyev and Tuset [34].

DEFINITION 2.30. The *H-equivariant* $K_0^H(\mathcal{A})$ group of the H-module algebra $\mathcal{A}$ is the Grothendieck group of H-equivariant stable isomorphism classes of H-equivariant finitely generated projective left modules over $\mathcal{A}$.

Clearly this group is not empty as all free modules over H-module algebra are equivariant.

3. Equivariant Fredholm modules. The basic roots for the K-homology lie in the index theorem of Atiyah and Singer—when it was understood that the correct objects, which pair with the K-theory of topological spaces are elliptic pseudodifferential operators. This led Kasparov [26] to the abstract notion of *generalised elliptic operators*. It was developed into the theory of Fredholm modules and the K-homology and finally into KK-theory of C^*-algebras by generalising the results of Brown, Douglas and Fillmore. To fix the notation we briefly review the fundamental definitions. The basic ingredients to construct a Fredholm module are rather simple: an involutive algebra $\mathcal{A}$, its representation π on a separable Hilbert space $\mathcal{H}$ and an operator F. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of bounded operators on $\mathcal{H}$.

DEFINITION 3.1. The data $(\mathcal{A}, \pi, \mathcal{H}, F)$ is a *Fredholm module* if $F = F^*$, $F^2 = 1$ on $\mathcal{H}$ and for every $a \in \mathcal{A}$, $\pi(a) \in \mathcal{B}(\mathcal{H})$ and for every $a \in \mathcal{A}$ the commutator $[F, \pi(a)]$ is compact on $\mathcal{H}$. If there exists a grading $\gamma = \gamma^*$, $\gamma^2 = 1$ and $\gamma F = -F\gamma$ then we have an *even* Fredholm module, otherwise we have an *odd* Fredholm module.

As we have mentioned, the prototype of a Fredholm module is given by the sign of an elliptic differential operator over a compact manifold M. The Hilbert space is then the the space of summable sections of a vector bundle over M, on which the operator is acting. With an appropriate definition of the equivalence relations (which we shall not enter into) and with the natural addition we obtain the abelian groups of K^0 (in the even case) and K^1 (in the odd case) of K-homology of the underlying algebra.

The known passage from the "differential geometry" data to K-homology motivated the search for noncommutative counterparts of well known objects from classical commutative geometry. This led to the introduction of the notion of K-cycles or unbounded Fredholm modules:

DEFINITION 3.2 (Connes, [12], p. 310). An *unbounded Fredholm module (K-cycle)* is given by an involutive algebra $\mathcal{A}$, its star representation π as bounded operators on a Hilbert space and an unbounded selfadjoint operator D with compact resolvent such that $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$.

There exists a canonical assignment of a Fredholm module to the K-cycle given by $F = \mathrm{sign}(D)$ outside ker D. On the finite dimensional kernel of D one takes an arbitrary isometry (see [12], p. 310 for details).

One of the most interesting problems in noncommutative geometry is an explicit construction of the inverse map, that is, an assignment of a K-cycle to a given Fredholm module. It seems to be a rather delicate task and up to now an explicit construction is possible only in special cases. It is known, however, (even in a more general setup of KK-theory) that all the elements of K-homology arise from appropriate constructions of unbounded "K-cycles" [2].

We are now ready for the definition of H-equivariant Fredholm modules. The notion of Hopf algebra equivariance (using coaction) was already introduced some time ago by Baaj and Skandalis [3] in the setup of Kasparov KK theory. Here, we rather follow the well-established methods of G-equivariant theory. The definition of equivariant spectral triples suggested in [36] goes along the same lines. First let us define:

Definition 3.3. Let H be a Hopf algebra and $\mathcal{A}$ an H-module algebra. We call the data $(\mathcal{A}, \pi, F, \mathcal{H})$ an *equivariant Fredholm module* (respectively odd or even) if it is a Fredholm module and there exists a dense linear subspace $V \subset \mathcal{H}$ such that V is an H-equivariant $\mathcal{A}$ module, and for every $h \in H$, $[F, h] = 0$ (on the subspace V). In the even case we shall also require that H respects the grading, that is, $\gamma h = h\gamma$ on V.

Note that in case of Hopf algebras, which have bounded representations on the Hilbert space we might simply take $V = \mathcal{H}$. In such a case we shall have a bounded representation of the cross-product algebra $\mathcal{A} \rtimes H$.

The equivalence classes of equivariant Fredholm modules give rise to the respective odd (even) H-equivariant K-homology of $\mathcal{A}$.

Consider as an example the simplest case of $\mathcal{A} = \mathbb{C}$ with the unique trivial action $h \triangleright 1 = \epsilon(h)$. Then, each equivariant Fredholm module gives simply a representation of H, so the equivariant K-homology of $\mathbb{C}$ is the representation ring of H.

A nontrivial noncommutative example of an equivariant Fredholm module is given by the construction of Masuda, Nakagami and Watanabe [33] of the Fredholm module on the Podleś sphere. It is $\mathcal{U}_q(su_2)$-equivariant in the sense of Definition 3.3. We state this here without proof, which will be given elsewhere.

3.1. *Cycles and equivariant cycles.* A *cycle* is a noncommutative generalisation of the basic structure in differential geometry, given by a differential algebra and a closed, graded linear functional (integral) on it.

Definition 3.4 (Connes, see [12], p. 183). A *cycle of dimension* n over $\mathcal{A}$ is a graded differential algebra Ω with a homomorphism $i : \mathcal{A} \to \Omega^0$ and a closed graded trace $\int : \Omega \to E$, where E is a linear space.

Using cycles one might easily construct cyclic cocycles. In particular, for $E = \mathbb{C}$, the *character of the cycle*, which is defined as a linear map, $\rho : \mathcal{A}^{\otimes(n+1)} \to \mathbb{C}$,

$$\rho(a_0, \ldots, a_n) = \int i(a_0)\, d(i(a_1)) \, \ldots \, d(i(a_n)). \tag{28}$$

is a cyclic cocycle (see Connes [12], p. 186)

Given a Fredholm module (3.1) one easily defines a graded differential algebra $\Omega(\mathcal{A})$ as the subalgebra generated by operators of the type:

$$a_0[F, a_1] \cdots [F, a_n], \tag{29}$$

with the natural grading. The differential d is given naturally by the graded commutator with F. If the commutators $[F, a]$ fall into the Schatten ideal of p-summable compact operators, i.e. operators such that $\mathrm{Tr}\, |T|^p$ is finite (see [9], p. 310), one easily defines the closed graded trace of dimension $p-1$ given (in the odd case) by:

$$\int \omega = \tfrac{1}{2}\mathrm{Tr}\,(\omega + F\omega F), \tag{30}$$

where ω is a Fredholm module p-form.

Now, let us proceed with the Hopf algebra symmetries:

DEFINITION 3.5. An *H-equivariant cycle over $\mathcal{A}$* is a cycle over an H-module algebra such that $\Omega^*(\mathcal{A})$ is an H-equivariant differential algebra, the inclusion i is an equivariant map, E is an H-module and the closed graded trace (valued in E) is equivariant:

$$\int (h \triangleright \omega) = h \triangleright \int \omega, \tag{31}$$

for any $h \in H$ and $\omega \in \Omega^n$.

The character of the equivariant cycle is an invariant cyclic cocycle valued in E, where the invariance means:

$$\rho(h_{(1)} \triangleright a_0, \dots, h_{n+1} \triangleright a_n) = h \triangleright \rho(a_0, \dots, a_n). \tag{32}$$

Note that in case of a $\mathbb{C}$-valued cocycle ($E = \mathbb{C}$) the action must be trivial, so then the Hochschild cycle is in fact H-invariant.

