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ON PROPERTY (β) OF ROLEWICZ IN MUSIELAK-ORLICZ SEQUENCE SPACES EQUIPPED WITH THE ORLICZ NORM

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Abstract. We prove that the Musielak-Orlicz sequence space with the Orlicz norm has property (β) iff it is reflexive. It is a generalization and essential extension of the respective results from [3] and [5]. Moreover, taking an arbitrary Musielak-Orlicz function instead of an N-function we develop new methods and techniques of proof and we consider a wider class of spaces than in [3] and [5].

1. Introduction. Throughout this paper $(X, \|\cdot\|_X)$ is a real Banach space. As usual, $S(X)$ and $B(X)$ stand for the unit sphere and the unit ball of X, respectively. For any subset A of X, we denote by $\operatorname{conv}(A)$ the convex hull of A.

The Banach space X is said to be *uniformly convex* ($X \in (UC)$ for short), if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in S(X)$ the inequality $\|x - y\|_X \geq \varepsilon$ implies $\|x + y\|_X \leq 2(1 - \delta)$ (see [2]).

Define for any $x \notin B(X)$ the *drop* $D(x, B(X))$ determined by x by $D(x, B(X)) = \operatorname{conv}(\{x\} \cup B(X))$.

Recall that for any subset C of X, the *Kuratowski measure of non-compactness* of C is the infimum $\alpha(C)$ of $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ε.

Rolewicz has proved that $X \in (UC)$ iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\|_X < 1 + \delta$ implies $\operatorname{diam}(D(x, B(X)) \setminus B(X)) < \varepsilon$ (see [20]). In connection with this he has introduced in [21] the following property.

A Banach space X has the *property* (β) ($X \in (\beta)$) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(D(x, B(X)) \setminus B(X)) < \varepsilon$ whenever $1 < \|x\|_X < 1 + \delta$.

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A Banach space is *nearly uniformly convex* ($X \in (NUC)$) if for every $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $\{x_n\}$ in $B(X)$ with $\operatorname{sep}\{x_n\} > \varepsilon$, we have $\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset$. Rolewicz proved the following implications: $(UC) \Rightarrow (\beta) \Rightarrow (NUC)$ (see [21]). Moreover, the class of Banach spaces with an equivalent norm with property (β) coincides neither with that of superreflexive spaces nor with the class of nearly uniformly convexifiable spaces (see [5] for references). Although property (β) was introduced during studies on well-posed problems in optimization theory (see [19], [21]), it has been widely and intensively developed from the geometric point of view (see [5], [13] and [14] for references). One of the reasons that property (β) is important is the fact that if a Banach space X has property (β), then both X and X^* have the fixed point property (FPP). The first fact follows from the implications $(\beta) \Rightarrow (NUC)$ and $(NUC) \Rightarrow (FPP)$ (see [6] and [21]). Moreover, if $X \in (\beta)$, then X^* has normal structure (see [17]). On the other hand, Kirk proved that normal structure implies the weak fixed point property $(WFPP)$ (see [6]). Since $(WFPP)$ and (FPP) coincide in reflexive spaces and property (β) implies reflexivity, property (β) implies also the fixed point property for the dual space.

A sequence $\{x_n\} \subset X$ is *ε-separated* for some $\varepsilon > 0$ if $\operatorname{sep}\{x_n\} = \inf\{\|x_n - x_m\|_X : n \neq m\} > \varepsilon$.

Although the primary definition of property (β) uses the *Kuratowski measure of noncompactness*, more convenient in our considerations is the following equivalent condition proved by Kutzarova in [16].

THEOREM 1. *A Banach space X has property (β) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}\{x_n\} \geq \varepsilon$ there is an index k for which $\|x + x_k\|_X \leq 2(1-\delta)$.*

Denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of natural, real and non-negative real numbers, respectively. Let $(\mathbb{N}, 2^{\mathbb{N}}, m)$ be the counting measure space and $l_0 = l_0(m)$ the linear space of all real sequences.

