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# VIRTUAL KNOT INVARIANTS ARISING FROM PARITIES

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Abstract. In [12, 15] it was shown that in some knot theories the crucial role is played by *parity*, i.e. a function on crossings valued in  $\{0, 1\}$  and behaving nicely with respect to Reidemeister moves. Any parity allows one to construct functorial mappings from knots to knots, to refine many invariants and to prove minimality theorems for knots. In the present paper, we generalise the notion of parity and construct parities with coefficients from an abelian group rather than  $\mathbb{Z}_2$  and investigate them for different knot theories. For some knot theories we show that there is the universal parity, i.e. such a parity that any other parity factors through it. We realise that in the case of flat knots all parities originate from homology groups of underlying surfaces and, at the same time, allow one to "localise" the global homological information about the ambient space at crossings.

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We prove that there is only one non-trivial parity for free knots, the Gaussian parity. At the end of the paper we analyse the behaviour of some invariants constructed for some modifications of parities.

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1. Introduction. In [15], the second named author introduced the notion of *parity* into the study of different knot theories, especially virtual knots: one distinguishes between two types of crossings, those which are *even* and those which are *odd* in a way compatible with the Reidemeister moves so that the parity allows one to refine many invariants, and construct new invariants. In some sense, odd crossings are responsible for non-triviality of link diagrams, and one can prove many minimality and non-triviality theorems starting with some parity. For every concrete parity, one gets explicit counterparts of most of theorems proved in [15].

One goal of the present paper is to generalise the notion of parity and construct the parity with coefficients from an abelian group. Another goal is to classify parities for different knot theories.

The paper is organised as follows. In the next section we recall the definitions of different "knot theories" (i.e. free knots, flat knots, virtual knots, classical knots, etc.) and the main constructions which will be used within the paper.

In Section 3 we introduce the notion of parity with coefficients in an abelian group. In this section we give the main examples of parities for different knot theories. We also describe how one can construct parities from homology classes and indicate how one can construct characteristic homology classes from a knot itself; these classes lead to concrete parities.

Section 4 is devoted to the universal parity. We deduce some basic properties of parity from the parity axioms and show that for some knot theories any parity can be obtained from one parity, the universal parity.

We conclude the paper with some applications of parity. Firstly, we construct a functorial map from knots to knots which allows us to extend some invariants. Secondly, we extend the parity bracket [12] to the parity bracket for any parity valued in  $\{0, 1\}$ . Acknowledgments. The authors are grateful to V. V. Chernov, A. T. Fomenko, L. Kauffman, O. V. Manturov for their interest to this work. The first and third authors were partially supported by grants of the Russian Government 11.G34.31.0053, RF President NSh – 1410.2012.1, RFBR 12-01-31432, 13-01-00664-a and 13-01-00830-a, the second author was partially supported by grants of the Russian Government 11.G34.31.0053, RF President NSh – 1410.2012.1, RFBR 12-01-31432 and 13-01-00830-a.

# 2. Basic definitions

**2.1. Framed 4-graphs and chord diagrams.** By a *graph* we always mean a finite graph; loops and multiple edges are allowed.

Let G be a graph with the set of vertices V(G) and the set of edges E(G). We think of an edge as an equivalence class of the two half-edges forming the edge. From now on, by a 4-graph we mean the following generalisation of a four-valent graph: a one-dimensional complex, with each connected component being homeomorphic either to the circle (with no matter how many 0-cells) or to a four-valent graph; by a vertex we shall mean only vertices of those components which are homeomorphic to four-valent graphs, and by edges we mean either edges of four-valent-graph-components or circular components; the latter will be called *cyclic edges*.

We say that a 4-graph is *framed* if for every vertex of it, the four emanating half-edges are split into two pairs. We call half-edges from the same pair *opposite*. We shall also apply the term opposite to edges containing opposite half-edges. By an *isomorphism* of framed 4-graphs we assume a framing-preserving homeomorphism. All framed 4-graphs are considered up to isomorphism. Denote by  $G_0$  the framed 4-graph homeomorphic to the circle. By a *unicursal component* of a framed 4-graph we mean either its connected component homeomorphic to the circle or an equivalence class of its edges, where the equivalence is generated by the relation of being opposite.

DEFINITION 2.1. By a *chord diagram* we mean a cubic graph consisting of one selected Hamiltonian cycle (a cycle passing through all vertices of the graph). We call this cycle the *core circle* of the chord diagram. Edges of the graph that do not belong to the core circle are called *chords*. A chord diagram is *oriented* whenever its core circle is oriented. Edges belonging to the core circle are called *arcs* of the chord diagram. One distinguishes between *oriented* and *non-oriented* chord diagrams depending on whether an orientation of the core circle is given or not. A chord diagram is depicted on the plane as the Euclidean circle with a collection of chords connecting end points of chords.

For a chord diagram D, the corresponding framed 4-graph G(D) (defined up to isomorphism) with a unique unicursal component is constructed as follows. If the set of chords of D is empty then the corresponding graph will be  $G_0$ . Otherwise, the edges of the graph are in one-to-one correspondence with the arcs of the chord diagram, and the vertices are in one-to-one correspondence with chords of D. The arcs incident to the same chord end correspond to the (half)edges which are formally opposite at the vertex corresponding to the chord.



Fig. 1. Two smoothings of a vertex of a framed graph

The inverse procedure (of constructing a chord diagram from a framed 4-graph with one unicursal component) is evident. In this situation every connected framed 4-graph can be considered as a topological space obtained from the circle by identifying some pairs of points. Thinking of the circle as the core circle of a chord diagram, where the pairs of identified points will correspond to chords, one obtains a chord diagram. The chord diagram obtained from a framed 4-graph with one unicursal component in this way is called a *Gauss diagram*.

DEFINITION 2.2. We say that two chords a and b of a chord diagram D are *linked* if the ends of the chord b belong to two different connected components of the complement to the ends of a in the core circle of D. Otherwise we say that chords are *unlinked*.

We say that two vertices of a framed 4-graph G are *linked* if the corresponding chords of its Gauss diagram are linked.

Define an operation on framed 4-graphs.

DEFINITION 2.3. By a smoothing of a framed 4-graph G at a vertex v we mean any of the two framed 4-graphs obtained from G by removing v and repasting the edges, see Fig. 1. The rest of the graph (together with all framings at vertices except v) remains unchanged.

Note that we may consider further smoothings at *several* vertices. Later on, by a *smoothing* we mean a sequence of smoothings at several vertices.

**2.2.** Virtual knots, flat knots and free knots. In this subsection we consider some knot theories. Let us give main definitions.

A virtual diagram is a framed 4-graph immersed in  $\mathbb{R}^2$  with a finite number of intersections of edges. Moreover, each intersection is a transverse double point which we call a virtual crossing and mark by a small circle, and each vertex of the graph is endowed with the classical crossing structure (with a choice for underpass and overpass specified). The vertices of the graph with that additional structure are called *classical crossings* or just crossings.



Fig. 2. The detour move

A virtual link is an equivalence class of virtual diagrams modulo generalised Reidemeister moves. The latter consist of the usual Reidemeister moves referring to classical crossings and the *detour move* that replaces one arc containing only virtual (self-)intersections by another arc of such sort in any other place of the plane, see Fig. 2.

When drawing framed graphs on the plane, we always assume that the framing is induced from the plane. In figures depicting moves we always take into consideration that each side of the move shows a small area of the diagram homeomorphic to a disc.

REMARK 2.1. If we consider embeddings of framed 4-graphs with the classical crossing structure at each vertex and the usual Reidemeister moves on them, then we get classical diagrams and classical links.

A flat diagram is defined as an immersion of a framed 4-graph in  $\mathbb{R}^2$  (without overcrossing and undercrossing structure at the vertices). We can then define an equivalence relation on flat diagrams specified using the flattened Reidemeister moves and detour move (these moves appear if we flatten the classical crossings in the Reidemeister moves and the detour move to double points, i.e. we just disregard over/undercrossing information). As a result we get a new object — a *flat knot*. It is easy to see that flat knots are equivalence classes of virtual knots modulo transformation swapping over/undercrossing structure.

J. S. Carter, S. Kamada and M. Saito showed that we can consider virtual knots as equivalence classes of embedded framed 4-graphs on compact oriented surfaces [3], where two knots are equivalent if there exists a finite sequence of stabilisations and Reidemeister moves transforming one knot to the other. The same is true for flat knots.

Let K be a virtual diagram, and let S be a closed oriented 2-surface. We call the pair P = (S, K) a canonical link surface diagram (CLSD) if there exists an embedding of the underlying framed 4-graph of K into S such that the complement to the image of this embedding is a disjoint union of 2-cells. Denote by  $\tilde{S}$  a neighbourhood of the embedding of K in S. For a CLSD, P = (S, K), if there exists an orientation preserving embedding  $f : \tilde{S} \to M$  into a closed oriented surface M, we call f(K) a diagram realisation of K in M. Two CLSD's P = (S, K) and P' = (S', K') are related by an abstract Reidemeister move if there is a closed oriented surface M and diagram realisations of K and K' in M which are related by a Reidemeister move in M. Two CLSD's are equivalent if they are related by a finite sequence of abstract Reidemeister moves. Following N. Kamada and



Fig. 3. The local structure

S. Kamada [10] one can construct a bijection

 $\psi$ : {virtual link diagrams}  $\rightarrow$  {CLSD's}.