The action of the Hopf algebra on the differential one-forms obtained from the equivariant Fredholm module is given as:

$$h \triangleright ([F, \pi(a)]) = h_{(1)}[F, \pi(a)](Sh_{(2)}),$$

where both sides are understood as operators on the dense subspace V. The right-hand side defines the adjoint action of H on the restrictions of the operators to the subspace V. Note that for the Hopf algebras, which have bounded representations on $\mathcal{H}$, one can easily observe that an equivariant Fredholm module gives rise to an equivariant cycle over $\mathcal{A}$ only if $S^2 = \mathrm{id}_H$. Only then (when $S = S^{-1}$) the tracial property of the closed graded trace defined in (30) gives the desired equivariance (31), then, of course, the cyclic cocycle obtained this way is H-invariant (32). This, however, is still far from having an equivariant version of Connes-Chern map, since neither coboundary nor cyclicity is well-defined on invariant cochains. Therefore an alternative approach was proposed recently [34].

4. Equivariant spectral triples

4.1. *Real spectral triples.* The basic data of a spectral triple is almost the same as that of a K-cycle. That was the original formulation, which appeared in earlier papers by Connes. The extra structures, which were introduced later (see the textbook [9], chapter 10 and the review [14] for a detailed account and literature, and [12] for an account of K-cycles) were motivated by the need for description of the real spin structure on commutative and noncommutative manifolds and by some arguments from theoretical physics [11].

Let us recall the definition:

DEFINITION 4.1. An *algebraic real* (*even*) *spectral triple* is given by the data $(\mathcal{A}, \pi, \mathcal{H}, D, J, \gamma)$, where $\mathcal{A}$ is an involutive algebra, π its faithful bounded star representation on a Hilbert space $\mathcal{H}$, D a selfadjoint operator with compact resolvent such that $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$, γ is a hermitian $\mathbb{Z}_2$ grading, $D\gamma = -\gamma D$, and J is an antilinear isometry such that:

$$[J\pi(a)J^{-1}, \pi(b)] = 0, \quad \forall a, b \in \mathcal{A}, \tag{33}$$

and

$$\left[J\pi(a)J^{-1}, [D, \pi(b)]\right] = 0, \quad \forall a, b \in \mathcal{A}. \tag{34}$$

The latter requirement is called the *order-one condition.* The dimension of the real spectral triple is defined as an integer n such that there exists an n-Hochschild cycle with coefficients in the bimodule $\mathcal{A} \otimes \mathcal{A}^{op}$,

$$a_0 \otimes b_0 \otimes a_1 \otimes \cdots \otimes a_n = c \in Z_0(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op}),$$

for which

$$\pi(c) = \pi(a_0)\left(J\pi(b_0)J^{-1}\right)[D, \pi(a_1)] \cdots [D, \pi(a_n)] = \gamma. \tag{35}$$

Moreover, one assumes further relations:

$$DJ = \epsilon JD, \quad J^2 = \epsilon', \quad J\gamma = \epsilon''\gamma J. \tag{36}$$

where $\epsilon, \epsilon', \epsilon''$ are ± 1 depending on n modulo 8 according to the following rules:

n mod 8	0	1	2	3	4	5	6	7
ϵ	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ'	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ''	$+$		$-$		$+$		$-$	

If we do not assume existence of J, we have a spectral triple without real structure.

We restrict ourselves only to the algebraic requirements and we refer the reader to the textbook [9] for details on further analytic requirements like regularity, summability, finiteness conditions as well as the Poincaré duality.

It is reasonable to assume always that the subalgebra of elements of $\mathcal{A}$ which commute with D is $\mathbb{C}$ (in case of a unital $\mathcal{A}$). Otherwise the differential algebra defined by D will be degenerate, that is, there will be a nontrivial kernel of d in $\mathcal{A}$. Spectral triples such that $[D, \pi(a)] = 0$ implies $a \in \mathbb{C}$ will be called *non-degenerate.*

4.2. *Equivariance.* In this section we shall review the applications of Hopf algebra symmetries to spectral triples. The definition and results are based on ideas presented in the papers [36] and [40].

Our aim is to regard isometries of spectral triples in the same sense as one takes the isometries of manifolds. Basically, each real (algebraic) spectral triple defines a fundamental class in K-homology. What we want to study are the noncommutative symmetries and respective equivariant representatives of this K-homology class.

DEFINITION 4.2. Suppose that we have an algebraic spectral triple (as defined earlier) over an H-module algebra $\mathcal{A}$. We say that the *spectral triple* $(\mathcal{A}, \pi, \mathcal{H}, D)$ *is H-equivariant* if there is a dense subspace $V \subset \mathcal{H}$ such that V is an H-equivariant $\mathcal{A}$-module, i.e.:

$$h(\pi(a)v) = \pi(h_{(1)} \triangleright a)h_{(2)}v, \quad \forall a \in \mathcal{A}, h \in H, v \in V,$$

and for every $h \in H$ the Dirac operator is equivariant, i.e. $[D, h] = 0$, on an intersection of V and the domain of D, which is dense in $\mathcal{H}$. If the triple is even, then we require that $[\gamma, h] = 0$. H will be called the *isometry* of the spectral triple.

The easiest example of a Hopf algebra acting on the algebra $\mathcal{A}$ is the group algebra $\mathbb{C}\mathcal{U}(\mathcal{A})$ of the unitary group of $\mathcal{A}$, with the adjoint action (6). However, it is clear that there is no non-degenerate spectral triple for which this would be an isometry.

We define the bimodule $\Omega^1(\mathcal{A})$ as the linear space spanned by all operators of $\pi(a)[D, \pi(b)]$, $a, b \in \mathcal{A}$, with the bimodule structure given by operator multiplication. The elements $[D, a]$ are denoted as da.

LEMMA 4.3. *Let $(\mathcal{A}, \mathcal{H}, D)$ be an H-equivariant spectral triple. Then the first order differential bimodule is an H-equivariant differential bimodule.*

Proof. Let us simply set $(h \triangleright da) = [D, (h \triangleright \pi(a))]$ and extend it on the entire bimodule of one-forms through:

$$h \triangleright (adb) = (h_{(1)} \triangleright a) d(h_{(2)} \triangleright b), \tag{37}$$

It is an easy exercise to check that the definition is correct. First, assume that for some $a_i, b_i \in \mathcal{A}$ we have $\sum_i a_i\, db_i = 0$. Calculate, for any $h \in H$ $\pi(h \triangleright (\sum_i a_i\, db_i))v$ for $v \in V$:

$$\begin{aligned}
\pi\Big(h \triangleright \Big(\sum_i a_i\, db_i\Big)\Big) v &= \sum_i \pi(h_{(1)} \triangleright a_i)[D, \pi(h_{(2)} \triangleright b_i)]v \\
&= \sum_i \pi(h_{(1)} \triangleright a_i)[D, h_{(2)}\pi(b_i)(Sh_{(3)})]v \\
&= \sum_i h_{(1)}\pi(a_i)(Sh_{(2)})h_{(3)}[D, \pi(b_i)](Sh_{(4)})v \\
&= h_{(1)}\Big(\sum_i \pi(a_i)[D, \pi(b_i)]\Big)(Sh_{(2)})v = 0.
\end{aligned}$$

It is obvious that d is an equivariant map and to see that $\Omega^1(\mathcal{A})$ is an equivariant bimodule it is sufficient to use the Leibniz rule:

$$a_1\, db\, a_2 = a_1\, \left(d(ba_2) - b\, da_2\right).$$

Then using the definition (37) we get:

$$\begin{aligned}
h \triangleright (a_1\, db\, a_2) &= (h_{(1)} \triangleright a_1)(d((h_{(2)} \triangleright b)(h_{(3)} \triangleright a_2)) - (h_{(2)} \triangleright b) d(h_{(3)} \triangleright a_2)) \\
&= (h_{(1)} \triangleright a_1)(d(h_{(2)} \triangleright b))(h_{(3)} \triangleright a_2). \quad \blacksquare
\end{aligned}$$

Next, we extend the equivariance property to the case of real spectral triples.

