A GENERALIZED PROJECTION DECOMPOSITION IN ORLICZ-BOCHNER SPACES

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Abstract. In this paper, a precise projection decomposition in reflexive, smooth and strictly convex Orlicz-Bochner spaces is given by the representation of the duality mapping. As an application, a representation of the metric projection operator on a closed hyperplane is presented.

1. Introduction. It is well known that if $K$ is a closed convex cone (resp. closed linear subspace) in a Hilbert function space, we have the Moreau (resp. Riesz) decomposition theorem $x = P_K(x) + P_{K^0}(x)$ (resp. $x = P_K(x) + P_{K^\perp}(x)$), but the decomposition does not hold in arbitrary Banach function spaces. Many authors have attempted to generalize it. In 1995, Y. W. Wang and Z. W. Li [15] (resp. in 2001, Y. W. Wang and H. Wang [16]) obtained a decomposition by using the metric projection operator (i.e. projector $\pi_L$)

$$x = \pi_L(x) + x_2, \quad \forall x \in X,$$

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where $L$ is a closed convex cone (resp. a Chebyshev subspace) of a real reflexive strictly convex Banach space (resp. a general Banach space) $X$, $x_2 \in J^{-1}L^\perp$ and $x_2$ is not definite. In 1998, Ya. I. Alber [1] obtained another decomposition in a reflexive strictly convex smooth Banach space $X$:

$$x = J^{-1} \Pi_{K^0} Jx + w,$$

where $K$ is a closed convex cone in $X$, $J : X \to X^*$ is the duality mapping of $X$, $w \in K$ and $w$ is not definite, so their decompositions are semi-definite. W. Song and Z. J. Cao [14] investigated this problem in a more precise and general form. The aim of this paper is to give a precise representation of such a decomposition in Orlicz-Bochner spaces $L_\Phi(X)$.

2. Definitions and preliminary lemmas. We denote by $(G, \Sigma, \mu)$ a measure space in the $n$-dimensional Euclidean space $\mathbb{R}^n$ with $0 < \mu G < \infty$, by $\mathbb{R}$ the set of real numbers, by $(X, \|\cdot\|_X)$ a reflexive real Banach space, by $(X^*, \|\cdot\|_{X^*})$ the dual space of $X$, by $(x^*, x)$ the dual pairing of $x^* \in X^*$ and $x \in X$ and by $L^0(G, X)$ the linear space of all $\mu$-equivalent classes of strongly measurable functions $x : G \to X$.

A convex and even function $\Phi : \mathbb{R} \to \mathbb{R}_+$ is called an Orlicz function if $\Phi(0) = 0$, $\Phi(u) > 0$ for $u \neq 0$, and

$$\lim_{u \to 0} \frac{\Phi(u)}{|u|} = 0, \quad \lim_{u \to \infty} \frac{\Phi(u)}{|u|} = \infty.$$

For any Orlicz function $\Phi$, we define its complementary function $\Psi : \mathbb{R} \to \mathbb{R}_+$ by the formula

$$\Psi(v) = \sup_{u > 0} \{u|v| - \Phi(u)\}$$

for every $v \in \mathbb{R}$. The function $\Psi$ is also an Orlicz function (see [8], [4]).

We say that an Orlicz function $\Phi$ satisfies the $\Delta_2$-condition (write $\Phi \in \Delta_2$) if there exist constants $K > 1$ and $u_0 > 0$ such that

$$\Phi(2u) \leq K\Phi(u) \quad \text{for any } u \geq u_0.$$

We say that an Orlicz function $\Phi$ satisfies the $\nabla_2$-condition (write $\Phi \in \nabla_2$) if its complementary function $\Psi$ satisfies the $\Delta_2$-condition.

Denote by small letters $\varphi$ and $\psi$ the right hand side derivatives of the Orlicz functions $\Phi$ and $\Psi$, respectively.

