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DECOMPOSABLE SETS AND MUSIELAK-ORLICZ SPACES OF MULTIFUNCTIONS

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Abstract. We introduce the Musielak-Orlicz space of multifunctions $X_{m,\varphi}$ and the set S_F^φ of φ-integrable selections of F. We show that some decomposable sets in Musielak-Orlicz space belong to $X_{m,\varphi}$. We generalize Theorem 3.1 from [6]. Also, we get some theorems on the space $X_{m,\varphi}$ and the set S_F^φ.

1. Introduction. Decomposability is a basic concept in Multivalued Analysis (see [7], p. 174). A notion of decomposibility has been introduced by Rockafellar in [14]. A similar but different notion has been introduced in [6] and [7] and we will use this notion. The Musielak-Orlicz spaces of multifunctions were introduced and studied in [8]-[11]. The Musielak-Orlicz space of multifunctions $X_{m,\varphi}$ has been introduced in [11]. The aim of this note is to obtain a generalization of Theorem 3.1 from [6] and Theorem 3.8, Chapter 2 from [7]. All definitions and theorems connected with Musielak-Orlicz spaces can be found in [12]. Definitions and theorems connected with multifunctions can be found in [1]-[7], [13] and [14].

Let (Ω, Σ, μ) be a measure space with a nonnegative, nontrivial σ-finite and complete measure μ. Let φ be a φ-function, i.e., $\varphi : \Omega \times R \to R_+, \varphi(t,u)$ is an even, continuous function of u, equal to zero iff $u = 0$ and nondecreasing for $u \geq 0$ for every $t \in \Omega$, is a measurable function of $t \in \Omega$ for every $u \in R$ and $\lim_{u\to\infty} \varphi(t,u) = \infty$ for μ-a.e. $t \in \Omega$. Moreover, if $\varphi(t,\cdot)$ is a convex function for every $t \in \Omega$, then we shall say that the φ-function φ is convex. Let $L^\varphi(\Omega, \Sigma, \mu)$ be the Musielak-Orlicz function space generated by the modular

$$\rho(x) = \int_\Omega \varphi(t, x(t))d\mu.$$

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Let $\|\cdot\|_\varphi^L$ denote the Luxemburg norm in $L^\varphi(\Omega,\Sigma,\mu)$ if φ is convex. Let Y be a real separable Banach space with the norm $\|\cdot\|_Y$. Let Θ denote the zero element of Y. If $A, B \subset Y$ are nonempty then we denote

$$H(A,B) = \max(\sup_{x\in A}\inf_{y\in B}\|x-y\|_Y,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|_Y).$$

Denote by $E(Y)$ the set of all nonempty and closed subsets of Y. Let

$$X = \{F : \Omega \to 2^Y : F(t) \in E(Y) \text{ for every } t \in \Omega\}.$$

Two multifunctions $F, G \in X$ such that $F(t) = G(t)$ for μ-a.e. $t \in \Omega$ will be treated as the same element of X.

Now we introduce the function $\mathbf{d}(F,G)$ by the formula:

$$\mathbf{d}(F,G)(t) = H(F(t),G(t)) \text{ for all } F,G \in X \text{ and } t \in \Omega.$$

Let $\mathbf{N}$ be the set of all positive integers. Let $\mathbf{0} \in X$ be such that $\mathbf{0}(t) = \{\Theta\}$ for every $t \in \Omega$. Denote $|F| = \mathbf{d}(F,\mathbf{0})$ for every $F \in X$.

2. On the space $X_{m,\varphi}$ and the set S_F^φ

DEFINITION 1. We say that $F \in X$ is a *step multifunction* if

$$F(t) = \sum_{k=1}^n \chi_{A_k}(t)B_k \text{ for every } t \in \Omega$$

where χ_A is the characteristic function of the set A, $B_k \in E(Y)$ for $k = 1,\dots,n$, $\Omega = \bigcup_{k=1}^n A_k$, $A_k \in \Sigma$ for $k = 1,\dots,n$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

DEFINITION 2. We say that $F \in X$ is *measurable* if there exists a sequence of step multifunctions $F_n \in X$ for every $n \in \mathbf{N}$ such that $\lim_{n\to\infty}\mathbf{d}(F,F_n)(t) = 0$ for μ-a.e. $t \in \Omega$.

Denote:

$$X_m = \{F \in X : F \text{ is measurable}\}, \qquad X_{m,\varphi} = \{F \in X_m : |F| \in L^\varphi(\Omega,\Sigma,\mu)\},$$

It is easy to see that $\mathbf{d}(F,G) \in L^\varphi(\Omega,\Sigma,\mu)$ if $F,G \in X_{m,\varphi}$.