DEFINITION 4.4. The algebraic spectral triple $(\mathcal{A}, \pi, \mathcal{H}, D, J, \gamma)$ is a *real H-equivariant algebraic spectral triple* if H is a Hopf algebra with an involution and $\mathcal{A}$ is an involutive H-module algebra (in the sense of (21)), $(\mathcal{A}, \pi, \mathcal{H}, D, \gamma)$ is an H-equivariant spectral triple, and:

$$J^{-1} h J = (Sh)^*. \tag{38}$$

on a dense subspace $V \subset \mathcal{H}$.

To see that the definition is self-consistent let us verify that we have a well-defined action of H^{cop} on the opposite algebra $\mathcal{A}^{op}$. Using the faithful representation π and the reality operator J we can identify $\mathcal{A}^{op}$ with $J\pi(\mathcal{A})J^{-1}$. Indeed, $a \mapsto J\pi(a^*)J^{-1}$ is a real representation of $\mathcal{A}^{op}$:

$$(J\pi(a)^*J^{-1})(J\pi(b)^*J^{-1}) = J(ba)^*J^{-1}, \quad \forall a, b \in \mathcal{A}.$$

We check the equivariance:

$$\begin{aligned} J^{-1}\pi(h \triangleright b)^* J &= J^{-1}(h_{(1)}\pi(b)(Sh_{(2)}))^* J \\ &= J^{-1}(Sh_{(2)})^*\pi(b)^* h^*_{(1)} J \\ &= \left(J^{-1}(Sh_{(2)})^* J\right)\left(J^{-1}\pi(b)^* J\right) J^{-1}(h^*_{(1)} J) \\ &= h_{(2)}(J^{-1}\pi(b)^* J)(Sh^*_{(1)})^* \\ &= h_{(2)}\pi(b^{op})(S^{-1}h_{(1)}), \end{aligned}$$

so $(h \triangleright b)^{op} = (h \triangleright' b^{op})$, where $\triangleright'$ is the action of H on the operators (acting on the dense subspace V) defined as $O \mapsto h_{(2)}O(S^{-1}h_{(1)})$. Since S^{-1} is the antipode for H with the coopposite algebra structure (see p.9, [31]) we obtain the desired result. Note that if one restricts oneself to cocommutative algebras, then there is no difference between these two actions.

Finally, observe that if $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is an H-equivariant spectral triple then the data $(\mathcal{A}, \mathcal{H}, \operatorname{sign}(D), \gamma)$ gives an H-equivariant Fredholm module. Since both 1 and γ are evidently H-invariant (by definition), the image of the Hochschild cycle $c \in Z_n(\mathcal{A} \otimes \mathcal{A}^{op}, \mathcal{A})$ is H-invariant (that is, it commutes with H). Note that this does not mean that the cycle itself is equivariant.

5. Examples of isometries of spectral triples. Before we present examples which will illustrate the notion of equivariant spectral triples, let us mention what are the advantages of this notion. Using spectral properties of Hopf algebra representations (similarly to the case of groups) we might easily compute simple examples of spectral triples. In particular, we shall use the isometries to calculate the spectrum of the (equivariant) Dirac operator.

There are three basic categories of examples: finite spectral triples, isospectral deformations, in particular the noncommutative torus, and a generalisation of the latter.

5.1. *Quantum symmetries of the noncommutative torus.* The spectral triple of the noncommutative torus is the best known noncommutative object. We shall show that this (real) spectral triple is equivariant and investigate its slightly bigger quantum symmetry from the point of view of equivariance of spectral triples.

Let us briefly recall the definition of the noncommutative torus, for details we refer to the book of Connes [12].

DEFINITION 5.1. Consider the Hilbert space $l^2(\mathbb{Z}^2)$ with the orthonormal basis $\{|n, m\rangle, n, m \in \mathbb{Z}\}$ and the unitary operators:

$$\begin{aligned} U|n, m\rangle &= |n + 1, m\rangle, \\ V|n, m\rangle &= \lambda^{-n}|n, m + 1\rangle, \end{aligned}$$

where λ is a complex number with $|\lambda| = 1$. The algebra generated by these operators will be called the algebra of functions on the noncommutative torus.

Note that we defined so far the algebra of polynomials and we might complete it either to a Fréchet algebra or a C^*-algebra.

PROPOSITION 5.2. *Let $u(1) \times u(1)$ be the universal enveloping algebra of the Lie algebra generated by two derivations on the noncommutative torus:*

$$\delta_1 \triangleright U = U, \qquad \delta_2 \triangleright U = 0,$$
$$\delta_1 \triangleright V = 0, \qquad \delta_2 \triangleright V = V.$$

Then, with the representation:

$$\delta_1 |n, m\rangle = n|n, m\rangle,$$
$$\delta_2 |n, m\rangle = m|n, m\rangle,$$

we have the representation of the cross-product algebra of the functions on the noncommutative torus by the symmetry algebra. Here, we take as the dense subspace V the linear space spanned by the basis $|n, m\rangle$, $n, m \in \mathbb{Z}$.

To construct the real spectral triple we need a grading γ (which just doubles the Hilbert space) and the antilinear isometry J, which we recover from the Tomita-Takesaki theory:

$$J_0 |n, m\rangle = \lambda^{-nm} | -n, -m\rangle.$$

To obtain J, which satisfies the algebraic requirements of the real spectral triple, we still use this Hilbert space but we have to tensor J_0 with a suitable matrix from $M_2(\mathbb{C})$. Taking γ to be block diagonal we have:

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

Being selfadjoint and anticommuting with γ, any operator D must be of the form

$$D = \begin{pmatrix} 0 & \partial \\ \partial^* & 0 \end{pmatrix}.$$

PROPOSITION 5.3. *Every Dirac operator D, for which $u(1) \times u(1)$ is an isometry must be of the form given above, with ∂:*

$$\partial |n, m, -\rangle = d_{n,m} |n, m, +\rangle, n, m \in \mathbb{Z}.$$

This follows directly from the requirement $[D, \delta_i] = 0$, $i = 1, 2$. As we shall see, this assumption, together with other algebraic requirements fixes ∂ up to a normalisation factor.

LEMMA 5.4. *Any Dirac operator D, which has $u(1) \times u(1)$ as an isometry and which is order-one (see (34)), is defined by the set of complex coefficients $d_{n,m}$ obeying the following recursion relations:*

$$d_{n+2,m} = 2d_{n+1,m} - d_{n,m},$$
$$d_{n,m+2} = 2d_{n,m+1} - d_{n,m}.$$

The solution (up to a multiplicative constant) is:

$$d_{n,m} = n + m\tau, \qquad \tau \in \mathbb{C}.$$

Proof. Direct calculation of the order-one condition. ∎

COROLLARY 5.5. *The above-defined Dirac operator gives a real spectral triple over the noncommutative torus: for the proof of the smoothness and finiteness properties see [13] or [9] p. 540.*

The result obtained gives the "standard" Dirac operator on the noncommutative torus. One should stress that as compared to usual constructions, we have derived it here from the order-one condition using additionally the principle of equivariance.