The space

$$L_\Phi(X) = \left\{ x \in L^0(G, X) : \exists k > 0 \text{ s.t. } \rho_\Phi(kx) = \int_G \Phi(k\|x(t)\|_X)dt < \infty \right\}$$

equipped with the so called Orlicz norm

$$\|x\|_\Phi^0 = \sup \left\{ \left| \int_G \langle y(t), x(t) \rangle dt \right| : y \in L_\Psi(X^*), \rho_\Psi(y) \leq 1 \right\}$$

or with the Luxemburg norm

$$\|x\|_\Phi = \inf \left\{ k > 0 : \rho_\Phi \left( \frac{x}{k} \right) \leq 1 \right\}$$
is said to be an Orlicz-Bochner space (see [7]). In the following $L_{\Phi}(X)$ (resp. $L_{\Phi}^0(X)$) denotes the Orlicz-Bochner space equipped with the Luxemburg norm (resp. equipped with the Orlicz norm). If $X = \mathbb{R}$, the Orlicz-Bochner spaces become the classical Orlicz spaces (see [10] or [17]) and they are denoted by $L_{\Phi}$ and $L_{\Phi}^0$, respectively.

The following H"older inequalities
\[
\left| \int_G \langle y(t), x(t) \rangle dt \right| \leq \|x\|_{\Phi} \|y\|_{\Psi},
\]
\[
\left| \int_G \langle y(t), x(t) \rangle dt \right| \leq \|x\|_{\Phi}^0 \|y\|_{\Psi}
\]
hold for any $x \in L_{\Phi}(X)$ and $y \in L_{\Psi}(X^*)$.

If $\Phi \in \Delta_2$, then $(L_{\Phi}(X))^* = L_{\Psi}^0(X^*)$, $(L_{\Phi}^0(X))^* = L_{\Psi}(X^*)$ and the spaces $L_{\Phi}(X)$ and $L_{\Phi}^0(X)$ are reflexive if and only if $\Phi \in \Delta_2 \cap \nabla_2$ (see [4] or [12]).

The Amemiya formula for the Orlicz norm
\[
\|x\|_{\Phi}^0 = \inf_{k>0} \frac{1}{k} [1 + \rho_{\Phi}(kx)]
\]
holds for every $x \in L_{\Phi}(X)$. Moreover, for every $x \in L_{\Phi}(X) \setminus \{0\}$ there exists $k > 0$ such that
\[
\|x\|_{\Phi}^0 = \frac{1}{k} [1 + \rho_{\Phi}(kx)].
\]

If there exists $k > 0$ such that
\[
\int_G \Psi [\varphi (k\|x(t)\|_X)] dt = 1,
\]
then
\[
\|x\|_{\Phi}^0 = \int_G \|x(t)\|_X \varphi (k\|x(t)\|_X) dt = \frac{1}{k} [1 + \rho_{\Phi}(kx)]
\]
(see [11]).

Now, we recall some geometric concepts in Banach spaces.

For any Banach space $X$ denote by $S(X)$ the unit sphere of $X$. The multi-valued mapping $\Lambda_X : X \setminus \{0\} \to S(X^*)$ defined by the formula
\[
\Lambda_X(x) = \{ x^* \in S(X^*) : \langle x^*, x \rangle = \|x\|_X \}
\]
for any $x \in X \setminus \{0\}$ is called the support mapping of $X$. The multi-valued mapping $F_X : X \to X^*$ defined by the formula
\[
F_X(x) = \{ x^* \in X^* : \langle x^*, x \rangle = \|x^*\|_X^2 = \|x\|_X^2 \}
\]
for any $x \in X$ is called the duality mapping of $X$. A relationship between the support mapping $\Lambda_X$ and the duality mapping $F_X$ can be expressed by the following formula:
\[
F_X(x) = \|x\|_X \Lambda_X(x) \quad \forall x \in X \setminus \{0\} \text{ and } F_X(0) = 0.
\]

The properties of the duality mapping are closely related to the geometric properties of the space. The following results may be found in [3]: $F_X$ is homogeneous; $F_X$ is surjective iff $X$ is reflexive; $F_X$ is injective iff $X$ is strictly convex; $F_X$ is single-valued iff $X$ is smooth.