Let $E = (E, \leq, \|\cdot\|_E)$ be a Banach sequence lattice over the measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$, that is, E is a Banach space which is a subspace of l_0 endowed with the natural coordinatewise semi-order relation, and E satisfies the conditions:

(i) if $x \in E, y \in l_0, |y| \leq |x|$, i.e. $|y(i)| \leq |x(i)|$ for every $i \in \mathbb{N}$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$,

(ii) there exists a sequence x in E that is positive on the whole $\mathbb{N}$ (see [11] and [18]).

Banach sequence lattices are often called *Köthe sequence spaces.*

A Köthe space E is called *order continuous* ($E \in (OC)$) if for every $x \in E$ and each sequence $(x_m) \in E$ such that $0 \leq x_m \leq |x|$ and $x_m \to 0$ we have $\|x_m\|_E \to 0$ (see [11] and [18]).

A function φ is called an *Orlicz function* if $\varphi\colon \mathbb{R} \to [0,\infty]$ is convex, even, $\varphi(0) = 0$ and φ is not identically equal to zero and infinity. A sequence $\varphi = (\varphi_i)$ of Orlicz functions φ_i is called a *Musielak-Orlicz function.* We will write $\varphi > 0$ if $\varphi_i(u) = 0$ iff $u = 0$ for every $i \in \mathbb{N}$. Given a Musielak-Orlicz function φ we will denote by φ^* the sequence $(\varphi_i^*)_{i=1}^{\infty}$ of functions $\varphi_i^*\colon \mathbb{R} \to [0,\infty]$ that are *complementary to φ_i in the sense of Young,* i.e.

$\varphi_i^*(v) = \sup_{u\geq 0}\{u|v| - \varphi_i(u)\}$ for every $v \in \mathbb{R}$ and $i \in \mathbb{N}$. Define on l_0 a convex modular I_φ by $I_\varphi(x) = \sum_{i=1}^\infty \varphi_i(x(i))$ for any $x \in l_0$. By the *Musielak-Orlicz space* l_φ we mean

$$l_\varphi = \{x \in l_0 : I_\varphi(cx) < \infty \text{ for some } c > 0\}.$$

This space is usually considered with the *Luxemburg norm* $\|x\|_\varphi = \inf\{\varepsilon > 0 : I_\varphi(x/\varepsilon) \leq 1\}$ (we write $l_\varphi = (l_\varphi, \|\cdot\|_\varphi)$) or with the equivalent *Orlicz norm* $\|x\|_\varphi^O = \sup\{|\sum_{i=1}^\infty x(i)y(i)| : I_{\varphi^*}(y) \leq 1\}$ (we write $l_\varphi^O = (l_\varphi, \|\cdot\|_\varphi^O)$). We consider this space with the Amemiya norm $\|x\|_\varphi^A = \inf_{k>0}\{\frac{1}{k}(1+I_\varphi(kx))\}$ (we write $l_\varphi^A = (l_\varphi, \|\cdot\|_\varphi^A)$) which seems to be equal to the Orlicz norm (but there is no proof of this fact in general). In the case of Orlicz spaces we have $\|x\|_\varphi^A = \|x\|_\varphi^O$ for an arbitrary Orlicz function (see [9]). However, the analogous argument as in the proof Theorem 1 in [9] gives

LEMMA 1. *Let φ be a finitely valued Musielak-Orlicz function such that $\varphi_i(u)/u \to \infty$ as $u \to \infty$ for each $i \in \mathbb{N}$. Then $\|x\|_\varphi^A = \|x\|_\varphi^O$ for any $x \in l_\varphi$.*

We say that a Musielak-Orlicz function φ satisfies *the δ_2-condition* ($\varphi \in \delta_2$) if there are constants $k_0, a_0 > 0$ and a sequence $(c_i^0)_{i=1}^\infty$ of positive reals with $\sum_{i=1}^\infty c_i^0 < \infty$ such that $\varphi_i(2u) \leq k_0\varphi_i(u) + c_i^0$ for each $i \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\varphi_i(u) \leq a_0$.