This is, however, not the end of the story. One can find a bigger symmetry of the noncommutative torus and, in fact, attempt to construct the corresponding equivariant spectral triple. The only difference is that we have to employ multiplier Hopf algebras instead of Hopf algebras. Recall that a multiplier Hopf algebra is a nonunital algebra with a coproduct valued in the multiplier of the tensor square of the algebra, satisfying certain conditions (for precise definition and properties see [19, 20])

Let us consider a nonunital algebra $\mathcal{U}_2$ (over complex numbers) with the basis $P^{i,k}$, $R^{j,l}$, $i,j,k,l \in \mathbb{Z}$ and relations:

$$\begin{aligned} P^{i,k}P^{j,l} &= \delta^{ij}\delta^{kl}P^{i,k}, \\ P^{i,k}R^{j,l} &= \delta^{ij}\delta^{kl}R^{i,k}, \\ R^{i,k}P^{j,l} &= \delta^{il}\delta^{kj}R^{i,k}, \\ R^{i,k}R^{j,l} &= \delta^{il}\delta^{kj}P^{i,k}. \end{aligned} \tag{39}$$

We now state:

PROPOSITION 5.6. *The algebra $\mathcal{U}_2$ is a multiplier Hopf algebra with the coproduct, counit and the antipode given by:*

$$\begin{aligned} \Delta P^{i,k} &= \sum_{j,l\in\mathbb{Z}} P^{(i-j),(k-l)} \otimes P^{j,l}, \\ \Delta R^{i,k} &= \sum_{j,l\in\mathbb{Z}} e^{\pi i\theta(kj-il)} R^{(i-j),(k-l)} \otimes R^{j,l}, \end{aligned} \tag{40}$$

$$\epsilon(P^{i,j}) = \delta^{i0}\delta^{j0}, \quad \epsilon(R^{i,j}) = \delta^{i0}\delta^{j0}, \quad S(P^{i,j}) = P^{-i,-j}, \quad S(R^{i,j}) = R^{-j,-i}.$$

for all $i,k \in \mathbb{Z}$ and where $0 \le \theta < 1$ is a real parameter. We define: $q = e^{2\pi i\theta}$.

Proof. It is easy to see that the product in $\mathcal{U}_2$ is non-degenerate, it is also an easy technical exercise to check the coassociativity of the coproduct (in the sense of multiplier Hopf algebras) and the axioms concerning the counit and the antipode. For example, to check the coassociativity, one must verify that for every $a,b,c \in \mathcal{U}_2$,

$$(a\otimes 1\otimes 1)(\Delta\otimes \mathrm{id})(\Delta(b)(1\otimes c)) = (\mathrm{id}\otimes\Delta)((a\otimes 1)\Delta(b))(1\otimes 1\otimes c).$$

Taking $a = P^{i,k}$, $b = R^{j,l}$, $c = P^{m,n}$ we have on the left-hand side:

$$\begin{aligned} &(P^{i,k}\otimes 1\otimes 1)(\Delta\otimes\mathrm{id})(e^{\pi i\theta(lm-jn)}R^{(j-m),(l-n)}\otimes R^{m,n}) \\ &\quad = e^{\pi i\theta(lm-jn)}e^{\pi i\theta(k(j-m-i)-i(l-n-k))}R^{i,k}\otimes R^{(j-m-i),(l-n-k)}\otimes R^{m,n} \end{aligned}$$

$$= e^{\pi i\theta(lm-jn+kj-km-il+in)} R^{i,k} \otimes R^{(j-m-i),(l-n-k)} \otimes R^{m,n},$$

whereas on the right-hand side:

$$\begin{aligned}
&(\mathrm{id}\otimes\Delta)(e^{\pi i\theta(kj-il)} R^{i,k} \otimes R^{(j-i),(l-k)})(1\otimes 1\otimes P^{m,n}) \\
&\quad = e^{\pi i\theta(kj-il)} e^{\pi i\theta(m(l-n-k)-n(j-m-i)} R^{i,k} \otimes R^{(j-m-i),(l-n-k)} \otimes R^{m,n} \\
&\quad = e^{\pi i\theta(kj+ml-mk-il+in-nj)} R^{i,k} \otimes R^{(j-m-i),(l-n-k)} \otimes R^{m,n},
\end{aligned}$$

and we can clearly see that they are the same.

We shall verify here the remaining condition, which is that the maps:

$$T_1(a\otimes b) = \Delta(a)(1\otimes b), \quad T_2(a\otimes b) = (a\otimes 1)\Delta(b),$$

are one-to-one and have their range equal to $\mathcal{U}_2 \otimes \mathcal{U}_2$.

We calculate explicitly on the basis:

$$\begin{aligned}
T_1(P^{i,k}\otimes P^{j,l}) &= P^{(i-j),(k-l)} \otimes P^{j,l}, \\
T_1(P^{i,k}\otimes R^{j,l}) &= P^{(i-j),(k-l)} \otimes R^{j,l}, \\
T_1(R^{i,k}\otimes P^{j,l}) &= e^{\pi i\theta(kj-il)} R^{(i-j),(k-l)} \otimes R^{j,l}, \\
T_1(R^{i,k}\otimes R^{j,l}) &= e^{\pi i\theta(kj-il)} R^{(i-j),(k-l)} \otimes P^{j,l}, \\
T_2(P^{i,k}\otimes P^{j,l}) &= P^{i,k} \otimes P^{(j-i),(l-k)}, \\
T_2(P^{i,k}\otimes R^{j,l}) &= e^{\pi i\theta(kj-il)} R^{i,k} \otimes R^{(j-i),(l-k)}, \\
T_2(R^{i,k}\otimes P^{j,l}) &= R^{i,k} \otimes P^{(j-i),(l-k)}, \\
T_2(R^{i,k}\otimes R^{j,l}) &= e^{\pi i\theta(kj-il)} P^{i,k} \otimes R^{(j-i),(l-k)},
\end{aligned}$$

and it becomes evident that the range of both maps is in $\mathcal{U}_2 \otimes \mathcal{U}_2$ and that both maps are one-to-one. ∎

For other details on this algebra see [21], where it is derived as the dual to the quantum double-torus (the latter constructed in [23]). Note that there is no problem in defining the action of this multiplier Hopf algebra on polynomials in U, V from the algebra of functions on the noncommutative torus: Indeed, we have:

$$\begin{aligned}
P^{i,k} \triangleright U &= \delta^{i1}\delta^{k0}\, U, & R^{i,k} \triangleright U &= \delta^{i0}\delta^{k1}\, V, \\
P^{i,k} \triangleright V &= \delta^{i0}\delta^{k1}\, V, & R^{i,k} \triangleright V &= \delta^{i1}\delta^{k0}\, U.
\end{aligned} \tag{41}$$

for all $i, k \in \mathbb{Z}$.

We check explicitly that the above defined action is compatible with the relation $UV = \lambda VU$. For any $i, k \in \mathbb{Z}$:

$$P^{i,k} \triangleright (UV - \lambda VU) = \delta^{i1}\delta^{k1}(UV) - \lambda\delta^{i1}\delta^{k1}(VU) = 0,$$

and:

$$\begin{aligned}
R^{i,k} \triangleright (UV - \lambda VU) &= e^{\pi i\theta}\delta^{i1}\delta^{k1}(VU) - \lambda e^{-\pi i\theta}\delta^{i1}\delta^{k1}(UV) \\
&= \delta^{i1}\delta^{k1}(e^{\pi i\theta} VU - e^{\pi i\theta}\lambda UV) = \delta^{i1}\delta^{k1} e^{-\pi i\theta} VU(e^{2\pi i\theta} - \lambda^2).
\end{aligned}$$

so, the necessary and sufficient condition for the action (41) to be well-defined on the algebra of polynomials on the noncommutative torus is that $e^{2\pi i\theta} = \lambda^2$.

It is easy to find a bounded representation of the defined multiplier Hopf algebra on the Hilbert space on which we had the representation of the noncommutative torus. It is sufficient to have:

$$P^{i,k}|n,m\rangle = \delta^{in}\delta^{km}|n,m\rangle, \qquad (42)$$
$$R^{i,k}|n,m\rangle = \delta^{im}\delta^{kn}|m,n\rangle,$$

for any $i,k,m,n \in \mathbb{Z}$.

