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GENERALIZED M-NORMS ON ORDERED NORMED SPACES

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Abstract. L-norms and M-norms on Banach lattices, unit-norms and base norms on ordered vector spaces are well known. In this paper m- and $m_{\le}$-norms are introduced on ordered normed spaces. They generalize the notions of the M-norm and the order-unit norm, possess also some interesting properties and can be characterized by means of the constants of reproducibility of cones. In particular, the dual norm of an ordered Banach space with a closed cone turns out to be additive on the dual cone if and only if the norm of the Banach space is an $m_{\le}$-norm and, on the other hand, the norm of an ordered normed space with a reproducing cone is an L-norm if and only if the dual norm is an $m_{\le}$-norm. Conditions are given for the operator norm to be an $m_{\le}$- or an L-norm.

Introduction and elementary properties. Duality relations in AL- and AM-spaces as well as in base-norm and order-unit spaces are of special interest. To some extent they can be interpreted as properties of the cone and its dual wedge (see [AB85], [BR84], [MN91]). For the used terminology see [MN91] and [KLS89].

If not stated otherwise, in this paper $(X, X_+, \|\cdot\|)$ denotes an ordered normed space. The notion of the M-norm (i.e. $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for any $x, y \in X_+$) is defined in a vector lattice. We extend this definition to arbitrary ordered normed vector spaces $(X, X_+, \|\cdot\|)$ in the following way. Let X_+ be a reproducing cone. The norm $\|\cdot\|$ will be called *$m_{\le}$-norm* (or $\|\cdot\|$ is said to be $m_{\le}$) if for any $x, y \in X_+$

$$\inf\{\|v\| : x, y \le v\} \le \max\{\|x\|, \|y\|\}. \tag{1}$$

If equality holds in (1) then the norm $\|\cdot\|$ is called a *generalized M-norm* or *m-norm* (or is said to be m)[1].

A norm $\|\cdot\|$ that is additive on the cone will be called an *L-norm.*

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[1] The notion of m-norm is already useed for some special norm of majorizing maps between Banach spaces and Banach lattices (see [Sch74], chapt. IV.3 and [MN91], chapt. 2.8). Nevertheless we shall use this notion throughout this paper only in the sense of (1) with equality.

It is easy to see that the m-norms are exactly the monotone $m_{\leq}$-norms and that, in general, not every $m_{\leq}$-norm is an m-norm, as the space $C^1([0,1])$ with the usual norm shows.

Some geometric condition on the unit ball guarantees that $\|\cdot\|$ is an $m_{\leq}$-norm.

THEOREM 1. *Let $(X, X_+, \|\cdot\|)$ be an ordered normed space and $u \in X_+$ an element such that $\|u\| = 1$ and $B_X \subseteq (1+\varepsilon)[-u, u]$ for all $\varepsilon > 0$. Then $\|\cdot\|$ is an $m_{\leq}$-norm.*

Proof. Because u is an order unit the cone X_+ is reproducing (see e.g. [Vul77], §II). Let be $x, y \in X_+$. Without loss of generality we may assume $\|x\| \geq \|y\|$. It suffices to consider only the case $x \neq 0$. For $\varepsilon > 0$ one has $0 \leq \frac{1}{\|x\|}x, \frac{1}{\|x\|}y \leq (1+\varepsilon)u$ and so $x, y \leq (1+\varepsilon)\|x\|\, u$, i.e.

$$\inf\{\|v\| : x, y \leq v\} \leq (1+\varepsilon)\|x\|\,\|u\| = (1+\varepsilon)\|x\| = (1+\varepsilon)\max\{\|x\|, \|y\|\}.$$

Then $\inf\{\|v\| : x, y \leq v\} \leq \max\{\|x\|, \|y\|\}$ follows. ∎

In particular, $\|\cdot\|$ is an $m_{\leq}$-norm if $B_X \subseteq [-u, u]$. So, as a corollary, we get that any order-unit norm is an $m_{\leq}$-norm, and, due to its monotonicity, also an m-norm. Furthermore, it is easy to see that in the case of a normed lattice the properties of the norm to be M, m and $m_{\leq}$ are equivalent.

It is natural to expect that for an ordered normed space $(X, X_+, \|\cdot\|)$ with an $m_{\leq}$- or an m-norm the inequality or equality in (1) are satisfied not only for two positive vectors but also for any finite set of positive vectors. This can be easily proved by induction.

A similar relation holds for any finite set of vectors, even if they are not positive.

