KNOTS IN POLAND III
BANACH CENTER PUBLICATIONS, VOLUME 100
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

ON STABILITY OF ALEXANDER POLYNOMIALS OF KNOTS AND LINKS (SURVEY)

MIKAMI HIRASAWA

Department of Mathematics, Nagoya Institute of Technology
Nagoya, Aichi 466-8555, Japan
E-mail: hirasawa.mikami@nitech.ac.jp

KUNIO MURASUGI

Department of Mathematics, University of Toronto
Toronto, Ontario M5S 2E4, Canada
E-mail: murasugi@math.toronto.edu

Abstract. We study distribution of the zeros of the Alexander polynomials of knots and links in S^3. After a brief introduction of various stabilities of multivariate polynomials, we present recent results on stable Alexander polynomials.

Contents

2010 *Mathematics Subject Classification*: 57M25, 57M27, 26C10.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc100-0-5 © Instytut Matematyczny PAN, 2014

Introduction. Let $\mathcal{H} \subset \mathbb{C}$ be an open right half-plane, i.e. $\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$, or an open upper half-plane, i.e. $\{\alpha \in \mathbb{C} : \mathrm{Im}(\alpha) > 0\}$. Let $f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial in n variables, $z_1, \ldots, z_n$. We say that $f(z_1, \ldots, z_n)$ is *$\mathcal{H}$-stable* if for any values $\alpha_j \in \mathcal{H}$, $1 \leq j \leq n$, $f(\alpha_1, \ldots, \alpha_n) \neq 0$. If $\mathcal{H}$ is an open right half-plane, then f is called *Hurwitz-stable.* If $\mathcal{H}$ is an open upper half-plane, then f is called a *stable polynomial.* The theory of stable polynomials has a long history, but the recent development of this theory is very impressive, and is summarized in a remarkable survey article [Wa'11].

The purpose of this report is to introduce a recent study on various stabilities of the Alexander polynomials of knots or links in S^3. The study was motivated by our desire to answer a question (later called conjecture) posed by Jim Hoste about ten years ago. He asks whether the zeros of the Alexander polynomial $\Delta_K(t)$ of an alternating knot K have real parts greater than -1. It is closely related to the question whether $\Delta_K(-(t+1))$ is Hurwitz-stable for an alternating knot K. (See Conjecture 4.2.) The question leads us to other problems on stabilities of the Alexander polynomial of a (not necessarily alternating) knot. For example, since the sequence of the coefficients of a stable univariate real polynomial under a certain condition is unimodal, we see immediately that stable Alexander polynomials of alternating knots satisfy the Trapezoidal Conjecture, one of the outstanding conjectures that still remains open.

In [LM'11], it is shown that for many 2-bridge knots or links, Hoste's question has the affirmative answer. Further, a few more subtle theorems on Hurwitz-stability and stability of the Alexander polynomials of 2-bridge knots are proven.

In this paper, knots are not necessarily alternating, and we discuss stabilities of the Alexander polynomials of knots, and further, we discuss the third stability, called *circular stability*, of the Alexander polynomials of knots.

This paper consists of two parts. The first part, consisting of Sections 1–3, is a quick review of various types of stable polynomials. Almost all material in this part is taken from various known sources and hence proofs are completely omitted. Section 4 is the second part, where we study the various types of stabilities of the Alexander polynomials of knots or links in S^3. However, most of the proofs of new theorems in this part are also omitted, since the details will appear elsewhere (cf. [HM'13]).

1. Half-plane property. Let $\mathcal{H} \subset \mathbb{C}$ be an open half-plane such that $\partial\overline{\mathcal{H}}$ contains the origin. Let $f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial in n variables.

Definition 1.1 ([Br'07, p. 303]). $f \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be *$\mathcal{H}$-stable* if for any values $\alpha_j \in \mathcal{H}$, $1 \leq j \leq n$, $f(\alpha_1, \ldots, \alpha_n) \neq 0$. If $f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$ is $\mathcal{H}$-stable for some open half-plane, we say f has a *half-plane property.*

There are two special cases.

Definition 1.2 ([Br'07, p. 303]).

(1) Let $\mathcal{H}$ be the right half-plane, i.e. $\mathcal{H} = \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$. Then an $\mathcal{H}$-stable polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called *Hurwitz-stable*. In other words, f is Hurwitz-stable if for any $\alpha_j \in \mathbb{C}$, $1 \leq j \leq n$, such that $\mathrm{Re}(\alpha_j) > 0$, $f(\alpha_1, \ldots, \alpha_n) \neq 0$.

(2) Let $\mathcal{H}$ be the upper half-plane, i.e. $\mathcal{H} = \{\alpha \in \mathbb{C} : \mathrm{Im}(\alpha) > 0\}$. Then an $\mathcal{H}$-stable polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ is called a *stable polynomial.*

REMARK 1.3. If a real polynomial $f \in \mathbb{R}[z_1, \dots, z_n]$ is stable, f is called *real stable*. For convenience, we regard the zero polynomial as stable.

From the definitions, we see immediately:

PROPOSITION 1.4. *Let $f(z) \in \mathbb{R}[z]$ be a real univariate polynomial. Then*

(1) *$f(z)$ is real stable if and only if $f(z)$ has only real zeros.*
(2) *$f(z)$ is Hurwitz-stable if and only if for any zero α of $f(z)$,* $\mathrm{Re}(\alpha) \leq 0$.

The theorem below is elementary, but useful.

