DECOMPOSABLE SETS AND MUSIELAK-ORLICZ SPACES OF MULTIFUNCTIONS

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Abstract. We introduce the Musielak-Orlicz space of multifunctions $X_{m,\varphi}$ and the set $S_F^\varphi$ of $\varphi$-integrable selections of $F$. We show that some decomposable sets in Musielak-Orlicz space belong to $X_{m,\varphi}$. We generalize Theorem 3.1 from [6]. Also, we get some theorems on the space $X_{m,\varphi}$ and the set $S_F^\varphi$.

1. Introduction. Decomposability is a basic concept in Multivalued Analysis (see [7], p. 174). A notion of decomposibility has been introduced by Rockafellar in [14]. A similar but different notion has been introduced in [6] and [7] and we will use this notion. The Musielak-Orlicz spaces of multifunctions were introduced and studied in [8]-[11]. The Musielak-Orlicz space of multifunctions $X_{m,\varphi}$ has been introduced in [11]. The aim of this note is to obtain a generalization of Theorem 3.1 from [6] and Theorem 3.8, Chapter 2 from [7]. All definitions and theorems connected with Musielak-Orlicz spaces can be found in [12]. Definitions and theorems connected with multifunctions can be found in [1]-[7], [13] and [14].

Let $(\Omega, \Sigma, \mu)$ be a measure space with a nonnegative, nontrivial $\sigma$-finite and complete measure $\mu$. Let $\varphi$ be a $\varphi$-function, i.e., $\varphi : \Omega \times R \rightarrow R_+, \varphi(t, u)$ is an even, continuous function of $u$, equal to zero iff $u = 0$ and nondecreasing for $u \geq 0$ for every $t \in \Omega$, is a measurable function of $t \in \Omega$ for every $u \in R$ and $\lim_{u \rightarrow \infty} \varphi(t, u) = \infty$ for $\mu$-a.e. $t \in \Omega$. Moreover, if $\varphi(t, \cdot)$ is a convex function for every $t \in \Omega$, then we shall say that the $\varphi$-function $\varphi$ is convex. Let $L^\varphi(\Omega, \Sigma, \mu)$ be the Musielak-Orlicz function space generated by the modular

$$\rho(x) = \int_{\Omega} \varphi(t, x(t))d\mu.$$
Let $\| \cdot \|_\varphi^*$ denote the Luxemburg norm in $L^\varphi(\Omega, \Sigma, \mu)$ if \varphi is convex. Let $Y$ be a real separable Banach space with the norm $\| \cdot \|_Y$. Let $\Theta$ denote the zero element of $Y$. If $A, B \subset Y$ are nonempty then we denote

$$H(A, B) = \max(\sup_{x \in A} \inf_{y \in B} \| x - y \|_Y, \sup_{y \in B} \inf_{x \in A} \| x - y \|_Y).$$

Denote by $E(Y)$ the set of all nonempty and closed subsets of $Y$. Let

$$X = \{ F : \Omega \to 2^Y : F(t) \in E(Y) \text{ for every } t \in \Omega \}.$$  

Two multifunctions $F, G \in X$ such that $F(t) = G(t)$ for $\mu$-a.e. $t \in \Omega$ will be treated as the same element of $X$.

Now we introduce the function $d(F, G)$ by the formula:

$$d(F, G)(t) = H(F(t), G(t)) \text{ for all } F, G \in X \text{ and } t \in \Omega.$$  

Let $N$ be the set of all positive integers. Let $0 \in X$ be such that $0(t) = \{ \Theta \}$ for every $t \in \Omega$. Denote $|F| = d(F, 0)$ for every $F \in X$.