Before we proceed, we need to define an H-module algebra $\mathcal{A}$ over a multiplier Hopf algebra H.

DEFINITION 5.7. If H is a multiplier Hopf algebra, we say that $\mathcal{A}$ is an *H-module algebra* if it is a module over the multiplier algebra $M(H)$ and for all $a,b \in \mathcal{A}$ and $h \in H$ we have

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b).$$

Using this we extend the definition of equivariant spectral triples 4.2 to include multiplier Hopf algebras as symmetries just by replacing *Hopf algebra* with *multiplier Hopf algebra.*

Then we can observe that the spectral data of the noncommutative torus is equivariant with respect to the action of the multiplier Hopf algebra of the dual quantum double-torus $\mathcal{U}_2$ (40-41), if the eigenvalues of the operator ∂ (which defines the Dirac operator) are (up to a multiplicative constant):

$$d_{n,m} = n + m. \qquad (43)$$

However, such data does not define a real two-dimensional spectral triple, as the dimension axiom (existence of a Hochschild cycle) cannot be fulfilled. First, observe that when working within the framework of multiplier algebras, we may easily identify a subalgebra of the multiplier of $\mathcal{U}_2$ as $u(1) \times u(1)$ Lie algebra by taking

$$\delta_1 = \sum_{n,m\in\mathbb{Z}} nP^{n,m}, \;\; \delta_2 = \sum_{n,m\in\mathbb{Z}} mP^{n,m}. \qquad (44)$$

This allows us to use some results of Lemma 5.4, in particular the already calculated form of the equivariant Dirac operator. It is easy to see that for such a Dirac operator, the requirement that it commutes with all $R^{i,k}$, $i,k \in \mathbb{Z}$, gives for the operator ∂ (as defined in Proposition 5.3):

$$R^{i,k}\partial|n,m,-\rangle = (n+m\tau)\delta^{im}\delta^{kn}|m,n,+\rangle,$$
$$\partial R^{i,k}|n,m,-\rangle = (m+n\tau)\delta^{im}\delta^{kn}|m,n,+\rangle.$$

Hence τ must be 1.

Now, if $d_{n+m} = n+m$ we can check that the representation images of one-forms $U^{-1}[D,U]$ and $V^{-1}[D,V]$ are equal to each other and their square is 1. Therefore any expression of the form $\pi(a_0)J\pi(b_0)J^{-1}[D,\pi(a_1)][D,\pi(a_2)]$ with a_0,a_1,a_2,a_3 from the algebra of functions on the noncommutative torus cannot be γ, since both the representation of the functions on the noncommutative torus as well as the opposite algebra are diagonal (that is, they are identical on both copies of the Hibert space).

In fact, we have shown that the requirement of a bigger symmetry is incompatible with the assumed form of the spectral triple of dimension 2. We thus conclude:

COROLLARY 5.8. *The multiplier Hopf algebra $\mathcal{U}_2$ cannot be an isometry of the noncommutative torus.*

5.2. *Isometries of isospectral deformations.* The isospectral deformation has appeared originally in the construction of the examples of noncommutative 4-spheres, which have the same instanton bundle as the "classical" sphere. One of the initial questions posed by the construction was whether the constructed *spheres* are still symmetric, i.e. whether the natural $SO(5)$ symmetries (or, respectively, $SO(4)$ for the 3-sphere) are preserved (in the form of Hopf algebras) and whether the constructed spectral triples are equivariant.

The answer about symmetries was provided independently by Varilly [41], Sitarz [40] and Connes, Dubois-Violette [17]. In the first and last approaches the group point view was taken, whereas our approach was focused on the dual version of symmetries (given by actions and not coactions) and the symmetry was described in terms of the deformation of the universal enveloping algebra (cocycle twists) acting on the deformation of the algebra of functions on the sphere.

Let us state here the main result:

THEOREM 5.9. *Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma)_H$ be the data defining an equivariant real spectral triple (of dimension n). Let Ψ (called a cocycle) be an invertible element in $H \otimes H$ such that:*

$$\Psi_{12}(\Delta \otimes \mathrm{id})\Psi = \Psi_{23}(\mathrm{id} \otimes \Delta)\Psi, \tag{45}$$

$$(\epsilon \otimes \mathrm{id})\Psi = 1 = (\mathrm{id} \otimes \epsilon)\Psi, \tag{46}$$

and Ψ acts naturally on $\mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A} \otimes V$, where V is the dense subspace of $\mathcal{H}$. Then the spectral triple $(\mathcal{A}_\Psi, \mathcal{H}, D, J_\Psi, \gamma)$ is equivariant with respect to the Hopf algebra H_Ψ, where H_Ψ is the Drinfeld twist of the Hopf algebra H.

For more details on Drinfeld twist we refer to [10], pp. 130. Before we proceed with the sketch of the proof and the corollaries let us recall the definition of $\mathcal{A}_\Psi$ and its representation.

DEFINITION 5.10. The algebra $\mathcal{A}_\Psi$ is a vector space $\mathcal{A}$ with the product $m_\psi : \mathcal{A}_\Psi \otimes \mathcal{A}_\Psi \to \mathcal{A}_\Psi$

$$m_\Psi\left(i_\Psi(a) \otimes i_\Psi(b)\right) = i_\Psi\left(m\left(\Psi \triangleright (a \otimes b)\right)\right), \tag{47}$$

for any $a, b \in \mathcal{A}$. We denote the identity map $\mathcal{A} \to \mathcal{A}_\Psi$ as i_Ψ, m denotes the standard product on $\mathcal{A}$.

PROPOSITION 5.11. *With the assumption as in the proposition above, the map:*

$$m^\pi_\Psi : (i_\Psi(a), v) \to m^\pi\left(\Psi \triangleright (a \otimes v)\right),$$

$a \in \mathcal{A}, v \in V$, defines a representation of $\mathcal{A}_\Psi$ on V.

In this proposition the action of Ψ on $a \otimes v$ is through the action of H on $\mathcal{A}$ in the first leg and the H-module structure of V in the second leg. The map m^π is $m^\pi(a, v) = \pi(a)v$.

We have:

LEMMA 5.12. *The algebra $\mathcal{A}_\Psi$ is an H_Ψ-module algebra and the representation m^π_Ψ of $\mathcal{A}_\Psi$ defined above is H_Ψ-equivariant.*

Proof. Of course, we work on the dense subspace V. The coproduct in H_Ψ is given by:

$$\Delta_\psi h = \Psi^{-1}\Delta h\Psi, \quad h \in H_\Psi. \tag{48}$$

The action of H_Ψ on $\mathcal{A}_\Psi$ is defined as:

$$h \triangleright_\Psi i_\Psi(a) = i_\Psi(h \triangleright a), \quad h \in H_\Psi, a \in \mathcal{A}. \tag{49}$$

Then $\mathcal{A}_\Psi$ is still an H_Ψ-module (since H and H_Ψ are isomorphic as algebras), it is also an H_Ψ-module algebra because:

$$\begin{aligned} h \triangleright (i_\Psi(a)i_\Psi(b)) &= i_\Psi\left(h \triangleright m\left(\Psi \triangleright (a \otimes b)\right)\right) \\ &= i_\Psi\left(m\left(((\Delta h)\Psi) \triangleright (a \otimes b)\right)\right) \\ &= i_\Psi\left(m\left((\Psi\Psi^{-1}(\Delta h)\Psi) \triangleright (a \otimes b)\right)\right) \\ &= i_\Psi\left(m\left(\Psi \triangleright (\Delta_\Psi h) \triangleright (a \otimes b)\right)\right) \\ &= m_\Psi\left((\Delta_\Psi h) \triangleright (a \otimes b)\right). \end{aligned}$$

Similarly one verifies that the representation from Lemma 5.11 is equivariant. ■

We shall need also the star structure on the algebra $\mathcal{A}_\Psi$. It is not difficult to observe that if H is a star Hopf algebra then the twisted algebra H_Ψ is also a star Hopf algebra provided that $\Psi^* = \Psi^{-1}$. The following lemma shows that there exists a star structure on $\mathcal{A}_\Psi$ compatible with the action of H_Ψ.