PROPOSITION 2. *Let $\|\cdot\|$ be an $m_{\leq}$-norm. Then the following properties hold:*

1. *X_+ is approximately absolutely dominating, i.e. for any $x \in X$ and any $M > 1$ exists $v \in X$ with $-v \leq x \leq v$ and $\|v\| \leq M\|x\|$.*
2. *For any $n \in \mathbb{N}$ and any $x_i \in X$ $(i = 1, \dots, n)$*

$$\inf\{\|v\| :\ -v \leq x_i \leq v,\ i = 1, \dots, n\} \leq \max\{\|x_i\| : i = 1, \dots, n\}.$$

3. *X_+ is non-flat.*

Proof. 1. Due to the reproducibility of X_+ for an arbitrary $x \in X$ one can find an element $v \in X_+$ that satisfies the inequalities $-v \leq x \leq v$. If $\|v\| \leq \|x\|$ then nothing has to be proved. Therefore suppose $\|x\| + \varepsilon = \|v\| > 0$ for some $\varepsilon > 0$. If there is a vector $\tilde{v}$ with $-\tilde{v} \leq x \leq \tilde{v}$ and $\|\tilde{v}\| \leq \|x\| + \frac{3}{4}\varepsilon$, then the assertion follows.

Obviously, $v - x,\ v + x \geq 0$. Because $\|\cdot\|$ is $m_{\leq}$, there exists some $\hat{v}$ such that

$$v - x,\ v + x \leq \hat{v} \qquad \text{and} \qquad \|\hat{v}\| \leq \max\{\|v - x\|, \|v + x\|\} + \frac{1}{2}\varepsilon. \tag{2}$$

Then $\hat{v} - 2x \geq \hat{v} - 2x - (v - x) = \hat{v} - (v + x) \geq 0$ and $2x + \hat{v} \geq \hat{v} + 2x - (v + x) = \hat{v} - (v - x) \geq 0$ yield $-\frac{1}{2}\hat{v} \leq x \leq \frac{1}{2}\hat{v}$ and the estimates $\|v + x\| \leq \|x\| + \|v\| = 2\|v\| - \varepsilon$ and $\|v - x\| \leq 2\|v\| - \varepsilon$ yield $\max\{\|v + x\|, \|v - x\|\} \leq 2\|v\| - \varepsilon$. So, (2) implies $\|\hat{v}\| \leq 2\|v\| - \frac{1}{2}\varepsilon$. For $\tilde{v} = \frac{1}{2}\hat{v}$ we get $\|\tilde{v}\| = \frac{1}{2}\|\hat{v}\| \leq \|v\| - \frac{1}{4}\varepsilon = \|x\| + \frac{3}{4}\varepsilon$.

2. Let $\{x_1, \dots, x_n\}$ be any finite set of vectors in X and $\varepsilon > 0$ arbitrary. By statement 1 for every $i = 1, \dots, n$ there is a positive vector v_i satisfying $-v_i \leq x_i \leq v_i$ and $\|v_i\| \leq \|x_i\| + \frac{1}{2}\varepsilon$. As mentioned before Proposition 2 there is a positive vector v with $v_i \leq$

v, $(i = 1, \ldots, n)$ and $\|v\| \leq \max\{\|v_i\| : i = 1, \ldots, n\} + \frac{1}{2}\varepsilon$. In particular, $-v \leq x_i \leq v$ $(i = 1, \ldots, n)$ and $\|v\| \leq \max\{\|x_i\| : i = 1, \ldots, n\} + \varepsilon$.

3. By statement 1 every $x \in X$ can be represented as $x = v - (v - x)$ with $0, x \leq v$ and $\|v\| \leq 2\|x\|$. Then the inequality $\|v - x\| \leq 3\|x\|$ completes the proof. ∎

The next proposition states that under an additional condition the property of the norm $\|\cdot\|$ to be $m_{\leq}$ is preserved under passing to the closure of the cone.

Proposition 3. *Let $\|\cdot\|$ be a semi-monotone $m_{\leq}$-norm. Then $\|\cdot\|$ is also an $m_{\leq}$-norm on $(X, \overline{X_+})$. In particular, if $\|\cdot\|$ is m, then it is also m on $(X, \overline{X_+})$.*

Proof. First remark that the semi-monotonicity of $\|\cdot\|$ is equivalent to the normality of X_+ and since the closure $\overline{X_+}$ of a normal cone X_+ is again a normal cone ([Vul77], §4.1), it follows that $(X, \overline{X_+})$ is also an ordered space. Then Lemma 3.5 in [TW03] can be applied, where it is proved that an m-norm on (X, X_+) remains an m-norm on $(X, \overline{X_+})$. Since the monotonicity of the norm is not used for $\|\cdot\|$ to be $m_{\leq}$ on $(X, \overline{X_+})$ the general result holds. ∎

