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NEW A PRIORI ESTIMATES FOR NONDIAGONAL STRONGLY NONLINEAR PARABOLIC SYSTEMS

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Abstract. We consider nondiagonal elliptic and parabolic systems of equations with quadratic nonlinearities in the gradient. We discuss a new description of regular points of solutions of such systems. For a class of strongly nonlinear parabolic systems, we estimate locally the Hölder norm of a solution. Instead of smallness of the oscillation, we assume local smallness of the Campanato seminorm of the solution under consideration. Theorems about *quasireverse* Hölder inequalities proved by the author are essentially used. We study systems under the Dirichlet boundary condition and estimate the Hölder norm of a solution up to the boundary (up to the parabolic boundary of the prescribed cylinder in the parabolic case).

1. Introduction. We consider the Cauchy-Dirichlet problem for nonlinear parabolic systems.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with sufficiently smooth boundary $\partial\Omega$, let $Q = \Omega \times (0,T)$ with any fixed $T > 0$, and $u : Q \mapsto \mathbb{R}^N$, $u = (u^1, \dots, u^N)$, $N > 1$, be a solution of the problem

$$u_t^k - (A_{kl}^{\alpha\beta}(z,u)u_{x_\beta}^l)_{x_\alpha} + b^k(z,u,u_x) = 0, \quad k = 1,\dots,N, \quad z \in Q, \tag{1}$$

$$u|_\Gamma = 0, \quad u|_{t=0} = \phi(x), \tag{2}$$

where $u_x = \{u_{x_\alpha}^k\}$, $\Gamma = \partial\Omega \times (0,T)$.

It is assumed that the matrix $A = \{A_{kl}^{\alpha\beta}(\cdot,\cdot)\}_{k,l\leq N}^{\alpha,\beta\leq n}$ is defined on the set $\mathcal{M} = \overline{Q} \times \mathbb{R}^N$, the function $b = \{b^k(\cdot,\cdot,\cdot)\}^{k\leq N}$ is a Carathéodory function on $\mathcal{M} \times \mathbb{R}^{nN}$. Moreover, we

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suppose the following:

a) the strong parabolicity condition holds, i.e., there exist numbers ν and $\mu > 0$ such that for every $(z,u) \in \mathcal{M}$, $\xi \in \mathbb{R}^{nN}$,

$$(A(z,u)\xi, \xi) \geq \nu|\xi|^2, \tag{3}$$

$$\sup_{\mathcal{M}} \|A(z,u)\| \leq \mu, \tag{4}$$

b) $A_{kl}^{\alpha\beta}$ are uniformly continuous functions, more exactly, there is a function $\omega(s,\tau)$ defined and continuous on $[0,\infty) \times [0,\infty)$ which is bounded, nondecreasing, concave in τ for any fixed s, $\omega(0,0) = 0$, and such that

$$\|A(z,u) - A(\zeta,v)\| \leq \omega(|z-\zeta|^2, |u-v|^2) \quad z,\zeta \in \overline{Q}, \quad u,v \in \mathbb{R}^N, \tag{5}$$

c) the function b satisfies the growth condition

$$|b(z,u,p)| \leq b_0|p|^2 + \mu, \quad (z,u) \in \mathcal{M},\ p \in \mathbb{R}^{nN},\ b_0 = const > 0, \tag{6}$$

d) Ω is a strictly Lipschitz domain, $\phi \in \overset{0}{W}{}_2^1(\Omega)$.

Condition (6) defines strongly nonlinear term $b(z,u,u_x)$ in system (1).

It is known that in the case of smooth data and under compatibility conditions, there exists a classical solution of (1), (2) on some time interval $[0,T_0)$ (see, for example, Theorem 1, [15] or [1]). A singular set $\sigma \subset \overline{\Omega}$ for $t = T_0$ is estimated in [2]. It is proved in [2] that the $(n-2)$-Hausdorff measure of σ is finite ($\mathcal{H}_{n-2}(\sigma) < +\infty$).

Strong nonlinearity (6) does not allow to apply well-known abstract theorems to state weak global solvability of the problem under consideration. Global solvability of this problem was proved only for a particular case.