LEMMA 5.13. *Let H be a Hopf star algebra, $\mathcal{A}$ an H-module algebra with a star structure, Ψ a cocycle satisfying $\Psi^* = \Psi^{-1}$, H_Ψ the twisted Hopf algebra and $\mathcal{A}_\Psi$ the twisted H_Ψ-module algebra. Let $u = (\Psi^{-1})_1(S(\Psi^{-1})_2)$, where we use the notation $\Psi = \Psi_1 \otimes \Psi_2$ (and similarly for its inverse; remember, however, that neither is necessarily a simple tensor). Then*

$$(i_\Psi(a))^\star = i_\Psi((u^* \triangleright a)^*)$$

gives a star structure on $\mathcal{A}_\Psi$, compatible with the H_Ψ action.

We use here $\star$ to denote the star structure on $\mathcal{A}_\Psi$ to distinguish it from the star $(*)$ on $\mathcal{A}$.

Proof. First, let us check that the defined star is an antihomomorphism (it is clear that it is antilinear).

$$\begin{aligned} (i_\Psi(a)i_\Psi(b))^\star &= i_\Psi(u^* \triangleright ((\Psi_1 \triangleright a)(\Psi_2 \triangleright b)))^* \\ &= (u^{-1})^* \triangleright ((\Psi_2 \triangleright b)^*(\Psi_1 \triangleright a)^*) \\ &= m(\Delta(u^{-1})^* \triangleright ((S\Psi_2)^* \otimes (S\Psi_1)^*) \triangleright (b^* \otimes a^*)) \\ &= m(\Delta(u^{-1})^*\Delta(u^*)\Psi((u^{-1})^* \otimes (u^{-1})^*) \triangleright (b^* \otimes a^*)) \\ &= m(\Psi \triangleright ((u^{-1})^* \otimes (u^{-1})^*) \triangleright (b^* \otimes a^*)) \\ &= m_\Psi(((u^{-1})^* \otimes (u^{-1})^*) \triangleright (b^* \otimes a^*)) \\ &= m_\Psi((i_\Psi(b))^\star \otimes (i_\Psi(a))^\star) = (i_\Psi(b))^\star(i_\Psi(a))^\star, \end{aligned}$$

where we have used $S(u^*) = u^{-1}$ and the following identity (after taking $*$ of both sides):

$$(u \otimes u)(S\Psi_2 \otimes S\Psi_1) = \Psi^{-1}(\Delta(u)).$$

The relations and identities, quoted here, can be found in [31], pp. 58–60.

Finally, let us check that $\star \circ \star$ is identity:

$$\begin{aligned}((i_\Psi(a))^\star)^\star &= i_\Psi(u^* \triangleright (i_\Psi(a)^\star))^* \\ &= i_\Psi(u^* \triangleright (u^* \triangleright a)^*)^* = i_\Psi((u^{-1})^* \triangleright (u^* \triangleright a)) \\ &= i_\Psi((u^{-1})^* u^* \triangleright a)) = i_\Psi(1 \triangleright a) = i_\Psi(a),\end{aligned}$$

where we have again used that $S(u^*) = u^{-1}$. ■

To have the full algebraic picture of a real spectral triple we need to define J and γ. Leaving the grading γ (in the even case, of course) intact, we set:

$$J_\Psi = J u^*. \tag{50}$$

It is easy to check that this agrees with the requirement of equivariance (38), as for any $h \in H$ we have:

$$\begin{aligned}(J_\Psi)^{-1} h (J_\Psi) = (J u^*)^{-1} h (J u^*) &= (u^{-1})^* (J^{-1} h J) u^* \\ &= (u^{-1})^* (Sh)^* u^* = (u S(h) u^{-1})^* = (S_\Psi h)^*.\end{aligned}$$

We have used here the definition of the twisted antipode: $S_\Psi(h) = u S(h) u^{-1}$.

The dimension condition, that is, the existence of a certain Hochschild cycle, is a more involved problem and we postpone the analysis of the behaviour of Hochschild homology under deformation induced by twisting to future research. Although it is known that Hochschild homology of both algebras is different (e.g. take the noncommutative torus), one may conjecture that the top dimension does not change. However, at the moment we need to assume that the cycle of the initial data is Ψ-invariant, in other words that the iterated action of Ψ on c does not change c and it still remains a Hochschild cycle over $\mathcal{A}_\Psi$. This is, for example, the case in the later discussed situation of isospectral deformations based on a twist defined over a Cartan subalgebra. Then, since D commutes with the action of H, we get that the image of c is again γ or 1 (for even and odd dimensions, respectively).

Note that apart from the dimension axiom (existence of an appropriate Hochschild cocycle) and assuming that the action of the Hopf algebra and the cocycle Ψ is well-defined, we have shown that our construction shall always give a deformed spectral triple with the same Dirac operator, thus it justifies the name of "isospectral" deformations.

Let us consider a concrete example, with the cocycle construction based on the Cartan subalgebra. Let H be a Hopf algebra which is an isometry of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and let δ_1, δ_2 be the elements from the Cartan subalgebra of $\mathcal{C} \subset H$ (we assume that it is at least of dimension 2), $\Delta\delta_i = 1 \otimes \delta_i + \delta_i \otimes 1$, $i = 1, 2$.

We take Ψ_c to be a cocycle taken from the multiplier of the Hopf algebra, which is closely related to this Cartan algebra. To be more precise, this is a multiplier Hopf algebra which is a subalgebra of the one considered in Proposition 5.6. We take its subalgebra which is the subspace with the basis $\{P^{i,j}, i, j \in \mathbb{Z}\}$. It is a multiplier Hopf algebra such that the generators of the Cartan subalgebra $\mathcal{C}$ can be identified as the elements in its multiplier (see 44). We set:

$$\Psi = \sum_{m,n,k,l \in \mathbb{Z}} \lambda^{-mn} P^{m,k} \otimes P^{l,n}, \tag{51}$$

where λ is a complex number of module 1. Note that identifying δ_1, δ_2 as the elements in the multiplier we can write

$$\Psi_c = \lambda^{-\delta_1 \otimes \delta_2}, \tag{52}$$

where the expression has a well-defined meaning in the multiplier of the tensor square of the nonunital algebra generated by the elements $P^{m,n}$. Indeed, taking into account (44) and setting $\lambda = e^{2\pi i\theta}$ we have:

$$\begin{aligned}\lambda^{-\delta_1 \otimes \delta_2}(P^{m,k} \otimes P^{l,n}) &= \sum_{s=0}^{\infty} \frac{(2\pi i\theta)^s(-1)^s m^s n^s}{s!}(P^{m,k} \otimes P^{l,n}) \\ &= \lambda^{-mn}(P^{m,k} \otimes P^{l,n}).\end{aligned}$$

Although the action of δ_1, δ_2 and their powers is well defined on $\mathcal{A}$ (for example, $\mathcal{A}$ is an algebra of smooth functions on a compact manifold and each δ_i is a generator of the action of a circle) we must show that the twisting by the action of the cocycle defined above is well-defined. Only then the deformed product m_{Ψ_c} and the deformed representation $m^{\pi}_{\Psi_c}$ can make sense. We shall show that it is possible to take a smaller algebra (and a smaller subspace of $\mathcal{H}$) for which this is always possible.