The idea of this map is illustrated in Fig. 3. Having a virtual link diagram K, we take all classical crossings of it and associate with a neighbourhood of a crossing two crossing bands — a 'piece of 2-surface', and with a virtual crossing we associate a pair of skew bands (when drawing on the plane it does not matter which band is over and which one is under). If we connect these crossings and bands by (non-overtwisted) bands going along edges, we get a 2-surface with boundary. Gluing its boundary components by discs, we get an orientable closed 2-surface. We call  $\psi(L)$  a *CLSD associated with a virtual diagram L*.

We have defined virtual knots and flat knots by using their diagrams which are obtained by immersions of framed 4-graphs in the plane. Let us now consider abstract framed 4-graphs and define the equivalence relation between two graphs using moves analogous to the Reidemeister moves. Recall that in figures depicting moves on diagrams we draw only the changing parts; the stable part will be omitted. In the case of one unicursal component a move can be represented on a Gauss diagram; it changes the diagram on some set of arcs; we shall not draw those chords away from the Reidemeister move being performed; the arcs having no ends of chords taking part in the move will be depicted by dotted lines.



Fig. 4. The first Reidemeister move and its chord diagram version

DEFINITION 2.4. The first Reidemeister move is an addition/removal of a loop, see Fig. 4. The second Reidemeister move is an addition/removal of a bigon formed by a pair of edges which are adjacent (not opposite) in each of the two vertices, see Fig. 5.

The third Reidemeister move is shown in Fig. 6.



Fig. 5. The second Reidemeister move and its chord diagram version



Fig. 6. The third Reidemeister move and its chord diagram version

REMARK 2.2. In the cases of the second Reidemeister move and third Reidemeister move we have one picture for a framed 4-graph and several pictures for chord diagrams. The number of the pictures for chord diagrams depends on ways of joining the ends of edges for framed 4-graphs.

DEFINITION 2.5. A *free link* is an equivalence class of framed 4-graphs modulo Reidemeister moves. It is evident that the number of components of a framed 4-graph does not change after applying a Reidemeister move, so, it makes sense to talk about the *number of components* of a free link.

By a *free knot* we mean a free link with one unicursal component. Free knots can be treated as equivalence classes of Gauss diagrams by a finite sequence of Reidemeister moves (on chord diagrams).

The free unknot (resp., the free *n*-component unlink) is the free knot (link) represented by  $G_0$  (resp., by *n* disjoint copies of  $G_0$ ).

The exact statement connecting virtual knots and free knots sounds as follows:

LEMMA 2.1. A free knot is an equivalence class of virtual knots modulo two transformations: classical crossing switches and virtualisations.

A virtualisation is a local transformation shown in Fig. 7.



Fig. 7. The virtualisation move

One may think of a virtualisation as way of changing the immersion of a framed 4-graph in plane.

#### 3. The definition of the parity

**3.1. Category of knot diagrams.** Let  $\mathcal{K}$  be a knot. We shall use the notion of 'knot' in one of the following situations:

- 1. a free knot;
- 2. a homotopy class of curves immersed in a given surface;
- 3. a flat knot;
- 4. a virtual knot.

Let us define the category  $\mathfrak{K}$  of diagrams of the knot  $\mathcal{K}$ . The objects of  $\mathfrak{K}$  are diagrams of  $\mathcal{K}$  and morphisms of the category  $\mathfrak{K}$  are (formal) compositions of *elementary* morphisms. By an elementary morphism we mean:

- an isotopy of diagram;
- a Reidemeister move.

Isotopy of diagrams is defined as an isotopy the surface where the diagrams lie ( $\mathbb{R}^2$  for diagrams of flat or virtual knots). For free knots we consider isomorphisms of framed 4-graphs.

DEFINITION 3.1. A partial bijection of sets X and Y is a triple  $(\widetilde{X}, \widetilde{Y}, \phi)$ , where  $\widetilde{X} \subset X$ ,  $\widetilde{Y} \subset Y$  and  $\phi : \widetilde{X} \to \widetilde{Y}$  is a bijection.

REMARK 3.1. Since the number of vertices of a diagram may change under Reidemeister moves, there is no bijection between the sets of vertices of two diagrams connected by a sequence of Reidemeister moves. To construct any connection between two sets of vertices we have introduced the notion of a partial bijection which means just the bijection between the subsets of vertices corresponding to each other in the two diagrams.

Let us denote by  $\mathcal{V}$  the vertex functor on  $\mathfrak{K}$ , i.e. a functor from  $\mathfrak{K}$  to the category  $\mathfrak{S}$ , the objects of which are finite sets and whose morphisms are partial bijections. For each diagram K we define  $\mathcal{V}(K)$  to be the set of classical crossings of K, i.e. the vertices of the underlying framed 4-graph. Any elementary morphism  $f: K \to K'$  naturally induces a partial bijection  $f_*: \mathcal{V}(K) \to \mathcal{V}(K')$ .

**3.2.** A parity. Now we are going to define a parity with coefficients in an arbitrary abelian group. In [12, 13, 15, 18] the parity with coefficients in  $\mathbb{Z}_2$  was defined. We extend that notion to the case with an abelian group. Note that one can define a parity with a non-abelian group, see, for example, [17].

Let A be an abelian group.

DEFINITION 3.2. A parity p on diagrams of a knot  $\mathcal{K}$  with coefficients in A is a family of maps  $p_K : \mathcal{V}(K) \to A, K \in ob(\mathfrak{K})$ , such that for any elementary morphism  $f : K \to K'$  the following holds:

- 1.  $p_{K'}(f_*(v)) = p_K(v)$  provided that  $v \in \mathcal{V}(K)$  and there exists  $f_*(v) \in \mathcal{V}(K')$ ;
- 2.  $p_K(v_1) + p_K(v_2) = 0$  if f is a decreasing second Reidemeister move and  $v_1, v_2$  are the disappearing crossings;
- 3.  $p_K(v_1) + p_K(v_2) + p_K(v_3) = 0$  if f is a third Reidemeister move and  $v_1, v_2, v_3$  are the crossings participating in this move.

REMARK 3.2. Note that each knot can have its own group A, and, therefore, different knots generally have different parities.

LEMMA 3.1. Let p be any parity and K be a diagram. Then  $p_K(v) = 0$  if f is a decreasing first Reidemeister move applied to K and v is the disappearing crossing of K.

*Proof.* Let us apply the second Reidemeister move g to the diagram K as is shown in Fig. 8. We have

$$p_{K'}(v_1) + p_{K'}(v_2) = 0,$$
  $p_{K'}(g_*(v)) + p_{K'}(v_1) + p_{K'}(v_2) = p_K(v) = 0.$ 

Let us consider some examples of parities for some knot theories.

**3.2.1.** Gaussian parity for free, flat and virtual knots. Let  $A = \mathbb{Z}_2$  and K be a virtual (flat) knot diagram (resp., a framed 4-graph with one unicursal component).

Define the map  $gp_K : \mathcal{V}(K) \to \mathbb{Z}_2$  by putting  $gp_K(v) = 0$  if the number of vertices linked with v is even (an even crossing), and  $gp_K(v) = 1$  otherwise (an odd crossing).



Fig. 8. Reduction of the first Reidemeister move to the second and third Reidemeister moves

LEMMA 3.2 ([15]). The map gp is a parity for free, flat and virtual knots.

DEFINITION 3.3. The parity gp is called the *Gaussian parity*.

**3.3. Parity and homology.** In this section we consider parities with coefficients in  $\mathbb{Z}_2$  (unless otherwise stated).

A natural source of parities comes from one-dimensional  $\mathbb{Z}_2$ -(co)homology classes of the underlying surface of a (virtual) knot. We shall see that if we consider curves in a given closed 2-surface then (modulo some restrictions) these homology classes will lead to well-defined parities for knots on such surfaces (the same works for virtual knots in the thickening of this surface). The inverse statement is also true: if we take a given parity on a given surface, then it will lead to a certain  $\mathbb{Z}_2$ -homology class of the surface.

So, when we have a knot and a fixed surface associated with it, this gives us a universal receipt of constructing parities and leads us to the universal parity, see ahead.

However, when passing to virtual knots by means of the stabilisation, this causes the following trouble: the surface is not fixed any more and there is no canonical coordinate system on this surface. Thus, for example, if we work on a concrete torus, we may fix a coordinate system on it and take the parity corresponding to the 'meridian'. However, when we stabilise and destabilise, we may destroy the coordinate system on the surface, so it will be impossible to recover the initial (co)homology class.

To this end, we introduce the notion of a characteristic class for underlying surfaces corresponding to virtual knots (see rigorous definition ahead). This is a class which does not depend on anything except a given virtual knot and behaves nicely on surfaces coming from diagrams, in particular, under stabilisations/destabilisations.

We give some concrete examples of constructing characteristic classes.

As we shall see later, this approach does not always help: for the flat knot diagram (in Fig. 9) on the surface of genus 2 (the surface is represented as a decagon with opposite sides identified) is so symmetric, that every characteristic class of it is trivial (see Example 3.1), though when we restrict ourselves to this concrete surface of genus 2, there will be non-trivial parities which have non-zero values on the crossings of the flat knot diagram.