THEOREM 1.5 ([Wa'11, Lemma 2.4]). *The following operations preserve $\mathcal{H}$-stability in $\mathbb{C}[z_1, \dots, z_n]$.*

(a) *Permutation*: *For any permutation $\sigma \in S_n$, $f \to f(z_{\sigma(1)}, \dots, z_{\sigma(n)})$.*
(b) *Scaling*: *For any $c \in \mathbb{C}$, and $(a_1, \dots, a_n) \in \mathbb{R}_+^n$ (i.e. $a_j > 0$, $1 \leq j \leq n$), $f \to cf(a_1 z_1, \dots, a_n z_n)$.*
(c) *Diagonalization*: *For $\{i, j\}$, $1 \leq i, j \leq n$, $f \to f(z_1, \dots, z_n)|_{z_i = z_j}$.*
(d) *Differentiation* (*or contraction*): *$f \to \frac{\partial}{\partial z_1} f(z_1, \dots, z_n)$.*

2. Hurwitz-stable polynomials. There are two basic tools to show Hurwitz-stability of a real univariate polynomial.

2.1. Hurwitz–Routh criterion. Let $f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \in \mathbb{R}[z]$ be a real polynomial, where $a_0 > 0$, $a_j \in \mathbb{R}$, $0 \leq j \leq n$. Define an $n \times n$ matrix H_n as follows:

$$H_n = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ & & \ddots & & & \\ \vdots & & & & & \vdots \\ & & & & \ddots & \\ a_{2n-1} & a_{2n-2} & & \cdots & a_{n+1} & a_n \end{bmatrix}, \tag{2.1}$$

where $a_j = 0$ if $j > n$.

For $1 \leq k \leq n$, let H_k be the first $k \times k$ principal submatrix of H_n. Namely, H_k is the $k \times k$ submatrix consisting of the first k rows and columns of H_n.

For example, $H_1 = [a_1]$ and $H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}$.

We say that $f(z)$ is *strongly Hurwitz-stable* (or simply *s-Hurwitz-stable*) if any zero of $f(z)$ has a negative real part.

THEOREM 2.1 (Hurwitz–Routh criterion [La'69, Theorem 8.8.1]). *A real polynomial $f(z) = \sum_{j=0}^{n} a_j z^{n-j}$, $a_0 > 0$, $a_j \in \mathbb{R}$, $1 \leq j \leq n$, is strongly Hurwitz-stable if and only if* $\det H_k > 0$ *for* $1 \leq k \leq n$.

Using Theorem 2.1, we can characterize strongly Hurwitz-stable polynomials with small degrees.

Example 2.2.

(1) $f(z) = a_0 z + a_1$, $a_0 > 0$, is s-Hurwitz-stable if and only if $a_1 > 0$.
(2) $f(z) = a_0 z^2 + a_1 z + a_2$, $a_0 > 0$, is s-Hurwitz-stable if and only if $a_1, a_2 > 0$.
(3) $f(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$, $a_0 > 0$, is s-Hurwitz-stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_0 a_3$.
(4) $f(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$, $a_0 > 0$, is s-Hurwitz-stable if and only if (i) $a_1, a_2, a_3, a_4 > 0$, (ii) $a_1 a_2 > a_0 a_3$, and (iii) $a_3(a_1 a_2 - a_0 a_3) > a_1^2 a_4$.

2.2. Lyapunov matrix. There is another important tool to study Hurwitz-stability of a real univariate polynomial given by Lyapunov. Let $f(z)$ be a real polynomial of degree n. Let M be a companion matrix of $f(z)$.

Theorem 2.3 (Lyapunov [La'69, Theorem 8.7.2]). *$f(z)$ is strongly Hurwitz-stable if and only if there exist two real positive definite (symmetric) matrices V and W such that*

$$VM + M^T V = -W. \tag{2.2}$$

For convenience, we call V a *Lyapunov matrix* associated to M. It is often quite difficult to find a Lyapunov matrix even if $f(z)$ is known to be Hurwitz-stable.

Example 2.4.

(1) $f(z) = z + a_1$. Then $M = [-a_1]$. If $a_1 < 0$, Lyapunov matrix does not exist, since M is positive definite. If $a_1 > 0$, then $V = E$ is a Lyapunov matrix associated to M and $f(z)$ is s-Hurwitz-stable.
(2) Let $f(z) = z^2 + a_1 z + a_2$. If $a_1, a_2 > 0$, then we know $f(z)$ is s-Hurwitz-stable, see Example 2.2 (2). For example, if $a_1 = 3$ and $a_2 = 4$, i.e., $M = \begin{bmatrix} 0 & -4 \\ 1 & -3 \end{bmatrix}$, then $V = \begin{bmatrix} 7/12 & -1/2 \\ -1/2 & 5/6 \end{bmatrix}$ is a Lyapunov matrix and $W = E$.

In graph theory, this concept appears in literature. We mention one example.

Example 2.5 ([COSW'04, Theorem 1.1] and [BB'08, p. 208]). The spanning-tree polynomial of a connected finite graph is Hurwitz-stable and also stable.

3. Stable polynomial

3.1. Multivariate stable polynomials. First, we state two basic properties of stable polynomials.

Theorem 3.1 ([Wa'11, Lemma 2.4]). *The following operations preserve stability in $\mathbb{C}[z_1, \dots, z_n]$.*

(a) *Specialization: For any $a \in \mathbb{C}$ with $\mathrm{Im}(a) \geq 0$, $f \to f(a, z_2, \dots, z_n)$.*
(b) *Inversion: If $\deg_{z_1}(f) = d$, $f \to z_1^d f(-z_1^{-1}, z_2, \dots, z_n)$.*

Next, the following theorems give us systematic ways to construct stable polynomials.

THEOREM 3.2 ([BB'08, Proposition 2.4]). *Let A_i, $1 \le i \le n$, be complex, semi-positive definite $m \times m$ matrices and B be an $m \times m$ Hermitian matrix. Then, $f(z_1, \dots, z_n) = \det[z_1 A_1 + \dots + z_n A_n + B]$ is stable.*

As a consequence of Theorem 3.2, we have:

THEOREM 3.3 ([Br'07, p. 308]). *Let $Z = \mathrm{diag}(z_1, \dots, z_n)$ be a diagonal matrix. If A is an $n \times n$ Hermitian matrix, then both $\det(Z + A)$ and $\det(E + AZ)$ are stable.*

If $n = 2$, then the converse of Theorem 3.2 holds for a real stable polynomial.

