ORLICZ CENTENARY VOLUME II
BANACH CENTER PUBLICATIONS, VOLUME 68
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

A GENERALIZED PROJECTION DECOMPOSITION IN ORLICZ-BOCHNER SPACES

HENRYK HUDZIK

Faculty of Mathematics and Computer Science, Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, Poland
E-mail: hudzik@amu.edu.pl

RYSZARD PŁUCIENNIK

Institute of Mathematics, Poznań University of Technology
Piotrowo 3A, 60-965 Poznań, Poland
E-mail: rplucien@math.put.poznan.pl

YUWEN WANG

School of Mathematics and Computer Science, Harbin Normal University
Harbin 150080, P.R. China
E-mail: wangyuwen2003@sohu.com

Abstract. In this paper, a precise projection decomposition in reflexive, smooth and strictly convex Orlicz-Bochner spaces is given by the representation of the duality mapping. As an application, a representation of the metric projection operator on a closed hyperplane is presented.

1. Introduction. It is well known that if K is a closed convex cone (resp. closed linear subspace) in a Hilbert function space, we have the Moreau (resp. Riesz) decomposition theorem $x = P_K(x) + P_{K^0}(x)$ (resp. $x = P_K(x) + P_{K^\perp}(x)$), but the decomposition does not hold in arbitrary Banach function spaces. Many authors have attempted to generalize it. In 1995, Y. W. Wang and Z. W. Li [15] (resp. in 2001, Y. W. Wang and H. Wang [16]) obtained a decomposition by using the metric projection operator (i.e. projector π_L)

$$x = \pi_L(x) + x_2,\ \forall x \in X,$$

2000 *Mathematics Subject Classification*: 46B20, 46E30.

Key words and phrases: Orlicz-Bochner space, projection decomposition, duality mapping, hyperplane.

Research of Y. Wang was supported by the National Science Foundation Grant (10471023) of China.

The paper is in final form and no version of it will be published elsewhere.

where L is a closed convex cone (resp. a Chebyshev subspace) of a real reflexive strictly convex Banach space (resp. a general Banach space) X, $x_2 \in J^{-1}L^{\perp}$ and x_2 is not definite. In 1998, Ya. I. Alber [1] obtained another decomposition in a reflexive strictly convex smooth Banach space X:

$$x = J^{-1}\Pi_{K^0}Jx + w,$$

where K is a closed convex cone in X, $J : X \to X^*$ is the duality mapping of X, $w \in K$ and w is not definite, so their decompositions are semi-definite. W. Song and Z. J. Cao [14] investigated this problem in a more precise and general form. The aim of this paper is to give a precise representation of such a decomposition in Orlicz-Bochner spaces $L_\Phi(X)$.

2. Definitions and preliminary lemmas. We denote by (G, Σ, μ) a measure space in the n-dimensional Euclidean space $\mathbb{R}^n$ with $0 < \mu G < \infty$, by $\mathbb{R}$ the set of real numbers, by $(X, \|.\|_X)$ a reflexive real Banach space, by $(X^*, \|.\|_{X^*})$ the dual space of X, by $\langle x^*, x\rangle$ the dual pairing of $x^* \in X^*$ and $x \in X$ and by $L^0(G, X)$ the linear space of all μ-equivalent classes of strongly measurable functions $x : G \to X$.

A convex and even function $\Phi : \mathbb{R} \to \mathbb{R}_+$ is called an *Orlicz function* if $\Phi(0) = 0$, $\Phi(u) > 0$ for $u \neq 0$, and

$$\lim_{u\to 0} \frac{\Phi(u)}{|u|} = 0, \qquad \lim_{u\to\infty} \frac{\Phi(u)}{|u|} = \infty.$$

For any Orlicz function Φ, we define its *complementary function* $\Psi : \mathbb{R} \to \mathbb{R}_+$ by the formula

$$\Psi(v) = \sup_{u>0}\{u|v| - \Phi(u)\}$$

for every $v \in \mathbb{R}$. The function Ψ is also an Orlicz function (see [8], [4]).