2. On the space $X_{m, \varphi}$ and the set $S_F^\varphi$

**Definition 1.** We say that $F \in X$ is a step multifunction if

$$F(t) = \sum_{k=1}^n \chi_{A_k}(t)B_k \text{ for every } t \in \Omega$$  

where $\chi_A$ is the characteristic function of the set $A$, $B_k \in E(Y)$ for $k = 1, \ldots, n$, $\Omega = \bigcup_{k=1}^n A_k$, $A_k \in \Sigma$ for $k = 1, \ldots, n$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

**Definition 2.** We say that $F \in X$ is measurable if there exists a sequence of step multifunctions $F_n \in X$ for every $n \in N$ such that $\lim_{n \to \infty} d(F, F_n)(t) = 0$ for $\mu$-a.e. $t \in \Omega$.

Denote:

$$X_m = \{ F \in X : F \text{ is measurable} \}, \quad X_{m, \varphi} = \{ F \in X_m : |F| \in L^\varphi(\Omega, \Sigma, \mu) \},$$

It is easy to see that $d(F, G) \in L^\varphi(\Omega, \Sigma, \mu)$ if $F, G \in X_{m, \varphi}$.

By [7], Chapter 2, Theorem 1.35, if $F \in X_m$ then $F$ is measurable and graph measurable in the sense of [7], Chapter 2, Definition 1.1.

The space $X_{m, \varphi}$ will be called the Musielak-Orlicz spaces of multifunctions.

By $L^\varphi((\Omega, \Sigma, \mu), Y)$ we will denote the set of all strongly measurable functions $f : \Omega \to Y$ such that $\| f(\cdot) \|_Y \in L^\varphi(\Omega, \Sigma, \mu)$.

In [11] the following was proved:

**Theorem 1.** Let $F_n \in X_{m, \varphi}$ for every $n \in N$. If for every $\epsilon > 0$ and every $a > 0$ there exists $K > 0$ such that $\int_\Omega \varphi(t, a d(F_m, F_n)(t))d\mu < \epsilon$ for all $m, n > K$, then there exists $F \in X_{m, \varphi}$ such that $\int_\Omega \varphi(t, a d(F, F_n)(t))d\mu \to 0$ as $n \to \infty$ for every $a > 0$.

**Corollary 1.** Let the $\varphi$-function $\varphi$ be convex, then the function

$$D_\varphi(F, G) = \| d(F, G) \|_{\varphi}^L$$

for all $F, G \in X_{m, \varphi}$ is a metric in $X_{m, \varphi}$, so $< X_{m, \varphi}, D_\varphi >$ is a complete metric space.
Let $F \in X$. Denote
\[ S_F^\varphi = \{ f \in L^\varphi((\Omega, \Sigma, \mu), Y) : f(t) \in F(t) \mu \text{ a.e.} \} . \]

**Definition 3.** The $\varphi$-function $\varphi$ will be called **locally integrable** if $\int_A \varphi(t, u) d\mu < \infty$ for every $u > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$.

Applying the proof of Proposition 3.3, Proposition 2.17 and Remark 3.4 Chapter 2 from [7] we easily obtain the following:

**Lemma 1.** Let the $\varphi$-function $\varphi$ be locally integrable and fulfil the condition $\Delta_2$, then for every $F \in X_m$ such that $S_F^\varphi \neq \emptyset$ there exists a sequence $\{f_n\} \subset L^\varphi((\Omega, \Sigma, \mu), Y)$ such that $F(t) = \overline{\{f_n(t)\}}$ for $\mu$-a.e. $t \in \Omega$.

**Corollary 2.** Let the $\varphi$-function $\varphi$ be locally integrable and fulfil the condition $\Delta_2$. Let $F, G \in X_m$ be such that $S_F^\varphi = S_G^\varphi \neq \emptyset$, then $F(t) = G(t)$ for $\mu$-a.e. $t \in \Omega$.

