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ORDER-BOUNDED OPERATORS FROM VECTOR-VALUED FUNCTION SPACES TO BANACH SPACES

MARIAN NOWAK

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Szafrana 4A, 65-516 Zielona Góra, Poland
E-mail: M.Nowak@wmie.uz.zgora.pl

Abstract. Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ). For a real Banach space $(X, \|\cdot\|_X)$ let $E(X)$ be a subspace of the space $L^0(X)$ of μ-equivalence classes of strongly Σ-measurable functions $f\colon \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. Let $E(X)^\sim$ stand for the order dual of $E(X)$. For $u \in E^+$ let D_u $(= \{f \in E(X)\colon \|f(\cdot)\|_X \le u\})$ stand for the order interval in $E(X)$. For a real Banach space $(Y, \|\cdot\|_Y)$ a linear operator $T\colon E(X) \to Y$ is said to be order-bounded whenever for each $u \in E^+$ the set $T(D_u)$ is norm-bounded in Y. In this paper we examine order-bounded operators $T\colon E(X) \to Y$. We show that T is order-bounded iff T is $(\tau(E(X), E(X)^\sim), \|\cdot\|_Y)$-continuous. We obtain that every weak Dunford-Pettis operator $T\colon E(X) \to Y$ is order-bounded. In particular, we obtain that if a Banach space Y has the Dunford-Pettis property, then T is order-bounded iff it is a weak Dunford-Pettis operator.

1. Introduction and preliminaries. P. G. Dodds [D] introduced and examined order-bounded operators from a vector lattice E to a Banach space Y. Recall that a linear operator $T\colon E \to Y$ is called order-bounded if the set $T([-u, u])$ is norm-bounded in Y for every $0 \le u \in E$. M. Duboux [Du] extended some of Dodd's results to the setting of Y being a locally convex space. Next, Z. Ercan [E] obtained some properties of order-bounded operators from a vector lattice E to a topological vector space Y.

In this paper we consider order-bounded operators from a vector-valued function space $E(X)$ to a Banach space Y.

For the terminology concerning vector lattices and function spaces we refer to [AB_1], [AB_3] and [KA]. Given a topological vector space (L, τ) by $(L, \tau)^*$ we will denote its

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topological dual. We denote by $\sigma(L,K)$ and $\tau(L,K)$ the weak topology and the Mackey topology for a dual system (L,K) resp. By $\mathbb{N}$ and $\mathbb{R}$ we will denote the sets of all natural and real numbers.

Throughout the paper we assume that (Ω,Σ,μ) is a complete σ-finite measure space and L^0 denotes the corresponding space of μ-equivalence classes of all Σ-measurable real valued functions. Let χ_A stand for the characteristic function of a set A. Let E be an ideal in L^0 with supp $E=\Omega$ and let $E^\sim$ stand for the order dual of E.

Let $(X,\|\cdot\|_X)$ be a real Banach space and let S_X stand for the unit sphere of X. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f\colon\Omega\to X$. For $f\in L^0(X)$ let us set $\widetilde{f}(\omega):=\|f(\omega)\|_X$ for $\omega\in\Omega$. Let

$$E(X)=\{f\in L^0(X)\colon \widetilde{f}\in E\}.$$

A subset H of $E(X)$ is said to be *solid* whenever $\widetilde{f_1}\le\widetilde{f_2}$ and $f_1\in E(X)$, $f_2\in H$ imply $f_1\in H$. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets. A linear topology τ on $E(X)$ that is at the same time locally solid and locally convex will be called a *locally convex-solid* topology on $E(X)$ (see [N_1]).

Recall that the algebraic tensor product $E\otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u\otimes x$, $(u\otimes x)(\omega)=u(\omega)x$, where $u\in E$, $x\in X$.

For each $u\in E^+$ the set $D_u=\{f\in E(X)\ \widetilde{f}\le u\}$ will be called an *order interval* in $E(X)$ (see [BL]).

Following [D] we are now ready to define some class of linear operators.