For two ordered normed spaces $(X, X_+, \|\cdot\|)$ and $(Y, Y_+, \|\cdot\|)$ denote by $(\mathfrak{L}(X,Y), \mathfrak{L}_+, \|\cdot\|)$ the vector space of all linear and continuous operators $T\colon X \to Y$ equipped with the operator norm. If the cone X_+ is reproducing then the wedge $\mathfrak{L}_+$ of all positive operators $T \in \mathfrak{L}(X,Y)$ is a cone (see [Vul78] §VI) and so $(\mathfrak{L}(X,Y), \mathfrak{L}_+, \|\cdot\|)$ turns out to be an ordered normed space. The norm dual of an ordered normed space is denoted by $(X', X'_+, \|\cdot\|')$, where X'_+ is the wedge of all positive functionals in X'.

Proposition 4. *If $\|\cdot\|$ on X is an $m_{\leq}$-norm and the norm on Y is absolutely monotone, i.e. $-v \leq x \leq v$ implies $\|x\| \leq \|v\|$, then the norm of each positive operator $T\colon X \to Y$ is positively attained, i.e. $\|T\| = \sup\{\|Tx\|\colon x \in B_X \cap X_+\}$. In particular, the norm of each functional $f \in X'_+$ is positively attained.*

Proof. Because the norm on X is approximately absolutely dominating (Proposition 2) the result follows from Proposition 1.7.8 in [BR84][2]. ∎

The Dedekind completion. Next we show that for every ordered normed space with a semi-monotone $m_{\leq}$-norm its Dedekind completion exists. For that we need some elementary facts on ordered normed spaces $(X, X_+, \|\cdot\|)$:

(i) If the cone X_+ is closed then the ordered normed space $(X, X_+, \|\cdot\|)$ is Archimedean, i.e. $x, y \in X$ and $nx \leq y$ for all $n \in \mathbb{N}$ (and so $x \leq \frac{1}{n}y$) imply $x \leq 0$.

(ii) Based on the classical Yudin's Theorem (see e.g. [Vul67], chapt. IV.11), any Archimedean ordered vector space with a reproducing cone possesses the Dedekind completion, that is, there exists a minimal Dedekind complete vector lattice $\widehat{X}$ such that X is algebraically and order isomorphic to some linear subspace of $\widehat{X}$, where the embedding preserves the exact bounds of subsets of X ([Vul78], §V.3).

[2] One should mention that in [BR84] $(X, X_+, \|\cdot\|)$ and $(Y, Y_+, \|\cdot\|)$ are Banach spaces, but the same proof can also be applied in the general case.

THEOREM 5. *Let $\|\cdot\|$ be a semi-monotone $m_{\leq}$-norm and the cone X_+ closed. Then the Dedekind completion $\widehat{X}$ of X exists and on $\widehat{X}$ there is an M-norm that is equivalent to the norm $\|\cdot\|$ on X.*

Proof. The existence of the Dedekind completion $\widehat{X}$ of X is guaranteed by (i) and (ii). Furthermore there exists a monotone norm $\|\cdot\|_{\widehat{X}}$ on $\widehat{X}$ which on X is equivalent to the norm $\|\cdot\|$ (see [Vul78], th. V.3.1). This norm is defined for all $y \in \widehat{X}$ by $\|y\|_{\widehat{X}} := \inf\{\|x\| : x \in X_+,\, x \geq |y|\}$. We show that $\|\cdot\|_{\widehat{X}}$ is an M-norm. For that let be $0 \leq y, y' \in \widehat{X}$. Then in view of (1) one has

$$\begin{aligned}
&\max\{\|y\|_{\widehat{X}}, \|y'\|_{\widehat{X}}\} \\
&\quad= \max\big\{\inf\{\|x\| : x \in X_+,\, x \geq y\}, \inf\{\|x'\| : x' \in X_+,\, x' \geq y'\}\big\} \\
&\quad= \inf\big\{\max\{\|x\|, \|x'\|\} \colon x, x' \in X_+,\, x \geq y \text{ and } x' \geq y'\big\} \\
&\quad\geq \inf\big\{\inf\{\|v\| : v \in X_+,\, v \geq x, x'\} \colon x, x' \in X_+,\, x \geq y \text{ and } x' \geq y'\big\} \\
&\quad\geq \inf\big\{\inf\{\|v\| : v \in X_+,\, v \geq y, y'\} \colon x, x' \in X_+,\, x \geq y \text{ and } x' \geq y'\big\} \\
&\quad= \inf\{\|v\| : v \in X_+,\, v \geq y, y'\} = \inf\{\|v\| : v \in X_+,\, v \geq y \vee y'\} = \|y \vee y'\|_{\widehat{X}}\,.
\end{aligned}$$