The symbol p_i stands for the right derivative of φ_i. For each $i \in \mathbb{N}$ denote

$$s_i = \sup\{u \geq 0 : \varphi_i^*(p_i(u)) \leq 1\} \text{ and } a_i = \sup\{u \geq 0 : \varphi_i^*(u) < \infty\}.$$

REMARK 1. If $\varphi_i^*(a_i) > 1$ for any $i \in \mathbb{N}$, then, without loss of generality, we may assume that all functions φ_i, for $u \geq s_i$, are square functions. Indeed, given a Musielak-Orlicz function φ with $\varphi_i^*(a_i) > 1$, $i \in \mathbb{N}$, take numbers w_i with $\varphi_i^*(w_i) = 1$. Obviously $w_i \leq p_i(s_i)$. Take

$$\overline{p}_i(t) = \begin{cases} p_i(t) & \text{for } 0 \leq t < s_i \\ \frac{w_i}{s_i}t & \text{for } t \geq s_i \end{cases} \quad \text{and} \quad \psi_i(u) = \int_0^u \overline{p}_i(t)dt. \tag{1}$$

The spaces l_φ^O and l_ψ^O are isometric, because $I_{\psi^*}(y) \leq 1$ if and only if $I_{\varphi^*}(y) \leq 1$. Notice that the right derivative $\overline{p}_i$ of ψ_i is nondecreasing, hence ψ_i is convex on the whole $\mathbb{R}_+$ (the situation of the respective isometry for the Luxemburg norm is different, see [10]). **In the whole paper we shall always assume that the Musielak-Orlicz function φ satisfies the condition $\varphi_i^*(a_i) > 1$, $i \in \mathbb{N}$, and we shall consider the function (ψ_i) instead of (φ_i) according to the formula (1). Note that for the function (ψ_i) we have $\overline{p}_i(s_i) = w_i$.**

Then, clearly, for each $i \in \mathbb{N}$ a function φ_i is finitely valued and $\varphi_i(u)/u \to \infty$ as $u \to \infty$. Hence $\|x\|_\varphi^A = \|x\|_\varphi^O$ for any $x \in l_\varphi$, by Lemma 1. Moreover, for any $x \in l_\varphi\backslash\{0\}$ the set $\{k_x > 0 : \|x\|_\varphi^A = \frac{1}{k_x}(1 + I_\varphi(k_x x))\}$ is nonempty and bounded (see [1] and [7]).

2. Results

LEMMA 2. *Assume that $\varphi^* \in \delta_2$ and $M = \sup_i p_i(s_i)s_i < \infty$. Then for every $\eta \in (0,1)$ there are $\gamma = \gamma(\eta) \in (0,1)$ and a sequence $h = (h_i)$ of positive numbers with $\sum_{i\in\mathbb{N}} \varphi_i(h_i) < \infty$ such that for any $\alpha \in (0,\eta]$ the inequality $\varphi_i(\alpha u) \leq (1-\gamma)\alpha\varphi_i(u)$ holds for all $i \in \mathbb{N}$ and $u \geq h_i$.*

