ORLICZ CENTENARY VOLUME II
BANACH CENTER PUBLICATIONS, VOLUME 68
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

NON-ARCHIMEDEAN K-SPACES

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Abstract. We study Banach spaces over a non-spherically complete non-Archimedean valued field K. We prove that a non-Archimedean Banach space over K which contains a linearly homeomorphic copy of l^∞ (hence l^∞ itself) is not a K-space. We discuss the three-space problem for a few properties of non-Archimedean Banach spaces.

1. Introduction. Throughout this paper, K will denote a non-Archimedean complete valued field with a non-trivial valuation $|.|$. A valued field K is said to be *spherically complete* if every shrinking sequence of closed balls in K has a non-empty intersection. Clearly, the spherical completeness implies completeness; the converse is not true (see [13]).

Normed (quasinormed) spaces over K are defined in a natural way. We say that a norm (quasinorm) $\|.\|$ on a vector space E is *non-Archimedean* if it satisfies 'the strong triangle inequality', i.e. $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in E$. We say that a normed (quasinormed) space is *non-Archimedean* if its topology is defined by a non-Archimedean norm (quasinorm). Note that there exist Banach spaces over K for which the normed topology cannot be defined by a non-Archimedean norm (e.g. l^p for $p \geq 1$, cf. [7]). We call a subset X of a Banach space E over K a *base* if for every $a \in E$, there exists a unique map $f_a : X \to K$ such that $a = \sum_{x \in X} f_a(x)x$.

Real and complex K-spaces were introduced by N. J. Kalton and N. T. Peck in 1979 ([5], see also [6]). These objects were studied in connection with the lifting theorems. The question whether every locally convex F-space is a K-space has been solved negatively by Kalton ([4]), Ribe ([10]) and Roberts ([11]), who proved that the real Banach space l^1 is not a K-space.

J. Martinez-Maurica and C. Perez-Garcia developed K-spaces on the non-Archimedean ground. They proved (Theorem 5 of [8]) that all non-Archimedean Banach spaces over

2000 *Mathematics Subject Classification*: Primary 46S10.
Key words and phrases: non-Archimedean Banach spaces, K-spaces, three-space property.
The paper is in final form and no version of it will be published elsewhere.

a spherically complete K are K-spaces and that the local K-convexity is a three-space property in the category of complete locally bounded spaces over spherically complete valued fields (Theorem 6 of [8]). The same authors showed that every Banach space over K which has a base is a K-space (Theorem 3 of [7]). In particular, in contrast to the classical results, the sequence space l^1 over K is a K-space. However, there are known quotients of l^1 which are not K-spaces (see Proposition 7 of [7]).

In this paper we study non-Archimedean Banach spaces over a non-spherically complete K. Assuming that $card(K)$ is nonmeasurable, we prove that the sequence space l^∞ is not a K-space (Theorem 5), and that every Banach space over K which contains a linearly homeomorphic copy of l^∞ is not a K-space, either (Corollary 6). Additionally, we discuss the three-space problem for a few properties of non-Archimedean Banach spaces (Proposition 8). In particular, we give the answer to the question of van Rooij and Schikhof (Problem 8 of [14]) concerning the reflexivity of non-Archimedean Banach spaces.

2. Preliminaries. From now on, in this paper all vector spaces are over K. The notion of F-space has the usual meaning, i.e. an *F-space* is a complete metrizable topological vector space (Definition 3.46 of [9]). By E' we denote the topological dual of an F-space E. We say that an F-space E is a *K-space* if, whenever X is an F-space and L is a one-dimensional subspace of X such that $X/L \approx E$ (i.e. X/L and E are linearly homeomorphic), then L is complemented in X (Chapter 5 of [6]). Following [7] and [8], we recall that if X is a locally bounded space and Y is an F-space such that $Y/L \approx X$, then Y must be locally bounded and quasinormable (local boundeness implies quasinormability as in the usual case). Hence, an equivalent condition for a Banach space to be a K-space is the following: a Banach space E is a *K-space* if whenever X is a quasi-Banach space and L is a subspace of X with dimension one, such that $X/L \approx E$, then L is complemented in X (see Definition 1 of [7]).