To overcome this difficulty, we enlarge the notion of parity. Instead of a parity valued in  $\mathbb{Z}_2$ , we introduce the universal parity valued in some linear space over  $\mathbb{Z}_2$  which is



Fig. 9. A knot in a surface of genus two



Fig. 10. The graphs  $K_{v,1}$  and  $K_{v,2}$ 

closely related to knot diagrams (the  $\mathbb{Z}_2$ -homology group of the underlying space with a fixed basis) and see that all previously known  $\mathbb{Z}_2$ -valued parities factor through this universal parity.

This parity allows one to work with examples where characteristic classes and their corresponding parities fail.

First of all we describe a connection between a parity and the homologies of a surface.

**3.3.1.** Homological parity for homotopy classes of curves generically immersed in a surface. Let S be a connected closed surface. We consider a free homotopy class  $\mathcal{K}$  of curves generically immersed in S.

Let  $A = H_1(S, \mathbb{Z}_2)/[\mathcal{K}]$ , where  $[\cdot]$  denotes a homological class.

Let K be a framed 4-graph embedded in S representing a curve from  $\mathcal{K}$ . For each vertex v we have two halves of the graph,  $K_{v,1}$  and  $K_{v,2}$ , obtained by smoothing at this vertex, see Fig. 10.

Define the map  $hp_K : \mathcal{V}(K) \to A$  by putting  $hp_K(v) = [K_{v,1}]$ .

LEMMA 3.3 ([15]). The map hp is a parity for homotopy classes of curves generically immersed in S.

*Proof.* From the definition of A it follows that hp does not depend on the choice of a half for a vertex.

Let  $f: K \to K'$  be an elementary morphism.



Fig. 11. The second Reidemeister move



Fig. 12. The third Reidemeister move

1) Since Reidemeister moves are performed in a small area of S homeomorphic to a disc, we have  $hp_{K'}(f_*(v)) = hp_K(v)$  provided that  $v \in \mathcal{V}(K)$  and there exists  $f_*(v) \in \mathcal{V}(K')$ .

2) Let f be a decreasing second Reidemeister move, and let  $v_1, v_2$  be the disappearing crossings. Denote by  $K_{v_1,1}$  and  $K_{v_2,1}$  the two halves corresponding to the vertices  $v_1$  and  $v_2$ , see Fig. 11.

We have

$$hp_{K}(v_{1}) + hp_{K}(v_{2}) = [K_{v_{1},1}] + [K_{v_{2},1}] = [K_{v_{1},1}] + [K_{v_{2},1}] + [\gamma] = [K] = 0.$$

3) Let f be a third Reidemeister move, and let  $v_1, v_2, v_3$  be the crossings participating in this move. Denote by  $K_{v_1,1}$ ,  $K_{v_2,1}$  and  $K_{v_3,1}$  the three halves corresponding to  $v_1, v_2$ and  $v_3$  respectively, see Fig. 12 (we consider only one case depicted in Fig. 12, all other versions of the third Reidemeister move can be treated in the same way). We have

$$\begin{split} hp_K(v_1) + hp_K(v_2) + hp_K(v_3) &= [K_{v_1,1}] + [K_{v_2,1}] + [K_{v_3,1}] \\ &= [K_{v_1,1}] + [K_{v_2,1}] + [K_{v_3,1}] + [\gamma] = [K] = 0. \quad \blacksquare \end{split}$$

**3.3.2.** Characteristic classes for framed 4-graphs. Our next task is to understand the topological nature of parity. As we shall see, when we deal with curves on a fixed surface, all possible parities for such curves are closely connected with (co)homology classes with coefficients in  $\mathbb{Z}_2$ .

However, when we deal with virtual knots or knots in an abstract thickened surface, then there is no canonical choice of the coordinate system on the surface, so we cannot say what is a 'cohomology class dual to the longitude' or a 'cohomology class dual to the meridian'. Moreover, cohomology classes have to be chosen in a way compatible with stabilisations.

There is a partial remedy which deals with so-called characteristic classes. Roughly speaking, a characteristic class is a class on the surface corresponding to a knot diagram which can be recovered from the diagram itself. This will be discussed in Section 3.3.3.

Consider a framed 4-graph K with one unicursal component. The homology group  $H_1(K, \mathbb{Z}_2)$  is generated by halves corresponding to vertices. If the set of framed 4-graphs (possibly, with some further decorations at vertices) is endowed with a *parity*, then we can construct the following cohomology class h: for each of the halves  $K_{v,1}, K_{v,2}$  we set  $h(K_{v,1}) = h(K_{v,2}) = p_K(v)$ , where  $p_K(v)$  is the parity of the vertex v. Taking into account that every two halves for each vertex sum up to give the cycle generated by the whole graph, we have defined a "characteristic" cohomology class h from  $H_1(K, \mathbb{Z}_2)$ .

Collecting the properties of this cohomology class we see that:

- 1. For every framed 4-graph K we have h(K) = 0.
- 2. Let K' be obtained from K by a second Reidemeister move increasing the number of crossings by two. Then for every basis  $\{\alpha_i\}$  of  $H_1(K, \mathbb{Z}_2)$  there exists a basis in  $H_1(K', \mathbb{Z}_2)$  consisting of one "bigon"  $\gamma$ , the elements  $\alpha'_i$  naturally corresponding to  $\alpha_i$  and one additional element  $\delta$ , see Fig. 13, left. Then  $h(\alpha_i) = h(\alpha'_i), h(\gamma) = 0$ .
- 3. Let K' be obtained from K by a third Reidemeister move. Then there exists a graph K'' with one vertex of valency 6 and the other vertices of valency 4 which is obtained from either of K or K' by contracting the "small" triangle to the point. This generates the mappings  $i : H_1(K, \mathbb{Z}_2) \to H_1(K'', \mathbb{Z}_2)$  and  $i' : H_1(K', \mathbb{Z}_2) \to H_1(K'', \mathbb{Z}_2)$ , see Fig. 13, right.

We require the following to hold: the cocycle h is equal to zero for small triangles, besides that if for  $a \in H_1(K, \mathbb{Z}_2)$ ,  $a' \in H_1(K', \mathbb{Z}_2)$  we have i(a) = i'(a'), then h(a) = h(a').

Note that in 2 no restriction on  $h(\delta)$  is imposed.

Thus, every parity defined for all free knots generates some  $\mathbb{Z}_2$ -cohomology class for all framed 4-graphs with one unicursal component, and this class behaves nicely under the Reidemeister moves.



Fig. 13. The cohomology condition for Reidemeister moves

The converse is true as well. Assume we are given a certain "universal"  $\mathbb{Z}_2$ -cohomology class for all framed 4-graphs satisfying the conditions 1–3 described above (later we shall describe the exact definition of the universality). Then it originates from some *parity*. Indeed, it is sufficient to define the parity of every vertex to be the parity of the corresponding half. The choice of a particular half does not matter, since the value of the cohomology class on the whole graph is zero. One can easily check that parity axioms follow.

This point of view allows one to find parities for those knots lying in  $\mathbb{Z}_2$ -homologically nontrivial manifolds. For more details, see [19].

**3.3.3.** Characteristic parities for virtual knots. Let K be a virtual knot diagram, and let P = (S, K) be the CLSD associated with the diagram K. A checkerboard colouring of S with respect to K is a colouring of all the components of  $S \setminus K'$ , where K' is the image of the embedding of K, by two colours, say black and white, such that two components of  $S \setminus K'$  being adjacent by an edge of K' have always distinct colours.

We say that a virtual diagram *admits a checkerboard colouring* or it is *checkerboard colourable* if the associated CLSD admits a checkerboard colouring.

THEOREM 3.1 ([6]). If two virtual diagrams admitting a checkerboard colouring are equivalent in the category of virtual knots, then they are equivalent in the category of virtual knots admitting a checkerboard colouring (i.e. there is a sequence of moves that transforms one diagram into the other such that all the intermediate diagrams admit a checkerboard colouring).

We consider the category of virtual knots admitting a checkerboard colouring.

DEFINITION 3.4. A characteristic class of a knot  $\mathcal{K} = \{K\}$  is a homology class of the surface S associated with a diagram K such that this class depends only on  $\mathcal{K}$  and behaves nicely under the Reidemeister moves (in other words, the class can be reconstructed from any diagram K but it does not depend of the choice of diagram).

Consider the group  $H_1(S, \mathbb{Z}_2)$  and any element  $[\gamma] \in H_1(S, \mathbb{Z}_2)$ . We know that [K']=0. Define the map  $\chi_{K,\gamma} : \mathcal{V}(K) \to \mathbb{Z}_2$  by putting  $\chi_{K,\gamma}(v)$  to be equal to the intersection number of  $\gamma$  and  $K'_{v,1}$ , where  $K'_{v,1}$  is a half of K' corresponding to v.

Our aim is to construct a homology class of  $\gamma$ , which depends only on a virtual knot generated by K, and defines a parity on the virtual knot.

Consider the following cases.

1) Let  $\gamma_a$  be the sum of halves over all classical crossings (for each classical crossing we take only one half).

2) Let  $\mathcal{L}$  be an arbitrary non-trivial free link with two linked components. At each vertex of K we can consider a smoothing giving the link diagram with two components. We say that a classical crossing v of K leads to  $\mathcal{L}$  if after a smoothing it and considering the result just as a framing 4-graph we get a diagram of  $\mathcal{L}$ . Let us define

$$\gamma_{\mathcal{L}}(K) = \sum_{v} K'_{v,1},$$

where the sum is taken over all classical crossings giving a diagram of  $\mathcal{L}$ .