**Lemma 2.** Let the $\varphi$-function $\varphi$ be locally integrable, convex and fulfil the condition $\Delta_2$. Let $F \in X_m$ and the sequence $\{f_n\} \subset L^\varphi((\Omega, \Sigma, \mu), Y)$ be such that $F(t) = \overline{\{f_n(t)\}}$ for $\mu$-a.e. $t \in \Omega$. Then for every $f \in S_F^\varphi$, every $a > 0$, every $\epsilon > 0$, there exists a finite measurable partition $\{A_1, \ldots, A_n\}$ of $\Omega$ such that $\int_\Omega \varphi(t, a\|f(t) - \sum_{i=1}^n \chi_{A_i}(t)f_i(t)\|_Y) d\mu < \epsilon$.

**Proof.** We may assume that $f(t) \in F(t)$ for every $t \in \Omega$. Let $a, \epsilon > 0$ be arbitrary. Take a strictly positive $\delta \in L^1(\Omega, \Sigma, \mu)$ satisfying $\int_\Omega \delta d\mu < \frac{\epsilon}{2}$. Then there exists a countable measurable partition $\{B_i\}$ of $\Omega$ such that
\[ \varphi(t, a\|f(t) - f_n(t)\|_Y) < \delta(t) \text{ for every } t \in B_n. \]

Take an integer $n$ such that
\[ \sum_{k=n+1}^{\infty} \int_{B_k} \varphi(t, 2a\|f(t)\|_Y) d\mu < \frac{2}{3}\epsilon, \]
and define a finite measurable partition $\{A_1, \ldots, A_n\}$ as follows:
\[ A_1 = B_1 \cup \left( \bigcup_{i=n+1}^\infty B_i \right) \]
and $A_j = B_j$ for $j = 2, \ldots$. Then we have
\[
\begin{align*}
\int_\Omega \varphi(t, a\|f(t) - \sum_{m=1}^n \chi_{A_m}(t)f_m(t)\|_Y) d\mu &= \sum_{m=1}^n \int_{A_m} \varphi(t, a\|f(t) - f_m(t)\|_Y) d\mu \\
&= \sum_{m=1}^n \int_{B_m} \varphi(t, a\|f(t) - f_m(t)\|_Y) d\mu + \int_{B_n} \varphi(t, a\|f(t) - f_1(t)\|_Y) d\mu \\
&\leq \int_\Omega \delta(t) d\mu + \frac{1}{2} \sum_{k=n+1}^{\infty} \int_{B_k} \varphi(t, 2a\|f(t)\|_Y) d\mu + \frac{1}{2} \sum_{k=n+1}^{\infty} \int_{B_k} \varphi(t, 2a\|f_1\|_Y) d\mu < \epsilon. \quad \blacksquare
\end{align*}
\]

**Definition 4.** Let $M$ be a set of measurable functions $f : \Omega \to Y$. We call $M$ **decomposable** (with respect to $\Sigma$) if $f_1, f_2 \in M$ and $A \in \Sigma$ imply $\chi_A f_1 + \chi_{\Omega \setminus A} f_2 \in M$. 

Theorem 2. Let the $\varphi$-function $\varphi$ be locally integrable, convex and fulfill the condition $\Delta_2$. Let $M$ be a nonempty and closed subset of $L^\varphi((\Omega, \Sigma, \mu), Y)$. Then there exists an $F \in X_m$ such that $M = S^\varphi_F$ if and only if $M$ is decomposable.

Proof. It is clear that $S^\varphi_F$ is necessarily decomposable and closed (with respect to norm) in $L^\varphi((\Omega, \Sigma, \mu), Y)$. To prove the converse, let $M$ be a nonempty, closed, decomposable subset of $L^\varphi((\Omega, \Sigma, \mu), Y)$. By Lemma 1, there exists a sequence $\{f_n\} \subset L^\varphi((\Omega, \Sigma, \mu), Y)$ such that $\{f_n(t)\}$ is dense in $Y$ for each $t \in \Omega$. For each $i \in \mathbb{N}$ and $a > 0$, let

$$r_i(a) = \inf \left\{ \int_\Omega \varphi(t, a\|f_i(t) - g(t)\|_Y)d\mu : g \in M \right\}$$

and choose a sequence $\{g_{ij}\} \subset M$ such that

$$\int_\Omega \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu \rightarrow r_i(a).$$