A topological vector space (E, τ) is called *locally K-convex* (Definition 3.10 of [9]) if τ has a basis of neighborhoods of the origin, consisting of K-convex sets; a subset $U \subset E$ is said to be *K-convex* (Definition 3.8 of [9]) if $\lambda x + \mu y + \nu z \in U$, whenever $x, y, z \in E$, $\lambda, \mu, \nu \in K$ and $|\lambda| \leq 1, |\mu| \leq 1, |\nu| \leq 1, \lambda+\mu+\nu = 1$. Recall that every non-Archimedean Banach space is locally K-convex. The sequence space l^1 is an example of a Banach space which is not locally K-convex (cf. [7], [8]).

We say that a subspace D of an F-space E has the *weak extension property (WEP)* in E if for every continuous linear functional $f : D \to K$ there is a continuous linear extension $g : E \to K$ of f. If K is spherically complete, then every linear subspace D of a locally K-convex space E has the WEP (see Theorem 4.12 of [9]). For the case when K is not spherically complete, there are many examples of closed subspaces of Banach spaces without the WEP (e.g. c_0 does not have the WEP in l^∞, see Theorem 4.15 of [13]). Observe that if every closed linear subspace of E has the WEP, then every closed linear subspace of E is weakly closed and E has a separating dual (cf. [1]). A closed linear subspace $D \subset E$ is weakly closed if and only if E/D is dual-separating (Theorem 3.2 of [1]).

The following theorems, proved by Kalton and Peck for the real and complex case, work in our context:

THEOREM 1 (Theorem 5.2 of [5]). *If E is an F-space and D is a closed subspace of E such that E/D is a K-space, then D has the WEP in E.*

THEOREM 2 (Theorem 5.3 of [5]). *Let E be an F-space. If E is a K-space and $D \subset E$ is a closed subspace with the WEP, then E/D is a K-space.*

Let I be a set. Denote by $c_0(I)$ the linear space of bounded maps $x : I \to K$ such that for every $\varepsilon > 0$ there exist only finitely many elements i of I for which $|x(i)| \geq \varepsilon$. $c_0(I)$ is a Banach space under the sup-norm $\|.\|_\infty$, defined by $\|x\|_\infty := \sup\{|x(i)| : i \in I\}$, $x \in c_0(I)$. For every set I the space $c_0(I)$ has a base (cf. [13]). If K is non-spherically complete and $card(I)$ is nonmeasurable, then $c_0(I)$ is reflexive (Theorem 4.22 of [13]).

Let E be a non-Archimedean Banach space. Following van Rooij ([12]), we denote by $m(E)$ the smallest one among the cardinalities of those subsets X of E for which $\overline{[X]} = E$. By Theorem 3.4 of [12], for every set I with $card(I) \geq m(E)$ there exists a quotient map $c_0(I) \to E$.

REMARK 3. Note that the result obtained by J. Martinez-Maurica and C. Perez-Garcia (Theorem 5, [8]), which shows that every non-Archimedean Banach space over a spherically complete K is a K-space, follows directly from Theorem 3 of [8], Theorem 2, Theorem 3.4 of [12] and Ingleton's theorem (Theorem 4.12 of [9]):

Let E be a non-Archimedean Banach space over a spherically complete K. By Theorem 3.4 of [12], there exists a set I and a quotient map $\pi : c_0(I) \to E$. Since $c_0(I)$ has a base, it follows from Theorem 3 of [8] that $c_0(I)$ is a K-space. Next, by Theorem 4.12 of [9], we get that $\ker \pi$, a closed linear subspace of $c_0(I)$, has the WEP in $c_0(I)$. Thus, from Theorem 2 we conclude that E is a K-space.

For other notations and conventions that we will use in the sequel we refer the reader to [6], [9], [13].