THEOREM 3.2. The maps  $\chi_{K,\gamma_a}$  and  $\chi_{K,\gamma_{\mathcal{L}}}$  are parities for virtual knots with coefficients in  $\mathbb{Z}_2$ .

*Proof.* We consider only the map  $\chi_{K,\gamma_{\mathcal{L}}}$ .

Let  $f : K_1 \to K_2$  be an elementary morphism of two knot diagrams. Consider two CLSD's  $P_1 = (S_1, K_1)$  and  $P_2 = (S_2, K_2)$  associated with  $K_1$  and  $K_2$ , respectively. It is sufficient to consider two cases:

1) If  $S_1$  and  $S_2$  have the same genus, then the virtue of the claim follows from Lemma 3.3.

2) If the genus of  $S_2$  is smaller than the genus of  $S_1$  by 1, then f is a decreasing second Reidemeister move, see Fig. 14.

As  $\mathcal{L}$  is a free link then the classical crossings  $v_1$  and  $v_2$  participating in the move either simultaneously give the free link  $\mathcal{L}$  or do not give it.

Denote by  $K'_i$  the image of  $K_i$  in  $S_i$ . As any half of any classical crossing of  $K'_1$  intersects any half of a classical crossing distinct from  $v_1$  and  $v_2$  either at 0 or precisely two of  $v_1$ ,  $v_2$  and we can pick halves  $K'_{v_1,i}$  and  $K'_{v_2,j}$  in such a way that they are homotopic as curves on  $S_1$ , we get

$$\chi_{K_1,\gamma_{\mathcal{L}}}(v_1) + \chi_{K_1,\gamma_{\mathcal{L}}}(v_2) = 0,$$

and  $\chi_{K_2,\gamma_{\mathcal{L}}}(f_*(v)) = \chi_{K_1,\gamma_{\mathcal{L}}}(v)$  provided that  $v \in \mathcal{V}(K_1)$  and there exists  $f_*(v) \in \mathcal{V}(K_2)$ .

EXAMPLE 3.1. Consider the knot diagram K depicted in Fig. 9. It is not difficult to show that we have the non-trivial map  $hp_K : \mathcal{V}(K) \to H_1(S, \mathbb{Z}_2)/[\mathcal{K}]$ . The image of this map is the subgroup of  $H_1(S, \mathbb{Z}_2)/[\mathcal{K}]$  generated by 5 elements  $a_i = hp_K(v_i)$  with the relations  $a_1 + a_2 + a_3 + a_4 + a_5 = 0$ , cf. [17].

But if we want to construct a characteristic parity with the methods described above we shall fail. K is symmetric so that all five crossings have the same parity, say p. Since we have pentagon, we get 5p = 0 and, then, p = 0.



Fig. 14. A second Reidemeister move adds a handle

Let K be an oriented knot diagram. At each classical vertex we have one smoothing respecting the orientation on K. We can construct parity  $\chi_{K,\mathcal{L}}$  with an oriented free link  $\mathcal{L}$  having two unicursal components by taking the sum only over classical crossings whose smoothings give  $\mathcal{L}$ .

Let  $\mathcal{L}$  be a non-invertible free link with two unicursal components [14], see, for example, Fig. 15. If a vertex of an oriented knot leads to  $\mathcal{L}$ , then this vertex does most probably not lead to  $\overline{\mathcal{L}}$ , where  $\overline{\mathcal{L}}$  is the free link obtained from  $\mathcal{L}$  by reversion of the orientation. It means that a parity does feel an orientation on diagrams.

4. The universal parity. In Section 3, we have given a receipt how to construct parities from homology classes and indicated how to construct characteristic homology classes from the knot itself; these classes lead to concrete parities. However, when we apply such characteristic classes to the knot in Fig. 9, we see that all corresponding parities vanish. Nevertheless, the corresponding flat knot lies in a surface  $S_2$  of genus 2 and is not contractible. So, there are some homology classes (which are presumably not characteristic) which yield some parity for some coordinate system of  $S_2$  which is non-trivial on some vertices of the knot. The idea of the present section is to construct the universal parity, cf. [17], valued in a certain group related to the knot rather than the group  $\mathbb{Z}_2$ . This parity will be universal in the sense that any concrete parity on a given surface factors through the universal one.

DEFINITION 4.1. A parity  $p_u$  with coefficients in  $A_u$  is called a *universal parity* if for any parity p with coefficients in A there exists a unique homomorphism of groups  $\rho : A_u \to A$  such that  $p_K = \rho \circ (p_u)_K$  for any diagram K.

Let us describe a construction of the universal parity in general case.



Fig. 15. A non-invertible free link

Let K be a knot diagram. Denote by  $1_{K,v}$  the generator of the direct summand in the group  $\bigoplus_K \bigoplus_{v \in \mathcal{V}(K)} \mathbb{Z}$  corresponding to the vertex v of K.

Let  $A_u$  be the group

$$A_u = \left(\bigoplus_K \bigoplus_{v \in \mathcal{V}(K)} \mathbb{Z}\right) / \mathcal{R},$$

where  $\mathcal{R}$  is the set of relations of four types:

- 1.  $1_{K',f_*(v)} = 1_{K,v}$  if  $v \in \mathcal{V}(K)$  and there exists  $f_*(v) \in \mathcal{V}(K')$ ;
- 2.  $1_{K,v_1} + 1_{K,v_2} = 0$  if f is a decreasing second Reidemeister move and  $v_1, v_2$  are the disappearing crossings;
- 3.  $1_{K,v_1} + 1_{K,v_2} + 1_{K,v_3} = 0$  if f is a third Reidemeister move and  $v_1, v_2, v_3$  are the crossings participating in this move.

The map  $(p_u)_K$  for each diagram K is defined by the formula  $(p_u)_K(v) = 1_{K,v}$ ,  $v \in \mathcal{V}(K)$ .

If p is a parity with coefficients in a group A, one defines the map  $\rho: A_u \to A$  in the following way:

$$\rho\left(\sum_{K, v \in \mathcal{V}(K)} \lambda_{K,v} \mathbf{1}_{K,v}\right) = \sum_{K, v \in \mathcal{V}(K)} \lambda_{K,v} p_K(v), \quad \lambda_{K,v} \in \mathbb{Z}.$$

The examples below present an explicit description of the universal parity.

**4.1. Free knots.** In the present subsection we show that in the case of the free knot theory there exists only one non-trivial parity, the Gaussian parity.

THEOREM 4.1. Let  $\mathcal{K}$  be a free knot. Then the Gaussian parity (with coefficients in  $\mathbb{Z}_2$ ) on diagrams of  $\mathcal{K}$  is the universal parity.



Fig. 17. A hexagon

REMARK 4.1. Theorem 4.1 means that for each free knot and for each parity on it either all vertices are even or they have the Gaussian parity.

This theorem will follow from Lemmas 4.1, 4.2, 4.3.

We consider free knots as Gauss diagrams with an ordered collection of distinct chords  $\{a_1, \ldots, a_n\}$ . Let us choose a point distinct from ends of chords on the core circle of a chord diagram. When going around the circle from the chosen point counterclockwise order we will meet each chord end. Denoting each end of a chord by the same letter as the chord we will get a word, where each letter corresponds to a chord and occurs precisely twice.

DEFINITION 4.2. Let D be a chord diagram. We will say that an ordered collection of chords with numbers  $i_1, \ldots, i_k$  of D forms a *polygon*, if a word, corresponding to D for some basepoint on the core circle, contains the following sequences of distinct letters  $b_{2p-1}b_{2p}$ , where  $b_{2p-1}, b_{2p} \in \{a_{i_{\sigma(p)}}, a_{i_{\sigma(p-1)}}\}, p = 1, \ldots, k$ , for some permutation  $\sigma \in S_k$ .

The pairs  $(b_{2p-1}, b_{2p})$  of letters  $b_{2p-1}, b_{2p}$  from the definition of a polygon are said to be *sides* of polygon.

EXAMPLE 4.1. Consider the chord diagrams depicted in Fig. 16. The chords denoted by  $a_2, a_4, a_5, a_6, a_8$  form a pentagon on the left diagram and on the right one.

In Fig. 17 we depict a hexagon for a knot diagram. The knot diagram does not intersect the interior of the hexagon.

LEMMA 4.1. For every parity and any chord diagram the sum of the parities of chords forming a polygon is equal to 0.

REMARK 4.2. The claim of Lemma 4.1 can be taken as a definition of a parity, see [17].

*Proof.* Let p be an arbitrary parity on chord diagrams of the free knot  $\mathcal{K}$ , and let D be a chord diagram representing  $\mathcal{K}$ . Let us prove the claim of the lemma by induction over the number of sides of a polygon.

The induction base. The virtue of the claim for a loop, bigon, triangle follows from Lemma 3.1 and Definition 3.2, respectively.

The induction step. Assume that the claim is true for (k-1)-gons. Let us consider an arbitrary k-gon  $a_{i_1}a_{i_2}\ldots a_{i_k}$ .

Let us apply the second Reidemeister move to the chord diagram D by adding two chords b and c, see Fig. 18 (we have depicted there the three possibilities of applying the second Reidemeister move depending on the ends of chords  $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_k}$ ).