THEOREM 3.4 ([BB'10, Theorem 1.13], characterization of real stable polynomials with two variables). *Let $f(x, y) \in \mathbb{R}[x, y]$. Then f is real stable if and only if it can be written as*

$$f(x, y) = \pm \det[xA + yB + C], \tag{3.1}$$

where A and B are positive semi-definite matrices and C is a symmetric matrix of the same order.

The following theorem claims that the stability of multivariate polynomials can be reduced to the stability of univariate polynomials.

THEOREM 3.5 ([Wa'11, Lemma 2.3]). *A polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ is stable if and only if for any $(a_1, \dots, a_n) \in \mathbb{R}^n$ and $(b_1, \dots, b_n) \in \mathbb{R}_+^n$ (i.e. $b_j > 0$, $1 \le j \le n$), $f(a_1 + b_1 t, \dots, a_n + b_n t) \in \mathbb{C}[t]$ is stable.*

If a polynomial is of special type, the stability problem could be slightly simpler.

THEOREM 3.6 ([Br'07, Theorem 5.6]). *Let $f \in \mathbb{R}[z_1, \dots, z_n]$ be a multi-affine polynomial (i.e. each variable z_j has degree at most 1 in each term). Then f is stable if and only if for all $(x_1, \dots, x_n) \in \mathbb{R}^n$ and for $1 \le i, j \le n$, $\Delta_{ij}(f)(x_1, \dots, x_n) \ge 0$, where $\Delta_{ij}(f) = \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_j} - \frac{\partial^2 f}{\partial z_i \partial z_j} f$.*

REMARK 3.7. If f is not multi-affine, then in Theorem 3.6, the "only if" part holds, but the "if" part does not.

EXAMPLE 3.8 ([Br'07, Example 5.7]). Let $f = a_{00} + a_{01} y + a_{10} x + a_{11} xy$, $a_{ij} \in \mathbb{R}$. Then $\Delta_{12}(f) = -\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$. Therefore, f is stable if and only if $\det[a_{ij}] \le 0$.

THEOREM 3.9 ([WW'09, p. 1]). *Suppose $f \in \mathbb{C}[z_1, \dots, z_n]$ is homogeneous. Then, f is Hurwitz-stable if and only if f is stable.*

3.2. Real stable univariate polynomials. Real stable univariate polynomials have many deep properties. Since we are particularly interested in these polynomials, we will spend more space for them.

We begin with the following interesting theorem.

THEOREM 3.10 ([Br'07, p. 307]). *Let $f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n \in \mathbb{R}[z]$, $a_0 \neq 0$, $a_j \ge 0$, $0 \le j \le n$. Suppose $f(z)$ is real stable. If $a_i a_k \neq 0$ for $i < k$, then for any j, $i < j < k$, $a_j \neq 0$. Therefore, if $a_n \neq 0$, then all $a_j \neq 0$, for $1 \le j \le n$.*

Theorem 3.10 shows that a sequence of the coefficients of a real stable polynomial is worth studying.

Definition 3.11 ([Wi'90, p. 126]). A sequence $\{c_0, c_1, \ldots, c_n\}$ of positive numbers is called *unimodal* if there exist indices r, s such that

$$c_0 \leq c_1 \leq \ldots \leq c_r = c_{r+1} = \ldots = c_{r+s} \geq c_{r+s+1} \geq \ldots \geq c_n. \tag{3.2}$$

Further, $\{c_0, c_1, \ldots, c_n\}$ is called *log-concave* if

$$c_{j-1}c_{j+1} \leq c_j^2 \text{ for } j = 1, 2, \ldots, n-1. \tag{3.3}$$

If "$\leq$" is replaced by "$<$" in (3.3), then it is called *strictly log-concave.*

The following theorem is well-known.

Theorem 3.12 ([Wi'90, Proposition p. 127]). *If a positive sequence $\{c_0, c_1, \ldots, c_n\}$ is log-concave, then it is unimodal.*

Now we have an important result.

Theorem 3.13 ([Wi'90, p. 126]). *Let $f(z) = a_0z^n + a_1z^{n-1} + \ldots + a_n \in \mathbb{R}[z]$, $a_0 \neq 0$, $a_n \neq 0$. Suppose $a_j \geq 0$, $0 \leq j \leq n$. If f is real stable (and hence $a_j > 0$ for all $j \geq 0$), then $\{a_0, a_1, \ldots, a_n\}$ is strictly log-concave, and hence it is unimodal.*

The concept defined below is well-studied and plays an important role in the theory of stable polynomials.

Definition 3.14 ([Br'07, p. 310]). Let $f, g \in \mathbb{R}[z]$ be univariate polynomials. Suppose f, g are real stable (i.e. all zeros are real). Let $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_m$ be the zeros of f and g, respectively. Then we say that the zeros $\{\alpha_j\}$ and $\{\beta_k\}$ are *interlaced* (or we simply say that f and g are *interlaced*), if the following is satisfied:

(i) $|m-n| \leq 1$,
(ii) they can be ordered so that
(a) if $n = m$, then

$$\begin{aligned} &\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \alpha_n \leq \beta_n, \\ or \quad &\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \beta_n \leq \alpha_n; \end{aligned}$$

(b) if $n = m+1$, then

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \alpha_m \leq \beta_m \leq \alpha_{m+1} (= \alpha_n);$$

(c) if $m = n+1$, then

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \beta_{n+1} (= \beta_m).$$

Suppose that (the zeros of) f and g are interlaced. Let $\hat{g}_i = \frac{g(z)}{z-\beta_i}$, $1 \leq i \leq m$.