We say that an Orlicz function Φ *satisfies the* Δ_2*-condition* (write $\Phi \in \Delta_2$) if there exist constants $K > 1$ and $u_0 > 0$ such that

$$\Phi(2u) \le K\Phi(u) \ \text{ for any } u \ge u_0.$$

We say that an Orlicz function Φ *satisfies the* ∇_2*-condition* (write $\Phi \in \nabla_2$) if its complementary function Ψ satisfies the Δ_2-condition.

Denote by small letters φ and ψ the right hand side derivatives of the Orlicz functions Φ and Ψ, respectively.

The space

$$L_\Phi(X) = \left\{x \in L^0(G, X) : \exists k > 0 \text{ s.t. } \rho_\Phi(kx) = \int_G \Phi(k\|x(t)\|_X)dt < \infty\right\}$$

equipped with the so called Orlicz norm

$$\|x\|_\Phi^0 = \sup\left\{\left|\int_G \langle y(t), x(t)\rangle dt\right| : y \in L_\Psi(X^*),\ \rho_\Psi(y) \le 1\right\}$$

or with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{k > 0 : \rho_\Phi\left(\frac{x}{k}\right) \le 1\right\}$$

is said to be an *Orlicz-Bochner space* (see [7]). In the following $L_\Phi(X)$ (resp. $L^0_\Phi(X)$) denotes the Orlicz-Bochner space equipped with the Luxemburg norm (resp. equipped with the Orlicz norm). If $X = \mathbb{R}$, the Orlicz-Bochner spaces become the classical Orlicz spaces (see [10] or [17]) and they are denoted by L_Φ and L^0_Φ, respectively.

The following *Hölder inequalities*

$$\left|\int_G \langle y(t), x(t)\rangle dt\right| \leq \|x\|_\Phi \|y\|^0_\Psi,$$

$$\left|\int_G \langle y(t), x(t)\rangle dt\right| \leq \|x\|^0_\Phi \|y\|_\Psi$$

hold for any $x \in L_\Phi(X)$ and $y \in L_\Psi(X^*)$.

If $\Phi \in \Delta_2$, then $(L_\Phi(X))^* = L^0_\Psi(X^*)$, $(L^0_\Phi(X))^* = L_\Psi(X^*)$ and the spaces $L_\Phi(X)$ and $L^0_\Phi(X)$ are reflexive if and only if $\Phi \in \Delta_2 \cap \nabla_2$ (see [4] or [12]).

The *Amemiya formula* for the Orlicz norm

$$\|x\|^0_\Phi = \inf_{k>0} \frac{1}{k}\left[1 + \rho_\Phi(kx)\right]$$

holds for every $x \in L_\Phi(X)$. Moreover, for every $x \in L_\Phi(X)\backslash\{0\}$ there exists $k > 0$ such that

$$\|x\|^0_\Phi = \frac{1}{k}\left[1 + \rho_\Phi(kx)\right]. \tag{1}$$

If there exists $k > 0$ such that

$$\int_G \Psi\left[\varphi\left(k\|x(t)\|_X\right)\right] dt = 1,$$

then

$$\|x\|^0_\Phi = \int_G \|x(t)\|_X \varphi\left(k\|x(t)\|_X\right) dt = \frac{1}{k}\{1 + \rho_\Phi(kx)\}$$

(see [11]).

Now, we recall some geometric concepts in Banach spaces.