3. Results. We start with the following Proposition, which shows some properties of K-spaces:

PROPOSITION 4. *Let E be a non-Archimedean Banach space. If K is non-spherically complete, we assume that $card(m(E))$ is nonmeasurable. If E is a K-space, then*

1. *E has a separating dual;*
2. *every closed linear subspace of E with the WEP in E is weakly closed in E;*
3. *every closed linear subspace of E is a K-space.*

Proof. If K is spherically complete, then there is nothing to prove, as all non-Archimedean Banach spaces possess properties 1)-3). Hence, we assume that K is non-spherically complete. By Theorem 3.4 of [12], there exists a set I, where $card(I)$ is nonmeasurable, such that $E \approx c_0(I)/D$ for some closed linear subspace D of $c_0(I)$. It follows from Theorem 1 that D has the WEP in $c_0(I)$. By Theorems 4.22, 5.9 of [13] and by Lemma 2.2 of [15], D is weakly closed in $c_0(I)$. We get that $c_0(I)/D$, thus E, has a separating dual.

Let G be a closed linear subspace of E which has the WEP in E. From Theorem 2 we get that E/G is a K-space, and by 1) E/G has a separating dual. Thus, we conclude that G is weakly closed in E.

Let H be a closed linear subspace of E. Then there exists a closed linear subspace $H_0 \subset c_0(I)$ such that $D \subset H_0$ and $H \approx H_0/D$. Since D, by Theorem 1, has the WEP in $c_0(I)$, D has the WEP in H_0. By Theorem 5.9 of [13], $H_0 \approx c_0(J)$ for some set J with $card(J) \leq card(I)$ and by Theorem 3 of [7], H_0 is a K-space. Finally, it follows from Theorem 2 that H_0/D, thus H, is a K-space. ∎

THEOREM 5. *Let K be non-spherically complete and let $card(K)$ be nonmeasurable. Then l^∞ is not a K-space.*

Proof. Note that $card(l^\infty) = card(K)$ (see [3], proof of Lemma 2.1). Hence $m(l^\infty)$ is nonmeasurable. To prove that l^∞ is not a K-space, it is enough to find a proper closed and weakly dense linear subspace of l^∞ with the WEP in l^∞. Assume that $H \subset l^\infty$ is such a subspace and l^∞ is a K-space. Then, by Theorem 2, the quotient l^∞/H is a K-space and by Proposition 4 it has a separating dual. But $(l^\infty/H)' = \{0\}$, since H is weakly dense in l^∞, a contradiction.

We apply the subspace of l^∞ constructed by van Rooij, presented in Example 4.J of [13] but in a different context (see also Lemma 1.4.21 of [2]). Let $\mathcal{F}$ be the collection of all sets $M \subset N$ for which

$$\lim_{n\to\infty} \frac{card(M \cap [1,n])}{n} = 0.$$

Let D be the set of all $x \in l^\infty$ ($x = (x_n)_{n=1}^\infty$) such that for every $\varepsilon > 0$ we have $\{n : |x_n| \geq \varepsilon\} \in \mathcal{F}$. It is easy to see that D is a closed linear subspace of l^∞. Observe that if $x \in c_0$ (where $x = (x_n)_{n=1}^\infty$), then for every $\varepsilon > 0$ the set $\{n : |x_n| \geq \varepsilon\}$ is finite, thus it belongs to $\mathcal{F}$. Hence $c_0 \subset D$. Taking $f \in (l^\infty)'$ with $f(D) = 0$, we get $f(c_0) = 0$. By Theorem 4.15 of [13], c_0 is weakly dense in l^∞, thus $f = 0$ on l^∞. Hence, we conclude that D is weakly dense in l^∞.