Define $F \in X_m$ by $F(t) = \{g_{ij}(t)\}$. We shall prove $M = S^\varphi_F$. For each $f \in S^\varphi_F$, $\epsilon > 0$ and $a > 0$, by Lemma 2 we can take a finite measurable partition $\{A_1, \ldots, A_n\}$ of $\Omega$ and $\{h_1, \ldots, h_n\} \subset \{g_{ij}\}$ such that

$$\int_\Omega \varphi(t, a\|f(t) - \sum_{k=1}^n \chi_{A_k}h_k(t)\|_Y)d\mu < \epsilon.$$ 

Since $\sum_{k=1}^n \chi_{A_k}h_k \in M$, this implies $f \in M$. Hence $S^\varphi_F \subset M$. Now suppose $M \neq S^\varphi_F$. Then there exist an $f \in M$ and $A \in \Sigma$ with $0 < \mu(A) < \infty$, and a $\delta > 0$ such that

$$\inf_{i,j} \|f(t) - g_{ij}(t)\| \geq \delta,$$ 

for $t \in A$.

Take an integer $i$, fixed in the rest of the proof, such that the set

$$B = A \cap \{t \in \Omega : \|f(t) - f_i(t)\|_Y < \delta/3\}$$

has a positive measure, and let

$$g'_j = \chi_B f + \chi_{\Omega \setminus B}g_{ij}, \quad j \in \mathbb{N}.$$ 

Then, since $\{g'_j\} \subset M$ and for $t \in B$

$$\|f_i(t) - g_{ij}(t)\|_Y \geq \|f(t) - g_{ij}(t)\|_Y - \|f(t) - f_i(t)\|_Y > 2\delta/3$$

it follows that for $j \in \mathbb{N}$

$$\int_\Omega \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu - r_i(a)$$

$$\geq \int_B \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu - \int_B \varphi(t, a\|f_i(t) - g'_j(t)\|_Y)d\mu$$

$$= \int_B \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu - \int_B \varphi(t, a\|f_i(t) - (t)\|_Y)d\mu$$

$$\geq \int_B (\varphi(t, 2a\delta/3) - \varphi(t, a\delta/3))d\mu > 0,$$

because $\varphi$ is strictly increasing with respect to $u > 0$. Letting $j$ go to infinity, we have a contradiction. \blacksquare
We have for \( \varphi(t, u) = u^p \) for every \( t \in \Omega \), where \( 1 \leq p < \infty \), Theorem 3.1 from [6].

**Lemma 3.** Let the \( \varphi \)-function \( \varphi \) be locally integrable, convex and fulfil the condition \( \Delta_2 \). Let \( F \in X_m \) and \( S^\varphi_F \neq \emptyset \). Then for every \( a > 0 \)
\[
\sup \{ \rho(a\|f(\cdot)\|_Y) : f \in S^\varphi_F \} = \int_\Omega \sup \{ \varphi(t, a\|x\|_Y) : x \in F(t) \} d\mu.
\]

**Proof.** Let \( a > 0 \) be fixed. Denote
\[
m^a(t) = \sup \{ \varphi(t, a\|x\|_Y) : x \in F(t) \}
\]
for every \( t \in \Omega \). It is easy to see that \( m^a \) is measurable (see Proposition 2.24, Chapter 2 from [7]).

For every \( f \in S^\varphi_F \), \( \mu \)-a.e. \( t \in \Omega \) we have \( \varphi(t, a\|f(t)\|_Y) \leq m^a(t) \) so
\[
\sup \{ \rho(a\|f(\cdot)\|_Y) : f \in S^\varphi_F \} \leq \int_\Omega m^a(t) d\mu.
\]
If \( f_0 \in S^\varphi_F \) and \( \rho(a\|f_0(\cdot)\|_Y) = \infty \) we are done. Thus assume that \( \rho(a\|f_0(\cdot)\|_Y) \) is finite. If \( \int_\Omega m^a(t) d\mu = 0 \), then the proof is evident, so we can assume that \( \int_\Omega m^a(t) d\mu > 0 \). If \( m^a(t) = +\infty \) on the set of positive measure the proof is also evident, so we can assume that \( m^a(t) \) is finite \( \mu \)-a.e.