Lemma 3.15 ([Wa'11, p. 56]). *If $\deg(f) \leq \deg(g)$ and the zeros of g are simple, then there is a unique real sequence $\{a, b_1, \ldots, b_m\}$ such that $f = ag + b_1\hat{g}_1 + \ldots + b_m\hat{g}_m$.*

Theorem 3.16 ([Wa'11, Exercise 2.5]). *Let $f, g \in \mathbb{R}[z]$ be real stable polynomials such that fg has only simple zeros. Suppose $n = \deg f \leq \deg g = m$ and $\beta_1 < \beta_2 < \ldots < \beta_m$ are the zeros of g. Then the following are equivalent:*

(a) *The zeros of* f *and* g *are interlaced.*
(b) *The sequence* $\{f(\beta_1), f(\beta_2), \dots, f(\beta_m)\}$ *alternates in sign (strictly).*
(c) *In* $f = ag + \sum_{j=1}^{m} b_j \widehat{g}_j$, *all of* $b_1, \dots, b_m$ *have the same sign and are non-zero.*

DEFINITION 3.17 ([Wa'11, p. 56]). For $f, g \in \mathbb{C}[z]$, we define the *Wronskian* $W[f, g]$ as $W[f, g] = f'g - fg'$. For $f(\neq 0)$, $g(\neq 0) \in \mathbb{R}[z]$, we say that two real stable f, g are in *proper position* (denoted by $f \ll g$) if $W[f, g] \leq 0$ on all real values.

If the zeros of f and g are interlaced, then either $W[f, g] \leq 0$ or $W[f, g] \geq 0$ on all real values and hence $f \ll g$ or $g \ll f$ [Wa'11, p. 56].

THEOREM 3.18 ([Wa'11, p. 57], Hermite–Kakeya–Obreschkoff Theorem). *Let* $f, g \in \mathbb{R}[z]$. *Then all non-zero polynomials in* $\{af + bg : a, b \in \mathbb{R}\}$ *are real-rooted if and only if*

(1) f, g *are real stable and*
(2) $f \ll g$, $g \ll f$ *or* $f = g = 0$.

In graph theory, the concept of interlacedness has been used in [GR'01], and for example, the following theorem is reproved.

THEOREM 3.19 ([GR'01, p. 195]). *Peterson graph does not have Hamilton cycles.*

CONJECTURE 3.20 (See [GR'01, p. 354]). *The sequence of the coefficients of the chromatic polynomial is unimodal.*

This conjecture is still open. However, a similar conjecture for the Tutte polynomial is false [Sc'93].

4. Knot polynomials

4.1. Hoste's conjecture. In 2002, based on his extensive calculations of the zeros of the Alexander polynomials, Hoste made the following conjecture.

CONJECTURE 4.1 (J. Hoste, 2002). *Let* K *be an alternating knot and* $\Delta_K(t)$ *the Alexander polynomial of* K. *Then for any zero* α *of* $\Delta_K(t)$, $\mathrm{Re}(\alpha) > -1$.

One of the key observations is that Conjecture 4.1 is equivalent to

CONJECTURE 4.2. *Under the same assumption,* $\Delta_K(-(t+1)) \in \mathbb{R}[t]$ *is strongly Hurwitz-stable.*

Using Lyapunov matrices, the following theorem is proved.

THEOREM 4.3 ([LM'11, Theorem 1]). *Let* K *be a 2-bridge knot (or link). Then* $\Delta_K(-(t+3))$ *and* $\Delta_K(t+6)$ *are strongly Hurwitz-stable. Equivalently, any zero* α *of* $\Delta_K(t)$ *satisfies*

$$-3 < \mathrm{Re}(\alpha) < 6. \tag{4.1}$$

For other special results, see [LM'11, Theorems 3,4 and 5].

REMARK 4.4. A. Stoimenow proves in [St'11] that for a 2-bridge knot (or link) K, any zero α of $\Delta_K(t)$ satisfies

$$\left| \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right| \leq 2. \tag{4.2}$$

This implies

$$-1 \leq \mathrm{Re}(\alpha) \leq 3 + \sqrt{8} = 5.8284\ldots \tag{4.3}$$

It should be noted that for a non-alternating knot, neither a lower bound nor an upper bound of $\mathrm{Re}(\alpha)$ exist [LM'11, Examples 1 and 2]. Further, we think that an upper bound of $\mathrm{Re}(\alpha)$ does exist only for a family of 2-bridge knots or links. In fact, the following theorem holds.

Theorem 4.5. *There exists an infinite sequence of alternating (Montesinos) knots* $K_1, K_2, \ldots, K_m, \ldots$ *such that*

(1) *for any* $m \geq 1$*, the Alexander polynomial of* K_m *has only real zeros, and*
(2) *the maximal value of the zeros of* $\Delta_{K_m}(t)$ *is at least* $m+1$.

Therefore, in general, an upper bound of $\mathrm{Re}(\alpha)$ does not exist, even for alternating knots. However, an upper bound may exist for some family of the Alexander polynomials. For example, let A_n be the set of all Alexander polynomials of degree n of alternating knots.

Conjecture 4.6. *There exist a real number* $\delta_n > 0$ *such that for any zero* α *of* $\Delta_K(t)$ *in* A_n

$$\mathrm{Re}(\alpha) \leq \delta_n \tag{4.4}$$

It is known that Conjecture 4.6 is false for non-alternating knots [LM'11, Example 2].

If the Alexander polynomial of an alternating knot K is stable, then all zeros of $\Delta_K(t)$ are positive and hence Conjecture 4.1 holds. Therefore, in the next Subsection 4.2, we discuss stable Alexander polynomials of knots or links.

4.2. Real stable Alexander polynomials. We say that an oriented link (or knot) L is *special alternating* if L has an alternating diagram without nested Seifert circles.

We say that a knot (or link) K is *real stable*, or simply, *stable*, if $\Delta_K(t)$ is real stable.