One class of systems was studied by the author in the case of two spatial variables ([15], [3]–[5]). It is assumed in these papers that the elliptic operator of the system is of a variational structure, and conditions **a)**–**c)** are valid. Existence of a solution almost everywhere smooth in $\overline{Q}$ was proved under the Dirichlet and the Neumann boundary conditions. The solution may have at most finitely many singular points. It was a development of the main idea by M. Struwe [28].

In the situation $b_0 = 0$ in (6), weak solvability of (1), (2) is a consequence of the Monotone Operators theory. In this case, it is interesting to study the regularity problem. Counterexamples show that in the multidimensional case, one can expect only partial regularity of solutions even under all smooth data of the problem ([23], [22]). As for the two dimensional case, it is unknown up to now whether a solution of the simplest parabolic system

$$u_t^k - (A_{kl}^{\alpha\beta}(u)u_{x_\beta}^l)_{x_\alpha} = 0, \quad k \leq N, \quad z \in Q,$$

is smooth in $\overline{Q}$ for any $T > 0$ ($A_{kl}^{\alpha\beta}$, $u|_{\partial' Q}$ are smooth enough) or a singular set can appear for $t > 0$.

On the other hand, when $b_0 \neq 0$ and the principal matrix A is of the diagonal structure, i.e.,

$$A_{kl}^{\alpha\beta}(z,u) = a^{\alpha\beta}(z,u)\delta_{k,l}, \tag{7}$$

where $\delta_{k,l}$ is the Kronecker symbol, global solvability of (1), (2) was proved for some special cases. Structural restrictions on function $b(z,u,u_x)$ were formulated in the monograph [25] (Chapter 7, §7) to prove global classical solvability or existence in the class $L^2((0,T);\overset{0}{W}{}_2^1(\Omega))\cap \mathcal{C}^{\alpha,\frac{1}{2}\alpha}(\overline{Q})$, $\alpha\in(0,1)$ (see also [29], [30] for more sharp conditions for b).

The systems of the type (1), (7), $a^{\alpha\beta}=a^{\alpha\beta}(x)$, describe heat flows of harmonic maps. Existence of global almost everywhere smooth in $\overline{Q}$ solution (it has finite energy and satisfies the integral identity in the sense of distributions) was proved for the twodimensional case [28], and for the multidimensional case [17]. It was stated that the solution may have at most finitely many singular points in the case $n=2$, and $\mathcal{H}_n(\Sigma;\delta)<+\infty$ for $n>2$.

We also mention the corresponding stationary case of the problem (1), (2). For a more general class of the elliptic operators

$$L=\{L^k\}^{k\le N},\quad L^k=-(a^k_\alpha(x,u,u_x))_{x_\alpha}+b^k(x,u,u_x),$$

(b satisfies condition (6), $n=2$), the Dirichlet problem was studied by J. Frehse [18]. Under the so-called "one-side condition"

$$b(x,u,p)\cdot u\ge \nu_*|p|^2-\mu;\quad \nu_*<\nu,$$

existence of a solution $u\in \overset{0}{W}{}_2^1(\Omega)\cap\mathcal{C}^\alpha(\overline{\Omega})$, $\alpha\in(0,1)$, was proved in [18].

The regularity problem for elliptic and parabolic nondiagonal systems of equations with strongly nonlinear terms in the gradient was studied in [20], [21], [26], [27]. Partial regularity of *bounded* weak solutions was proved.

In the parabolic situation, under the assumption

$$2b_0\|u\|_{\infty,Q}<\nu, \tag{8}$$

it was proved that the solution u of system (1) is a Hölder continuous function in the vicinity of a point $z^0\in Q$ provided that

$$\liminf_{R\to 0}\frac{1}{R^n}\int_{Q_R(z^0)}|u_x|^2\,dz<\varepsilon_0^2, \tag{9}$$

where the number $\varepsilon_0>0$ depends on the data only. It means that condition (9) describes regular points of bounded weak solutions of (1) under restrictions (8) inside Q. Regularity of bounded weak solutions near the lateral surface Γ, $\Gamma=\partial\Omega\times(0,T)$, under the Dirichlet and Neumann boundary conditions and restriction (8) was studied in [6]. Partial regularity of the solutions up to Γ was proved, and the Hausdorff measure of the corresponding singular sets was estimated.

Note that to prove smoothness of a bounded solution u in a neighborhood of z^0, it is sufficient to assume that

$$\underset{Q_{R_0}(z^0)\cap\overline{Q}}{osc}u<\theta \tag{10}$$

for some $\theta>0$ and $R_0=R_0(z^0)>0$ (instead of conditions (8), (9)).