Now, we recall the concepts of the metric projection and the generalized projection.
Let $C$ be a convex subset of a normed linear space $X$. The multi-valued mapping 
$\pi(C|\cdot) : X \to C$ defined by the formula 
\[
\pi(C|x) = \{x_0 \in C : \|x - x_0\|_X = \inf_{z \in C} \|x - z\|_X\}
\]
for any $x \in X$ is called the metric projection onto $C$. If $\pi(C|\cdot)$ is single-valued, then it is 
called the metric projection operator or the best approximation operator and it is denoted by $\pi_C$ (see [13]).

In the following we assume that $X$ is a reflexive, strictly convex and smooth Banach space. Consider the problem of the attainability of
\[
\inf_{y \in C}\{\|x\|_X^2 - 2\langle F_X(x), y \rangle + \|y\|_X^2\}.
\]
We know that this problem has a unique solution (see [2]). The operator
\[
\Pi_C x := \{y_x \in C : W(x, y_x) = \min_{y \in C} W(x, y)\},
\]
where $W(x, y) = \|x\|_X^2 - 2\langle F_X(x), y \rangle + \|y\|_X^2$ for $x, y \in X$, is said to be the generalized projection of $x$ on $C$. Alber ([1]) obtained the following result:

**Theorem A.** Let $X$ be a reflexive strictly convex smooth real Banach space, $K$ be a 
nonempty closed convex cone in $X$ (i.e. $\lambda K \subset K$ for all $\lambda \geq 0$ and $K + K = K$). Then 
for every $x \in X$ and $x^* \in X^*$ there exist $\omega \in K$ and $\chi \in K^0$, satisfying 
\[
x = F_X^{-1}\Pi_{K^0}F_X(x) + \omega \text{ and } \langle \pi_{K^0}F_X(x), \omega \rangle = 0,
\]
\[
x^* = F_X\Pi_KF_X^{-1}(x^*) + \chi \text{ and } \langle \chi, F_X^{-1}(x^*) \rangle = 0,
\]
where $K^0 = \{x^* \in X^* : \langle x^*, x \rangle \leq 0 \ \forall x \in K\}$ is the polar cone of $K$.

3. A representation of the duality mapping

**Theorem 1.** Let $\Phi \in \Delta_2$, $\varphi$ be continuous and $X$ be a smooth Banach space. Then the 
duality mapping $F_{L_\Phi^0 (X)}$ of the Orlicz-Bochner space $L_\Phi^0 (X)$ can be represented by the 
formula
\[
F_{L_\Phi^0 (X)}(x)(t) = \|x\|_\Phi^0(\Lambda_X(x(t)))\varphi[k\|x(t)\|_X]
\]
for $\mu$-a.e. $t \in G$ and for any $x \in L_\Phi^0 (X)$, where $k$ satisfies
\[
\int_G \Psi [\varphi(k\|x(t)\|_X)] dt = 1.
\]

**Proof.** Let $Y = L_\Phi^0 (X)$. Then we know that $Y^* = L_\Psi (X^*)$. Since $\Phi \in \Delta_2$, $\varphi$ is continuous 
and $X$ is smooth, by Th. 4 in [11], $Y = L_\Phi^0 (X)$ is a smooth Banach space. Consequently, 
$F_Y : L_\Phi^0 (X) \to L_\Psi (X^*)$ is a single-valued mapping and, by (3), $F_Y(x) = \|x\|_\Phi^0(\Lambda_Y(x))$ for 
any $x \in L_\Phi^0 (X) \setminus \{0\}$. 