For any Banach space X denote by $S(X)$ the unit sphere of X. The multi-valued mapping $\Lambda_X : X\backslash\{0\} \to S(X^*)$ defined by the formula

$$\Lambda_X(x) = \{x^* \in S(X^*) : \langle x^*, x\rangle = \|x\|_X\}$$

for any $x \in X\backslash\{0\}$ is called the *support mapping of* X. The multi-valued mapping $F_X : X \to X^*$ defined by the formula

$$F_X(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|^2_X = \|x\|^2_X\} \tag{2}$$

for any $x \in X$ is called the *duality mapping of* X. A relationship between the support mapping Λ_X and the duality mapping F_X can be expressed by the following formula:

$$F_X(x) = \|x\|_X \Lambda_X(x) \quad \forall x \in X\backslash\{0\} \text{ and } F_X(0) = 0. \tag{3}$$

The properties of the duality mapping are closely related to the geometric properties of the space. The following results may be found in [3]: F_X is homogeneous; F_X is surjective iff X is reflexive; F_X is injective iff X is strictly convex; F_X is single-valued iff X is smooth.

Now, we recall the concepts of the metric projection and the generalized projection.

Let C be a convex subset of a normed linear space X. The multi-valued mapping $\pi(C|\cdot) : X \to C$ defined by the formula

$$\pi(C|x) = \{x_0 \in C : \|x - x_0\|_X = \inf_{z \in C} \|x - z\|_X\}$$

for any $x \in X$ is called the *metric projection onto* C. If $\pi(C|\cdot)$ is single-valued, then it is called the *metric projection operator* or the *best approximation operator* and it is denoted by π_C (see [13]).

In the following we assume that X is a reflexive, strictly convex and smooth Banach space. Consider the problem of the attainability of

$$\inf_{y \in C} \{\|x\|_X^2 - 2\langle F_X(x), y\rangle + \|y\|_X^2\}.$$

We know that this problem has a unique solution (see [2]). The operator

$$\Pi_C x := \{y_x \in C : W(x, y_x) = \min_{y \in C} W(x, y)\},$$

where $W(x, y) = \|x\|_X^2 - 2\langle F_X x, y\rangle + \|y\|_X^2$ for $x,\, y \in X$, is said to be the *generalized projection of* x on C. Alber ([1]) obtained the following result:

THEOREM A. *Let X be a reflexive strictly convex smooth real Banach space, K be a nonempty closed convex cone in X (i.e. $\lambda K \subset K$ for all $\lambda \geq 0$ and $K + K = K$). Then for every $x \in X$ and $x^* \in X^*$ there exist $\omega \in K$ and $\chi \in K^0$, satisfying*

$$x = F_X^{-1}\Pi_{K^0} F_X(x) + \omega \text{ and } \langle \pi_{K^0} F_X(x), \omega\rangle = 0,$$
$$x^* = F_X \Pi_K F_X^{-1}(x^*) + \chi \text{ and } \langle \chi, F_X^{-1}(x^*)\rangle = 0,$$

where $K^0 = \{x^* \in X^* : \langle x^*, x\rangle \leq 0 \ \ \forall x \in K\}$ is the polar cone of K.

3. A representation of the duality mapping

THEOREM 1. *Let $\Phi \in \Delta_2$, φ be continuous and X be a smooth Banach space. Then the duality mapping $F_{L_\Phi^0(X)}$ of the Orlicz-Bochner space $L_\Phi^0(X)$ can be represented by the formula*

$$F_{L_\Phi^0(X)}(x)(t) = \|x\|_\Phi^0 \Lambda_X(x(t))\varphi\,[k\|x(t)\|_X]$$

for μ-a.e. $t \in G$ and for any $x \in L_\Phi^0(X)$, where k satisfies

$$\int_G \Psi\,[\varphi\,(k\|x(t)\|_X)]\,dt = 1.$$

Proof. Let $Y = L_\Phi^0(X)$. Then we know that $Y^* = L_\Psi(X^*)$. Since $\Phi \in \Delta_2$, φ is continuous and X is smooth, by Th. 4 in [11], $Y = L_\Phi^0(X)$ is a smooth Banach space. Consequently, $F_Y : L_\Phi^0(X) \to L_\Psi(X^*)$ is a single-valued mapping and, by (3), $F_Y(x) = \|x\|_\Phi^0 \Lambda_Y(x)$ for any $x \in L_\Phi^0(X) \setminus \{0\}$.