Proof. Although we argue analogously as in the proof of Lemma 3 in [4] we present the proof for the sake of convenience. Since $\varphi^* \in \delta_2$, there are constants $a, k > 0$ and sequences $b = (b_i)$, $d = (d_i)$ with $\sum_{i\in\mathbb{N}} \varphi_i^*(b_i) < \infty$ and $\varphi_i^*(d_i) = a$ such that $\varphi_i^*(2u) \le k\varphi_i^*(u)$ for each i and $b_i \le u \le d_i$ (see [4]). We claim that there is $k_1 > 0$ such that $\varphi_i^*(2u) \le k_1\varphi_i^*(u)$ for each i and $u \ge p_i(s_i)$. We have $w_i = p_i(s_i)$ and $\varphi_i^*(u) = 1 + \frac{s_i}{2w_i}(u^2 - w_i^2)$ for $u \ge w_i$ (see Remark 1). Set $f_i(u) = \varphi_i^*(2u)/\varphi_i^*(u)$ for $u \ge w_i$. It is easy to check that f_i is increasing if $s_i w_i < 2$ and f_i is non-increasing if $s_i w_i \ge 2$. Moreover, for each $i \in \mathbb{N}$, $\lim_{u\to\infty} f_i(u) = 4$, $\varphi_i^*(2w_i) = 1 + \frac{3}{2}s_i w_i \le 1 + \frac{3}{2}M$ and $\varphi_i^*(w_i) = 1$. This proves the claim with $k_1 = \max\{4, 1 + \frac{3}{2}M\}$. Let $i \in \mathbb{N}$ and $u > d_i$. Consequently, if $d_i \ge p_i(s_i)$ and $u > d_i$, then $\varphi_i^*(2u) \le k_1\varphi_i^*(u)$. If $d_i < p_i(s_i)$ and $d_i < u < p_i(s_i)$, then $\varphi_i^*(2u) \le \varphi_i^*(2p_i(s_i)) \le k_1\varphi_i^*(p_i(s_i)) \le k_1\varphi_i^*(u)/a$. Hence

$$\varphi_i^*(2u) \le k_0\varphi_i^*(u)$$

for each i and $u \ge b_i$, where $k_0 = \max\{k_1, k_1/a, k\}$. Analogously as in Lemma 3 in [4] we prove that there exists $\xi > 1$ such that

$$\varphi_i\left(\frac{u}{2}\right) \le \frac{1}{2\xi}\varphi_i(u) + \varphi_i^*(b_i),\ \ i \in \mathbb{N}, u \in \mathbb{R}.$$

Taking numbers $\widetilde{b}_i \ge 0$ such that $\varphi_i^*(\frac{2\xi}{\sqrt{\xi}-1}b_i) = \varphi_i(\widetilde{b}_i)$ we get $\sum_{i\in\mathbb{N}} \varphi_i(\widetilde{b}_i) < \infty$ because $\varphi^* \in \delta_2$ and $\sum_{i\in\mathbb{N}} \varphi_i^*(b_i) < \infty$. Consequently, for each $i \in \mathbb{N}$ and $u \ge \widetilde{b}_i$, we get

$$\varphi_i\left(\frac{u}{2}\right) \le \frac{1}{2\xi}\varphi_i(u) + \frac{\sqrt{\xi}-1}{2\xi}\varphi_i^*\left(\frac{2\xi}{\sqrt{\xi}-1}b_i\right) = \frac{1}{2\xi}\varphi_i(u) + \frac{\sqrt{\xi}-1}{2\xi}\varphi_i(\widetilde{b}_i) \le \frac{1}{2\sqrt{\xi}}\varphi_i(u).$$

Modifying slightly the previous proof one can show that for each $\eta \in (0,1)$ there are $\gamma = \gamma(\eta) \in (0,1)$ and a sequence $h = (h_i)$ of positive numbers with $\sum_{i\in\mathbb{N}} \varphi_i(h_i) < \infty$ such that $\varphi_i(\eta u) \le (1-\gamma)\eta\varphi_i(u)$ for all $i \in \mathbb{N}$ and $u \ge h_i$. Applying the fact that for every $i \in \mathbb{N}$ the function $\varphi_i(u)/u$ is nondecreasing it is easy to finish the proof. ∎

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It is known that the equivalence $\|x_n\|_\varphi \to 0$ iff $I_\varphi(x_n) \to 0$ holds if and only if $\varphi \in \delta_2$ and $\varphi > 0$ (Theorem 0.1 in [10]). Dropping the assumption that $\varphi > 0$ we get

LEMMA 3 (Lemma 7 in [14]). *The following statements are equivalent:*

(i) $\|x_n\|_\varphi \to 0$ *if and only if* $I_\varphi(x_n) \to 0$ *for every sequence* (x_n) *in* l_φ *with elements* x_n *having pairwise disjoint supports.*

(ii) $\varphi \in \delta_2$.