We show that D has the WEP in l^∞. Obviously $D \neq l^\infty$. First, we prove that c_0 is $\sigma(D, D')$-dense in D. Take $f \in D'$ with $f(c_0) = 0$. Let $x \in D$ ($x = (x_n)_{n=1}^\infty$). Take $\varepsilon > 0$. We prove that $|f(x)| \leq \varepsilon \cdot \|f\|$. Observe that the set $S := \{n : |x_n| \geq \varepsilon\}$ belongs to $\mathcal{F}$. Denote by $D_0 := \{y \in l^\infty : y_n = 0 \text{ whenever } n \notin S\}$. Then D_0 is a subspace of D. Let $x_S = (x_S^n)_{n=1}^\infty$ be an element of l^∞, defined by

$$x_S^n := \begin{cases} 0 & \text{if } n \notin S, \\ x_n & \text{if } n \in S. \end{cases}$$

Then $x_S \in D_0$ and $dist(x, D_0) \leq \|x - x_S\| \leq \varepsilon$. Hence, we see that there exists $x_0 \in l^\infty$ such that $x = x_S + x_0$ and $\|x_0\| \leq \varepsilon$. Thus, if we prove that $f = 0$ on D_0, then using

$$\begin{aligned} |f(x)| &= |f(x_S + x_0)| \leq \max\{|f(x_S)|, |f(x_0)|\} \\ &\leq \max\{|f(x_S)|, \|f\| \cdot \|x_0\|\} \leq \varepsilon \cdot \|f\|, \end{aligned}$$

we obtain that $f = 0$ on D. If S is finite then, since $f(c_0) = 0$, we get that $f(x_S) = 0$. Assume that S is infinite, say $S = \{s_1, s_2, ...\}$ where $s_1 < s_2 < ....$ The map ϕ given by the following formula

$$\phi(u)_k := \begin{cases} 0 & \text{if } k \in N \setminus S, \\ u_j & \text{if } j \in N,\ k = s_j, \end{cases}$$

is an isometrical isomorphism $l^\infty \to D_0$. Thus $D_0/(c_0 \cap D_0) \cong l^\infty/c_0$ and, since $(l^\infty/c_0)' = \{0\}$ by Theorem 4.15 of [13], $(D_0/(c_0 \cap D_0))' = \{0\}$. Hence, $f = 0$ on D_0, and finally $f = 0$ on D.

Now we prove that there exists an isometrical isomorphism $(l^\infty)' \to D'$ (note that $c_0 \cong (l^\infty)'$, i.e. c_0 and $(l^\infty)'$ are isometrically isomorphic, by Exercise 3.Q of [13]). Let $y \in c_0$ ($y = (y_n)_{n=1}^\infty$). We define the map $y \mapsto f_y$ by $f_y(x) = \sum_{n=1}^\infty x_n y_n$, where $x = (x_n)_{n=1}^\infty \in D$. It is easy to see that this map is a linear isometry. We prove surjectivity. Let $f \in D'$. We are going to find $y \in c_0$ for which $f = f_y$. Since $c_0 \subset D$, for $x \in c_0$ ($x = (x_n)_{n=1}^\infty$) we have $f(x) = \sum_{n=1}^\infty x_n y_n$ for some $y = (y_n)_{n=1}^\infty \in l^\infty$. We prove that $y \in c_0$. If not, there exists $\varepsilon > 0$ such that the set $M := \{m : |y_m| \geq \varepsilon\}$ is infinite. Therefore, M contains an infinite subset $S := \{s_1, s_2, ...\} \in \mathcal{F}$, where $s_1 < s_2 < ....$ As we showed above, the map $\phi : l^\infty \to D$ is a linear isometry. But $f \circ \phi \in (l^\infty)'$ and therefore, since c_0 is reflexive by Corollary 4.18 of [13], $f \circ \phi$ has the form $u \mapsto \sum_{m=1}^\infty u_m a_m$, where $a = (a_m)_{m=1}^\infty \in c_0$, $u \in l^\infty$. By choosing for u successively the unit vectors, we obtain $|a_j| = |y_{s_j}| \geq \varepsilon$ for each j, a contradiction. Thus, $y = (y_n)_{n=1}^\infty \in c_0$ and the linear map

$$f_y : x \mapsto \sum_{n=1}^\infty x_n y_n, \ x \in D$$

is an element of D', coinciding with f on c_0. Since c_0 is $\sigma(D, D')$-dense in D, we obtain that $f_y = f$. Thus, it follows that D has the WEP in l^∞. ∎