By (1) and because of $\|\Lambda_Y(x)\|_\Psi = 1$, there exists $k > 0$ such that

$$\begin{aligned}\frac{1}{k}\left(1+\int_G \Phi(k\|x(t)\|_X)dt\right) &= \|x\|_\Phi^0 = \int_G \langle \Lambda_Y(x)(t), x(t)\rangle dt \\ &\le \frac{1}{k}\int_G k\|x(t)\|_X \|\Lambda_Y(x)(t)\|_{X^*} dt \\ &\le \frac{1}{k}\left(\int_G \Phi\left(k\|x(t)\|_X\right)dt + \int_G \Psi\left(\|\Lambda_Y(x)(t)\|_{X^*}\right)dt\right) \\ &\le \frac{1}{k}\left(\int_G \Phi(k\|x(t)\|_X)dt + 1\right).\end{aligned}$$

Hence

$$\int_G \Psi\left(\|\Lambda_Y(x)(t)\|_{X^*}\right)dt = 1 \tag{4}$$

and

$$\int_G \left[\Phi\left(k\|x(t)\|_X\|\right) + \Psi\left(\|\Lambda_Y(x)(t)\|_{X^*}\right) - k\|x(t)\|_X\|\Lambda_Y(x)(t)\|_{X^*}\right]dt = 0.$$

It follows from the Young inequality that

$$\Phi\left(k\|x(t)\|_X\|\right) + \Psi\left(\|\Lambda_Y(x)(t)\|_{X^*}\right) = k\|x(t)\|_X\|\Lambda_Y(x)(t)\|_{X^*}$$

for μ-a.e. $t \in G$. The fact that φ is continuous, and the condition for equality in the Young inequality yield that

$$\|\Lambda_Y(x)(t)\|_{X^*} = \varphi\left(k\|x(t)\|_X\right)$$

for μ-a.e. $t \in G$. Therefore, we have

$$\begin{aligned}\int_G \langle \Lambda_Y(x)(t), kx(t)\rangle dt &= \int_G k\|x(t)\|_X\|\Lambda_Y(x)(t)\|_{X^*} dt \\ &= \int_G k\|x(t)\|_X \varphi\left(k\|x(t)\|_X\right)dt \\ &= \int_G \langle \varphi\left(k\|x(t)\|_X\right)\Lambda_X(x(t)), kx(t)\rangle dt.\end{aligned}$$

Since the map $x \mapsto \Lambda_Y(x)$ is single-valued, we obtain

$$\Lambda_Y(x)(t) = \varphi\left(k\|x(t)\|_X\right)\Lambda_X(x(t)) \tag{5}$$

for μ-a.e. $t \in G$. Combining (4) and (5), we get

$$\int_G \Psi\left(\varphi\left(k\|x(t)\|_X\right)\right)dt = 1,$$

and from (5) and the relationship between the duality mapping and the support mapping, we have

$$F_{L_\Phi^0(X)}(x)(t) = \|x\|_\Phi^0 \varphi\left[k\|x(t)\|_X\|\right]\Lambda_X(x(t))$$

for μ-a.e. $t \in G$. ■

THEOREM 2. *Let $\Phi \in \Delta_2$, φ be continuous and X be a smooth Banach space. Then the duality mapping $F_{L_\Phi(X)}$ of the Orlicz-Bochner space $L_\Phi(X)$ can be represented by the* *formula*

$$F_{L_\Phi(X)}(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt} \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) \Lambda_X(x(t))$$

for μ-a.e. $t \in G$ and for every $x \in L_\Phi(X) \setminus \{0\}$.