By [7], Chapter 2, Theorem 1.35, if $F \in X_m$, then F is measurable and graph measurable in the sense of [7], Chapter 2, Definition 1.1.

The space $X_{m,\varphi}$ will be called the Musielak-Orlicz spaces of multifunctions.

By $L^\varphi((\Omega,\Sigma,\mu),Y)$ we will denote the set of all strongly measurable functions $f : \Omega \to Y$ such that $\|f(\cdot)\|_Y \in L^\varphi(\Omega,\Sigma,\mu)$.

In [11] the following was proved:

THEOREM 1. *Let $F_n \in X_{m,\varphi}$ for every $n \in \mathbf{N}$. If for every $\epsilon > 0$ and every $a > 0$ there exists $K > 0$ such that $\int_\Omega \varphi(t, a\mathbf{d}(F_m,F_n)(t))d\mu < \epsilon$ for all $m,n > K$, then there exists $F \in X_{m,\varphi}$ such that $\int_\Omega \varphi(t, a\mathbf{d}(F_n,F)(t))d\mu \to 0$ as $n \to \infty$ for every $a > 0$.*

COROLLARY 1. *Let the φ-function φ be convex, then the function*

$$D_\varphi(F,G) = \|\mathbf{d}(F,G)\|_\varphi^L$$

for all $F,G \in X_{m,\varphi}$ is a metric in $X_{m,\varphi}$, so $< X_{m,\varphi}, D_\varphi >$ is a complete metric space.

Let $F \in X$. Denote

$$S_F^\varphi = \{f \in L^\varphi((\Omega, \Sigma, \mu), Y) : f(t) \in F(t)\mu \text{ a.e.}\}.$$

DEFINITION 3. The φ-function φ will be called *locally integrable* if $\int_A \varphi(t,u)d\mu < \infty$ for every $u > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$.

Applying the proof of Proposition 3.3, Proposition 2.17 and Remark 3.4 Chapter 2 from [7] we easily obtain the following:

LEMMA 1. *Let the φ-function φ be locally integrable and fulfil the condition Δ_2, then for every $F \in X_m$ such that $S_F^\varphi \neq \emptyset$ there exists a sequence $\{f_n\} \subset L^\varphi((\Omega, \Sigma, \mu), Y)$ such that $F(t) = \overline{\{f_n\}(t)}$ for μ-a.e. $t \in \Omega$.*

COROLLARY 2. *Let the φ-function φ be locally integrable and fulfil the condition Δ_2. Let $F, G \in X_m$ be such that $S_F^\varphi = S_G^\varphi \neq \emptyset$, then $F(t) = G(t)$ for μ-a.e. $t \in \Omega$.*

LEMMA 2. *Let the φ-function φ be locally integrable, convex and fulfil the condition Δ_2. Let $F \in X_m$ and the sequence $\{f_n\} \subset L^\varphi((\Omega, \Sigma, \mu), Y)$ be such that $F(t) = \overline{\{f_n\}(t)}$ for μ-a.e. $t \in \Omega$. Then for every $f \in S_F^\varphi$, every $a > 0$, every $\epsilon > 0$, there exists a finite measurable partition $\{A_1, \dots, A_n\}$ of Ω such that $\int_\Omega \varphi(t, a\|f(t) - \sum_{i=1}^n \chi_{A_i}(t) f_i(t)\|_Y)d\mu < \epsilon$.*

Proof. We may assume that $f(t) \in F(t)$ for every $t \in \Omega$. Let $a, \epsilon > 0$ be arbitrary. Take a strictly positive $\delta \in L^1(\Omega, \Sigma, \mu)$ satisfying $\int_\Omega \delta d\mu < \frac{\epsilon}{3}$. Then there exists a countable measurable partition $\{B_i\}$ of Ω such that

$$\varphi(t, a\|f(t) - f_n(t)\|_Y) < \delta(t) \text{ for every } t \in B_n.$$

Take an integer n such that

$$\sum_{k=n+1}^\infty \int_{B_k} \varphi(t, 2a\|f(t)\|_Y)d\mu < \frac{2}{3}\epsilon, \qquad \sum_{k=n+1}^\infty \int_{B_k} \varphi(t, 2a\|f_1(t)\|_Y)d\mu < \frac{2}{3}\epsilon,$$

and define a finite measurable partition $\{A_1, \dots, A_n\}$ as follows:

$$A_1 = B_1 \cup \Big(\bigcup_{i=n+1}^\infty B_i \Big)$$

and $A_j = B_j$ for $j = 2, \dots$ Then we have

$$\int_\Omega \varphi(t, a\|f(t) - \sum_{m=1}^n \chi_{A_m}(t) f_m(t)\|_Y)d\mu = \sum_{m=1}^n \int_{A_m} \varphi(t, a\|f(t) - f_m(t)\|_Y)d\mu$$
$$= \sum_{m=1}^n \int_{B_m} \varphi(t, a\|f(t) - f_m(t)\|_Y)d\mu + \sum_{m=n+1}^\infty \int_{B_m} \varphi(t, a\|f(t) - f_1(t)\|_Y)d\mu$$
$$\leq \int_\Omega \delta(t)d\mu + \frac{1}{2}\sum_{k=n+1}^\infty \int_{B_k} \varphi(t, 2a\|f(t)\|_Y)d\mu + \frac{1}{2}\sum_{k=n+1}^\infty \int_{B_k} \varphi(t, 2a\|f_1\|_Y)d\mu < \epsilon. \blacksquare$$

DEFINITION 4. Let M be a set of measurable functions $f : \Omega \to Y$. We call M *decomposable* (with respect to Σ) if $f_1, f_2 \in M$ and $A \in \Sigma$ imply $\chi_A f_1 + \chi_{\Omega \setminus A} f_2 \in M$.

THEOREM 2. *Let the φ-function φ be locally integrable, convex and fulfil the condition Δ_2. Let M be a nonempty and closed subset of $L^\varphi((\Omega,\Sigma,\mu),Y)$. Then there exists an $F \in X_m$ such that $M = S_F^\varphi$ if and only if M is decomposable.*

Proof. It is clear that S_F^φ is necessarily decomposable and closed (with respect to norm) in $L^\varphi((\Omega,\Sigma,\mu),Y)$. To prove the converse, let M be a nonempty, closed, decomposable subset of $L^\varphi((\Omega,\Sigma,\mu),Y)$. By Lemma 1, there exists a sequence $\{f_n\} \subset L^\varphi((\Omega,\Sigma,\mu),Y)$ such that $\{f_n(t)\}$ is dense in Y for each $t \in \Omega$. For each $i \in \mathbf{N}$ and $a > 0$, let

$$r_i(a) = inf\left\{\int_\Omega \varphi(t, a\|f_i(t) - g(t)\|_Y)d\mu : g \in M\right\}$$

and choose a sequence $\{g_{ij}\} \subset M$ such that

$$\int_\Omega \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu \to r_i(a).$$

Define $F \in X_m$ by $F(t) = \overline{\{g_{ij}(t)\}}$. We shall prove $M = S_F^\varphi$. For each $f \in S_F^\varphi$, $\epsilon > 0$ and $a > 0$, by Lemma 2 we can take a finite measurable partition $\{A_1,\ldots,A_n\}$ of Ω and $\{h_1,\ldots,h_n\} \subset \{g_{ij}\}$ such that

$$\int_\Omega \varphi(t, a\|f(t) - \sum_{k=1}^n \chi_{A_k} h_k(t)\|_Y)d\mu < \epsilon.$$

Since $\sum_{k=1}^n \chi_{A_k} h_h \in M$, this implies $f \in M$. Hence $S_F^\varphi \subset M$. Now suppose $M \neq S_F^\varphi$. Then there exist an $f \in M$ and $A \in \Sigma$ with $0 < \mu(A) < \infty$, and a $\delta > 0$ such that

$$\inf_{i,j} \|f(t) - g_{ij}(t)\| \geq \delta, \text{ for } t \in A.$$

Take an integer i, fixed in the rest of the proof, such that the set

$$B = A \cap \{t \in \Omega : \|f(t) - f_i(t)\|_Y < \delta/3\}$$

has a positive measure, and let

$$g'_j = \chi_B f + \chi_{\Omega\setminus B} g_{ij}, \quad j \in \mathbf{N}.$$

Then, since $\{g'_j\} \subset M$ and for $t \in B$

$$\|f_i(t) - g_{ij}(t)\|_Y \geq \|f(t) - g_{ij}(t)\|_Y - \|f(t) - f_i(t)\|_Y > 2\delta/3$$

it follows that for $j \in \mathbf{N}$

$$\begin{aligned}
&\int_\Omega \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu - r_i(a) \\
&\qquad \geq \int_\Omega \varphi(a\|f_i(t) - g_{ij}(t)\|_Y)d\mu - \int_\Omega \varphi(a\|f_i(t) - g'_j(t)\|_Y)d\mu \\
&\qquad = \int_B \varphi(t, a\|f_i(t) - g_{ij}(t)\|_Y)d\mu - \int_B \varphi(t, a\|f_i(t) - (t)\|_Y)d\mu \\
&\qquad \geq \int_B (\varphi(t, 2a\delta/3) - \varphi(t, a\delta/3))d\mu > 0,
\end{aligned}$$

because φ is strictly increasing with respect to $u > 0$. Letting j go to infinity, we have a contradiction. ∎

We have for $\varphi(t,u) = u^p$ for every $t \in \Omega$, where $1 \le p < \infty$, Theorem 3.1 from [6].