Since the Orlicz and Luxemburg norms are equivalent, from Lemma 3 we conclude immediately

COROLLARY 1. *If* $\varphi \in \delta_2$, *then for every* $\varepsilon > 0$ *there exists* $\sigma = \sigma(\varepsilon) > 0$ *such that for every sequence* (x_n) *in* l_φ *with elements* x_n *having pairwise disjoint supports and satisfying* $\|x_n\|_\varphi^O \ge \varepsilon$ *for every* $n \in \mathbb{N}$ *the inequality* $I_\varphi(x_n) \ge \sigma$ *holds for almost every* $n \in \mathbb{N}$.

For each $p \in [0,1)$ define

$$\overline{k}(p) = \sup_{1-p \leq \|x\|_\varphi^A \leq 1} \left\{ k_x : \|x\|_\varphi^A = \frac{1}{k_x}(1 + I_\varphi(k_x x)) \right\}. \tag{2}$$

It appears that the condition $\overline{k}(0) < \infty$ plays a crucial role in many proofs concerning geometric properties of Musielak-Orlicz spaces with the Orlicz-Amemiya norm. However, the proof of this condition uses essentially additional assumptions on the function φ, that is: φ is an N-function, i.e. $\varphi > 0$ and $\varphi(u)/u \to 0$ as $u \to 0$ (see [1]). Although some of these assumptions have been weakened by several authors in different particular cases (see [8] and [15]), the assumption $\varphi > 0$ has not been dropped yet (as far as we know). Furthermore, the case $\overline{k}(0) < \infty$ for the Musielak-Orlicz sequence spaces has not been solved even for N-functions. It can be seen that the assumption $\varphi > 0$ is crucial in all proofs of the fact $\overline{k}(0) < \infty$. On the other hand, it seems that l_φ may have property (β) even when functions φ vanish outside zero (it has already been proved for the Luxemburg norm—see [14]). Thus it seems to be natural to try proving Theorem 2 below without the assumption that $\varphi > 0$. In order to do it we prove that Theorem 1.35 from [1] is true not only for N-functions but for arbitrary Musielak-Orlicz functions.

LEMMA 4. *If $\varphi^* \in \delta_2$ and $\sup_i p_i(s_i)s_i < \infty$, then $\overline{k}(p) < \infty$ for each $p \in [0,1)$.*

Proof. First we shall show that $\overline{k}(0) < \infty$. Take a sequence $h = (h_i)$ and a number $\gamma \in (0,1)$ from Lemma 2 for $\eta = 1/2$. Given a number $\sigma > 0$ define

$$k_1(\sigma) = \sup_{x \in B(\sigma)} \left\{ k : \|x\|_\varphi^A = \frac{1}{k}(1 + I_\varphi(kx)) \right\},$$

where

$$B(\sigma) = \{x \in l_\varphi^A : \|x\|_\varphi^A = 1 \text{ and } I_\varphi(2x\chi_{B_x}) \geq \sigma\},\ B_x = \{i \in \mathbb{N} : 2|x(i)| \leq h_i\}.$$