By (1) and because of \( \|\Lambda_Y(x)\|_\Psi = 1 \), there exists \( k > 0 \) such that
\[
\frac{1}{k} \left( 1 + \int_G \Phi(k\|x(t)\|_X) dt \right) = \|x\|^0 = \int_G \langle \Lambda_Y(x)(t), x(t) \rangle dt
\]
\[
\leq \frac{1}{k} \int_G k\|x(t)\|_X \|\Lambda_Y(x)(t)\|_{X^*} dt
\]
\[
\leq \frac{1}{k} \left( \int_G \Phi(k\|x(t)\|_X) dt + \int_G \Psi(\|\Lambda_Y(x)(t)\|_{X^*}) dt \right)
\]
\[
\leq \frac{1}{k} \left( \int_G \Phi(k\|x(t)\|_X) dt + 1 \right).
\]
Hence
\[
\int_G \Psi(\|\Lambda_Y(x)(t)\|_{X^*}) dt = 1 \quad (4)
\]
and
\[
\int_G [\Phi(k\|x(t)\|_X)] + \Psi(\|\Lambda_Y(x)(t)\|_{X^*}) - k\|x(t)\|_X \|\Lambda_Y(x)(t)\|_{X^*} dt = 0.
\]
It follows from the Young inequality that
\[
\Phi(k\|x(t)\|_X) + \Psi(\|\Lambda_Y(x)(t)\|_{X^*}) = k\|x(t)\|_X \|\Lambda_Y(x)(t)\|_{X^*}
\]
for \( \mu \)-a.e. \( t \in G \). The fact that \( \varphi \) is continuous, and the condition for equality in the Young inequality yield that
\[
\|\Lambda_Y(x)(t)\|_{X^*} = \varphi(k\|x(t)\|_X)
\]
for \( \mu \)-a.e. \( t \in G \). Therefore, we have
\[
\int_G \langle \Lambda_Y(x)(t), kx(t) \rangle dt = \int_G k\|x(t)\|_X \|\Lambda_Y(x)(t)\|_{X^*} dt
\]
\[
= \int_G k\|x(t)\|_X \varphi(k\|x(t)\|_X) dt
\]
\[
= \int_G \langle \varphi(k\|x(t)\|_X) \Lambda_X(x(t)), kx(t) \rangle dt.
\]
Since the map \( x \mapsto \Lambda_Y(x) \) is single-valued, we obtain
\[
\Lambda_Y(x)(t) = \varphi(k\|x(t)\|_X) \Lambda_X(x(t)) \quad (5)
\]
for \( \mu \)-a.e. \( t \in G \). Combining (4) and (5), we get
\[
\int_G \Psi(\varphi(k\|x(t)\|_X)) dt = 1,
\]
and from (5) and the relationship between the duality mapping and the support mapping, we have
\[
F_{L_\Phi(X)}(x)(t) = \|x\|^0 \varphi[k\|x(t)\|_X] \Lambda_X(x(t))
\]
for \( \mu \)-a.e. \( t \in G \). \[
\]
**Theorem 2.** Let \( \Phi \in \Delta_2 \), \( \varphi \) be continuous and \( X \) be a smooth Banach space. Then the duality mapping \( F_{L_\Phi(X)} \) of the Orlicz-Bochner space \( L_\Phi(X) \) can be represented by the
formula

\[ F_{L\Phi}(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_\Phi \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t)) \]

for \(\mu\)-a.e. \(t \in G\) and for every \(x \in L\Phi(X) \setminus \{0\}\).

\textbf{Proof}. Let \(Y = L\Phi(X)\). Then \(Y^* = L\Phi^0(X^*)\). Since \(\Phi \in \Delta_2\), \(\varphi\) is continuous, and \(X\) is smooth, by Th. 3 in [11], the space \(Y = L\Phi(X)\) is a smooth Banach space. Consequently, the duality mapping \(F_Y(\cdot) = \| \cdot \|_\Phi Y(\cdot)\) is single-valued.