Proof. Let $Y = L_\Phi(X)$. Then $Y^* = L_\Psi^0(X^*)$. Since $\Phi \in \Delta_2$, φ is continuous, and X is smooth, by Th. 3 in [11], the space $Y = L_\Phi(X)$ is a smooth Banach space. Consequently, the duality mapping $F_Y(\cdot) = \|\cdot\|_\Phi \Lambda_Y(\cdot)$ is single-valued.

Let $x \in L_\Phi(X) \setminus \{0\}$. Then $\Lambda_Y(x) \in S(L_\Psi^0)$ and

$$\|x\|_\Phi = \int_G \langle \Lambda_Y(x)(t), x(t) \rangle dt.$$

By (1), there is $k > 0$ such that

$$\begin{aligned} \|\Lambda_Y(x)\|_\Psi^0 = 1 &= \frac{1}{k}\left(1 + \int_G \Psi\left(k\|\Lambda_Y(x)(t)\|_{X^*}\right) dt\right) \\ &= \int_G \left\langle \Lambda_Y(x)(t), \frac{x(t)}{\|x\|_\Phi} \right\rangle dt \\ &\le \frac{1}{k}\int_G k\|\Lambda_Y(x)(t)\|_{X^*} \frac{\|x(t)\|_X}{\|x\|_\Phi} dt \\ &\le \frac{1}{k}\left(\int_G \Psi\left(k\|\Lambda_Y(x)(t)\|_{X^*}\right) dt + \int_G \Phi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt\right) \\ &\le \frac{1}{k}\left(1 + \int_G \Psi\left(k\|\Lambda_Y(x)(t)\|_{X^*}\right) dt\right). \end{aligned} \tag{6}$$

It follows that

$$k\|\Lambda_Y(x)(t)\|_{X^*} \frac{\|x(t)\|_X}{\|x\|_\Phi} = \Psi\left(k\|\Lambda_Y(x)(t)\|_{X^*}\right) + \Phi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right)$$

for μ-a.e. $t \in G$ and hence, by the continuity of φ and by the condition for equality in the Young inequality, we obtain

$$\|\Lambda_Y(x)(t)\|_{X^*} = \frac{1}{k}\varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right)$$

for μ-a.e. $t \in G$. By (6), we have

$$\begin{aligned} 1 &= \int_G \left\langle \Lambda_Y(x)(t), \frac{x(t)}{\|x\|_\Phi} \right\rangle dt \\ &= \int_G \|\Lambda_Y(x)(t)\|_{X^*} \frac{\|x(t)\|_X}{\|x\|_\Phi} dt \\ &= \int_G \frac{1}{k}\varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) \frac{\|x(t)\|_X}{\|x\|_\Phi} dt \\ &= \int_G \left\langle \frac{1}{k}\varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) \Lambda_X(x(t)), \frac{x(t)}{\|x\|_\Phi} \right\rangle dt. \end{aligned} \tag{7}$$

Hence, it follows that

$$\Lambda_Y(x)(t) = \frac{1}{k}\varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right)\Lambda_X(x(t))$$

for μ-a.e. $t \in G$. From (7), we see that

$$k = \frac{1}{\|x\|_\Phi}\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt.$$

Therefore, we obtain

$$F_{L_\Phi}(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt}\varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right)\Lambda_X(x(t))$$

for μ-a.e. $t \in G$. ■

4. A generalized projection decomposition

THEOREM 3. *Let $\Phi \in \Delta_2 \cap \nabla_2$, φ and ψ be continuous, X be a reflexive strictly convex smooth Banach space, K be a nonempty closed convex cone in the Orlicz-Bochner space $L_\Phi(X)$, $K^0 = \{y \in L_\Psi^0(X^*) : \int_G \langle y(t), x(t)\rangle dt \le 0 \ \forall x \in K\}$. Then for any $x \in L_\Phi(X)\setminus K$, we have the unique decomposition*