First we prove that

$$k_1(\sigma) < \infty \text{ for each } \sigma > 0. \tag{3}$$

Suppose that this is not true. Then there is $\sigma > 0$, a sequence (x_n) in $B(\sigma)$ and a sequence $k_n \to \infty$ with $\|x_n\|_\varphi^A = \frac{1}{k_n}(1 + I_\varphi(k_n x_n))$. Then $I_\varphi(2x_n\chi_{B_{x_n}}) \geq \sigma$, $n \in \mathbb{N}$. We claim that there is $i_0 \in \mathbb{N}$ and $\delta > 0$ such that $|x_n(i_0)| > \delta$ for infinitely many n. Otherwise $x_n \to 0$ pointwise. Then $y_n = 2x_n\chi_{B_{x_n}} \to 0$ pointwise. Moreover, $|y_n(i)| \leq h_i$ for each $i, n \in \mathbb{N}$ and $\sum_{i\in\mathbb{N}} \varphi_i(h_i) < \infty$, so $(\varphi_i(h_i))_{i=1}^\infty \in l_1$. Since $l_1 \in (OC)$, so $I_\varphi(y_n) = \|(\varphi_i(y_n(i)))_{i=1}^\infty\|_{l^1} \to 0$ as $n \to \infty$. This contradiction proves the claim. By Remark 1 we have $\frac{\varphi_i(u)}{u} \to \infty$ as $u \to \infty$ for any i. Consequently

$$1 = \|x_n\|_\varphi^A = \frac{1}{k_n}(1 + I_\varphi(k_n x_n)) \geq \frac{I_\varphi(k_n x_n)}{k_n} \geq \frac{\varphi_{i_0}(k_n x_n(i_0))}{k_n} \geq \frac{\varphi_{i_0}(k_n\delta)}{k_n\delta}\delta \to \infty.$$

This contradiction proves (3).

Take $x \in l_\varphi^A$ with $\|x\|_\varphi^A = 1$ and k such that $\|x\|_\varphi^A = \frac{1}{k}(1 + I_\varphi(kx))$. Then $I_\varphi(2x) \geq 1$. We consider two cases.

I. If $I_\varphi(2x\chi_{B_x}) \geq 1/2$, then $k \leq k_1(1/2)$, by (3).

II. Suppose that $I_\varphi(2x\chi_{\mathbb{N}\setminus B_x}) \geq 1/2$. Applying Lemma 2 it is easy to conclude that $\varphi_i(2u) \geq 2\xi\varphi_i(u)$ for every $u \geq h_i/2$, where $\xi = 1/(1-\gamma)$. Let $m \in \mathbb{N}$ be such that

$2^m \le k \le 2^{m+1}$. Consequently

$$1 = \frac{1}{k}(1 + I_\varphi(kx)) > \frac{1}{k} I_\varphi(kx\chi_{\mathbb{N}\setminus B_x}) \ge \frac{1}{2^{m+1}} I_\varphi(2^{m-1}2x\chi_{\mathbb{N}\setminus B_x}) \\ \ge \frac{1}{2^{m+1}}(2\xi)^{m-1} I_\varphi(2x\chi_{\mathbb{N}\setminus B_x}). \tag{4}$$

Thus $1 \ge \frac{\xi^{m-1}}{8}$, whence $k \le 2^{m+1} \le 2^{\log_\xi(8)+2}$. This proves $\overline{k}(0) < \infty$. Note that if $x \in l^A_\varphi$ and k is such that $\|x\|^A_\varphi = \frac{1}{k}(1 + I_\varphi(kx))$, then taking $y = \lambda x$ for $\lambda > 0$ we have $\|y\|^A_\varphi = \frac{1}{k_\lambda}(1 + I_\varphi(k_\lambda y))$, where $k_\lambda = k/\lambda$. Thus $\overline{k}(p) < \infty$ for each $p \in [0, 1)$. ■

REMARK 2. The assumption $\sup_i p_i(s_i)s_i < \infty$ in Lemma 4 cannot be dropped. Let

$$p_i(t) = \begin{cases} t & \text{for } 0 \le t < 1, \\ 1 & \text{for } 1 \le t < 2^i, \\ \frac{2^{i+1}+1}{2^{2i+1}} t & \text{for } t \ge 2^i. \end{cases}$$

Then $s_i = 2^i$ and $p_i(s_i) = 1 + \frac{1}{2^{i+1}}$, whence $\sup_i p_i(s_i)s_i = \infty$. Put $a_i = 1 + \frac{1}{2^{i+1}}$ and $x_i = \frac{1}{a_i} e_i$. Then for $k_i = 2^i + \frac{1}{2}$ we have $I_{\varphi^*}(p(k_i x_i)) = 1$ and consequently $\|x_i\|^O_\varphi = \frac{1}{k_i}\{1 + I_\varphi(k_i x_i)\} = 1$. Hence $\overline{k}(0) = \infty$. Note also that $\varphi, \varphi^* \in \delta_2$.