Definition. Let E be an ideal of L^0 and $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ real Banach spaces. A linear operator $T\colon E(X)\to Y$ is said to be *order-bounded* whenever for each $u\in E^+$ the set $T(D_u)$ is norm-bounded in Y.

Now we recall terminology concerning the duality theory of the function spaces $E(X)$ as set out in [B], [BL], [N_1], [N_2].

For a linear functional F on $E(X)$ let us put

$$|F|(f)=\sup\{|F(h)|\colon h\in E(X),\widetilde{h}\le\widetilde{f}\}\quad\text{for } f\in E(X).$$

The set

$$E(X)^\sim=\{F\in E(X)^\#\colon |F|(f)<\infty \text{ for all } f\in E(X)\}$$

will be called the *order dual* of $E(X)$ (here $E(X)^\#$ denotes the algebraic dual of $E(X)$).

For $F_1,F_2\in E(X)^\sim$ we will write $|F_1|\le|F_2|$ whenever $|F_1|(f)\le|F_2|(f)$ for all $f\in E(X)$. A subset A of $E(X)^\sim$ is said to be *solid* whenever $|F_1|\le|F_2|$ with $F_1\in E(X)^\sim$ and $F_2\in A$ imply $F_1\in A$.

Now we consider absolute weak topologies on $E(X)$ and $E(X)^\sim$. For each $F\in E(X)^\sim$ let

$$\rho_F(f)=|F|(f)\quad\text{for } f\in E(X).$$

We define the *absolute weak topology* $\sigma|(E(X),E(X)^\sim)$ on $E(X)$ as the locally convex topology generated by the family $\{\rho_F\colon F\in E(X)^\sim\}$ of seminorms. Then $|\sigma|(E(X),E(X)^\sim)$ is the topology of uniform convergence on sets of the form $\{G\in E(X)^\sim\colon |G|\le|F|\}$ for every $F\in E(X)^\sim$ (see [N_1, Section 4]).

For each $f \in E(X)$ let

$$\rho_f(F) = |F|(f) \quad \text{for } F \in E(X)^\sim.$$

We define the *absolute weak topology* $|\sigma|(E(X)^\sim, E(X))$ on $E(X)^\sim$ as the locally convex topology generated by the family $\{\rho_f\colon f \in E(X)\}$ of seminorms. Clearly $|\sigma|(E(X)^\sim, E(X))$ is the topology of uniform convergence on the family of all order intervals D_u, where $u \in E^+$.

In particular, if $(E, \|\cdot\|_E)$ is a Banach function space then the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is a Banach space and it is usually called the *Köthe-Bochner space*. It is well known that $(E(X), \|\cdot\|_{E(X)})^* = E(X)^\sim$ (see [BL, §3, Lemma 12]).

2. Characterization of order-bounded operators. It is known that on every vector lattice E one can define the so-called *order-bounded* topology τ_0 as the finest locally convex topology on E for which every order interval is a bounded set (see [Na]). A local base at zero for τ_0 is given by the class of all absolutely convex subsets of E that absorb all order bounded sets in E. It is known that τ_0 coincides with the Mackey topology $\tau(E, E^\sim)$ (see [F, 811(c)]).

Now let $\mathcal{B}_0$ be the family of all absolutely convex subsets of $E(X)$ that absorb every order interval in $E(X)$. Then $\mathcal{B}_0$ is a local base at zero for a locally convex topology τ_0 on $E(X)$, which will be called an *order-bounded topology* on $E(X)$.

We are ready to characterize order-bounded operators $T\colon E(X) \to Y$ in terms of the $(\tau_0, \|\cdot\|_Y)$-continuity of T.