$$x(t) = \pi_K(x)(t) + \|y\|_\Psi^0 \Lambda_{X^*}(y(t))\psi(k\|y(t)\|_{X^*})$$

for μ-a.e. $t \in G$, where $k > 0$ and $y \in L_\Psi^0(X^*)$ satisfy the conditions

$$\int_G \Phi\left[\psi\left(k\|y(t)\|_{X^*}\right)\right] dt = 1$$

and

$$y(t) = \Pi_{K^0}\left(\frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt}\varphi\left(\frac{\|x(.)\|_X}{\|x\|_\Phi}\right)\Lambda_X(x(.))\right)(t)$$

for μ-a.e. $t \in G$, where π_K is the metric projection operator from L_Φ onto K and Π_{K^0} is the generalized projection operator from $L_\Psi^0(X^*)$ onto K^0.

Proof. Let $Y = L_\Phi(X)$. Then $Y^* = L_\Psi^0(X^*)$ and both $L_\Phi(X)$ and $L_\Psi^0(X^*)$ are reflexive, strictly convex and smooth spaces. For any $x \in Y \setminus K$, by Theorem A, there exists a function $\omega \in K$ such that

$$x = \omega + F_Y^{-1}\Pi_{K^0}F_Y(x) \text{ and } \langle \Pi_{K^0}F_Y(x), \omega\rangle = 0. \tag{8}$$

Hence, we have

$$F_Y(x-\omega) = \Pi_{K^0}F_Y(x) \in K^0,$$
$$\langle F_Y(x-\omega), \omega\rangle = 0 \text{ and } \langle F_Y(x-\omega), w\rangle \le 0$$

for any $w \in K$. By Theorem 2, we obtain

$$\int_G \left\langle \varphi\left(\frac{\|x(t)-\omega(t)\|_X}{\|x-\omega\|_\Phi}\right)\Lambda_X(x(t)-\omega(t)), \omega(t)-w(t)\right\rangle dt \ge 0$$

for any $w \in K$. It follows from Theorem 6 in [11] that

$$\omega = \pi_K(x). \tag{9}$$

By Theorem 2, we get

$$F_Y(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt} \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) \Lambda_X(x(t))$$

and

$$y(t) = \Pi_{K^0} F_Y(x)(t) = \Pi_{K^0}\left(\frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt} \varphi\left(\frac{\|x(\cdot)\|_X}{\|x\|_\Phi}\right) \Lambda_X(x(\cdot))\right)(t)$$

for μ-a.e. $t \in G$. The fact that Y and Y^* are reflexive, strictly convex and smooth Banach spaces implies that $F_Y^{-1} = F_{Y^*}$ and hence, by Theorem 1, we have

$$F_Y^{-1}\Pi_{K^0} F_Y(x)(t) = F_{Y^*}(y)(t) = \|y\|_\Psi^o \Lambda_{X^*}(y(t))\psi\left(k\|y(t)\|_{X^*}\right) \tag{10}$$

for μ-a.e. $t \in G$ and

$$\int_G \Phi\left[\psi\left(k\|y(t)\|_{X^*}\right)\right] dt = 1.$$

Combining (8), (9) and (10), we finish the proof. ■

Corollary 1 (Moreau decomposition theorem). *Let X be a Hilbert space, $K \subset L^2(X)$ be a closed convex cone, $K^0 \subset L^2(X)$ be its polar cone. Then for every $x \in L^2(X) \setminus K$, there is a unique decomposition*

$$x = \pi_K(x) + \pi_{K^0}(x),$$

where π_K and π_{K^0} are the metric projection operators.