Let \( \beta < \int_\Omega m^a(t) d\mu \). We will produce an \( f \in S^\varphi_F \) such that \( \beta < \rho(a\|f(\cdot)\|_Y) \) and this will finish the proof. Let \( \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \) with \( \Omega_n \subset \Omega_{n+1} \) and \( \mu(\Omega_n) < \infty \) for every \( n \in \mathbb{N} \). Also let \( \delta : \Omega \rightarrow \mathbb{R}_+ \setminus \{0\} \) be an \( L^1(\Omega, \Sigma, \mu) \) function. Define \( A_n = \Omega_n \cap \{ t \in \Omega : \varphi(t, a\|f_0(t)\|_Y) \leq n \} \) and
\[
m^a_n(t) = \begin{cases} m^a(t) - \frac{\delta(t)}{n}, & \text{if } t \in A_n, m^a(t) \leq n, \\ n - \frac{\delta(t)}{n}, & \text{if } t \in A_n, m^a(t) > n, \\ \varphi(t, a\|f_0(t)\|_Y), & \text{if } t \in \Omega \setminus A_n. \end{cases}
\]
Evidently \( m^a_n \in L^1(\Omega, \Sigma, \mu) \) and \( m^a_n \uparrow m^a \) in \( \mu \)-measure. So passing to a subsequence if necessary, we may assume that \( m^a_n(t) \uparrow m^a(t) \) \( \mu \)-a.e. Thus by the monotone convergence theorem, we deduce that there exists \( n_0 \in \mathbb{N} \) such that \( \beta < \int_\Omega m^a_{n_0}(t) d\mu \). Let
\[
G_a(t) = F(t) \cap \{ x \in Y : \varphi(t, a\|x\|_Y) \geq m^a_{n_0}(t) \}
\]
for every \( t \in \Omega \). By modifying \( G_a \) on a \( \mu \)-null set, we may assume that \( G_a \neq \emptyset \) for every \( t \in \Omega \) and then \( G_a \) is graph-measurable so (see [7], Chapter 2, Theorems 2.1 and 2.14) there exists \( g : \Omega \rightarrow Y \) which is a strongly measurable selection of \( G_a \). Let
\[
C_n = \Omega_n \cap \{ t \in \Omega : \|g(t)\|_Y \leq n \}
\]
and \( f_n = \chi_{C_n} g + \chi_{\Omega \setminus C_n} f_0 \). It is easy to see that \( C_n \in \Sigma \). Since \( S^\varphi_F \) is decomposable, we have \( f_n \in S^\varphi_F \) and
\[
\rho(a\|f_n(\cdot)\|_Y) = \int_{C_n} \varphi(t, a\|g(t)\|_Y) d\mu + \int_{\Omega \setminus C_n} \varphi(t, a\|f_0(t)\|_Y) d\mu \\
\geq \int_\Omega m^a_{n_0}(t) d\mu + \int_{\Omega \setminus C_n} [\varphi(t, a\|f_0(t)\|_Y) - m^a_{n_0}(t)] d\mu.
\]
Note that \( \mu(\Omega \setminus C_n) \rightarrow 0 \) and \( \int_\Omega m^a_{n_0}(t) d\mu > \beta \), so for some \( n_1 \in \mathbb{N} \) we have
\[
\rho(a\|f_{n_1}(\cdot)\|_Y) > \beta.
\]
By Theorem 2 and Lemma 3 we obtain the following:

**Theorem 3.** Let the $\varphi$-function $\varphi$ be locally integrable, convex and fulfil the condition $\Delta_2$. Let $M$ be a nonempty, bounded, decomposable and closed subset of $L^\varphi((\Omega, \Sigma, \mu), Y)$. Then there exists an $F \in X_{m, \varphi}$ such that $M = S_F^\varphi$.