Let \(x \in L\Phi(X) \setminus \{0\}\). Then \(\Lambda_Y(x) \in S(L\Phi^0)\) and

\[ \|x\|_\Phi = \int_G \langle \Lambda_Y(x)(t), x(t) \rangle dt. \]

By (1), there is \(k > 0\) such that

\[ \|\Lambda_Y(x)\|_\Psi^0 = 1 = \frac{1}{k} \left( 1 + \int_G \Psi(k \|\Lambda_Y(x)(t)\|_{X^*}) dt \right) \]

\[ \leq \frac{1}{k} \int_G \langle \Lambda_Y(x)(t), \frac{x(t)}{\|x\|_\Phi} \rangle \varphi dt \]

\[ \leq \frac{1}{k} \left( \int_G \Psi(k \|\Lambda_Y(x)(t)\|_{X^*}) dt + \int_G \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \right) \]

\[ \leq \frac{1}{k} \left( 1 + \int_G \Psi(k \|\Lambda_Y(x)(t)\|_{X^*}) dt \right). \]

It follows that

\[ k \|\Lambda_Y(x)(t)\|_{X^*} \frac{\|x(t)\|_X}{\|x\|_\Phi} = \Psi(k \|\Lambda_Y(x)(t)\|_{X^*}) + \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \]

for \(\mu\)-a.e. \(t \in G\) and hence, by the continuity of \(\varphi\) and by the condition for equality in the Young inequality, we obtain

\[ \|\Lambda_Y(x)(t)\|_{X^*} = \frac{1}{k} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \]

for \(\mu\)-a.e. \(t \in G\). By (6), we have

\[ 1 = \int_G \langle \Lambda_Y(x)(t), \frac{x(t)}{\|x\|_\Phi} \rangle dt \]

\[ = \int_G \|\Lambda_Y(x)(t)\|_{X^*} \frac{\|x(t)\|_X}{\|x\|_\Phi} dt \]

\[ \leq \int_G \frac{1}{k} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \frac{\|x(t)\|_X}{\|x\|_\Phi} dt \]

\[ \leq \int_G \frac{1}{k} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t)), \frac{x(t)}{\|x\|_\Phi} dt. \]
Hence, it follows that
\[
\Lambda_Y(x)(t) = \frac{1}{k} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t))
\]
for \( \mu \)-a.e. \( t \in G \). From (7), we see that
\[
k = \frac{1}{\|x\|_\Phi} \int_G \|x(t)\|_X \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt.
\]
Therefore, we obtain
\[
F_{L_\Phi}(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t))
\]
for \( \mu \)-a.e. \( t \in G \). \( \blacksquare \)

4. A generalized projection decomposition

**Theorem 3.** Let \( \Phi \in \Delta_2 \cap \nabla_2 \), \( \varphi \) and \( \psi \) be continuous, \( X \) be a reflexive strictly convex smooth Banach space, \( K \) be a nonempty closed convex cone in the Orlicz-Bochner space \( L_\Phi(X), K^0 = \{ y \in L_\Phi^0(X^*) : \int_G \langle y(t), x(t) \rangle dt \leq 0 \ \forall x \in K \} \). Then for any \( x \in L_\Phi(X) \setminus K \), we have the unique decomposition
\[
x(t) = \pi_K(x(t)) + \|y\|_\Phi \Lambda_{X^*}(y(t)) \psi(k\|y(t)\|_{X^*})
\]
for \( \mu \)-a.e. \( t \in G \), where \( k > 0 \) and \( y \in L_\Phi^0(X^*) \) satisfy the conditions
\[
\int_G \Phi(\psi(k\|y(t)\|_{X^*})) dt = 1
\]
and
\[
y(t) = \Pi_{K^0} \left( \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t)) \right)(t)
\]
for \( \mu \)-a.e. \( t \in G \), where \( \pi_K \) is the metric projection operator from \( L_\Phi \) onto \( K \) and \( \Pi_{K^0} \) is the generalized projection operator from \( L_\Phi^0(X^*) \) onto \( K^0 \).