The resulting twisted algebra $\mathcal{A}_{\Psi_c}$ in this case (we shall call it $\mathcal{A}_\lambda$) is an isospectral deformation of $\mathcal{A}$, the latter as defined by Connes-Landi in [16]. Moreover the deformed data is a real spectral triple with the same Dirac operator.

First, notice that the cocycle action is well-defined for certain elements of $\mathcal{A}$ and $\mathcal{H}$. Indeed, if a, b are homogeneous elements of $\mathcal{A}$ with respect to the action of δ_1 and δ_2, respectively, that is, there exist numbers n, m such that:

$$\delta_1 \triangleright a = na, \qquad \delta_2 \triangleright b = mb,$$

then the twisted product m_{Ψ_c} for them simply becomes:

$$m_{\Psi_c}(a, b) = \lambda^{-nm} i_{\Psi_c}(ab).$$

Similar argument works for the representation of a homogeneous element of $\mathcal{A}$ on a homogeneous element of a dense subspace of $\mathcal{H}$. Next, taking the subalgebra of $\mathcal{A}$ generated by such homogeneous elements and restricting appropriately the dense subspace of $\mathcal{H}$ to the linear span of homogeneous elements is sufficient for the consistent definition of the twisted algebra and representation. One would like it still to be a dense subspace of the Hilbert space in order that the equivariance condition is satsfied on a dense subspace. In the case of $\mathcal{A}$ being smooth functions on a manifold the proof is in [16].

For this version of the isospectral deformation it is relatively easy to get the desired Hochschild cycle c. One might proceed as follows: using the assumption that the action of δ_i comes from the circle actions, one averages c over this action to get a cycle, which shall be Ψ_c invariant.

Remark 5.14. The noncommutative torus is an example of an isospectral deformation of a torus and its equivariant spectral triple could obtained from the construction presented above.

Note that the Drinfeld twist does not change $u(1) \times u(1)$ and the isometry remains the same, although the twisted $u(1) \times u(1)$ module algebra $\mathcal{A}_\lambda$ is different. The generators U, V of the algebra of functions on the two-torus are homogeneous with respect to the actions of δ_1 and δ_2, so the twist is well-defined on the algebra of polynomials (and one can easily verify that this could be extended to a larger class of functions). The same is true for the linear span of the basis $\{|n,m\rangle, n, m \in \mathbb{Z}\}$, since each element of the basis is again homogeneous, therefore there is no problem with the twisted representation on a dense subspace of the Hilbert space.

5.3. *Isometries of finite spectral triples.* The simplest possible examples of spectral triples as noncommutative geometries are given by finite-dimensional semisimple $*$-algebras. In such a case, we do not need to distinguish between spectral triples and algebraic spectral triples as most of the additional analytical axioms are void. The construction and complete characterisation of finite spectral triples was presented in [35]. Throughout this section H is always a finite Hopf algebra.

Let us begin with a simple observation. Consider a spectral triple over H and the adjoint action of H on itself. Then there is no finite spectral triple equivariant with respect to this action. Indeed, it is clear that any operator commuting with the representation of H (as a symmetry) must also commute with the representation of H (as an algebra). Hence the differential calculus must be trivial and the spectral triple is degenerate. Note that if we consider a proper sub-Hopf algebra $H_0 \subset H$ ($H_0 \neq \mathbb{C}$) and its adjoint action on H, the result will still be analogous and there cannot be a H_0-equivariant spectral triple over H. Therefore we need to look for external symmetries, like the action of the dual of H or its subalgebra.

Example 5.15. Let us consider an algebra $\mathcal{A}$ of complex functions on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group and a subgroup of its symmetry: $\mathbb{Z}_2$. The algebra $\mathcal{A}$ is generated by four mutually commuting idempotents, e_{ab} where a, b are labelled by elements of $\mathbb{Z}_2$. We choose the diagonal nontrivial action of $\mathbb{Z}_2$ on the algebra given by:

$$g \triangleright e_{ab} = e_{(ga)(gb)},$$

From the classification of real spectral triples (as derived in [35]) we know that the finite dimensional spectral triple is characterised entirely by its intersection matrix. This is an integer-valued symmetric square matrix, such that the number of columns (and rows) is the number of simple components of the finite dimensional algebra. The finite dimensional Hilbert space is split into subspaces relative to their bimodule structure over the algebra (the right-module multiplication is introduced through the reality operator J). The absolute value of the entry in n-th row and k-th column of the intersection matrix gives the multiplicity of the dimension of the subspace which is a left module over the n-th simple component and right module over the k-th simple component of the algebra. (The full dimension is this number multiplied by the dimensions of minimal modules for both simple subalgebras). The sign of this entry tells the eigenvalue of the γ operator. For details, we refer again to [35].

Knowing this, it is not difficult to satisfy the equivariance of the representation of the algebra. If $\mathcal{H}$ denotes the Hilbert space and π the representation of $\mathcal{A}$, the representation is equivariant provided that the Hilbert subspaces $\pi(e_{ab})\mathcal{H}$ and $\pi(e_{(ga)(gb)})\mathcal{H}$ are isomorphic to each other. In the finite-dimensional case this simply means that their dimensions are equal.

Moreover, from the requirement that the real structure is equivariant, we obtain the same condition for $J\pi(e_{(a)(b)})J\mathcal{H}$ and $J\pi(e_{(ga)(gb)})J\mathcal{H}$. The most general intersection form of such a spectral triple is given by the matrix:

$$\begin{pmatrix} n_1 & m_1 & k_1 & l_1 \\ m_1 & n_2 & m_2 & k_2 \\ k_1 & m_2 & n_3 & m_3 \\ l_1 & k_2 & m_3 & n_4 \end{pmatrix}.$$

and the equivariance forces:

$$\begin{aligned} |n_1| + |m_1| + |k_1| &= |k_2| + |m_3| + |n_4|, \\ |m_1| + |n_2| + |k_2| &= |k_1| + |n_3| + |m_3|. \end{aligned}$$

To find all admissible Dirac operators one needs to know only the signs in the intersection matrix (as D is chiral, it acts between the subspaces of different signs in the intersection form). We shall present here the lowest dimensional example of a possible spectral triple and the restriction on D (we assume again that the triple is non-degenerate, that is, $[D, a]$ is nonzero for all a different from a constant element). The intersection matrix is

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -2 & 0 & 1 & 0 \end{pmatrix}.$$

Let us see what is the requirement for the considered $\mathbb{Z}_2$-symmetry to be an isometry of the Dirac operator. First of all, from the intersection form and the analysis of admissible Dirac operators (again we refer to [35]) one sees that D shall only have components acting between $\mathcal{H}_{1,2}$ and $\mathcal{H}_{1,4}$ as well as $\mathcal{H}_{4,1}$ and $\mathcal{H}_{4,3}$. (Here $\mathcal{H}_{I,J}$ denotes the subspace of the Hilbert space corresponding to the I, J entry in the intersection matrix, and $D_{(I,J)(K,L)}$ denotes the part of D mapping $\mathcal{H}_{I,J}$ to $\mathcal{H}_{K,L}$.)

The action of $\mathbb{Z}_2$ interchanges $\mathcal{H}_{1,2}$ with $\mathcal{H}_{4,3}$, similarly for the second and third row, so in effect we have:

REMARK 5.16. *An isometric Dirac operator for the above spectral triple has the property that* $D_{(1,2)(1,4)} = D_{(4,3)(4,1)}$ *and* $D_{(2,1)(2,3)} = D_{(3,4)(3,2)}$.

Note that this relation is a straightforward consequence of the $\mathbb{Z}_2$ symmetry and would not appear in the general construction of the Dirac operator for the spectral triple under consideration.