Now the monotonicity of $\|\cdot\|_{\widehat{X}}$ yields the equation $\max\{\|y\|_{\widehat{X}}, \|y'\|_{\widehat{X}}\} = \|y \vee y'\|_{\widehat{X}}$. ∎

Some characterizations. By means of the so-called *constants of reproducibility* we are able to characterize the $m_{\leq}$- and m-norms on ordered normed spaces. For a reproducing cone X_+ and for all natural numbers $n = 1, 2, \ldots$ define

$$V(X_+, n) = \sup_{x_1,\ldots,x_n \in B_X} \inf_{v \geq x_1,\ldots,x_n} \|v\| \tag{3}$$

(see e.g. [Vul78]). In general $V(X_+, n) \in [0, +\infty]$. It is interesting that the conditions (i) X_+ is non-flat, (ii) $V(X_+, 2) < +\infty$ and (iii) $V(X_+, n) < +\infty$ for any $n \geq 2$ are equivalent ([Vul78], th. III.3.1). So, it follows from Proposition 2 that $V(X_+, n) < +\infty$ holds for all n provided $\|\cdot\|$ is an $\mathrm{m}_{\leq}$-norm. But we can state even more.

THEOREM 6. *The following two groups include equivalent statements:*

1. *$\|\cdot\|$ is an $m_{\leq}$-norm.*
2. *$V(X_+, k) \leq 1$ for some $k \geq 2$.*
3. *$V(X_+, n) \leq 1$ for all $n \geq 2$.*

and

4. *$\|\cdot\|$ is an m-norm.*
5. *The norm $\|\cdot\|$ is monotone and $V(X_+, k) = 1$ for some $k \geq 2$.*
6. *The norm $\|\cdot\|$ is monotone and $V(X_+, n) = 1$ for all $n \geq 2$.*

Proof. 2⇒1. Take $x, y \in X_+$. Without loss of generality $\|x\| \geq \|y\|$ and $x \neq 0$ may be assumed. Then $x, y \in \|x\|\, B_X$ and, by assumption, for some $k \geq 2$

$$\begin{aligned}
\max\{\|x\|, \|y\|\} &= \|x\| \geq \|x\| \cdot V(X_+, k) = \|x\| \cdot \sup_{x_1,\ldots,x_k \in B_X} \inf_{v \geq x_1,\ldots,x_k} \|v\| \\
&= \sup_{x_1,\ldots,x_k \in \|x\| B_X} \inf_{v \geq x_1,\ldots,x_k} \|v\| \geq \sup_{x_1,x_2 \in \|x\| B_X} \inf_{v \geq x_1,x_2} \|v\| \geq \inf\{\|v\| : x, y \leq v\}.
\end{aligned}$$

1⇒3. Proposition 2 implies that for a finite number of elements $x_1, \ldots, x_n \in B_X$ one has $\inf\{\|v\| : -v \le x_1, \ldots, x_n \le v\} \le \max\{\|x_1\|, \ldots, \|x_n\|\} \le 1$ and so

$$V(X_+, n) = \sup_{x_1,\ldots,x_n \in B_X} \inf_{v \ge x_1,\ldots,x_n} \|v\| \le \sup_{x_1,\ldots,x_n \in B_X} \inf_{v \ge x_1,\ldots,x_n \ge -v} \|v\| \le 1.$$

3⇒2. Is clear.

5⇒4. Follows from 2⇒1.

4⇒6. Since $\|\cdot\|$ is monotone, the inequality $\|v\| \ge \max\{\|x_1\|, \ldots, \|x_n\|\}$ holds for arbitrary vectors $x_1, \ldots, x_n \in B_X \cap X_+$ such that $v \ge x_1, \ldots, x_n$. Therefore

$$V(X_+, n) = \sup_{x_1,\ldots,x_n \in B_X} \inf_{v \ge x_1,\ldots,x_n} \|v\| \ge \sup_{x_1,\ldots,x_n \in B_X \cap X_+} \inf_{v \ge x_1,\ldots,x_n} \|v\| \ge 1.$$

The equation $V(X_+, n) = 1$ follows now from 1⇒3.