**Proof.** Let \( Y = L_\Phi(X) \). Then \( Y^* = L_\Phi^0(X^*) \) and both \( L_\Phi(X) \) and \( L_\Phi^0(X^*) \) are reflexive, strictly convex and smooth spaces. For any \( x \in Y \setminus K \), by Theorem A, there exists a function \( \omega \in K \) such that
\[
x = \omega + F_Y^{-1} \Pi_{K^0} F_Y(x) \text{ and } \langle \Pi_{K^0} F_Y(x), \omega \rangle = 0.
\]
Hence, we have
\[
F_Y(x - \omega) = \Pi_{K^0} F_Y(x) \in K^0,
\]
\[
\langle F_Y(x - \omega), \omega \rangle = 0 \text{ and } \langle F_Y(x - \omega), w \rangle \leq 0
\]
for any \( w \in K \). By Theorem 2, we obtain
\[
\int_G \varphi \left( \frac{\|x(t) - \omega(t)\|_X}{\|x - \omega\|_\Phi} \right) \Lambda_X(x(t) - \omega(t)), \omega(t) - w(t) \right) dt \geq 0
\]
for any \( w \in K \). It follows from Theorem 6 in [11] that
\[
\omega = \pi_K(x).
\]
By Theorem 2, we get
\[ F_Y(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t)) \]
and
\[ y(t) = \Pi_{K^0} F_Y(x)(t) = \Pi_{K^0} \left( \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t)) \right)(t) \]
for $\mu$-a.e. $t \in G$. The fact that $Y$ and $Y^*$ are reflexive, strictly convex and smooth Banach spaces implies that $F_Y^{-1} = F_{Y^*}$ and hence, by Theorem 1, we have
\[ F_Y^{-1} \Pi_{K^0} F_Y(x)(t) = F_{Y^*}(y)(t) = \|y\|_\Psi^0 \Lambda_{X^*}(y(t)) \psi(k\|y(t)\|_{X^*}) \]
(10)
for $\mu$-a.e. $t \in G$ and
\[ \int_G \Phi[\psi(k\|y(t)\|_{X^*})] dt = 1. \]

**Corollary 1 (Moreau decomposition theorem).** Let $X$ be a Hilbert space, $K \subset L^2(X)$ be a closed convex cone, $K^0 \subset L^2(X)$ be its polar cone. Then for every $x \in L^2(X) \setminus K$, there is a unique decomposition
\[ x = \pi_K(x) + \pi_{K^0}(x), \]
where $\pi_K$ and $\pi_{K^0}$ are the metric projection operators.

**Proof.** Let $\Phi(u) = |u|^2/2$. Then $\Psi(v) = |v|^2/2$. Since $X$ is a Hilbert space, $Y = L_\Phi(X) = L^2(X)$ and $Y^* = L^2_\Psi(X^*) = L^2(X)$. Moreover, for any $x \in L_\Phi(X)$ and for any $y \in L^2_\Psi(X^*)$, we have $\|x\|_\Phi = \|x\|_2/\sqrt{2}$ and $\|y\|_\Psi = \sqrt{2}\|y\|_2$. Consequently, for any $x \in L_\Phi(X) \setminus K$, we have
\[ F_Y(x)(t) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_\Phi} \right) \Lambda_X(x(t)) = \frac{\|x\|_\Phi^2}{\int_G \|x(t)\|_X^2 dt} \varphi \left( \frac{\|x(t)\|_X}{\|x\|_2} \right) \Lambda_X(x(t)) = \frac{1}{2} \|x(t)\|_X \Lambda_X(x(t)) = \frac{1}{2} x(t) \]
for $\mu$-a.e. $t \in G$. Since in any Hilbert space, the generalized projection operator $\Pi_{K^0}$ coincides with the metric projection operator $\pi_{K^0}$,
\[ y(t) = \Pi_{K^0} F_Y(x)(t) = \frac{1}{2} \pi_{K^0}(x)(t) \]
for $\mu$-a.e. $t \in G$. On the other hand, we also have
\[ \|y\|_\Psi^0 \psi[k\|y(t)\|_{X^*}] \Lambda_{X^*}(y(t)) = \sqrt{2}\|y\|_2 k\|y(t)\|_X \Lambda_X(y(t)) = \sqrt{2}\|y\|_2 ky(t). \]
From the condition
\[ 1 = \int_G \Phi[\psi(k\|y(t)\|_{X^*})] dt = \frac{k^2}{2} \int_G \|y(t)\|_X^2 dt \]
we get that $k\|y\|_2 = \sqrt{2}$, and so
\[ \|y\|_\Psi^0 \psi[k\|y(t)\|_{X^*}] \Lambda_{X^*}(y(t)) = 2y(t) = \pi_{K^0}(x)(t). \]
Hence and from Theorem 3, we get