LEMMA 3. *Let the φ-function φ be locally integrable, convex and fulfil the condition Δ_2. Let $F \in X_m$ and $S_F^\varphi \neq \emptyset$. Then for every $a > 0$*

$$\sup[\rho(a\|f(\cdot)\|_Y) : f \in S_F^\varphi] = \int_\Omega \sup\{\varphi(t, a\|x\|_Y) : x \in F(t)\} d\mu.$$

Proof. Let $a > 0$ be fixed. Denote

$$m^a(t) = \sup[\varphi(t, a\|x\|_Y) : x \in F(t)]$$

for every $t \in \Omega$. It is easy to see that m^a is measurable (see Proposition 2.24, Chapter 2 from [7]).

For every $f \in S_F^\varphi$, μ-a.e. $t \in \Omega$ we have $\varphi(t, a\|f(t)\|_Y) \le m^a(t)$ so

$$\sup[\rho(a\|f(\cdot)\|_Y) : f \in S_F^\varphi] \le \int_\Omega m^a(t) d\mu.$$

If $f_0 \in S_F^\varphi$ and $\rho(a\|f_0(\cdot)\|_Y) = \infty$ we are done. Thus assume that $\rho(a\|f_0(\cdot)\|_Y)$ is finite. If $\int_\Omega m^a(t)d\mu = 0$, then the proof is evident, so we can assume that $\int_\Omega m^a(t)d\mu > 0$. If $m^a(t) = +\infty$ on the set of positive measure the proof is also evident, so we can assume that $m^a(t)$ is finite μ-a.e.

Let $\beta < \int_\Omega m^a(t)d\mu$. We will produce an $f \in S_F^\varphi$ such that $\beta < \rho(a\|f(\cdot)\|_Y)$ and this will finish the proof. Let $\Omega = \bigcup_{n\in\mathbf{N}} \Omega_n$ with $\Omega_n \subset \Omega_{n+1}$ and $\mu(\Omega_n) < \infty$ for every $n \in \mathbf{N}$. Also let $\delta : \Omega \to R_+ \setminus \{0\}$ be an $L^1(\Omega, \Sigma, \mu)$ function. Define $A_n = \Omega_n \cap \{t \in \Omega : \varphi(t, a\|f_0(t)\|_Y) \le n\}$ and

$$m_n^a(t) = \begin{cases} m^a(t) - \frac{\delta(t)}{n}, & \text{if} \quad t \in A_n, m^a(t) \le n, \\ n - \frac{\delta(t)}{n}, & \text{if} \quad t \in A_n, m^a(t) > n, \\ \varphi(t, a\|f_0(t)\|_Y), & \text{if} \quad t \in \Omega \setminus A_n. \end{cases}$$

Evidently $m_n^a \in L^1(\Omega, \Sigma, \mu)$ and $m_n^a \uparrow m^a$ in μ-measure. So passing to a subsequence if necessary, we may assume that $m_n^a(t) \uparrow m^a(t)$ μ-a.e. Thus by the monotone convergence theorem, we deduce that there exists $n_0 \in \mathbf{N}$ such that $\beta < \int_\Omega m_{n_0}^a(t)d\mu$. Let

$$G_a(t) = F(t) \cap \{x \in Y : \varphi(t, a\|x\|_Y) \ge m_{n_0}^a(t)\}$$

for every $t \in \Omega$. By modifying G_a on a μ-null set, we may assume that $G_a \neq \emptyset$ for every $t \in \Omega$ and then G_a is graph-measurable so (see [7], Chapter 2, Theorems 2.1 and 2.14) there exists $g : \Omega \to Y$ which is a strongly measurable selection of G_a. Let

$$C_n = \Omega_n \cap \{t \in \Omega : \|g(t)\|_Y \le n\}$$

and $f_n = \chi_{C_n} g + \chi_{\Omega\setminus C_n} f_0$. It is easy to see that $C_n \in \Sigma$. Since S_F^φ is decomposable, we have $f_n \in S_F^\varphi$ and

$$\begin{aligned} \rho(a\|f_n(\cdot)\|_Y) &= \int_{C_n} \varphi(t, a\|g(t)\|_Y) d\mu + \int_{\Omega\setminus C_n} \varphi(t, a\|f_0(t)\|_Y) d\mu \\ &\ge \int_\Omega m_{n_0}^a(t) d\mu + \int_{\Omega\setminus C_n} [\varphi(t, a\|f_0(t)\|_Y) - m_{n_0}^a(t)] d\mu. \end{aligned}$$