Proof. Let $\Phi(u) = |u|^2/2$. Then $\Psi(v) = |v|^2/2$. Since X is a Hilbert space, $Y = L_\Phi(X) = L^2(X)$ and $Y^* = L_\Psi^0(X^*) = L^2(X)$. Moreover, for any $x \in L_\Phi(X)$ and for any $y \in L_\Psi^0(X^*)$, we have $\|x\|_\Phi = \|x\|_2/\sqrt{2}$ and $\|y\|_\Psi^0 = \sqrt{2}\|y\|_2$. Consequently, for any $x \in L_\Phi(X) \setminus K$, we have

$$\begin{aligned} F_Y(x)(t) &= \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) dt} \varphi\left(\frac{\|x(t)\|_X}{\|x\|_\Phi}\right) \Lambda_X(x(t)) \\ &= \frac{\|x\|_2^2}{\int_G \|x(t)\|_X^2 dt} \cdot \frac{\sqrt{2}\|x\|_2}{2\sqrt{2}} \cdot \frac{\|x(t)\|_X}{\|x\|_2} \Lambda_X(x(t)) = \frac{1}{2}\|x(t)\|_X \Lambda_X(x(t)) = \frac{1}{2}x(t) \end{aligned}$$

for μ-a.e. $t \in G$. Since in any Hilbert space, the generalized projection operator Π_{K^0} coincides with the metric projection operator π_{K^0},

$$y(t) = \Pi_{K^0} F_Y(x)(t) = \frac{1}{2}\pi_{K^0}(x)(t)$$

for μ-a.e. $t \in G$. On the other hand, we also have

$$\|y\|_\Psi^0 \psi\left[k\|y(t)\|_{X^*}\right] \Lambda_{X^*}(y(t)) = \sqrt{2}\|y\|_2 k\|y(t)\|_X \Lambda_X(y(t)) = \sqrt{2}\|y\|_2 k y(t).$$

From the condition

$$1 = \int_G \Phi\left[\psi\left(k\|y(t)\|_{X^*}\right)\right] dt = \frac{k^2}{2}\int_G \|y(t)\|_X^2 dt$$

we get that $k\|y\|_2 = \sqrt{2}$, and so

$$\|y\|_\Psi^0 \psi\left[k\|y(t)\|_{X^*}\right] \Lambda_{X^*}(y(t)) = 2y(t) = \pi_{K^0}(x)(t).$$

Hence and from Theorem 3, we get

$$x(t) = \pi_K(x)(t) + \pi_{K^0}(x)(t)$$

for μ-a.e. $t \in G$. ■

By Corollary 1, we obtain immediately the following

Corollary 2 (Riesz orthogonal decomposition theorem). *Let X be a Hilbert space, $L \subset L^2(X)$ be a closed linear subspace, $L^\perp \subset L^2(X)$ be its orthogonal complement. Then for every $x \in L^2(X) \setminus L$ there is a unique decomposition*

$$x = P_L(x) + P_{L^\perp}(x),$$

where P_L and $P_{L^\perp}$ are the orthogonal projection operators.

Now we will give an application of Theorem 3. Namely, we have the representation of the metric projection operator onto a closed hyperplane in Orlicz-Bochner spaces.

Theorem 4. *Let $\Phi \in \Delta_2 \cap \nabla_2$, φ and ψ be continuous, X be a reflexive, strictly convex and smooth Banach space. Let $L = \{x \in L_\Phi(X) : \int_G \langle x_0^*(t), x(t)\rangle dt = 0\}$ be a closed hyperplane in $L_\Phi(X)$, where $x_0^* \in L_\Psi^0(X^*) \setminus \{0\}$. Then for every $x \in L_\Phi(X) \setminus L$, we have*

$$\pi_L(x)(t) = x(t) - \frac{\int_G \langle x_0^*(t), x(t)\rangle dt}{\|x\|_\Psi^0} \psi\left[k\|x_0^*(t)\|_{X^*}\right] \Lambda_{X^*}(x_0^*(t)),$$

for μ-a.e. $t \in G$, where Λ_{X^*} is the support mapping of X^*.