THEOREM 2.1. *For a linear operator $T\colon E(X) \to Y$ the following statements are equivalent:*

(i) *T is order-bounded.*
(ii) *T is $(\tau_0, \|\cdot\|_Y)$-continuous.*

Proof. (i)⇒(ii). Assume that T is order bounded and let $r > 0$ be given. We shall show that there is $V \in \mathcal{B}_0$ such that $T(V) \subset B_Y(r)$ $(= \{y \in Y\colon \|y\|_Y \le r\})$. In fact, let $V = T^{-1}(B_Y(r)) = \{f \in E(X)\colon \|T(f)\|_Y \le r\}$. Since $T(V) = T(T^{-1}(B_Y(r))) \subset B_Y(r)$, it is enough to show that V absorbs every order interval in $E(X)$. Indeed, for given $u \in E^+$ there is $r_u > 0$ such that $T(D_u) \subset B_Y(r_u)$. Taking $\lambda_u = \frac{r}{r_u}$ for $f \in D_u$ we get $\|T(\lambda_u f)\|_Y \le r$, so $\lambda_n f \in V$. This means that $\lambda_u D_u \subset V$.

(ii)⇒(i). Assume that T is $(\tau_0, \|\cdot\|_Y)$-continuous. Then there is $V \in \mathcal{B}_0$ such that $T(V) \subset B_Y(1)$. Moreover, given $u \in E^+$ there is $\lambda_u > 0$ such that $\lambda_u D_u \subset V$. Hence $T(\lambda_u D_u) \subset B_Y(1)$, so $T(D_u) \subset B_Y(1/\lambda_u)$, as desired. ■

As an application of Theorem 2.1 we obtain:

COROLLARY 2.2. *The order-bounded topology τ_0 on $E(X)$ coincides with the Mackey topology $\tau(E(X), E(X)^\sim)$, i.e., $\tau_0 = \tau(E(X), E(X)^\sim)$.*

Proof. In view of Theorem 2.1, $(E(X), \tau_0)^* = E(X)^\sim$, so by the Mackey-Arens theorem $\tau_0 \subset \tau(E(X), E(X)^\sim)$.

To show that $\tau(E(X), E(X)^\sim) \subset \tau_0$, let $W \in \mathcal{B}_{\tau(E(X),E(X)^\sim)}$ (= the local base at zero for $\tau(E(X), E(X)^\sim)$). It is enough to show that W absorbs every order interval in $E(X)$.

Since the Mackey topology $\tau(E(X),E(X)^\sim)$ is locally solid (see [N_2, Theorem 3.7]), $W = {}^0A$ $(=\{f \in E(X)\colon |F(f)| \le 1$ for all $F \in A\})$, where A is an absolutely convex, solid and $\sigma(E(X)^\sim, E(X))$-compact subset of $E(X)^\sim$. But in view of [N_2, Theorem 3.5] A is $|\sigma|(E(X)^\sim, E(X))$-bounded, so for each $f \in E(X)$ we have:

$$(*)\qquad \sup\{|F|(f)\colon F \in A\} < \infty.$$

Observe that

$$^0A = \{f \in E(X)\colon |F|(f) \le 1 \text{ for all } F \in A\}.$$

In fact, let $f \in {}^0A$, i.e., $|F(f)| \le 1$ for all $F \in A$. It is enough to show that $|F|(f) \le 1$ for all $F \in A$. Given $F_0 \in A$, for each $G \in E(X)^\sim$ with $|G| \le |F_0|$ we have that $G \in A$ because A is a solid subset of $E(X)^\sim$. Hence $|G(f)| \le 1$. But by [N_1, Lemma 2.1] $F_0|(f) = \sup\{|G(f)|\colon G \in E(X)^\sim, |G| \le |F_0|\}$, so we obtain that $|F_0|(f) \le 1$, as desired.

Now, we are ready to show that W absorbs every order interval in $E(X)$. In fact, let $u \in E^+$ and $x_0 \in S_X$. Hence in view of $(*)$ $\sup\{|F|(u \otimes x_0)\colon F \in A\} = \lambda_u < \infty$. Then for $f \in D_u$ and all $F \in A$ we have:

$$|F|\left(\frac{1}{\lambda_u} f\right) = \frac{1}{\lambda_u}|F|(f) \le \frac{1}{\lambda_u}|F|(u \otimes x_0) \le 1,$$

so $\frac{1}{\lambda_u} f \in W$, i.e., $D_u \subset \lambda_u W$. ■

We say that a sequence (f_n) in $E(X)$ is *uniformly convergent* to $f \in E(X)$ (in symbols, $f_n \to f(ru)$), if there exist $r \in E^+$ and a sequence (ε_n) of positive numbers with $\varepsilon_n \downarrow 0$ such that $\|f_n(\omega) - f(\omega)\|_X \le \varepsilon_n r(\omega)$ μ-a.e. on Ω.