As a result we shall obtain the new chord diagram D' and the (k-1)-gon  $c a_{i_3} a_{i_4} \dots a_{i_k}$ and the triangle  $b a_{i_1} a_{i_2}$ . By the induction hypothesis, we have

$$p_{D'}(c) + \sum_{j=3}^{k} p_{D'}(a_{i_j}) = 0, \quad p_{D'}(b) + p_{D'}(a_{i_1}) + p(a_{i_2}) = 0, \quad p_{D'}(b) + p_{D'}(c) = 0.$$

Therefore,

$$\sum_{j=1}^k p_{D'}(a_{i_j}) = \sum_{j=1}^k p_D(a_{i_j}) = 0. \quad \bullet$$

REMARK 4.3. If we work with knot diagrams, then the corresponding picture for Lemma 4.1 looks like as is shown in Fig. 19.

Let us pass from the free knot theory to the flat knot theory and the virtual knot theory. Since bigons and triangles participating in Reidemeister moves can be spanned by discs we get the following

COROLLARY 4.1. For every parity and any flat (virtual) knot diagram the sum of the parities of crossings forming a polygon, which is spanned by a disc in the underlying surface, is equal to 0.

By using virtualisation moves we can transform any polygon to a polygon which is spanned by a disc in the underlying surface. As a result we get the following

COROLLARY 4.2. If we consider the theory of pseudo-knots, i.e. the theory of virtual knots modulo the virtualisation move, then Lemma 4.1 remains true in this theory too, that is the existence of the writhe number gives us no additional information.

LEMMA 4.2. For a free knot (pseudo-knot) with a diagram K and an arbitrary parity p we have  $p_K(a) = 0$  if  $gp_K(a) = 0$ .

*Proof.* Let p be a parity, and let a be a chord of a chord diagram D with  $gp_D(a) = 0$ . Let us consider the two halves of the core circle of D, which are obtained by removing the chord a. Since  $gp_D(a) = 0$  each half-circle corresponding to a contains an even number of ends of chords. Let us apply the induction over the number of ends of chords.



Fig. 18. The second Reidemeister move



Fig. 19. The second Reidemeister move

The induction base: If the number of ends on any half-circle is equal to 0, then  $p_D(a) = 0$  by using the property of the first Reidemeister move.

The induction step: Assume that for any chord d of D with  $gp_D(d) = 0$  such that a half-circle contains less than n = 2k ends of chords, we have  $p_D(d) = 0$ . Let us consider a chord a such that one of its half-circles,  $K_{a,1}$ , contains exactly n ends of chords and the other one,  $K_{a,2}$ , contains at least n ends.

Let us orient D in counterclockwise manner and consider the following two cases.



Fig. 20. The Gaussian parity zero



Fig. 21. The Gaussian parity zero

1) The first two ends in  $K_{a,1}$  belong to two distinct chords  $a_1, a_2$ , see Fig. 20. Apply the second increasing Reidemeister move by adding a pair of chords b, b' in such a way that the half-circle corresponding to b' would contain the set of ends lying in  $K_{a,1}$  minus the first ends of  $a_1, a_2$ , see Fig. 21 (above). Let us show that  $p_{D'}(a) + p_{D'}(b) = 0$  in the new chord diagram D'. Let us add the pair of chords c, c' to form the triangle  $a_1a_2c$ , see Fig. 21 (below). Then  $p_{D''}(a_1) + p_{D''}(a_2) + p_{D''}(c) = 0$  in D''. Moreover, we have the pentagon  $aa_1ca_2b$  and, therefore, the following equality holds (Lemma 4.1)

$$p_{D''}(a) + p_{D''}(a_1) + p_{D''}(c) + p_{D''}(a_2) + p_{D''}(b) = 0.$$

We get  $p_{D''}(a) + p_{D''}(b) = 0$  and  $p_{D'}(a) + p_{D'}(b) = 0$ . In the half-circle corresponding to b' the number of ends is less than the number of ends in the half-circle corresponding to a. By the induction hypothesis, we get  $p_{D'}(b) = p_{D'}(b') = 0$ , and  $p_D(a) = 0$ .

2) If the first two ends belong to the same chord c, then  $p_D(c) = 0$  (the first Reidemeister move) and c forms the triangle in D' with the chords a and b. Therefore,  $p_{D'}(a) + p_{D'}(b) + p_{D'}(c) = 0$ . By the induction hypothesis, we get  $p_{D'}(b) = p_{D'}(b') = 0$  and  $p_D(a) = p_{D'}(b) = 0$ .

LEMMA 4.3. Let p be an arbitrary parity (with coefficients from a group A) on diagrams of the free knot represented by a chord diagram D. Then for any two chords a, b such that  $gp_D(a) = gp_D(b) = 1$  we have  $p_D(a) = p_D(b) = x \in A$  and 2x = 0.



Fig. 22. The Gaussian parity one

*Proof.* Let  $c_1, \ldots, c_k$  be ends of chords lying between the nearest ends of a and b.

Apply k times the second Reidemeister moves as it is shown in Fig. 22 (in the centre). Let us show that  $p_{D'}(d_l) = (-1)^l x$ , where  $x = p_{D'}(a)$ . Apply the second Reidemeister move by adding two chords f, f' to form the triangle  $ad_1 f$ . We have

$$gp_{D''}(a) = gp_{D''}(d_1) = 1 \Longrightarrow gp_{D''}(f) = 0 \Longrightarrow p_{D''}(f) = 0$$
$$\Longrightarrow p_{D'}(d_1) = p_{D''}(d_1) = -x.$$

By the induction we can prove that  $p_{D'}(d_l) = (-1)^l x$  and  $p_D(b) = (-1)^{k+1} x$ .

Let us apply the third Reidemeister move to the triangle  $ad_1f$ . The parity p and the Gaussian parity of the chord a do not change but the parity of the number of ends of chords between a and b changes. Applying the previous trick we get  $p_D(b) = (-1)^k x$ , i.e. 2x = 0.

By using Lemmas 4.2, 4.3 for any parity p (with coefficients from a group A) on diagrams of the free knot having a diagram K we can construct the homomorphism  $\rho : \mathbb{Z}_2 \to A$  by taking  $\rho(1) = x$ , where  $p_K(a) = x$  and  $gp_K(a) = 1$ . This concludes the proof of Theorem 4.1.

REMARK 4.4. Let p be a parity on a free knot  $\mathcal{K}$ . It is not possible that there exist two diagrams  $K_1$  and  $K_2$  of  $\mathcal{K}$ , both having chords being odd in the Gaussian parity such

that p is trivial on  $K_1$ , and p is the Gaussian parity on  $K_2$ . It follows from the fact that there is a sequence of Reidemeister moves transforming  $K_1$  to  $K_2$  such that any diagram in this sequence has chords being odd in the Gaussian parity.

Before passing to classical knots, we should point out the following. It is known that classical knot and link theories embed in virtual knot and link theories [5, 11]. This means that if two classical knot (link) diagrams are virtually equivalent then they are isotopic (classically equivalent).

Nevertheless, the parity axiomatic applied to the classical knot theory as a part of the virtual knot theory and to the classical knot theory as it is, should be treated differently.

Namely, from the above we get the following

THEOREM 4.2. Any parity on virtual knots (one-component knots, not links) is trivial on any classical knots.

By itself, it does not guarantee that there is no non-trivial parity on classical knots: possibly, there might be some which does not extend to virtual knots? Indeed, for the classical knot theory as it is we are restricted only to those diagrams having classical crossings, and some "additional" crossing used to prove the above lemmas can make the diagram classical.

However, the following theorem holds as well.

THEOREM 4.3. For classical knot theory there exists a unique parity — the trivial parity.

The proof is indeed a slight modification of Theorem 4.1, which is based on Lemmas 4.2, 4.3. We just use classical knot diagrams on the plane and bear in mind Corollary 4.1.

**4.2. Homotopy classes of curves generically immersed in a surface.** In the previous subsection we have the situation when all polygons "are spanned" by discs on the plane. Now we are interested in those polygons which are spanned by discs in a surface. As a result we deal with the homology of the surface.

THEOREM 4.4. Let  $\mathcal{K}$  be a homotopy class of curves generically immersed in a surface S. Then the homological parity (with coefficients in  $H_1(S,\mathbb{Z}_2)/[\mathcal{K}]$ ) is the universal parity on curves of  $\mathcal{K}$ .

*Proof.* We start the proof of the theorem with the following general lemmas.

LEMMA 4.4. Let p be a parity, K be a curve on S and  $a \in \mathcal{V}(K)$ . Then  $2p_K(a) = 0$ .

*Proof.* By applying the second and third Reidemeister moves we get curves  $K_1$  and  $K_2$  (see Fig. 23). We have the equality  $p_{K_1}(a) + p_{K_1}(b) = 0$ . Then  $p_{K_2}(a) + p_{K_2}(b) = 0$ . We also have  $p_{K_2}(a) + p_{K_2}(c) + p_{K_2}(d) = 0$  and  $p_{K_2}(b) + p_{K_2}(c) + p_{K_2}(d) = 0$ . Hence,  $p_{K_2}(a) = p_{K_2}(b)$  and  $2p_{K_2}(a) = 0$ . Then  $2p_{K_1}(a) = 0$  and  $2p_{K}(a) = 0$ .

LEMMA 4.5 (cf. [17]). Let K be a framed 4-graph with one unicursal component. Consider K as a one-dimensional cell complex. Then  $H_1(K, \mathbb{Z}_2)/[K] \cong \bigoplus_{v \in \mathcal{V}(K)} \mathbb{Z}_2$ .