\[ x(t) = \pi_K(x)(t) + \pi_{K^0}(x)(t) \]

for \( \mu \)-a.e. \( t \in G \). ■

By Corollary 1, we obtain immediately the following

**Corollary 2** (Riesz orthogonal decomposition theorem). Let \( X \) be a Hilbert space, \( L \subset L^2(X) \) be a closed linear subspace, \( L^\perp \subset L^2(X) \) be its orthogonal complement. Then for every \( x \in L^2(X) \setminus L \) there is a unique decomposition

\[ x = P_L(x) + P_{L^\perp}(x), \]

where \( P_L \) and \( P_{L^\perp} \) are the orthogonal projection operators.

Now we will give an application of Theorem 3. Namely, we have the representation of the metric projection operator onto a closed hyperplane in Orlicz-Bochner spaces.

**Theorem 4.** Let \( \Phi \in \Delta_2 \cap \nabla_2 \), \( \varphi \) and \( \psi \) be continuous, \( X \) be a reflexive, strictly convex and smooth Banach space. Let \( L = \{ x \in L_\Phi(X) : \int_G \langle x_0^*(t), x(t) \rangle dt = 0 \} \) be a closed hyperplane in \( L_\Phi(X) \), where \( x_0^* \in L_\Phi^0(X^*) \setminus \{0\} \). Then for every \( x \in L_\Phi(X) \setminus L \), we have

\[ \pi_L(x)(t) = x(t) - \frac{\int_G \langle x_0^*(t), x(t) \rangle dt}{\|x\|_\Phi} \psi [k\|x_0^*(t)\|_{X^*}] \Lambda_{X^*}(x_0^*(t)), \]

for \( \mu \)-a.e. \( t \in G \), where \( \Lambda_{X^*} \) is the support mapping of \( X^* \).

**Proof.** By the assumptions, we have that the Orlicz-Bochner space \( Y = L_\Phi(X) \) is a reflexive, strictly convex and smooth Banach space. For the closed hyperplane \( L \), we know that

\[ L^0 = L^\perp = \{ \lambda x_0^* : \lambda \in R \} \subset L_\Phi^0(X^*) = Y^*. \]

For any \( x \in L_\Phi(X) \setminus L \), by Theorem 3, we have

\[ \pi_L(x)(t) = x(t) - F_{Y^*}(y)(t), \]

where \( y = \lambda x_0^* \in L^\perp \) for some \( \lambda \neq 0 \). Note that the duality mapping \( F_{Y^*} \) is homogeneous, so we obtain

\[ \pi_L(x)(t) = x(t) - \lambda F_{Y^*}^{-1}(x_0^*)(t). \tag{11} \]

Taking the value of the functional \( x_0^*(t) \) at the elements from both sides of (11) for every \( t \in G \) and then integrating them over \( G \) with respect to \( t \in G \), we get

\[ 0 = \int_G \langle x_0^*(t), x(t) \rangle dt - \lambda \int_G \langle x_0^*(t), F_{Y^*}^{-1}(x_0^*)(t) \rangle dt = \int_G \langle x_0^*(t), x(t) \rangle dt - \lambda (\|x_0^*\|^0_\Phi)^2. \]

Hence, it follows that

\[ \lambda = \frac{\int_G \langle x_0^*(t), x(t) \rangle dt}{(\|x_0^*\|^0_\Phi)^2}. \tag{12} \]

On the other hand, by Theorem 1, we also have

\[ F_{Y^*}^{-1}(x_0^*)(t) = F_{Y^*}(x_0^*)(t) = \|x_0^*\|^0_\Phi \psi [k\|x_0^*(t)\|_{X^*}] \Lambda_{X^*}(x_0^*(t)) \]

Combining (11), (12) and (13), we complete the proof. ■
References