Making use of Theorem 2.1 and Corollary 2.2 we get:

THEOREM 2.3. *For a linear operator* $T\colon E(X) \to Y$ *the following statements are equivalent:*

(i) T *is order bounded.*
(ii) T *is* $(\tau(E(X),E(X)^\sim), \|\cdot\|_Y)$*-continuous.*
(iii) T *is* $(\sigma(E(X),E(X)^\sim), \sigma(Y,Y^*))$*-continuous.*
(iv) *If* $\widetilde{f_n} \le u$ *for some* $u \in E^+$ *and all* $n \in \mathbb{N}$ *and* $f_n \to 0$ *for* $|\sigma|(E(X),E(X)^\sim)$*, then* $\sup_n \|T(f_n)\|_Y < \infty$.
(v) *If* $f_n \to 0(ru)$*, then* $\sup_n \|T(f_n)\|_Y < \infty$.

In particular, for a Banach function space $(E, \|\cdot\|_E)$ *the statements* (i)–(v) *are equivalent to the following:*

(vi) T *is* $(\|\cdot\|_{E(X)}, \|\cdot\|_Y)$*-continuous.*

Proof. (i)⇔(ii). It follows from Theorem 2.1 and Corollary 2.2.

(ii)⇔(iii). See [W, Corollary 11.1.3, Corollary 11.2.6].

(i)⇒(iv). It is obvious.

(iv)⇒(i). Assume that (iv) holds and (i) fails. Hence there is $u_0 \in E^+$ such that $\sup\{\|T(f)\|_Y\colon f \in E(X), \widetilde{f} \le u_0\} = \infty$. It follows that one can find a sequence (f_n) in $E(X)$ such that $f_n \le u_0$ for all $n \in \mathbb{N}$ and $\|T(f_n)\|_Y \ge n^2$ for all $n \in \mathbb{N}$. Hence putting $h_n = \frac{1}{n} f_n$ for $n \in \mathbb{N}$, we have $\widetilde{h_n} \le \frac{1}{n} u_0 \le u_0$ and $\|T(h_n)\|_Y \ge n$ for $n \in \mathbb{N}$. We shall show that $h_n \to 0$ for $|\sigma|(E(X), E(X)^\sim)$. Indeed, let $F \in E(X)^\sim$, and $x_0 \in S_X$. Then

$$\rho_F(h_n) = |F|(h_n) \le |F|\left(\frac{1}{n}(u_0 \otimes x_0)\right) = \frac{1}{n}|F|(u_0 \otimes x_0) \to 0.$$

Hence $h_n \to 0$ for $\sigma|(E(X), E(X)^\sim)$, so by (iv) $\sup_n \|T(h_n)\|_Y < \infty$. This contradiction establishes that (iv)$\Rightarrow$(i).

(i)$\Rightarrow$(v). Assume that (i) holds and (v) fails. Then there exists a sequence (f_n) in $E(X)$ such that $f_n \to 0$ (ru) and $\sup_n \|T(f_n)\|_Y = \infty$. This means that there exist $r \in E^+$ and a sequence (ε_n) with $\varepsilon_n \downarrow 0$ such that $\widetilde{f_n} \le \varepsilon_n r \le \varepsilon_n r$ for all $n \in \mathbb{N}$. Hence by (i) $\sup_n \|T(f_n)\|_Y < \infty$, and we get a contradiction.

(v)$\Rightarrow$(i). Assume that (i) fails. Hence there exists $u_0 \in E^+$ such that

$$\sup\{\|T(f)\|_Y\colon f \in E(X), \widetilde{f} \le u_0\} = \infty.$$

So, there exists a sequence (f_n) in $E(X)$ such that $\widetilde{f_n} \le u_0$ and $\|T(f_n)\|_Y \ge n^2$ for all $n \in \mathbb{N}$. Denoting $h_n = \frac{1}{n} f_n$ for $n \in \mathbb{N}$ we get $\widetilde{h_n} \le \frac{1}{n} u_0$ for $n \in \mathbb{N}$, i.e., $h_n \to 0$ (ru) and $\|T(h_n)\|_Y \ge n$ for all $n \in \mathbb{N}$. It follows that (v) does not hold.