Proof. By the assumptions, we have that the Orlicz-Bochner space $Y = L_\Phi(X)$ is a reflexive, strictly convex and smooth Banach space. For the closed hyperplane L, we know that

$$L^0 = L^\perp = \{\lambda x_0^* : \lambda \in R\} \subset L_\Psi^o(X^*) = Y^*.$$

For any $x \in L_\Phi(X) \setminus L$, by Theorem 3, we have

$$\pi_L(x)(t) = x(t) - F_{Y^*}(y)(t),$$

where $y = \lambda x_0^* \in L^\perp$ for some $\lambda \neq 0$. Note that the duality mapping F_{Y^*} is homogeneous, so we obtain

$$\pi_L(x)(t) = x(t) - \lambda F_Y^{-1}(x_0^*)(t). \tag{11}$$

Taking the value of the functional $x_0^*(t)$ at the elements from both sides of (11) for every $t \in G$ and then integrating them over G with respect to $t \in G$, we get

$$0 = \int_G \langle x_0^*(t), x(t)\rangle dt - \lambda \int_G \langle x_0^*(t), F_Y^{-1}(x_0^*)(t)\rangle dt = \int_G \langle x_0^*(t), x(t)\rangle dt - \lambda(\|x_0^*\|_\Psi^0)^2.$$

Hence, it follows that

$$\lambda = \frac{\int_G \langle x_0^*(t), x(t)\rangle dt}{(\|x_0^*\|_\Psi^0)^2}. \tag{12}$$

On the other hand, by Theorem 1, we also have

$$F_Y^{-1}(x_0^*)(t) = F_{Y^*}(x_0^*)(t) = \|x_0^*\|_\Psi^0 \psi\left[k\|x_0^*(t)\|_{X^*}\right] \Lambda_{X^*}(x_0^*(t)) \tag{13}$$

Combining (11), (12) and (13), we complete the proof. ■

References

[1] Ya. I. Alber, *Generalized projections decompositions and the Pythagorean-type theorem in Banach spaces*, Appl. Math. Lett. 11 (1998), 115-121.

[2] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: Theory and Applications of Nonlinear Operators of Monotone and Accretive Type, A. Kartsatos (ed.), Marcel Dekker, New York, 1996, 15–50.

[3] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces*, Ed. Aca. Rep. Soc. Romania, Bucureşti, 1978.

[4] S. Chen, *Geometry of Orlicz spaces*, Dissertationes Mathematicae 356 (1996).

[5] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Lecture Notes in Math. 485, Springer, New York, 1975.

[6] H. Hudzik, *Strict convexity of Musielak-Orlicz space with Luxemburg norm*, Bull. Polish Acad. Sci. Math. 29 (1981), 338–367.

[7] A. Kozek, *Orlicz spaces of functions with values in Banach spaces*, Comment. Math. 19 (1976), 259–288.

[8] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Noordhof, Groningen, 1961.

[9] J. J. Moreau, *Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires*, C. R. Acad. Sci. Paris 255 (1962), 238–240.

[10] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, 1983.

[11] R. Płuciennik and Y. W. Wang, *Metric projections and best approximants in Bochner-Orlicz spaces*, Revista Matemática Complutense 7 (1994), 57–77.

[12] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.

[13] I. Singer, *The Theory of Best Approximation and Functional Analysis*, Springer-Verlag, New York, 1970.

[14] W. Song and Z. J. Cao, *The generalized projection decomposition theorem in Banach spaces and its applications*, J. Approx. Theory 129 (2004), 167–181.

[15] Y. W. Wang and Z. W. Li, *The Moore-Penrose generalized inverse in Banach spaces and ill-posed problem*, J. Sys. Sci. 15 (1995), 175–185.

[16] Y. W. Wang and H. Wang, *Generalized orthogonal decomposition theorem in Banach spaces and generalized orthogonally complemented subspace*, Acta Math. Sinica 44 (2001), 1045–1050.

[17] C. X. Wu, T. F. Wang, S. T. Chen and Y. W. Wang, *Geometry of Orlicz Spaces*, Harbin (1986) (in Chinese).