Proposition 4.7. *For (non-trivial) special alternating knots,* $\Delta_K(-(t+1))$ *is strongly Hurwitz-stable, but* $\Delta_K(t)$ *is not real stable.*

In fact, $\Delta_K(t)$ has only zeros on the unit circle in $\mathbb{C}$.

Proposition 4.8. *If a knot* K *is stable, then the signature* $\sigma(K)$ *of* K *is zero.*

In fact, if the signature is not zero, $\Delta_K(t)$ has a zero on the unit circle. However, the converse of Proposition 4.8 is false.

The following Example 4.9 shows that for links, Proposition 4.8 does not hold.

Example 4.9. Let L be a pretzel link $P(2,4,4)$, oriented so that L is a special alternating 3-component link. Then the reduced Alexander polynomial $\Delta_L(t) = 8(t-1)^2$ that is stable while $\sigma(L) = 2$.

From now on, we assume that a knot or link is always oriented. First we consider 2-bridge knots and links.

Let $K(r)$ be a 2-bridge knot or link of type $r = \frac{\beta}{\alpha}$, $-\alpha \leq \beta \leq \alpha$. We may assume without loss of generality that one of α and β is even. Let $r = [2a_1, 2a_2, \ldots, 2a_m]$, $a_j \neq 0$, $1 \leq j \leq m$, be the even (negative) continued fraction expansion of r, e.g.,

$[a, b, c, d] = 1/(a - 1/(b - 1/(c - 1/d)))$. We assume 2-bridge links are oriented as in Figure 1.

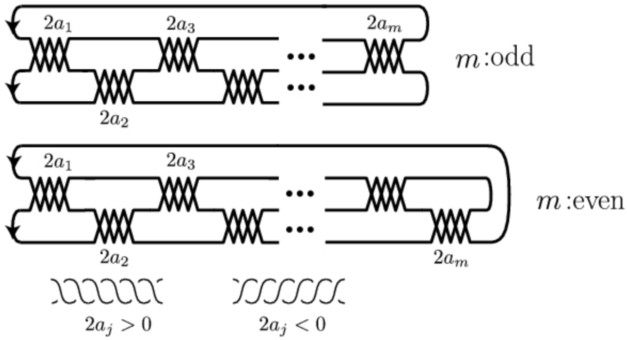

Fig. 1

By convention, $\frac{2}{3} = [2, 2]$ represents 3_1 and $\frac{2}{5} = [2, -2]$ represents 4_1, while $\frac{1}{4} = [4]$ represents a non-fibred link, $\frac{3}{4} = [2, 2, 2]$ represents a fibred torus link.

The first family of stable knots is given in the following theorem.

Theorem 4.10 ([LM'11, Theorem 2]). *If the sequence* $\{2a_1, 2a_2, \dots, 2a_m\}$ *alternates in sign, i.e.* $a_j a_{j+1} < 0$*, for* $1 \le j \le m - 1$*, then* $K(r)$ *is stable.*

The converse is not necessarily true.

Example 4.11. Let $r = [2, -2, -2a, 2b]$. Then $K(r)$ is stable if (i) $b = 1$ and $a \ge 4$ or (ii) $b \ge 2$ and $a \ge 3$. For example, if $r = \frac{32}{81} = [2, -2, -8, 2]$, then $\Delta_{K(r)}(t) = (2t^2 - 5t + 2)^2$, which is real stable.

In case of 2-bridge links, there are many real stable links corresponding to non-alternating even continued fractions.

Example 4.12. Let $r = [2a, 2b, -2c]$, $a, b, c > 0$. Then $\Delta_{K(r)}(t)$ is stable if and only if $a \ge c$.

Now let $r_m = [2a_1, -2a_2, \dots, (-1)^{m-1} 2a_m]$, $a_j > 0$, for $1 \le j \le m$. Since $K(r_m)$ is real stable, we have a sequence of real stable 2-bridge knots or links, $\{K(r_m), m \ge 1\}$. The following theorem shows that two consecutive knots $K(r_m)$ and $K(r_{m+1})$ have an interesting property.

Theorem 4.13. *Let* $r = [2a_1, -2a_2, \dots, (-1)^{m-1} 2a_m]$*,* $a_j > 0$*,* $1 \le j \le m$*, and* $r' = [2a_1, -2a_2, \dots, (-1)^{m-2} 2a_{m-1}]$*. Then the zeros of* $\Delta_{K(r)}(t)$ *and* $\Delta_{K(r')}(t)$ *are all simple, distinct and they are interlaced. To be more precise, let* $0 < \alpha_1 < \alpha_2 < \dots < \alpha_m$ *and* $0 < \beta_1 < \beta_2 < \dots < \beta_{m-1}$ *be the zeros of* $\Delta_{K(r)}(t)$ *and* $\Delta_{K(r')}(t)$*, respectively. Then* $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_{m-1} < \beta_{m-1} < \alpha_m$*.*

Remark 4.14. If $a_j = c > 0$ for all j, $1 \le j \le m$, Theorem 4.13 is implicitly proven in [LM'11, Theorem 5].

Further, we have:

THEOREM 4.15 ([LM'11, Remark 2]). *Let $s_k = [2, -2, \ldots, (-1)^{k-1}2]$. Let α_k be the maximal zero of $\Delta_{K(s_k)}(t)$. Then*

$$\lim_{k\to\infty} \alpha_k = 3 + \sqrt{8}. \tag{4.5}$$

Therefore, the upper bound of $\mathrm{Re}(\alpha)$ in (4.3) is the limit of the sequence of the largest zeros of $\Delta_{K(s_k)}(t)$. We notice that $K(s_k)$ is a fibred knot. Therefore, we propose the following conjecture stronger than Conjecture 4.6.

CONJECTURE 4.16. *The upper bound δ_n in* (4.4) *is attained by a fibred alternating stable knot of genus $\frac{n}{2}$.*

This is obviously true for $n = 2$.