6⇒5. Is clear again. ∎

The open unit ball $B = \{x \in X : \|x\| < 1\}$ of an ordered normed space $(X, X_+, \|\cdot\|)$ is called *upward directed* if for any $x, y \in B$ there exists an element $w \in B$ such that $x, y \le w$. This concept is examined, e.g., in [Ng67], and turns out to be equivalent for a norm to be $m_\le$.

Proposition 7. *The following conditions are equivalent:*

1. *$\|\cdot\|$ is an $m_\le$-norm.*
2. *The open unit ball B is upward directed.*

Proof. 1⇒2. For $x, y \in B$ there exists an $\varepsilon > 0$ such that $\|x\| + \varepsilon, \|y\| + \varepsilon < 1$. Proposition 2 guarantees the existence of an element v with the properties $x, y \le v$ and $\|v\| \le \max\{\|x\|, \|y\|\} + \varepsilon < 1$.

2⇒1. Let $x, y \in X_+,\ \varepsilon > 0$. Without loss of generality we assume $\|x\| \ge \|y\|$. Then $\tilde{x} = \frac{1}{\|x\|+\varepsilon} x$ and $\tilde{y} = \frac{1}{\|x\|+\varepsilon} y$ belong to B. By assumption there exists an element $\tilde{w}$ such that $\tilde{x}, \tilde{y} \le \tilde{w}$ and $\|\tilde{w}\| < 1$. For $w = (\|x\| + \varepsilon)\tilde{w}$ one has $x, y \le w$ and $\|w\| = (\|x\| + \varepsilon)\|\tilde{w}\| < \|x\| + \varepsilon = \max\{\|x\|, \|y\|\} + \varepsilon$.

In order to show that X_+ is reproducing notice that, by definition, for any $x \in B$ there exists an $y \in B$ that dominates x and 0. ∎

Duality results. Now we want to examine some duality properties for ordered normed spaces with L-, m- and $m_\le$-norms. We first characterize the dual norm to be L or $m_\le$.

Theorem 8. *Consider the two properties:*

1. *The norm $\|\cdot\|$ is an $m_\le$-norm.*
2. *The norm $\|\cdot\|'$ is an L-norm.*

Then 1⇒2 always holds. If, in addition, $(X, X_+, \|\cdot\|)$ is an ordered Banach space and the cone X_+ is closed then also 2⇒1.

Proof. 1⇒2. Take $f, g \in X'_+$ and $\varepsilon > 0$. It follows from Proposition 4 (with $Y = \mathbb{R}^1$) that there are elements $\tilde{x}, \tilde{y} \in B_X \cap X_+$ with $\|f\|' \le f(\tilde{x}) + \varepsilon$ and $\|g\|' \le g(\tilde{y}) + \varepsilon$. Because $\|\cdot\|$ is an $m_\le$-norm, there exists a vector $v \in X_+$ with $\tilde{x}, \tilde{y} \le v$ and $\|v\| \le$

$\max\{\|\tilde{x}\|, \|\tilde{y}\|\} + \varepsilon \leq 1 + \varepsilon$. Then

$$\|f\|' + \|g\|' \leq f(\tilde{x}) + g(\tilde{y}) + 2\varepsilon \leq f(v) + g(v) + 2\varepsilon = (f+g)(v) + 2\varepsilon$$
$$\leq \|f+g\|' \|v\| + 2\varepsilon \leq \|f+g\|' + \varepsilon(\|f+g\|' + 2).$$

By arbitrariness of $\varepsilon > 0$ there follows $\|f\|' + \|g\|' \leq \|f+g\|'$, i.e. $\|f\|' + \|g\|' = \|f+g\|'$.

2⇒1. By Theorem 4 from [Ng67] condition 2 implies that the open unit ball of X is upward directed. Then Proposition 7 completes the proof. ∎

It is natural to ask whether the role of $m_{\leq}$ and L-norms can be interchanged in Theorem 8. The answer will be given in Theorem 10. For its proof we need the following lemma.

LEMMA 9. *Let be X_+ a reproducing cone and the norm on X an L-norm. Let $(Y, Y_+, \|\cdot\|)$ be an arbitrary ordered normed space. For any fixed $\alpha \in \mathbb{R}$ and $y \in Y$ by means of*

$$F(x) = \alpha(\|x_1\| - \|x_2\|)y \qquad \text{with} \quad x_1, x_2 \in X_+,\ x = x_1 - x_2$$

a linear continuous operator $F : X \to Y$ is well defined and satisfies $\|F\| = |\alpha| \|y\|$. F is positive if and only if $\alpha y \geq 0$.