Indeed, from the integral identity for problem (1), (2), and assumption (10), it follows that

$$\frac{1}{R^n}\int_{Q_R(z^0)}|u_x|^2\,dz \le c_0\frac{(\theta^2+R^2)}{(\nu-b_0\theta)}, \tag{11}$$

where the constant c_0 does not depend on z^0, $R \le \frac{R_0}{2}$.

Evidently, the inequalities

$$\theta < \frac{\nu}{b_0}, \quad c_0\frac{(\theta^2+R_0^2)}{(\nu-b_0\theta)} < \varepsilon_0^2,$$

guarantee condition (9) and local variant of (8) ($b_0 osc_{Q_{R_0}}\, u < \nu$). Therefore, all mentioned results on the regularity are valid under condition (10). Moreover, this condition allows us to estimate the stronger norms of u in the vicinity of z^0. Unfortunately, description (10) of regular points does not allow to obtain a reasonable estimate of the set of singular points of the solution under consideration. At the same time, an appropriate information on the singular set can be helpful to study *solvability* of the problem (see, for example, [17]). There arises a question *how to relax* condition (10) in description of regular points.

The author studied this problem for the stationary case. We considered the Dirichlet problem for quasilinear elliptic systems with quadratic nonlinearities in the gradient. We proved in [7] and [8] that the assumption

$$\underset{\Omega_R(x^0)}{osc}\, u < \theta, \quad \Omega_R(x^0) = \Omega\cap B_R(x^0), \tag{12}$$

in a point $x^0 \in \overline{\Omega}$, supplying an estimate of the Hölder norm of u, can be relaxed to condition

$$[u]_{\mathcal{L}^{2,n}(\Omega_R(x^0))} + sup_{y^0\in\partial\Omega\cap B_R(x^0),\,\varrho\le R}|u_{\varrho,y^0}| < \theta. \tag{13}$$

We denote by $[\cdot]_{\mathcal{L}^{2,n}(\Omega_R)}$ the seminorm in the Campanato space $\mathcal{L}^{2,n}(\Omega_R)$. The second term in (13) is absent for points x^0 inside Ω.

Instead of (13), we can assume the condition

$$\|u_x\|_{L^{2,n-2}(\Omega_R(x^0))} < \theta_1. \tag{14}$$

θ and θ_1 in (13) and (14) depend on the data of the problem, $\theta_1 = c(n)\theta$.

(The same relaxations for q-nonlinear systems, $1 < q < 2$, can be found in [13].)

In the situations when (14) can be relaxed to the condition:

$$\frac{1}{R^{n-2}}\int_{\Omega(x^0)}|u_x|^2\,dx < \theta_1 \tag{15}$$

at the fixed point $x^0 \in \overline{\Omega}$ for sufficiently small R, we are able to estimate the Hausdorff measure of the singular set. In the case of the simplest quasilinear elliptic systems ($b_0 = 0$), condition (15) describes regular points of solutions. That is why we can say that (15) introduces an optimal description of regular points in the case $b_0 \neq 0$.

There arises a question of the possibility to relax condition (14) to condition (15). It is not difficult to see that the monotonicity of the function

$$\Phi(\varrho, y) = \frac{1}{\varrho^{n-2}}\int_{\Omega_\varrho(y)}|u_x|^2\,dx \tag{16}$$

in ϱ for any fixed $y \in \overline{\Omega}$ provides such transformation.

For one class of strongly nonlinear elliptic systems, the author proved monotonicity type inequality for the function Φ introduced by (16) [9]. This inequality permitted to relax condition (14) to the optimal regularity condition (15).

It is evident that in the stationary situation, condition (14) allowed us to transform the problem of the optimal description of the regular set to derivation of the monotonicity type inequality for the function $\Phi(\varrho,\cdot)$.

The same considerations have been undertaken by the author for the parabolic problem (1), (2). The question was how to relax condition (10) in description of regular points of solutions.