Theorem 3 in [3] states that l^O_φ is nearly uniformly convex iff $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$, where $\varphi = (\varphi_i)$ is a Musielak-Orlicz function with all φ_i being finitely valued N-functions, i.e. each function φ_i vanishes only at zero and satisfies two conditions: $\varphi_i(u)/u \to \infty$ as $u \to \infty$ and $\varphi_i(u)/u \to 0$ as $u \to 0$. The next theorem is an extension of this result. It also generalizes Theorem 2 from [5], which has been proved only for N-functions. Moreover, it is proved for essentially wider class of Musielak-Orlicz functions, since in our consideration functions φ_i not satisfying the conditions: $\varphi_i(u)/u \to 0$ as $u \to 0$, $\varphi_i(u)/u \to \infty$ as $u \to \infty$, $\varphi > 0$, are not excluded. As a consequence, in many parts of the proof new methods and techniques are developed.

THEOREM 2. *Suppose that* $\sup_i p_i(s_i)s_i < \infty$. *Then* $l^O_\varphi \in (\beta)$ *if and only if* l^O_φ *is reflexive, i.e.* $\varphi \in \delta_2$ *and* $\varphi^* \in \delta_2$.

Proof. Necessity. If $l^O_\varphi \in (\beta)$, then l^O_φ is reflexive and consequently $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$.

Sufficiency. Let $\varepsilon > 0$. Basing on Theorem 1 in [12], we conclude that property (β) can be equivalently considered on the unit sphere in place of the unit ball. Take $x, x_n \in S(l^o_\varphi), n = 1, 2, ...$ such that $\text{sep}\{x_n\}_{l^O_\varphi} \ge \varepsilon$. Let $\sigma = \sigma(\varepsilon/8)$ be from Corollary 1. Applying Lemma 2 take the sequence $(h_i)_{i=1}^\infty$ and the number $\gamma \in (0, 1)$ for $\eta = \overline{k}(1/4)/(1+\overline{k}(1/4))$, where $\overline{k}(1/4)$ is defined in (2). Next, to deduce inequalities (5) and (6) we apply the same methods as in the proof of Theorem 4 in [14]. Notice that $l^O_\varphi \in (OC)$, by $\varphi \in \delta_2$. Then there exists a set $A \subset \mathbb{N}$ with $\text{card}\, A < \infty$ such that

$$\|x\chi_{\mathbb{N}\setminus A}\|^O_\varphi < \min\{\gamma\sigma/4, 1/4\}. \tag{5}$$

Passing to a subsequence of (x_n) if necessary, we can find a sequence (A_n) of subsets of $\mathbb{N}$ such that $A_k \cap A_l = \emptyset$ for any $k \ne l$, $A_k \cap A = \emptyset$ for any k and $\|x_n\chi_{A_n}\|^O_\varphi \ge \varepsilon/8$ for each $n \in \mathbb{N}$. By Corollary 1 we get

$$I_\varphi(x_n\chi_{A_n}) \ge \sigma \tag{6}$$

for almost every $n \in \mathbb{N}$. Denote

$$A_n^1 = \{i \in A_n : |x_n(i)| \geq h_i\} \text{ and } A_n^2 = \{i \in A_n : |x_n(i)| < h_i\}.$$