Proof. It is shown in [TW03], Lemma 4.2, that $v : X \to \mathbb{R}$, $v(x) = \alpha(\|x_1\| - \|x_2\|)$ with $x_1, x_2 \in X_+$, $x = x_1 - x_2$ is a well defined linear continuous functional on X satisfying $\|v\|' = |\alpha|$. It is positive iff $\alpha \geq 0$. The representation $F(x) = v(x)y$ finishes the proof. ∎

THEOREM 10. *Let X_+ be a reproducing cone. Then the following statements are equivalent:*

1. *The norm $\|\cdot\|$ is an L-norm.*
2. *The norm $\|\cdot\|'$ is an $m_{\leq}$-norm.*

Proof. 1⇒2. Since the cone X_+ in a space with L-norm is normal, the dual cone X'_+ is reproducing. From Lemma 9 it follows that $e'(x) = \|x_1\| - \|x_2\|$ is an element of X'_+ satisfying the condition $\|e'\|' = 1$, where $x = x_1 - x_2$, $x_1, x_2 \in X_+$. For $f \in B_{X'}$ and $x \in X_+$ one has $|f(x)| \leq \|x\| = e'(x)$, i.e. $-e' \leq f \leq e'$. Statement 2 follows now from Theorem 1.

2⇒1. In view of Theorem 8 the norm $\|\cdot\|''$ of the bidual is an L-norm. Therefore the norm $\|\cdot\|$, as a restriction of $\|\cdot\|''$ on X, is an L-norm, too. ∎

As the proof of the implication 1⇒2 shows, the functional e' has the properties $\|e'\|' = 1$ and $B_{X'} \subseteq [-e', e']$. Therefore, for each $f \in X'$ one has $-\|f\|' e' \leq f \leq \|f\|' e'$, where $\left\|\|f\|' e'\right\|' = \|f\|'$. Due to Theorem 3.6.7 in [Jam70] the norm $\|\cdot\|$ is absolutely monotone (see Proposition 4).

For an ordered Banach space $(X, X_+, \|\cdot\|)$ with a closed cone X_+ the equivalence of the statements (i) X_+ is approximately dominating (i.e. for any $x \in X$ and any $\epsilon > 0$ there exists a vector $v \in X_+$ such that $x \leq v$ and $\|v\| \leq (1+\epsilon)\|x\|$) and (ii) the dual norm is monotone on X'_+, is known from Theorem 1.2.2 in [BR84], where the implication (i)⇒(ii) holds even in arbitrary ordered normed spaces. By combining this with the previous duality result, the next proposition follows directly.

PROPOSITION 11. *Let* $(X, X_+, \|\cdot\|)$ *be an ordered Banach space with a closed reproducing cone* X_+. *Then the following statements are equivalent:*

1. *The norm* $\|\cdot\|$ *is an* L*-norm and* X_+ *is approximately dominating.*
2. *The norm* $\|\cdot\|'$ *is an* m*-norm.*

The implication 1⇒2 also holds for any ordered normed space with a reproducing cone.

The assertion of Proposition 11 also holds in an arbitrary ordered normed space if the cone X_+ possesses interior points.

PROPOSITION 12. *Let* X_+ *satisfy the condition* $\operatorname{int}(X_+) \neq \emptyset$. *Then the following statements are equivalent:*

1. $\|\cdot\|$ *is an* L*-norm and* X_+ *is approximately dominating.*
2. *The norm* $\|\cdot\|'$ *is an* m*-norm.*

Proof. Notice that due to $\operatorname{int}(X_+) \neq \emptyset$ the cone X_+ is reproducing and that it suffices to prove only the implication 2⇒1. By Theorem 10 the norm $\|\cdot\|$ is an L-norm. To show that X_+ is approximately dominating, we assume the contrary. Suppose there exist a positive $\varepsilon > 0$ and a vector $x_0 \in X$, $x_0 \neq 0$ such that for all $y \in X_+$ with $y \geq x_0$ the inequality $\|y\| \geq (1 + \frac{\varepsilon}{\|x_0\|})\,\|x_0\|$ holds. Without loss of generality assume $\|x_0\| = 1$. Define the subsets $A_1(x_0)$, $A_2(\varepsilon) \subseteq X$ by

$$A_1(x_0) := x_0 + X_+ = \{y \in X \colon x_0 \leq y\} \qquad \text{and} \qquad A_2(\varepsilon) := X_+ \cap \left(1 + \frac{\varepsilon}{2}\right) B_X.$$

These sets are not empty and $\operatorname{int}(X_+) \neq \emptyset$ implies that $A_1(x_0)$ has a non-empty interior.