In the following example, we construct a series of stable knots which is a slight generalization of $K(s_k)$.

EXAMPLE 4.17. Let F_n be a Seifert surface obtained by applying Seifert's algorithm to the diagram in Figure 2, where $2m_i$ and $2\ell_j$ indicate the number of half twists in the bands. F_n is a Murasugi sum of two disks each attached with n twisted bands. F_n is denoted by $F_n(2m_1, \ldots, 2m_n|2\ell_1, \ldots, 2\ell_n)$ and its boundary by $K_n(2m_1, \ldots, 2m_n|2\ell_1, \ldots, 2\ell_n)$. For example, $K_1(2|2)$ is 3_1, $K_1(2|-2)$ is 4_1 and $K_2(2, 4|-2, -2)$ is 10_{13}.

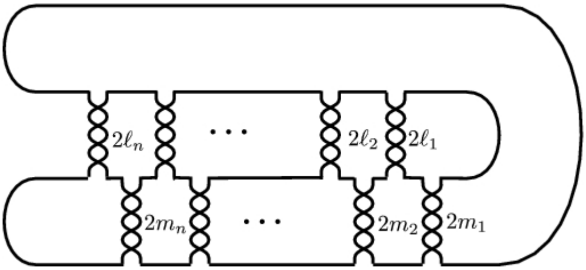

Fig. 2

THEOREM 4.18. *Suppose $m_j > 0$ and $\ell_j < 0$ for all j, $1 \le j \le n$. Then*

(1) *$K_n(2m_1, \ldots, 2m_n|2\ell_1, \ldots, 2\ell_n)$ is a real stable alternating knot of genus n.*
(2) *The maximal (real positive) value of the zero $\alpha(n)$ of the Alexander polynomial of a knot $K_n(2, \ldots, 2| -2, \ldots, -2)$ is larger than $n + 1$. Further, for $n \ge 2$, the zeros of $\Delta_{K_n(2,\ldots,2|-2,\ldots,-2)}(t)$ and $(t-1)\Delta_{K_{n-1}(2,\ldots,2|-2,\ldots,-2)}(t)$ are interlaced.*

Experimentally, $\alpha(n)$ is approximated by a quadratic polynomial $f(x)$:

$$f(x) = 0.405121x^2 + 0.411943x + 2.05373.$$

Finally, let $\Delta_K(t) = \sum_{j=0}^{2n}(-1)^j c_j t^{2n-j}$, $c_j > 0$, be the Alexander polynomial of an alternating knot K. Then the trapezoidal conjecture claims:

CONJECTURE 4.19 ([Fox'62]). *There is an integer k, $1 \le k \le n$, such that*

$$c_0 < c_1 < \ldots < c_k = c_{k+1} = \ldots = c_{2n-k} > c_{2n-k+1} \ldots > c_{2n}. \tag{4.6}$$

By applying Theorem 3.13, on $\Delta_K(-t)$, we obtain

Theorem 4.20. *The Alexander polynomial of a real stable alternating knot satisfies the trapezoidal conjecture. Therefore, the trapezoidal conjecture holds for* $K_n(2m_1, \dots, 2m_n | 2\ell_1, \dots, 2\ell_n)$ *if* $m_j > 0$ *and* $\ell_j < 0$ *for all* j.

4.3. Circular stable polynomials. In this subsection, we discuss another type of stability.

Definition 4.21 ([BB'09]). Let $\mathcal{D}$ be an open disk in $\mathbb{C}$. A polynomial $f(z) \in \mathbb{C}[z]$ is called *$\mathcal{D}$-stable* if $f(\alpha) \neq 0$ for any $\alpha \in \mathcal{D}$. In particular, if $f(z)$ is $\mathcal{D}$-stable with $\mathcal{D}$ the open unit disk, i.e. $\mathcal{D} = \{\alpha \in \mathbb{C} : |\alpha| < 1\}$, we say $f(z)$ is *circular stable* or *c-stable.*

Suppose $f(z)$ is reciprocal, i.e. $f(z) = \pm z^n f(z^{-1})$ for some n. If $f(z)$ is c-stable, then all zeros α of $f(z)$ are on the unit circle, i.e. $|\alpha| = 1$.

We say that a knot (or link) K is *c-stable* if $\Delta_K(t)$ is c-stable.

Proposition 4.22. *Let K be a knot and* $\deg \Delta_K(t) = 2n$. *If the signature of K is* $\pm 2n$, *then K is c-stable, i.e. all zeros of* $\Delta_K(t)$ *are on the unit circle. In particular, if K is a special alternating knot, then K is c-stable.*

The converse is not necessarily true. For example, 10_{130} is c-stable, but the signature is 0. The Alexander polynomial is $2t^4 - 4t^3 + 5t^2 - 4t + 2$.

By using the surface $F_n(2m_1, \dots, 2m_n | 2\ell_1, \dots, 2\ell_n)$, we can construct c-stable knots or links.

Theorem 4.23. *Consider the knot* $K_n(2m_1, \dots, 2m_n | 2\ell_1, \dots, 2\ell_n)$.

For $k = 1, 2, \dots, \left[\frac{n+1}{2}\right]$, *if*

$$\ell_{2k-1}, \ell_{2k}, m_{s-2k}, m_{s-2k+1} \geq k \tag{4.7}$$

then $K_n(2m_1, \dots, 2m_n | 2\ell_1, \dots, 2\ell_n)$ *is c-stable, where* $s = 2\left[\frac{n+1}{2}\right] + 2$.

Example 4.24. $K_1(2|2)$ is 3_1, $K_1(2|4)$ is 5_2, $K_2(2,2|2,2)$ is 5_1 and $K_2(2,4|2,2)$ is 7_3.

Now, using Möbius transformation, we can find a correspondence between circular stable knots and real stable links, and real stable knots and circular stable links.

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a Möbius transformation defined by $\varphi(z) = \frac{1-zi}{z-i}$.