*Proof.* Let C be the chord diagram corresponding to K. Then C and K are homotopy equivalent as topological spaces. Let C' (resp., K') be the topological space obtained by



Fig. 23. The second and third Reidemeister moves

gluing to C (resp., K) a 2-disc along the core circle of C. Then C' and K' are homotopy equivalent also and  $H_1(C', \mathbb{Z}_2) \cong H_1(K', \mathbb{Z}_2) = H_1(K, \mathbb{Z}_2)/[K]$ . On the other hand, C' is homotopy equivalent to the bouquet of circles corresponding to the cords of the diagram C, i.e. the crossings of K. Hence,  $H_1(C', \mathbb{Z}_2) \cong \bigoplus_{v \in \mathcal{V}(K)} \mathbb{Z}_2$ .

The isomorphism of the lemma identifies the generator of the group  $\mathbb{Z}_2$  corresponding to a vertex  $v \in \mathcal{V}(K)$  with the homology class  $[K_{v,1}] = [K_{v,2}] \in H_1(K, \mathbb{Z}_2)/[K]$ .

LEMMA 4.6. Let  $\omega$  be a closed path on the curve K which turns at points  $v_1, v_2, \ldots, v_k$ . Then  $[\omega] = \sum_{i=1}^{k} [K_{v_i}] \in H_1(K, \mathbb{Z}_2)/[K]$ .

*Proof.* By attaching a half  $K_{v_i,j}$  for each vertex  $v_i$  to the path  $\omega$  we get a closed path without turning points, i.e. a multiple of K. Thus,

$$[\omega] + \sum_{i=1}^{k} [K_{v_i}] = m[K] = 0. \quad \bullet$$

Let us return now to the proof of Theorem 4.4.

Let p be a parity with coefficients in a group A on curves of a homotopy class  $\mathcal{K}$  on a closed 2-surface S.

Let K be a curve from  $\mathcal{K}$  on the surface S. Assume that K splits the surface into a union of 2-cells. Arguing as above in Lemma 4.1, we obtain the following:

LEMMA 4.7. Let e be a cell in  $S \setminus K$  with vertices  $v_1, \ldots, v_k$  (not necessarily distinct). Then  $\sum_{i=1}^k p_K(v_i) = 0$ .

Let us show that the map  $\rho_K : H_1(S, \mathbb{Z}_2)/[\mathcal{K}] \to A$  given by the formula  $\rho([K_{v,1}]) = p_K(v), v \in \mathcal{V}(K)$ , is well defined.

The group  $H_1(S, \mathbb{Z}_2)/[K]$  is the first homology group of the topological space S' obtained from S by gluing a disc along K. S' can also be considered as the result of gluing cells  $e \in S \setminus K$  to the space K' of Lemma 4.5. Hence,

$$H_1(S, \mathbb{Z}_2)/[K] = (H_1(K', \mathbb{Z}_2)/[K]) / ([\partial e], \ e \in S \setminus K)$$
$$= \bigoplus_{v \in \mathcal{V}(K)} \mathbb{Z}_2[K_{v,1}] / \left(\sum_{v \in e \cap \mathcal{V}(K)} [K_{v,1}] = 0, \ e \in S \setminus K\right)$$
$$= \bigoplus_{v \in \mathcal{V}(K)} \mathbb{Z} \mathbf{1}_{K,v} / \left(2 \cdot \mathbf{1}_{K,v} = 0, \ v \in \mathcal{V}(K); \ \sum_{v \in e \cap \mathcal{V}(K)} \mathbf{1}_{K,v} = 0, \ e \in S \setminus K\right).$$

The second equality follows from Lemmas 4.5, 4.6.

On the other hand, due to Lemmas 4.4 and 4.7 we have identities  $2p_K(v) = 0$ ,  $v \in \mathcal{V}(K)$ , and  $\sum_{v \in e \cap \mathcal{V}(K)} p_K(v) = 0$ ,  $e \in S \setminus K$ , which imply that the map  $\rho$  is well defined epimorphism of groups.

Let  $f: K \to K'$  be an elementary morphism (an isotopy or a Reidemeister move) and the diagram K' splits the surface into cells. Then for any vertex  $v' \in \mathcal{V}(K')$  such that  $v' = f_*(v)$  for some  $v \in \mathcal{V}(K)$  we have  $[K_{v,1}] = [K'_{v',1}]$  and  $p_K(v) = p_{K'}(v')$ . Since the elements  $[K'_{v',1}]$  for such vertices v' generate the group  $H_1(S,\mathbb{Z}_2)/[\mathcal{K}]$  the maps  $\rho_K$  and  $\rho_{K'}$  coincide. Hence, the map  $\rho = \rho_K$  does not depend on a choice of the diagram K and  $p_K = \rho \circ hp_K$  for any diagram which splits the surface into cells.

If  $S \setminus K$  is not a union of cells, then we can apply second Reidemeister moves several times and obtain a diagram K' splitting the surface into cells. Indeed, if there is a component U of  $S \setminus K$  which is not a cell, then there exists a curve  $\gamma$  in U such that  $U \setminus \gamma$ is connected. If we apply the second Reidemeister move along  $\gamma$  we get a diagram  $K_{\gamma}$ . The components of  $S \setminus K_{\gamma}$  are homeomorphic to the corresponding components of  $S \setminus K$ except a component U' homeomorphic to  $U \setminus \gamma$  (that corresponds to the component U of  $S \setminus K$ ) and an additional small disk (see Fig. 24). After repeating this operation several times we get a diagram that splits the surface into cells.



Fig. 24. Component reduction of the complement of the knot diagram with a second Reidemeister move

By properties of the parities hp and p we have  $[K_{v,1}] = [K'_{f_*(v),1}]$  and  $p_K(v) = p_{K'}(f_*(v))$  for any  $v \in \mathcal{V}(K)$ . Therefore  $p_K(v) = p_{K'}(f_*(v)) = \rho \circ hp_{K'}(f_*(v)) = \rho \circ hp_{K'}(f_*(v)) = \rho \circ hp_K(v)$ .

Thus,  $p_K = \rho \circ hp_K$  for any diagram K, so the homological parity hp is universal.

The homological parity remains universal if we pass from the category of homotopy classes of curves on a given surface S to the category of knots on S (to be precise, knots in the thickened surface). The following lemma shows that in some sense parity does not feel the over- and undercrossing structure.

LEMMA 4.8. Let p be a parity on the category of knots on a surface S, and let K be a diagram of a knot on S. If vertices  $a, b \in \mathcal{V}(K)$  form a bigon in S, then  $p_K(a) + p_K(b) = 0$ . If vertices  $v_1, v_2, v_3 \in \mathcal{V}(K)$  form a triangle in S, then  $p_K(a) + p_K(b) + p_K(c) = 0$ .

*Proof.* We prove the lemma for a triangle, the proof for a bigon is analogous. Let the vertices  $a, b, c \in \mathcal{V}(K)$  form a triangle. If one can apply the third Reidemeister move to the triangle, the identity  $p_K(a) + p_K(b) + p_K(c) = 0$  follows from definition of parity.



Fig. 25. An alternating triangle

Otherwise the vertices constitute an alternating triangle. By applying three second and one third Reidemeister moves we get the diagram K' (see Fig. 25), where the following equalities hold:

$$p_{K'}(b) + p_{K'}(c) + p_{K'}(d) = 0,$$
  

$$p_{K'}(e) + p_{K'}(f) + p_{K'}(g) = 0,$$
  

$$p_{K'}(a) + p_{K'}(f) + p_{K'}(g) = 0,$$
  

$$p_{K'}(e) + p_{K'}(d) = 0.$$

Then we have  $p_{K'}(a) = p_{K'}(e) = p_{K'}(d) = p_{K'}(b) + p_{K'}(c)$  (we do not need signs because Lemma 4.4 remains true in the category of knots). Therefore,  $p_K(a) + p_K(b) + p_K(c) = 0$ .

The claim above ensures that Lemma 4.7 holds in the current situation too. Hence, one can repeat the proof of Theorem 4.4 and get the following result.

THEOREM 4.5. Let  $\mathcal{K}$  be a knot on a surface S. Then the homological parity (with coefficients in  $H_1(S,\mathbb{Z}_2)/[\mathcal{K}]$ ) is the universal parity on diagrams of  $\mathcal{K}$ .

COROLLARY 4.3. Any parity on classical knots is trivial.

*Proof.* Any classical knot  $\mathcal{K}$  is represented by diagrams on  $S^2$ . But  $H_1(S^2, \mathbb{Z}_2) = 0$ , so the universal parity group as well as any parity is trivial.

5. Applications of parity. Let us briefly summarise some theorems from [15] reformulating them for parities with coefficients from an abelian group.

5.1. The functorial mapping f. Let  $\mathcal{K}$  be a virtual, flat or free knot and  $\mathfrak{K}$  be the corresponding category of its diagrams.

Let us consider any family of maps  $\tilde{p}_K : \mathcal{V}(K) \to \mathbb{Z}_2, K \in \mathrm{ob}(\mathfrak{K})$ , that possesses all the properties of Definition 3.2 except for the property 3. Instead of it we impose the condition: if  $v_1, v_2, v_3$  are crossings participating in a third Reidemeister move then the number of vertices v among  $v_1, v_2, v_3$  such that  $p_K(v) = 1$  is not equal to 1. We call such a family a *pseudoparity*  $\tilde{p}$  of  $\mathcal{K}$  with coefficients in  $\mathbb{Z}_2$ . The following statement follows directly from the definition.