The sets $A_1(x_0)$ and $A_2(\varepsilon)$ are disjoint. Indeed, if there is a vector $y \in A_1(x_0) \cap A_2(\varepsilon)$ then the relations $y \geq x_0$ and $y \in X_+$ imply $\|y\| \geq 1+\varepsilon$, which contradicts $y \in (1+\frac{\varepsilon}{2})B_X$, i.e. $\|y\| \leq 1 + \frac{\varepsilon}{2}$.

The convexity of both sets $A_1(x_0)$, $A_2(\varepsilon)$ is immediate. Consequently, the sets $A_1(x_0)$ and $A_2(\varepsilon)$ can be separated by a linear continuous functional $f \neq 0$, i.e. there is a number $\lambda \in \mathbb{R}$ such that

$$f(z) \leq \lambda \leq f(y) \qquad \text{for all } y \in A_1(x_0),\ z \in A_2(\varepsilon). \tag{4}$$

The functional f is positive. Indeed, since $x_0 + \alpha x \in A_1(x_0)$ for each $x \in X_+$ and $\alpha > 0$, one has $\lambda \leq f(x_0 + \alpha x) = f(x_0) + \alpha f(x)$ and so, $\frac{\lambda - f(x_0)}{\alpha} \leq f(x)$ for all $\alpha > 0$, which implies $0 \leq f(x)$.

Since X_+ is reproducing there is a vector $z \in X_+$ such that $\|z\| = 1$ and $f(z) > 0$. Obviously, z belongs to $A_2(\varepsilon)$, which in view of the inequalities (4) implies $\lambda > 0$. Let be $x \in X_+$. Then $\frac{1}{\|x\|}(1 + \frac{\varepsilon}{2})x \in A_2(\varepsilon)$ and, according to (4), $f(\frac{1+\frac{\varepsilon}{2}}{\|x\|\lambda}x) \leq 1$. Therefore $f(\frac{1+\frac{\varepsilon}{2}}{\lambda}x) \leq \|x\| \leq e'(x)$ for all $x \in X_+$, where e' is the functional $e' \in X'_+$ defined according to Lemma 9 by $e'(x) = \|x_1\| - \|x_2\|$ with $\|e'\|' = 1$, where $x = x_1 - x_2$, $x_1, x_2 \in X_+$. The functional $g = \frac{1+\frac{\varepsilon}{2}}{\lambda} f$ satisfies the condition $g \leq e'$.

On the other hand from $x_0 \in A_1(x_0)$, $\|x_0\| = 1$ and (4) it follows that $f(x_0) \geq \lambda$, which yields

$$\|g\|' \geq g(x_0) = \frac{1 + \frac{\varepsilon}{2}}{\lambda} f(x_0) \geq \frac{1 + \frac{\varepsilon}{2}}{\lambda}\lambda = 1 + \frac{\varepsilon}{2} > \|e'\|'.$$

This contradicts the monotonicity of the norm $\|\cdot\|'$ on X'. ∎

Properties of the operator norm. Let $(X, X_+, \|\cdot\|)$ and $(Y, Y_+, \|\cdot\|)$ be two ordered normed spaces with reproducing cones X_+, Y_+ and $(\mathfrak{L}(X,Y), \mathfrak{L}_+, \|\cdot\|)$ the space of all continuous linear operators between X and Y. We are now interested in some order properties of $(\mathfrak{L}(X,Y), \mathfrak{L}_+, \|\cdot\|)$ and obtain results similar to those of Theorem 8 and Theorem 10.

A norm $\|\cdot\|$ on an ordered normed space $(X, X_+, \|\cdot\|)$ is called an *$m_\leq$-norm with unit* if there is an order unit $e_X \in X$ such that $\|e_X\| = 1$ and $B_X \subseteq [-e_X, e_X]$ (cf. Theorem 1).

THEOREM 13. *If the norm on X is an L-norm then the following assertions are equivalent:*

1. *The norm on Y is an $m_\leq$-norm with unit.*
2. *The norm on $\mathfrak{L}(X,Y)$ is an $m_\leq$-norm with unit.*

Proof. 1⇒2. If e_Y is the order unit in Y corresponding to the $m_\leq$-norm in Y then define the operator $E\colon X \to Y$ by $Ex = (\|x_1\| - \|x_2\|)e_Y$, where $x = x_1 - x_2$ is any representation of x as a difference of two positive elements of X. By Lemma 9 E is a well defined positive linear continuous operator, i.e. an element of $\mathfrak{L}_+$ with $\|E\| = 1$.