Remark 4.25. φ maps the interior of the unit disk to the upper half-plane. In particular, the unit circle (resp. the real line) is mapped on the real line (resp. the unit circle). Also, $\varphi(0) = i$, $\varphi(-1) = -1$ and $\varphi(1) = 1$.

Theorem 4.26. *Let K be a c-stable knot* (*resp. real stable knot*). *Let* $\left\{\alpha_1, \frac{1}{\alpha_1}, \dots, \alpha_n, \frac{1}{\alpha_n}\right\}$ *be the zeros of* $\Delta_K(t)$, *i.e.* $\Delta_K(t) = c_0 \prod_{j=1}^n (t - \alpha_j) \prod_{j=1}^n \left(t - \frac{1}{\alpha_j}\right)$. *Let* $\varphi(\alpha_j) = \beta_j$, *for* $1 \leq j \leq n$. *Then* $\varphi\left(\frac{1}{\alpha_j}\right) = \frac{1}{\beta_j}$, $1 \leq j \leq n$, *and*

$$a \prod_{j=1}^n (t - \beta_j) \prod_{j=1}^n \left(t - \frac{1}{\beta_j}\right) = \widehat{\Delta}_K(t), \quad \textit{where } a = c_0 \prod_{j=1}^n \left(\alpha_j + \frac{1}{\alpha_j}\right),$$

is an integer polynomial of degree $2n$, *and it is stable* (*resp. c-stable*).

Proposition 4.27. $\widehat{\Delta}_K(t)$ *is reciprocal, and* $|\widehat{\Delta}_K(1)| = 2^n$. *Therefore* $\widehat{\Delta}_K(t)$ *is a Hosokawa polynomial of some link* (*of multi-components*). *Further we have* $|\widehat{\Delta}_K(-1)| = 2^n |\Delta_K(-1)|$.

EXAMPLE 4.28.

(1) Let $\Delta_K(t) = t^2 - t + 1$, which is c-stable. Since $\alpha = \frac{1-\sqrt{3}i}{2}$ and $\frac{1}{\alpha} = \frac{1+\sqrt{3}i}{2}$, $\beta = \frac{1}{2+\sqrt{3}}$ and $\frac{1}{\beta} = 2+\sqrt{3}$. Therefore, $\widehat{\Delta}_K(t) = (t-\beta)(t-\frac{1}{\beta}) = t^2 - 4t + 1$, where $a = \alpha + \frac{1}{\alpha} = 1$.
(2) $\Delta_K(t) = mt^2 - (2m-1)t + m$, $m > 0$, is c-stable, and $\widehat{\Delta}_K(t) = (2m-1)t^2 - 4mt + (2m-1)$ is stable.
(3) $\Delta_K(t) = t^4 - t^3 + t^2 - t + 1$ is c-stable, and $\widehat{\Delta}_K(t) = t^4 + 4t^3 - 14t^2 + 4t + 1$ is stable.
(4) $\Delta_K(t) = 2t^4 - 4t^3 + 5t^2 - 4t + 2$ is c-stable, and $\widehat{\Delta}_K(t) = t^4 - 16t^3 + 34t^2 - 16t + 1$ is stable.
(5) Let $\Delta_K(t) = t^6 - t^5 + t^3 - t + 1$ (i.e. the Alexander polynomial of the torus knot of type $(3,4)$), which is c-stable. Then $\widehat{\Delta}_K(t) = 3t^6 - 12t^5 - 7t^4 + 40t^3 - 7t^2 - 12t + 3$. The zeros of $\widehat{\Delta}_K(t)$ are all real, and two of them are negative.

EXAMPLE 4.29.

(1) $\Delta_K(t) = t^2 - 3t + 1$ is stable, and $\widehat{\Delta}_K(t) = 3t^2 - 4t + 3$ is c-stable.
(2) $\Delta_K(t) = mt^2 - (2m+1)t + m$, $m > 0$, is stable, and $\widehat{\Delta}_K(t) = (2m+1)t^2 - 4mt + (2m+1)$ is c-stable.
(3) $\Delta_K(t) = 2t^4 - 12t^3 + 21t^2 - 12t + 2$ is stable, and $\widehat{\Delta}_K(t) = 17t^4 - 48t^3 + 66t^2 - 48t + 17$ is c-stable.

4.4. Alexander polynomials of links. The Alexander polynomial of a link is a multivariate real polynomial. The stability problem for links is also an interesting problem, but it may be much harder. In this subsection, we discuss this problem.

Let $\Delta_L(t_1, \dots, t_n)$ be the Alexander polynomial of an n-component link L.

EXAMPLE 4.30. Let $\Delta_{L(r)}(x, y)$ be the Alexander polynomial of a 2-bridge link $L(r)$. If $r = [2, -2k, 2]$, $k > 0$, then $\Delta_{L(r)}(x, y)$ is real stable. In fact, $\Delta_{L(r)}(x, y) = k - (k+1)(x+y) + kxy$, $k > 0$. This is stable by Example 3.8.

As a slight generalization of Theorem 4.10, we asked the following question.

QUESTION 4.31. Let $r = [2a_1, 2a_2, \dots, 2a_m]$, m odd. If $a_j a_{j+1} < 0$, $1 \le j \le m-1$, then is $\Delta_{L(r)}(x, y)$ real stable?

The answer is No. One negative example is the following.

EXAMPLE 4.32. Let $r = [4, -2, 2]$. Then $\Delta_{L(r)}(x, y) = (x+y) - (2x^2 + 3xy + 2y^2) + (xy^2 + x^2y)$. Although $\Delta_{L(r)}(t, t) = 2 - 7t + 2t^2$ is stable, $\Delta_{L(r)}(x, y)$ is not stable. Because, if $\Delta_{L(r)}(x, y)$ is stable, then $f(x, y) = x^2 \Delta_{L(r)}(-x^{-1}, y)$ must be stable (Theorem 3.1(b)). Since $f(x, y) = -2 - (x - y) + 3xy - (xy^2 - x^2y) - 2x^2y^2$, it follows that $f(x, x) = -2 + 3x - 2x^2$ must be stable. But, $f(x, x)$ is obviously not stable.