LEMMA 5.1. If p is a parity (with coefficients in a group A), then the formula

$$\widetilde{p}_K(v) = \begin{cases} 1, & p_K(v) \neq 0, \\ 0, & p_K(v) = 0 \end{cases}$$

defines a pseudoparity on  $\mathcal{K}$ .

Let  $\tilde{p}$  be a pseudoparity on a knot  $\mathcal{K}$  and K be a diagram of  $\mathcal{K}$ . We call a classical crossing v of K an *odd crossing* if  $\tilde{p}_K(v) = 1$  and an *even crossing* if  $\tilde{p}_K(v) = 0$ . Let  $f_{\tilde{p}}(K)$  be the diagram obtained from K by making all odd crossings virtual. In other words, we remove all odd chords of the corresponding chord diagram.

THEOREM 5.1. The map  $f_{\tilde{p}}$  defines a functor from the category of diagrams of a virtual (resp., flat, free) knot  $\mathcal{K}$  with the pseudoparity  $\tilde{p}$  to the category of diagrams of the virtual (resp., flat, free) knot  $\mathcal{K}' = f_{\tilde{p}}(\mathcal{K})$ .

*Proof.* The map  $f_{\widetilde{p}}$  determines how a functor should act on objects of the category  $\mathfrak{K}$ . We need to show that for any elementary morphism  $h: K_1 \to K_2$  between two diagrams of  $\mathcal{K}$  there exists an elementary morphism  $f_{\widetilde{p}}(h)$  connecting the diagrams  $f_{\widetilde{p}}(K_1)$  and  $f_{\widetilde{p}}(K_1)$ .

If h is an isotopy, then the diagrams  $f_{\tilde{p}}(K_1)$  and  $f_{\tilde{p}}(K_1)$  are isotopic and we can take this isotopy for  $f_{\tilde{p}}(h)$ . If h is a detour move, the diagrams  $f_{\tilde{p}}(K_1)$  and  $f_{\tilde{p}}(K_1)$  are also related by a detour move.

If h is a first Reidemeister move and the vertex v of the move is even, then the diagrams  $f_{\tilde{p}}(K_1)$  and  $f_{\tilde{p}}(K_1)$  differ by a first Reidemeister move. If v is odd, the diagrams are connected by a detour move.

If h is a second Reidemeister move and the vertices  $v_1, v_2$  of the move are even, then  $f_{\tilde{p}}(h)$  is a second Reidemeister move. If the vertices are odd, then we can connect the diagrams with a detour move.

If h is a third Reidemeister move then depending on the (pseudo)parity of the vertices of the move, we can take for the map  $f_{\tilde{p}}(h)$  either a third Reidemeister move (if all the vertices of the move are even) or a detour move (if there are odd vertices).

REMARK 5.1. The mapping "deleting" all odd classical crossings is a mapping into itself, i.e. we do not go out from the category. If we had had a non-trivial parity in the category of classical knots, then we could have gone out from the category to the category of virtual knots.

COROLLARY 5.1. For any pseudoparity  $\tilde{p}$  on  $\mathcal{K}$  the isotopy class of the diagram  $f_{\tilde{p}}(K)$ does not depend on the choice of a diagram K of the knot  $\mathcal{K}$ . In other words, the knot  $f_{\tilde{p}}(\mathcal{K})$  is correctly defined.

In the case of the trivial pseudoparity  $\widetilde{p}$  (i.e.  $\widetilde{p}_K(v) = 0$  for any  $v \in \mathcal{V}(K)$ ) we have  $f_{\widetilde{p}}(\mathcal{K}) = \mathcal{K}$ .

As an example showing the power of the notion of parity we present the following theorem.

THEOREM 5.2 ([12]). Let K be a framed 4-graph with one unicursal component such that all vertices of K are odd and no decreasing second Reidemeister move can be applied to K. Then K is a minimal diagram of the corresponding free knot in the following strong sense: for any diagram K' equivalent to K there is a smoothing of K' isomorphic to the graph K.

**5.2. The parity bracket.** A particular case of the parity bracket firstly appeared in [12]. That bracket was constructed for the Gaussian parity and played a significant role in proving minimality theorems. Also the bracket was generalised for the case of graph-links, see [9], and allowed the authors to prove the existence of non-realisable graph-links, for more details see [9].

In this subsection we consider the parity bracket for any parity valued in  $\mathbb{Z}_2$ . This bracket is a generalisation of the bracket from [12].

Let  $\mathfrak{G}$  be the set of all equivalence classes of framed graphs with one unicursal component modulo second Reidemeister moves. Consider the linear space  $\mathbb{Z}_2\mathfrak{G}$ .

Let  $\mathcal{K}$  be a virtual (resp., flat, free) knot, p be a parity on diagrams of  $\mathcal{K}$  with coefficients from the group  $\mathbb{Z}_2$ , and K be a diagram of  $\mathcal{K}$  with  $\mathcal{V}(K) = \{v_1, \ldots, v_n\}$ . For each element  $s \in \{0, 1\}^n$  we define  $K_s$  to be equal to the sum of all graphs obtained from K by a smoothing at each vertex  $v_i$  if  $s_i = 1$ . Let |s| be the number of ones in s. If |s| = l,  $K_s$  contains  $2^l$  summands. Define  $q_{K,s}(v_i) = p_K(v_i)$  if  $s_i = 0$ , and  $q_{K,s}(v_i) = 1 - p_K(v_i)$ if  $s_i = 1$ .

Consider the following sum (the *parity bracket*)

$$[K] = \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K,s}(v_i) K_s \in \mathbb{Z}_2 \mathfrak{G},$$

where only those summands with one unicursal component are taken into account.

THEOREM 5.3. If K and K' represent the same knot, then the following equality holds in  $\mathbb{Z}_2\mathfrak{G}$ : [K] = [K'].

*Proof.* Let us check the invariance  $[K] \in \mathbb{Z}_2 \mathfrak{G}$  under the three Reidemeister moves.

1) Let K' differ from K by a first Reidemeister move, and  $\mathcal{V}(K') = \{v_1, v_2, \dots, v_{n+1}\},$  $\mathcal{V}(K) = \{v_1, v_2, \dots, v_n\}.$  We have  $p_{K'}(v_{n+1}) = 0$  and

$$[K'] = \left[ \bigcirc \right] = \sum_{s \in \{0,1\}^{n+1}} \prod_{i=1}^{n+1} q_{K',s}(v_i) K'_s$$
$$= \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K',s}(v_i) \left( p_{K'}(v_{n+1}) \bigotimes + (1 - p_{K'}(v_{n+1})) \left( \bigotimes + \bigotimes \right) \right)$$
$$= \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K',s}(v_i) \bigotimes = [K].$$

2) Let K' be obtained from K by a second Reidemeister move adding two vertices, where  $\mathcal{V}(K') = \{v_1, v_2, \ldots, v_{n+1}, v_{n+2}\}$  and  $\mathcal{V}(K) = \{v_1, v_2, \ldots, v_n\}$ . We have

$$p_{K'}(v_{n+1}) + p_{K'}(v_{n+2}) = 0, \text{ i.e. } p_{K'}(v_{n+1}) = p_{K'}(v_{n+2}) = 0 \text{ or } p_{K'}(v_{n+1}) = p_{K'}(v_{n+2}) = 1, \text{ and}$$

$$[K'] = \left[\left(\bigotimes\right)\right] = \sum_{s \in \{0,1\}^n} \prod_{i=1}^{n+2} q_{K',s}(v_i) K'_s$$

$$= \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K',s}(v_i) \left(p_{K'}(v_{n+1})p_{K'}(v_{n+2})\bigotimes\right)$$

$$+ p_{K'}(v_{n+1})(1 - p_{K'}(v_{n+2})) \left(\bigotimes\right) + \bigotimes\right)$$

$$+ (1 - p_{K'}(v_{n+1}))p_{K'}(v_{n+2}) \left(\bigotimes\right) + \bigotimes\right)$$

$$+ (1 - p_{K'}(v_{n+1}))(1 - p_{K'}(v_{n+2})) \left(\bigotimes\right) + \bigotimes\right) + \bigotimes\right)$$

$$= \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K',s}(v_i) \bigotimes\right) = [K].$$

We recall that we consider the resulting graphs up to second Reidemeister moves.