For a solution u of (1), (2), a local L_p-estimate, $p>2$, of the gradient was obtained under condition

$$[u]_{\mathcal{L}^{2,n+2}(Q_R(z^0))} + sup_{\zeta\in\Gamma\cap Q_R,\,\varrho\le R}|u_{\varrho,\zeta}| < \theta, \tag{17}$$

or

$$\|u_x\|_{L^{2,n}(Q_R(z^0))} < \theta_1, \quad z^0\in Q\cup\Gamma, \tag{18}$$

$\theta_1 = c(n)\theta$ (see [10], Theorem 2.1 and Remark 2.3).

We note that the L_p - estimate of u_x is useful to estimate the Hölder norm of u locally by the so-called direct method.

We explain in this paper how to estimate u in $\mathcal{C}^{\alpha,\frac{\alpha}{2}}$-norm, $\alpha\in(0,1)$, in a neighborhood of a fixed point $z^0\in Q\cup\partial' Q$ provided that condition (17) (or (18)) holds. It will be done for one special class of nondiagonal strongly nonlinear parabolic systems.

Finally, we would like to remark that some smoothness of a solution is assumed in this work. We intend to apply all a priori information about the solution to investigate in the future a regularization of the problem.

We adopt the following notation:

$$\begin{aligned}
&\Lambda=(0,T),\quad Q=\Omega\times(0,T),\ B_R(x^0)=\{x\in\mathbb{R}^n:\,|x-x^0|<R\},\\
&\Omega_R(x^0)=\Omega\cap B_R(x^0),\quad \Lambda_R(t^0)=(t^0-R^2,t^0+R^2),\\
&\mathbb{P}_R(z^0)=B_R(x^0)\times\Lambda_R(t^0),\quad Q_R(z^0)=\mathbb{P}_R(z^0)\cap Q,\\
&\Gamma_R(z^0)=\mathbb{P}_R(z^0)\cap\Gamma,\quad \Omega^{(0)}(x^0)=\mathbb{P}_R(z^0)\cap\{t=0\},
\end{aligned}$$

$\partial' Q_R(z^0)$ is the parabolic boundary of $Q_R(z^0)$, $|D| = meas_{n+1}D$ for a Lebesgue measurable set in $\mathbb{R}^{n+1}$, $d_0 = diam\,\Omega$,

$$g_{r,z^0} = -\!\!\!\!\!\!\int_{Q_r(z^0)} g\,dz = \frac{1}{|Q_r|}\int_{Q_r(z^0)} g\,dz,\quad |Q_r| = 2\omega_n r^{n+2},\ \omega_n = meas_n B_1(0),$$

$$\neq\!\!\!\!\!\!\int_{Q_r(z^0)} g\,dz = \frac{1}{r^n}\int_{Q_r(z^0)} g\,dz,\quad \neq\!\!\!\!\!\!\int_{\Omega_r(x^0)}\psi\,dx = \frac{1}{r^{n-2}}\int_{\Omega_r(x^0)}\psi\,dx,$$

$\|u\|_{m,D}$ is the norm of a function u in the space $L_m(D)$, $m\in[1,\infty]$,

$$\delta(z^1,z^2) = \max\{|x^1-x^2|,|t^1-t^2|^{\frac12}\},\quad z^i=(x^i,t^i)\in\mathbb{R}^{n+1},\ i=1,2,$$

$$< u >_Q^{(\alpha,\frac{\alpha}{2})} = \sup_{z,z'\in\overline{Q},\,z\ne z'} \frac{|u(z)-u(z')|}{\delta(z,z')^\alpha}.$$

Let the Lipschitz characteristics of $\partial\Omega$ be estimated by the constant l_Γ.

2. L_p-estimate of the gradient of a solution. Let conditions **a), c),** and **d)** of the Introduction hold. By Theorem 2.1 and Remark 2.3 [10], there exists a number $\theta_0 > 0$ depending on the data only such that condition (17) (or (18)) with $\theta \le \theta_0$ for a fixed point $z^0 \in Q \cup \Gamma$ ensures the estimate

$$\left(\fint_{Q_\varrho(\xi)} |u_x|^p \, dz\right)^{\frac{2}{p}} \le c_1 \fint_{Q_{a\varrho}(\xi)} (1+|u_x|)^2 \, dz \tag{19}$$

for all $\xi \in Q_{\frac{R}{2}}(z^0)$, $\varrho \le \frac{R}{2a}$, $a \ge 2$ is an absolute constant. Here $p = p(\nu, \mu, n) > 2$, $c_1 = c_1(\nu, \mu, n, b_0, l_\Gamma)$.