A 2-bridge link with an alternating continued fraction has a very interesting property. In fact, the following theorem is proven.

THEOREM 4.33. *Let $r = [2a_1, 2a_2, \dots, 2a_m]$, m odd. Suppose that $a_j a_{j+1} < 0$, for $1 \le j \le m-1$. Reverse the orientation of one component of $L(r)$ and denote the result by $L'(r)$. Then the reduced Alexander polynomial of $L(r)$ is stable and that of $L'(r)$ is* *c-stable. The Alexander polynomial of $L(r)$ may be unstable as a multivariate polynomial.* (*See Example* 4.32.)

Furthermore, there are many non-2-bridge links with this property. One of such links is shown in Figure 3. The (2 variable) Alexander polynomial $\Delta_L(x,y)$ is $\Delta_L(x,y) = 8 - 12(x+y) + (6x^2 + 31xy + 6y^2) - (x^3 + 24x^2y + 24xy^2 + y^3) + (6x^3y + 31x^2y^2 + 6xy^3) - 12(x^3y^2 + x^2y^3) + 8x^3y^3$. Now the reduced Alexander polynomial of L is $\Delta_L(t) = (t-1)\Delta_L(t,t) = (t-1)(8 - 24t + 43t^2 - 50t^3 + 43t^4 - 24t^5 + 8t^6)$, and $\Delta_L(t)$ is c-stable. On the other hand, let L' be the link obtained from L by reversing the orientation of one component. Then $\Delta_{L'}(t) = (t-1)t^3\Delta_{L'}(t,t^{-1}) = (t-1)(1 - 12t + 48t^2 - 78t^3 + 48t^4 - 12t^5 + t^6) = (t-1)(t^2-6t+1)(t^2-3t+1)^2$ is stable. However, $\Delta_L(x,y)$ is not stable as a multivariate polynomial, since $[y^3\Delta_L(x,-y^{-1})]_{x=y} = 1 - x^6$, that is not stable.

We note that the links we considered above are alternating.

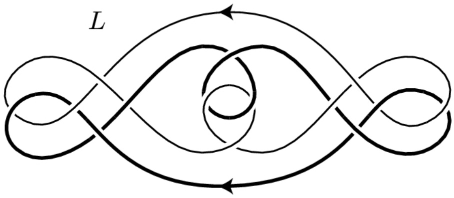

Fig. 3

Question 4.34. Does the stability of the multivariate Alexander polynomial of a link have some connection with this property?

4.5. Open questions

Question 4.35. To what extent does the stability property of the Alexander polynomial of an alternating knot K reflect topological properties of K?

Question 4.36. Characterize stable alternating knots or c-stable alternating knots.

Question 4.37. Let K be a c-stable knot and L be a stable link obtained by Möbius transformation. What can we say about L? Does there exist a geometric way to construct L from K?

Question 4.38. If the zeros of $\Delta_{K_1}(t)$ and $\Delta_{K_2}(t)$ are interlaced, how are K_1 and K_2 related geometrically?

Note added in proof. Conjecture 3.20 on the unimodality for chromatic polynomials has been solved by June Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc. 25 (2012), 907–927.

References

[BB'08] J. Borcea, P. Brändén, *Applications of stable polynomials to mixed determinants: Johnson's conjectures, unimodality and symmetrized Fisher products*, Duke Math. J. 143 (2008), 205–223.

[BB'09] J. Borcea, P. Brändén, *Pólya–Schur master theorems for circular domains and their boundaries*, Ann. of Math. (2) 170 (2009), 465–492.

[BB'10] J. Borcea, P. Brändén, *Multivariate Pólya–Schur classification problems in the Weyl algebra*, Proc. London Math. Soc. (3) 101 (2010), 73–104.

[Br'07] P. Brändén, *Polynomials with the half-plane property and matroid theory*, Adv. Math. 216 (2007), 302–320.

[COSW'04] Y. B. Choe, J. G. Oxley, A. D. Sokal, D. G. Wagner, *Homogeneous multivariate polynomials with the half-plane property*, Adv. in Appl. Math. 32 (2004), 88–187.

[Fox'62] R. H. Fox, *Some problems in knot theory*, In: Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, NJ, 1962, 168–176.

[GR'01] C. Godsil, G. Royle, *Algebraic Graph Theory*, Grad. Texts in Math. 207, Springer, New York, 2001.

[Ha'95] W. H. Haemers, *Interlacing eigenvalues and graphs*, Linear Algebra Appl. 226/228 (1995), 593–616.

[HM'13] M. Hirasawa, K. Murasugi, *Various stabilities of the Alexander polynomials of knots and links*, arXiv:1307.1578v1.

[La'69] P. Lancaster, *Theory of Matrices*, Academic Press, New York, 1969.

[LM'11] L. Lyubich, K. Murasugi, *On zeros of the Alexander polynomial of an alternating knot*, Topology Appl. 159 (2012), 290–303.

[Sc'93] W. Schwarzler, *The coefficients of the Tutte polynomial are not unimodal*, J. Combin. Theory Ser. B 58 (1993), 240–242.

[St'11] A. Stoimenow, *Hoste's conjecture for* 2*-bridge knot*, draft, 2011.

[Wa'11] D. G. Wagner, *Multivariate stable polynomials: theory and applications*, Bull. Amer. Math. Soc. 48 (2011), 53–84.

[WW'09] D. G. Wagner, Y. Wei, *A criterion for the half-plane property*, Discrete Math. 309 (2009), 1385–1390.

[Wi'90] H. S. Wilf, *generatingfunctionology*, Academic Press, Boston, 1990.