In view of the assumption for any $x \in X_+$ and $F \in \mathfrak{L}(X,Y)$ with $\|F\| \leq 1$ the inclusions $Fx \in \|x\|\, B_Y \subseteq \|x\|\, [-e_Y, e_Y] = [-Ex, Ex]$ hold. This means $-E \leq F \leq E$. So, $B_{\mathfrak{L}} \subseteq [-E, E]$. By the remark to Theorem 1 the implication is proved.

2⇒1. Let $E_{\mathfrak{L}}$ be the order unit in $\mathfrak{L}(X,Y)$ corresponding to the $m_\leq$-norm on $\mathfrak{L}(X,Y)$. Since $\|E_{\mathfrak{L}}\| = 1$, for any $x \in X_+$ with $\|x\| = 1$ one has $\|E_{\mathfrak{L}}x\| \leq 1$. Fix $y \in B_Y$ and define $F_y\colon X \to Y$ by $F_y x = (\|x_1\| - \|x_2\|)y$, where $x \in X$, $x = x_1 - x_2$ with $x_1, x_2 \in X_+$. Lemma 9 shows again that F_y is a well defined positive linear continuous operator, satisfying $\|F_y\| = \|y\| \leq 1$. By assumption $-E_{\mathfrak{L}} \leq F_y \leq E_{\mathfrak{L}}$ and so, due to $F_y x = y$ for $x \in X_+$ and $\|x\| = 1$, one obtains $-E_{\mathfrak{L}}x \leq y \leq E_{\mathfrak{L}}x$ and, since $y \in B_Y$ was arbitrary, also $B_Y \subseteq [-E_{\mathfrak{L}}x, E_{\mathfrak{L}}x]$. ∎

THEOREM 14. *If the norm on X is an $m_\leq$-norm then the following assertions are equivalent:*

1. *The norm on Y is an L-norm.*
2. *The norm on $\mathfrak{L}(X,Y)$ is an L-norm.*

Proof. 1⇒2. By the remark to Theorem 10 assumption 1 implies that the norm on Y is absolutely monotone. Let $S, T \in \mathfrak{L}(X,Y)$ be two positive operators and $\varepsilon > 0$. Then there are two elements $x, y \in X_+ \cap B_X$ with $\|S\| \leq \|Sx\| + \varepsilon$ and $\|T\| \leq \|Ty\| + \varepsilon$ (see Proposition 4). Since the norm on X is $m_\leq$, we conclude that there exists $z \in X_+ \cap (1+\varepsilon)B_X$ such that $z \geq x, y$. The positivity of S and T implies $Sz \geq Sx$, $Tz \geq Ty$ and, together with the monotonicity of the norm on Y, one has $\|Sz\| \geq \|Sx\|$ and $\|Tz\| \geq \|Ty\|$. Because the norm on Y is an L-norm, it follows that $\|Sz\| + \|Tz\| = \|Sz + Tz\|$. A similar argument as in the proof of Theorem 8 leads to

$$\begin{aligned} \|S\| + \|T\| &\leq \|Sx\| + \|Ty\| + 2\varepsilon \leq \|Sz\| + \|Tz\| + 2\varepsilon = \|(S+T)z\| + 2\varepsilon \\ &\leq \|S+T\|\,\|z\| + 2\varepsilon \leq \|S+T\| + \varepsilon(\|S+T\| + 2), \end{aligned}$$

for arbitrary $\varepsilon > 0$. One concludes $\|S\| + \|T\| \leq \|S+T\|$, and so $\|S\| + \|T\| = \|S+T\|$.

2⇒1. Let $y_1, y_2 \in Y_+$ and $f \in X'_+$ with $\|f\|' = 1$. Define $S, T\colon X \to Y$ by $Sx = f(x)y_1$ and $Tx = f(x)y_2$. Then $0 \le S, T \in \mathcal{L}(X,Y)$ and $\|S\| = \|y_1\|$, $\|T\| = \|y_2\|$. Therefore

$$\begin{aligned}\|y_1\| + \|y_2\| &= \|S\| + \|T\| = \|S+T\| \\ &= \sup_{x\in B_X} \|(S+T)x\| = \sup_{x\in B_X} \|f(x)y_1 + f(x)y_2\| \\ &= \sup_{x\in B_X} |f(x)|\,\|y_1+y_2\| = \|y_1+y_2\|\,,\end{aligned}$$

i.e. $\|y_1\| + \|y_2\| = \|y_1 + y_2\|$. ∎

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