Remark 1. To derive (19), it was assumed in [10] that $u \in L_\infty(\Lambda; L_2(\Omega)) \cap L_2(\Lambda; \overset{0}{W}{}_2^1(\Omega))$, and there exists $m > 2$ such that $u_x \in L_m(Q_R(z^0))$. Moreover, it follows from (17) or (18) that the corresponding characteristics of u should be finite. We note that condition (17) (or (18)) does not guarantee the higher integrability of $|u_x|$, but only estimate (19). Certainly, $p \le m$ in (19).

Further, we say that "u is a suitable solution in $Q_R(z^0)$" provided that all mentioned in Remark 1 characteristics of u are finite in $Q_R(z^0)$.

Remark 2. In (19) and below we denote by c_i different constants that may depend on the parameters of the data. Dependence on other parameters is marked explicitly. For example, $c_i = c_i(\varepsilon), \varepsilon > 0$.

We did not discuss in [10] the situation $\mathbb{P}_R(z^0) \cap \{t = 0\} \ne \emptyset$. The corresponding analysis was done earlier in [11]. We studied in [11] bounded weak solutions u of (1), (2), $u \in V = \{v : \operatorname{ess\,sup}_\Lambda \|v(\cdot, t)\|_{2,\Omega} + \|v_x\|_{2,Q} < +\infty\}$ and assumed condition (8). The reverse Hölder inequalities for $|u_x(z)|$ with additional terms depending on the initial function ϕ were derived in [11].

Let now $\Omega_R^{(0)}(z^0) \ne \emptyset$ and condition (17) or (18) hold with $\theta \le \theta_0$. Using the same ideas as in [10] and [11], we are able to deduce in our situation the *quasireverse* Hölder inequalities for $g(z) = (1+|u_x|)^{\frac{2}{q}}$, $q = \frac{n+2}{n}$, $l > 1$:

$$\begin{gathered}\fint_{Q_\varrho(\xi)} g^q \, dz \le \varepsilon_1 \fint_{Q_{a\varrho}(\xi)} g^q \, dz + c_2(\varepsilon_1)\left(\fint_{Q_{a\varrho}(\xi)} g \, dz\right)^q \\ + c_3(l)\theta b_0 \left(\fint_{Q_{a\varrho}(\xi)} g^{ql} \, dz\right)^{\frac{1}{l}} + c_4\left(\fint_{\Omega^{(0)}_{a\varrho}(x^*)} |\phi_x|^{\frac{2}{q}} \, dx\right)^q, \\ \xi = (x^*, t^*),\ \xi \in \overline{\Omega_{\frac{R}{2}}(x^0)} \times \left[0, t^0 + \left(\frac{R}{2}\right)^2\right],\ \varrho \le \frac{R}{2a},\ a = const \ge 2.\end{gathered} \tag{20}$$

We follow [11], [14], and [10] to assert that the inequality

$$\left(\fint_{Q_\varrho(\xi)} |u_x|^p \, dz\right)^{\frac{2}{p}} \le c_5 \fint_{Q_{a\varrho}(\xi)} ((1+|u_x|)^2 \, dzg + c_6\left(\fint_{\Omega^{(0)}_{a\varrho}(x^*)} |\phi_x|^{\frac{2}{q}+(p-2)} \, dx\right)^{\frac{2}{\frac{2}{q}+(p-2)}}$$

holds with some $p = p(\nu, \mu, n) > 2$ and the same ξ and ϱ as in (20) provided that θ and l are fixed appropriately. We may suppose that $p < \frac{2(n+4)}{n+2}$, apply the Hölder inequality to the last term in the last inequality, and derive the following estimate:

$$\left(\fint_{Q_\varrho(\xi)} |u_x|^p\,dz\right)^{\frac{2}{p}} \le c_5 \fint_{Q_{a\varrho}(\xi)} (1+|u_x|)^2\,dz + c_6 \fint_{\Omega^{(0)}_{a\varrho}(x^*)} |\phi_x|^2\,dx,\ \varrho \le \frac{R}{2a}. \tag{21}$$

In (21) $\xi = (x^*, t^*) \in \overline{\Omega_{\frac{R}{2}}(x^0)} \times [0, t^0 + (\frac{R}{2})^2]$ for the case $t^0 \le R^2$, and $\xi \in \overline{Q_{\frac{R}{2}}(z^0)}$ in the other case. We recall that (21) is derived under assumption (17) (or (18)) with $\theta \le \theta_0$, where θ_0 is fixed by the data.