**Proof.** By Theorem 2, $F \in X_m$, by Lemma 3 we have $|F| \in L^\varphi((\Omega, \Sigma, \mu)$, so $F \in X_{m, \varphi}$. ■

**Corollary 3.** Let the $\varphi$-function $\varphi$ be locally integrable, convex and fulfil the condition $\Delta_2$. Let $M$ be a nonempty, bounded, decomposable and closed subset of $L^\varphi((\Omega, \Sigma, \mu), Y)$ and let $M(t) = \{f(t) : f \in M\}$ be a closed subset of $Y$ for every $t \in \Omega$. Then there exists an $F \in X_{m, \varphi}$ such that $M(t) = F(t)$ $\mu$-a.e.

**Proof.** Denote $S_F^\varphi(t) = \{f(t) : f \in S_F^\varphi\}$ for every $t \in \Omega$. By Lemma 1 we have $S_F^\varphi(t) \subset F(t) \subset S_F^\varphi(t)$ $\mu$-a.e. So by the assumptions we have $F(t) = S_F^\varphi(t)$ $\mu$-a.e. ■

**Remark 1.** Let the $\varphi$-function $\varphi$ be locally integrable, convex and fulfil the condition $\Delta_2$. If $F \in X_{m, \varphi}$, then $S_F^\varphi$ is a bounded and closed subset of $L^\varphi((\Omega, \Sigma, \mu), Y)$.

**Theorem 4.** Let the $\varphi$-function $\varphi$ be locally integrable, convex and fulfils the $\Delta_2$ condition. Let $F_1, F_2 \in X_m$ and $S_{F_1}^\varphi, S_{F_2}^\varphi \neq \emptyset$. Let $F(t) = F_1(t) + F_2(t)$ for every $t \in \Omega$, then $S_F^\varphi = S_{F_1}^\varphi + S_{F_2}^\varphi$.

**Proof.** It is easy to see that $F \in X_m$, so $S_F^\varphi$ is closed, hence $S_{F_1}^\varphi + S_{F_2}^\varphi \subset S_F^\varphi$. On the other hand by Lemma 1 we may find $\{f_{1n}\} \subset S_{F_1}^\varphi$ and $\{f_{2m}\} \subset S_{F_2}^\varphi$ such that $F_1(t) = \{f_{1n}(t)\}$ and $F_2(t) = \{f_{2m}(t)\} \mu$-a.e. Evidently $F(t) = \{f_{1n}(t) + f_{2m}(t)\}$ $\mu$-a.e. By Lemma 2 for $f \in S_{F_2}^\varphi$ and $\epsilon > 0$ we can find $\{A_1, \ldots, A_I\}$ a finite $\Sigma$-partition of $\Omega$ and positive integers $n_1, \ldots, n_I, m_1, \ldots, m_I$ such that

$$\left\| \left\| f(\cdot) - \sum_{k=1}^{I} \chi_{A_k}(f_{1n_k}(\cdot) + f_{2m_k}(\cdot)) \right\|_Y \right\|_\varphi < \epsilon.$$ 

Hence $f \in S_{F_1}^\varphi + S_{F_2}^\varphi$, so $S_F^\varphi = S_{F_1}^\varphi + S_{F_2}^\varphi$. ■

For $\varphi(t, u) = u^p$ for every $t \in \Omega$, where $1 \leq p < +\infty$, we have Proposition 3.28, Chapter 2 from [7].

**3. Final remark.** The results of this paper can be extended to the case that the $\varphi$-function $\varphi$ is not convex but only strictly increasing with respect to $u$. Clearly we must then use the $F$-norm in Musielak-Orlicz space.

**References**