Note that $\mu(\Omega \setminus C_n) \to 0$ and $\int_\Omega m_{n_0}^a(t)d\mu > \beta$, so for some $n_1 \in \mathbf{N}$ we have

$$\rho(a\|f_{n_1}(\cdot)\|_Y) > \beta. \ \blacksquare$$

By Theorem 2 and Lemma 3 we obtain the following:

THEOREM 3. *Let the φ-function φ be locally integrable, convex and fulfil the condition Δ_2. Let M be a nonempty, bounded, decomposable and closed subset of $L^\varphi((\Omega,\Sigma,\mu),Y)$. Then there exists an $F \in X_{m,\varphi}$ such that $M = S_F^\varphi$.*

Proof. By Theorem 2, $F \in X_m$, by Lemma 3 we have $|F| \in L^\varphi(\Omega,\Sigma,\mu)$, so $F \in X_{m,\varphi}$. ■

COROLLARY 3. *Let the φ-function φ be locally integrable, convex and fulfil the condition Δ_2. Let M be a nonempty, bounded, decomposable and closed subset of $L^\varphi((\Omega,\Sigma,\mu),Y)$ and let $M(t) = \{f(t) : f \in M\}$ be a closed subset of Y for every $t \in \Omega$. Then there exists an $F \in X_{m,\varphi}$ such that $M(t) = F(t)$ μ-a.e.*

Proof. Denote $S_F^\varphi(t) = \{f(t) : f \in S_F^\varphi\}$ for every $t \in \Omega$. By Lemma 1 we have $S_F^\varphi(t) \subset F(t) \subset \overline{S_F^\varphi(t)}$ μ-a.e. So by the assumptions we have $F(t) = S_F^\varphi(t)$ μ-a.e. ■

REMARK 1. Let the φ-function φ be locally integrable, convex and fulfil the condition Δ_2. If $F \in X_{m,\varphi}$, then S_F^φ is a bounded and closed subset of $L^\varphi((\Omega,\Sigma,\mu),Y)$.

THEOREM 4. *Let the φ-function φ be locally integrable, convex and fulfils the Δ_2 condition. Let $F_1, F_2 \in X_m$ and $S_{F_1}^\varphi, S_{F_2}^\varphi \neq \emptyset$. Let $F(t) = \overline{F_1(t) + F_2(t)}$ for every $t \in \Omega$, then $S_F^\varphi = \overline{S_{F_1}^\varphi + S_{F_2}^\varphi}$.*

Proof. It is easy to see that $F \in X_m$, so S_F^φ is closed, hence $\overline{S_{F_1}^\varphi + S_{F_2}^\varphi} \subset S_F^\varphi$. On the other hand by Lemma 1 we may find $\{f_{1n}\} \subset S_{F_1}^\varphi$ and $\{f_{2m}\} \subset S_{F_2}^\varphi$ such that $F_1(t) = \overline{\{f_{1n}(t)\}}$ and $F_2(t) = \overline{\{f_{2m}(t)\}}$ μ-a.e. Evidently $F(t) = \overline{\{f_{1n}(t) + f_{2m}(t)\}}$ μ-a.e. By Lemma 2 for $f \in S_F^\varphi$ and $\epsilon > 0$ we can find $\{A_1,\ldots,A_I\}$ a finite Σ-partition of Ω and positive integers $n_1,\ldots,n_I,m_1,\ldots,m_I$ such that

$$\Big\|\Big\|f(\cdot) - \sum_{k=1}^{I} \chi_{A_k}(f_{1n_k}(\cdot) + f_{2m_k}(\cdot))\Big\|_Y\Big\|_\varphi^L < \epsilon.$$

Hence $f \in \overline{S_{F_1}^\varphi + S_{F_2}^\varphi}$, so $S_F^\varphi = \overline{S_{F_1}^\varphi + S_{F_2}^\varphi}$. ■

For $\varphi(t,u) = u^p$ for every $t \in \Omega$, where $1 \le p < +\infty$, we have Proposition 3.28, Chapter 2 from [7].

3. Final remark. The results of this paper can be extended to the case that the φ-function φ is not convex but only strictly increasing with respect to u. Clearly we must then use the F-norm in Musielak-Orlicz space.

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