COROLLARY 6. *Every non-Archimedean Banach space over non-spherically complete K with nonmeasurable card(K), which contains a linearly homeomorphic copy of l^∞ is not a K-space.*

Proof. This follows directly from Proposition 4 and Theorem 5. ∎

Let (P) be some property of an F-space. We say that the property (P) is a *three-space property* if, whenever E is an F-space with a closed linear subspace D such that both D and E/D have property (P), then E has property (P). The results obtained by Kalton ([4]), Ribe ([10]) and Roberts ([11]) show that the local convexity is not a three-space property in the category of real or complex F-spaces. Surprisingly, however, J. Martinez-Maurica and C. Perez-Garcia proved that the local K-convexity is a three-space property in the category of locally bounded spaces over spherically complete K (Theorem 6 of [8]).

Let K be non-spherically complete. Although there exist non-Archimedean Banach spaces which are not K-spaces (even dual-separating, e.g. l^∞), we do not know any example of a locally bounded (or Banach) space E which is not locally K-convex, but which possesses a locally K-convex closed linear subspace D such that E/D is a non-Archimedean Banach space. We leave as an open problem the following question:

PROBLEM 7. *Is the local K-convexity a three-space property in the category of locally bounded spaces (or Banach spaces) over a non-spherically complete K?*

Observe that if K is not spherically complete, then the reflexivity is not a three-space property:

PROPOSITION 8. *In the category of non-Archimedean Banach spaces over a non-spherically complete K with card(K) nonmeasurable, the properties for a Banach space E:*

- *E' separates the points of E,*
- *E is reflexive,*

are not three-space properties.

Proof. Take $E := l^\infty$. There exists a set I, where $card(I)$ is nonmeasurable, and a quotient map $\pi : c_0(I) \to E$. Since E has a separating dual, $D := \ker \pi$ is a weakly closed linear subspace of $c_0(I)$. By Theorem 5, E is not a K-space and it follows from Theorem 2 that D does not have the WEP in $c_0(I)$. Thus, there exists $f \in D'$ without any continuous linear extension to the whole of $c_0(I)$. Let $D_0 := \ker f$. We can easily show that D_0 is not weakly closed in $c_0(I)$; in fact, if $g \in (c_0(I))'$ is a linear functional with $g(D_0) = 0$ and $g(x_0) \neq 0$ for $x_0 \in D \backslash D_0$, we can construct a continuous linear extension of f on $c_0(I)$, a contradiction. Take $Z := c_0(I)/D_0$, $L := D/D_0$. Then Z' does not separate the points of Z, although Z/L ($E \cong Z/L$) and L both have the separating duals. Obviously, Z is not reflexive, however Z/L and L are reflexive (Theorem 4.21 of [13]). ∎

REMARK 9. Clearly, Proposition 8 is not true if K is spherically complete. In this case every non-Archimedean Banach space is dual-separating, it follows from Theorem 4.12 of [9], and every infinite-dimensional non-Archimedean Banach space is non-reflexive, see Theorem 4.16 of [13].

Note that the proof of Proposition 8 gives the negative answer to the question formulated by A. van Rooij and W. Schikhof (Problem 8 of [14]):

PROBLEM 10. *Let E be a non-Archimedean Banach space and $a \in E$, $a \neq 0$. If $E/[a]$ is reflexive, must E itself be reflexive?*

In general, there are not known necessary and sufficient conditions for a non-Archimedean Banach space to be a K-space. All examples of non-Archimedean Banach spaces over a non-spherically complete K which are K-spaces have a base. This fact leads us to the following question:

PROBLEM 11. *Let K be non-spherically complete and let E be a non-Archimedean Banach space which is a K-space. Must E have a base?*

It is easy to observe that this question is equivalent to the following one:

PROBLEM 12. *Let K be non-spherically complete and let I be a set. Let D be a closed linear subspace of $c_0(I)$. Does D have the WEP in $c_0(I)$ if and only if D is complemented in $c_0(I)$?*

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