Fig. 26. A third Reidemeister move

3) Let K' be obtained from K by a third Reidemeister move applied to vertices  $v_1, v_2, v_3$  in K. Denote by  $v'_1, v'_2, v'_3 \in \mathcal{V}(K')$  the vertices corresponding to  $v_1, v_2, v_3$ , see Fig. 26 (here  $\mathcal{V}(K') = \{v_1, v_2, \dots, v_n\}$  and  $\mathcal{V}(K) = \{v'_1, v'_2, \dots, v'_n\}$ ). We have

$$p_K(v_1) + p_K(v_2) + p_K(v_3) = 0, \quad p_{K'}(v'_1) + p_{K'}(v'_2) + p_{K'}(v'_3) = 0,$$

and

$$[K] = \left[\bigotimes^{n}\right] = \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K,s}(v_i) K_s$$
$$= \sum_{s \in \{0,1\}^{n-3}} \prod_{i=4}^n q_{K,s}(v_i) \left(\underbrace{p_K(v_1)p_K(v_2)p_K(v_3)}_{=0} \bigotimes^{n}\right)$$
$$+ p_K(v_1)p_K(v_2)(1 - p_K(v_3)) \left(\bigotimes^{n} + \bigotimes^{n}\right)$$
$$+ (1 - p_K(v_1))p_K(v_2)p_K(v_3) \left(\bigotimes^{n} + \bigotimes^{n}\right)$$

$$+ p_{K}(v_{1})(1 - p_{K}(v_{2}))p_{K}(v_{3})\left(\bigotimes + \bigotimes \right)$$

$$+ \underbrace{(1 - p_{K}(v_{1}))(1 - p_{K}(v_{2}))p_{K}(v_{3})}_{=0}\left(\bigotimes + \bigotimes + \bigotimes + \bigotimes \right)$$

$$+ \underbrace{(1 - p_{K}(v_{1}))p_{K}(v_{2})(1 - p_{K}(v_{3}))}_{=0}\left(\bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes \right)$$

$$+ \underbrace{p_{K}(v_{1})(1 - p_{K}(v_{2}))(1 - p_{K}(v_{3}))}_{=0}\left(\bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes \right)$$

$$+ (1 - p_{K}(v_{1}))(1 - p_{K}(v_{2}))(1 - p_{K}(v_{3}))\left(\bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes + \bigotimes \right)$$

$$+ \underbrace{(1 - p_{K}(v_{1}))(1 - p_{K}(v_{3}))}_{=0}\left(\bigotimes + \bigotimes + \bigotimes \right)$$

$$+ (1 - p_{K}(v_{1}))p_{K}(v_{2})p_{K}(v_{3})\left(\bigotimes + \bigotimes \right)$$

$$+ (1 - p_{K}(v_{1}))p_{K}(v_{2})p_{K}(v_{3})\left(\bigotimes + \bigotimes \right)$$

$$+ (1 - p_{K}(v_{1}))(1 - p_{K}(v_{2}))(1 - p_{K}(v_{3}))\left(\bigotimes + \bigotimes \right)$$

$$+ (1 - p_{K}(v_{1}))(1 - p_{K}(v_{2}))(1 - p_{K}(v_{3}))\left(\bigotimes + \bigotimes \right)$$

$$+ (1 - p_{K}(v_{1}))(1 - p_{K}(v_{2}))(1 - p_{K}(v_{3}))\left(\bigotimes + \bigotimes \right)$$

$$\begin{split} [K'] &= \left[\bigotimes\right] = \sum_{s \in \{0,1\}^n} \prod_{i=1}^n q_{K',s}(v'_i)K'_s \\ &= \sum_{s \in \{0,1\}^{n-3}} \prod_{i=4}^n q_{K',s}(v'_i) \left(\underbrace{p_{K'}(v'_1)p_{K'}(v'_2)p_{K'}(v'_3)}_{=0} \bigotimes\right) \\ &+ p_{K'}(v'_1)p_{K'}(v_2)(1 - p_{K'}(v'_3)) \left(\bigcup\right) + \bigotimes\right) \\ &+ (1 - p_{K'}(v'_1))p_{K'}(v'_2)p_{K'}(v'_3) \left(\bigotimes\right) + \bigotimes\right) \\ &+ p_{K'}(v'_1)(1 - p_{K'}(v'_2))p_{K'}(v'_3) \left(\bigotimes\right) + \bigotimes\right) \\ &+ \underbrace{(1 - p_{K'}(v'_1))(1 - p_{K'}(v'_2))p_{K'}(v'_3)}_{=0} \left(\bigotimes\right) + \bigotimes\right) \\ &+ \underbrace{(1 - p_{K'}(v'_1))(1 - p_{K'}(v'_2))p_{K'}(v'_3)}_{=0} \left(\bigotimes\right) + \bigotimes\right) \\ &+ \underbrace{(1 - p_{K'}(v'_1))p_{K'}(v'_2)(1 - p_{K'}(v'_3))}_{=0} \left(\bigotimes\right) + \bigotimes\right) \\ &+ \underbrace{(1 - p_{K'}(v'_1))p_{K'}(v'_2)(1 - p_{K'}(v'_3))}_{=0} \left(\bigotimes\right) + \bigotimes\right) \\ &+ \underbrace{(1 - p_{K'}(v'_1))p_{K'}(v'_2)(1 - p_{K'}(v'_3))}_{=0} \left(\bigotimes\right) + \bigotimes\right) \\ &+ \underbrace{(1 - p_{K'}(v'_1))p_{K'}(v'_2)(1 - p_{K'}(v'_3))}_{=0} \left(\bigotimes\right) + \underbrace{(1 - p_{K'}(v'_1)p_{K'}(v'_2)(1 - p_{K'}(v'_3))}_{=0} \left(\bigotimes\right) + \underbrace{(1 - p_{K'}(v'_1)p_{K'}(v'_2)(1 - p_{K'}(v'_3))}_{=0} \left(\bigotimes\right) + \underbrace{(1 - p_{K'}(v'_1)p_{K'}(v'_1)p_{K'}(v'_2)}_{=0} \left(\bigotimes\right) + \underbrace{(1 - p_{K'}(v'_1)p_{K'}(v'_2)p_{K$$

$$+ \underbrace{p_{K'}(v_1')(1 - p_{K'}(v_2'))(1 - p_{K'}(v_3'))}_{=0} \left( \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} \right) \\ + (1 - p_{K'}(v_1'))(1 - p_{K'}(v_2'))(1 - p_{K'}(v_3')) \left( \underbrace{\bigcirc} + \underbrace{(1 - p_{K'}(v_1'))(1 - p_{K'}(v_2'))(1 - p_{K'}(v_3'))} \left( \underbrace{\bigcirc} + \underbrace{)} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{\bigcirc} + \underbrace{$$

As we consider  $\mathbb{Z}_2 \mathfrak{G}$  (i.e. up to second Reidemeister moves), we have



Therefore, [K] = [K'].

#### References

- D. M. Afanas'ev, Refining virtual knot invariants by means of parity, Mat. Sb. 201 (2010), no. 6, 3–18; English transl.: Sb. Math. 201 (2010), 785–800.
- J. S. Carter, Closed curves that never extend to proper maps of disks, Proc. Amer. Math. Soc. 113 (1991), 879–888.
- [3] J. S. Carter, S. Kamada, M. Saito, Stable equivalence of knots on surfaces and virtual knot cobordisms, J. Knot Theory Ramifications 11 (2002), 311-322.
- [4] A. Gibson, Homotopy invariants of Gauss words, Math. Ann. 349 (2011), 871–887.
- [5] M. Goussarov, M. Polyak, O. Viro, *Finite type invariants of classical and virtual knots*, Topology 39 (2000), 1045–1068.
- [6] D. P. Ilyutko, V. O. Manturov, *Introduction to graph-link theory*, J. Knot Theory Ramifications 18 (2009), 791-823.
- D. P. Ilyutko, V. O. Manturov, *Graph-links*, Dokl. Akad. Nauk 428 (2009), 591-594; English transl.: Dokl. Math. 80 (2009), 739–742.
- [8] D. P. Ilyutko, V. O. Manturov, *Cobordisms of free knots*, Dokl. Akad. Nauk 429 (2009), 439-441; English transl.: Dokl. Math. 80 (2009), 844–846.

- [9] D. P. Ilyutko, V. O. Manturov, *Graph-links*, in: Introductory Lectures on Knot Theory, Ser. Knots Everything, World Sci. Publ., Hackensack, NJ, 2012, 135–161.
- [10] N. Kamada, S. Kamada, Abstract link diagrams and virtual knots, J. Knot Theory Ramifications 9 (2000), 93–109.
- [11] L. H. Kauffman, Virtual knot theory, European J. Combin. 20 (1999), 663–690.
- [12] V. O. Manturov, On free knots, arXiv:math.GT/0901.2214v2.
- [13] V. O. Manturov, On free knots and links, arXiv:math.GT/0902.0127.
- [14] V. O. Manturov, Free knots are not invertible, arXiv:math.GT/0909.2230.
- [15] V. O. Manturov, Parity in knot theory, Mat. Sb. 201:5 (2010), 65–110; English transl.:
   Sb. Math. 201 (2010), 693–733.
- [16] V. O. Manturov, Parity, free knots, groups, and invariants of finite type, Tr. Mosk. Mat. Obs. 72 (2011), 207-222; English transl.: Trans. Moscow Math. Soc. 72 (2011), 157–169.
- [17] V. O. Manturov, Parity and projection from virtual knots to classical knots, J. Knot Theory Ramifications 22 (2013), 1350044.
- [18] V. O. Manturov, Free knots and parity, in: Introductory Lectures on Knot Theory, Ser. Knots Everything, World Sci. Publ., Hackensack, NJ, 2012, 321–345.
- [19] V. O. Manturov, O. V. Manturov, Free knots and groups, J. Knot Theory Ramifications 19 (2010), 181–186.
- [20] V. O. Manturov, O. V. Manturov, Free knots and groups, Dokl. Akad. Nauk 434:1 (2010), 25-28; English transl.: Dokl. Math. 82 (2010), 697–700.
- [21] V. G. Turaev, Virtual open strings and their cobordisms, preprint (2004), arXiv:math.GT/0311185v5.
- [22] V. G. Turaev, Cobordisms of words, arXiv:math.CO/0511513v2.
- [23] V. G. Turaev, Cobordisms of knots on surfaces, arXiv:math.GT/0703055v1.