Now assume that $(E, \|\cdot\|_E)$ is a Banach function space. Then $(E(X), \|\cdot\|_{E(X)})^* = E(X)^\sim$ and the Mackey topology $\tau(E(X), E(X)^\sim)$ coincides with the $\|\cdot\|_{E(X)}$-norm topology. Hence (ii)$\Leftrightarrow$(vi). ∎

Recall that a Banach space Y is said to have the *Dunford-Pettis property* if for sequences (y_n) in Y and (y_n^*) in Y^*, $y_n^*(y_n) \to 0$ whenever $y_n \xrightarrow{(w)} 0$ in Y and $y_n^* \xrightarrow{(w)} 0$ in Y^* (see [AB$_1$, Section 19]).

Following ([AB$_2$], [AB$_3$, Section 19]) we say that a linear operator $T : E(X) \to Y$ is *a weak Dunford-Pettis operator* if $f_n \to 0$ in $E(X)$ for $\sigma(E(X), E(X)^\sim)$ and $y_n^* \xrightarrow{(w)} 0$ in Y^* imply $y_n^*(T(f_n)) \to 0$.

As an application of Theorem 2.3 we get (see [E, Theorem 1.4]):

THEOREM 2.4.

(i) *Every weak Dunford-Pettis operator $T\colon E(X) \to Y$ is order-bounded.*
(ii) *If Y has the Dunford-Pettis property, then every order-bounded operator $T\colon E(X) \to Y$ is a weak Dunford-Pettis operator.*

Proof. (i) Assume that $T\colon E(X) \to Y$ is a weak Dunford-Pettis operator and it is not order-bounded. Hence by Theorem 2.3 there exists a sequence (f_n) in $E(X)$ such that $f_n \to 0$ (ru) and $\sup_n \|T(f_n)\|_Y = \infty$. This means that there exist $u_0 \in E^+$ and a sequence (ε_n) with $\varepsilon_n \downarrow 0$ such that $\widetilde{f_n} \le \varepsilon_n u_0$ for all $n \in \mathbb{N}$ and the set $\{T(f_n)\colon n \in \mathbb{N}\}$ in Y is not weakly bounded. Hence there exist $y_0^* \in Y^*$ and a subsequence (f_{k_n}) of (f_n) such that

$$(*) \qquad |y_0^*(T(f_{k_n}))| > n \qquad \text{for all } n \in \mathbb{N}.$$

One can observe that $f_n \to 0$ for $\sigma(E(X), E(X)^\sim)$. Indeed, let $F \in E(X)^\sim$ and $x_0 \in S_X$. Then

$$|F(f_n)| \le |F|(f_n) \le |F|(\varepsilon_n(u_0 \otimes x_0)) = \varepsilon_n |F|(u_0 \otimes x_n),$$

so $F(f_n) \to 0$, as desired. On the other hand, since $\frac{1}{n}y_0^* \xrightarrow{(w)} 0$ in Y^* and T is supposed to be a weak Dunford-Pettis operator, we get $\frac{1}{n}y_0^*(T(f_{k_n})) \to 0$, which contradicts $(*)$.

(ii) Assume that $T\colon E(X) \to Y$ is order-bounded. Then by Theorem 2.3 T is $(\sigma(E(X), E(X)^\sim), \sigma(Y, Y^*))$-continuous. Let $f_n \to 0$ for $\sigma(E(X), E(X)^\sim)$ and $y_n^* \xrightarrow{(w)} 0$ in Y^*. Then $T(f_n) \xrightarrow{(w)} 0$ in Y, and since Y is supposed to have the Dunford-Pettis property, we conclude that $y_n^*(T(f_n)) \to 0$. This means that T is a weak Dunford-Pettis operator. ∎

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