We claim that $I_\varphi(x_{n_0}\chi_{A^1_{n_0}}) \geq \sigma/2$ for some $n_0 \in \mathbb{N}$. Suppose that

$$I_\varphi(x_n\chi_{A^1_n}) < \sigma/2 \text{ for every } n \in \mathbb{N}. \tag{7}$$

We have $I_\varphi(x_n\chi_{A^2_n}) \leq \sum_{i\in A^2_n} \varphi_i(h_i) \to 0$ as $n \to \infty$, because $\sum_{i=1}^\infty \varphi_i(h_i) < \infty$ and $A_k \cap A_l = \emptyset$ for any $k \neq l$. Then $I_\varphi(x_n\chi_{A^2_n}) < \sigma/2$ for sufficiently large n. Then, in view of (6) and (7), we get a contradiction, which proves the claim. Denote $x_0 = x\chi_A$. Let numbers k_0 and k_{n_0} be such that

$$\|x_0\|^O_\varphi = \frac{1}{k_0}\{1 + I_\varphi(k_0x_0)\} \quad \text{and} \quad \|x_{n_0}\|^O_\varphi = \frac{1}{k_{n_0}}\{1 + I_\varphi(k_{n_0}x_{n_0})\}.$$

Since $\|x_0\|^O_\varphi \geq 3/4$, by (5), so $k_{n_0}, k_0 \in (1, \overline{k}(1/4))$, in view of Lemma 4. We have

$$\frac{k_0}{k_0 + k_{n_0}} < \frac{\overline{k}(1/4)}{1 + \overline{k}(1/4)}.$$

It follows by Lemma 2 that

$$\frac{k_0 + k_{n_0}}{k_0k_{n_0}}\varphi_i\left(\frac{k_0k_{n_0}}{k_0 + k_{n_0}}|x_{n_0}(i)|\right) \leq (1 - \gamma)\left(\frac{\varphi_i(k_{n_0}|x_{n_0}(i)|)}{k_{n_0}}\right) \tag{8}$$

for every $i \in A^1_{n_0}$. Notice that the function $f(u) = \varphi(u)/u$ is nondecreasing. Hence, by the convexity of φ_i for every $i \in \mathbb{N}$ and inequality (8), we get

$$\begin{aligned}
\|x_0 + x_{n_0}\|^O_\varphi &\leq \frac{k_0 + k_{n_0}}{k_0k_{n_0}}\left\{1 + I_\varphi\left(\frac{k_0k_{n_0}}{k_0 + k_{n_0}}(x_0 + x_{n_0})\right)\right\} \\
&= \frac{k_0 + k_{n_0}}{k_0k_{n_0}}\left[1 + I_\varphi\left(\frac{k_0k_{n_0}}{k_0 + k_{n_0}}(x_0 + x_{n_0})\chi_{A^1_{n_0}}\right) + I_\varphi\left(\frac{k_0k_{n_0}}{k_0 + k_{n_0}}(x_0 + x_{n_0})\chi_{\mathbb{N}\setminus A^1_{n_0}}\right)\right] \\
&\leq \frac{k_0 + k_{n_0}}{k_0k_{n_0}} + \frac{I_\varphi(k_0x_0)}{k_0} + \frac{I_\varphi(k_{n_0}x_{n_0})}{k_{n_0}} - \gamma\frac{I_\varphi(k_{n_0}x_{n_0}\chi_{A^1_{n_0}})}{k_{n_0}} \\
&\leq \frac{1}{k_0}\{1 + I_\varphi(k_0x_0)\} + \frac{1}{k_{n_0}}\{1 + I_\varphi(k_{n_0}x_{n_0})\} - \gamma I_\varphi(x_{n_0}\chi_{A^1_{n_0}}) \leq 2 - \gamma\sigma/2.
\end{aligned}$$

Finally, by (5), $\|x+x_{n_0}\|^O_\varphi \leq 2-\gamma\sigma/2+\gamma\sigma/4 = 2-\gamma\sigma/4$. Hence $l^O_\varphi \in (\beta)$, by Theorem 1. ∎

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