In the context of applications to physical models (gauge theory) one might interpret the relation as the equality of masses of the fermions present in the model. Instead of four massive fermions with a priori arbitrary masses we see that the $\mathbb{Z}_2$ symmetry enforces (as was to be expected) the degeneracy: we have two pairs of massive fermions with the same masses within each pair.

The example based on the same algebra was discussed also (but rather in a different context of gravity) in [32].

The above example is a particular case of a much general picture, which is the invariance under the canonical action of the dual Hopf algebra H^* on H given in Example 2.4. We can have either the strong version (with the full dual algebra) or a smaller one, with a Hopf subalgebra of the dual being an isometry. Note that in the case discussed earlier we had exactly such a situation: the action of the diagonal subgroup of the dual Hopf algebra (which is a group algebra) was an isometry of the spectral triple.

Let us turn to another example, now with the algebra of functions on a finite group.

EXAMPLE 5.17. Take a commutative algebra of functions on a group, $\mathbb{C}(G)$, with its basis $\{e_g, g \in G\}$, $e_g e_h = \delta_{g,h} e_g$ and its dual, the group algebra $\mathbb{C}G$ with the basis $\{g, g \in G\}$. Then the action of $\mathbb{C}G$ on $\mathbb{C}(G)$, expressed using the above basis becomes:

$$g \triangleright e_h = e_{hg^{-1}}.$$

The cross product algebra is generated by orthogonal projectors e_g and g, with the cross-commutation rules:

$$g e_h = e_{hg^{-1}} g. \tag{53}$$

To study the equivariance of this spectral triple we need first to determine the representation of the cross product. Using the projectors e_h we split the Hilbert space $\mathcal{H}$ into subspaces $\mathcal{H}_{gh}$, defining:

$$\mathcal{H}_{gh} = \pi(e_g) J \pi(e_h) J^{-1} \mathcal{H}, \quad g, h \in G.$$

Further, using the rule (53) we see that for each $i \in G$, g acts as a linear map (remember that $\mathcal{H}$ is $\mathbb{C}G$ module), $g : \oplus_j \mathcal{H}_{ij} \to \oplus_j \mathcal{H}_{(ig^{-1})j}$.

Let us denote its restriction to $\mathcal{H}_{ij}$, which maps it to $\mathcal{H}_{(ig^{-1})l}$, by $g_{ij,l}$. (Note that, since g is invertible, $\mathcal{H}_{ij}$ and $\mathcal{H}_{(ig^{-1})l}$ must be of the same dimension if $g_{ij,l} \neq 0$.) Introducing the Dirac operator D, which commutes with every g, we obtain:

$$\sum_m D_{kl,(ig^{-1})m} g_{ij,m} = \sum_m g_{(kg)m,l} D_{(kg)m,ij}, \quad k, l, m, i, j, g \in G.$$

After taking into account other restriction on D (from the order-one condition) this gives:

$$D_{kj,(ig^{-1})j} g_{ij,j} = g_{(kg)j,j} D_{(kg)j,ij},$$

and

$$\sum_m D_{(ig^{-1})l,(ig^{-1})m} g_{ij,m} = \sum_m g_{im,l} D_{im,ij},$$

for all $k, l, m, i, j, g \in G$.

This is the most general form of restrictions on possible Dirac operators on the spectral triple built over the algebra of functions on a finite group, with the group algebra as its symmetry.

EXAMPLE 5.18. Let us now consider the dual situation: a spectral triple over the group algebra $\mathbb{C}G$, with the equivariance given by the action of the dual. The notation is the same as in the above example, here, however:

$$e_h \triangleright g = \delta_{h,g} g, \quad h, g \in G.$$

The cross product algebra in this dual picture is again generated by orthogonal projectors e_g and g, with the cross-commutation rules:

(54) $$e_h g = g e_{g^{-1}h}.$$

Here, instead of studying the matrix elements of the Dirac operator we shall investigate the structure of the differential calculus set by the Dirac operator D and find the restrictions posed by the requirement of equivariance.

First of all, consider the universal calculus. As a left-module it is generated by forms $\chi^g = g^{-1}\,dg$, $g \neq e$, (here e denotes the neutral element of G) the right and left module structures are related through:

(55) $$h\,\chi^g = (\chi^{gh^{-1}} - \chi^{h^{-1}})h, \quad h, g \in G, g \neq e.$$

Now, let us take any $\mathbb{C}(G)$ equivariant calculus. Every such calculus is always given by a representation of the group G. Indeed, assume that an element $\omega = \sum_{e\neq g,h\in G} c_{g,h} h\chi^g \in \Omega^1_u(\mathbb{C}G)$ is in the subbimodule which defines the first order calculus. Then from the required invariance of the ideal we have:

$$e_p \triangleright \omega = \sum_{e\neq g\in G} c_{g,p} p\chi^g.$$

Therefore, multiplying the result by p^{-1} we get that $\sum_{e\neq g\in G} c_{g,p}\chi^g$ is also in the ideal. Hence, the ideal is generated (as a bimodule) by a subspace of invariant forms χ^g. Therefore the equivariant differential bimodule which we want to study is always of the form:

$$\Omega^1(\mathbb{C}G) \sim V \otimes \mathbb{C}G$$

where V is a representation space of G. If we denote the representation by ρ, the bimodule structure is given by:

$$h\,(v \otimes g)\,h' = \rho(h)v \otimes (hgh'), \quad g, h, h' \in G, v \in V.$$

Note that in the group algebra situation the covariance under the action of the dual gives the same characterisation of the differential calculi as left covariance (or bicovariance, since we are dealing with a cocommutative coproduct) for the coaction, which were studied in [35].

Now, using the above characterisation of equivariant differential calculi we may show an interesting no-go statement: not every equivariant first-order differential calculus is obtained from a spectral triple.

To see this, it is sufficient to note that for certain (adjoint) representations of the group G we shall have:

$$\rho(g)v_h = v_{ghg^{-1}},$$

and therefore the element $\sum_{e\neq g\in G}(v_g \otimes g)$ would be central in the bimodule $\Omega^1(\mathbb{C}G)$. Since the latter is incompatible with a real finite spectral triple ([35], observation 6) we get immediately our result. A concrete example of $\mathbb{C}S_3$ was studied in [35].

6. Outlook. Having presented here a review of the definitions and a couple of illustrative examples let us present several questions and problems, which should be considered in future research.

First of all, one would like to have more examples with genuine quantum group type symmetries. Some steps in this directions were already done. In fact, it is not difficult (see [8]) to construct the example of an $U_q(su(2))$ equivariant Laplace-type operators for $SU_q(2)$ or the Podleś sphere, the real difficulty lies in the reality condition and the deep relations between K-cycles and cyclic cohomology. An important background has been set by the work of Schmüdgen on the cross product algebras for quantum groups [38]. Some results on the construction of the algebraic data of a real spectral triple for the standard quantum Podleś sphere are in this volume [18].

A further most welcomed input would be to provide an equivariant version of Chern-Connes pairing between equivariant Fredholm modules and a version of equivariant cyclic cohomology. Note that some attempts in this direction in the formalism of J-L-O cocycles appeared recently [22].

To extend the notion of isometry one may consider equivariant spectral triples, with Hopf algebra symmetries such that D does not necessarily commute with H but each commutator is bounded. (An infinitesimal version of this, which could be the natural version used for the Fredholm modules should require the commutator $[F, h]$ to be compact). One could consider the action of the Hopf algebra H (by the usual adjoint action) on the space of allowed Dirac operators. Clearly, this might be formulated only in some special cases unless we change the axioms of spectral triples.

From the physical point of view, however, the most interesting seems to be the search for symmetries in examples of finite spectral triples, as related to the quest for "hidden symmetries" in the fundamental theories.

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