3. Energy estimates and the main statement. First, we remark that the global energy estimate for a *bounded* solution $u \in L_\infty(\Lambda; L_2(\Omega)) \cap L_2(\Lambda; \overset{0}{W}{}^1_2(\Omega))$ can be deduced provided that condition (8) holds. Instead of (8), we can assume "one-side condition" mentioned earlier, and derive both global and local variant of the energy estimate:

$$\sup_{\Lambda_R(t^0)} \|u(\cdot,t)\|^2_{2,\Omega(x^0)} + (\nu - \nu_*)\|u_x\|^2_{2,Q_R(z^0)}$$
$$\le \mu|Q_{2R}| + \frac{c}{R^2}\int_{Q_{2R}(z^0)} |u|^2\,dz + \|\phi\|^2_{2,\Omega^{(0)}_{2R}(x^0)}, \quad z^0 \in \overline{Q}.$$

Nevertheless, we are forced to assume validity of the stronger energy estimate for our future considerations. In [6], a local estimate of the Hölder norm of solution u was derived under condition (8) and assumption that $u_t \in L_2(Q)$. In general, the last condition is very strong but for parabolic systems with elliptic operators of the variational structure, the following estimate holds:

$$\|u_t\|^2_{2,Q} + \sup_{\Lambda} \|u_x(\cdot,t)\|^2_{2,\Omega} \le c_7(1 + \|\phi_x\|^2_{2,\Omega}). \tag{22}$$

More exactly, let L be the Euler operator for the quadratic functional

$$E[u] = \int_\Omega \left[\frac{1}{2}A^{\alpha\beta}_{kl}(x,u)u^l_{x_\beta}u^k_{x_\alpha} + f^k(x)u^k\right]dx. \tag{23}$$

Then

$$b^k(x,u,u_x) = \frac{1}{2}(A^{\alpha\beta}_{ml}(x,u))'_{u^k}u^l_{x_\beta}u^m_{x_\alpha} + f^k(x)$$

in system (1), and in (6)

$$b_0 = \frac{1}{2}\sup_{x\in\Omega,\,u\in\mathbb{R}^N} \sum_{\alpha,\beta\le n;\,m,k,l\le N} |(A^{\alpha\beta}_{ml}(x,u))'_{u^k}| < \infty,$$
$$\mu = \|f\|_{\infty,\Omega} \quad \text{for } f \in L_\infty(\Omega).$$

For the system $u_t + Lu = 0$ with the described operator L (under conditions (2)), inequality (22) can be obtained immediately. Moreover, the local energy estimate

$$\sup_{\Lambda_r(\hat t)}\left(\int_{\Omega_r(\hat x)} |u_x(x,t)|^2\,dx\right) + \int_{Q_r(\hat z)} |u_t|^2\,dz$$
$$\le c_8\left\{\frac{1}{r^2}\int_{Q_{2r}(\hat z)} (1+|u_x|)^2\,dz + \int_{\Omega^{(0)}_{2r}(\hat x)} |\phi_x|^2\,dx\right\}, \quad \hat z = (\hat x, \hat t) \in \overline{Q}, \quad r \le d_0, \tag{24}$$

also holds.

In particular, it follows from (24) that

$$\fint_{Q_r(\hat z)} |u-u_{r,\hat z}|^2\,dz \le c_9\Big(\fint_{Q_{2r}(\hat z)} (1+|u_x|)^2\,dz + \fint_{\Omega^{(0)}_{2r}(\hat x)} |\phi_x|^2\,dx\Big). \tag{25}$$

Further we assume that inequality (24) holds for the solution under investigation.

Now we are ready to formulate the main result of the paper.

Theorem. *Let conditions* **a)–d)** *hold, let u be a suitable solution of (1), (2) in $Q_{R_0}(z^0)\subset Q$ (see Remark 1), and satisfy inequality (24) in all cylinders of $Q_{R_0}(z^0)$. Let $\gamma_{R_0}(x^0)=B_{R_0}(x^0)\cap\partial\Omega\in\mathcal{C}^1$ and $\phi_x\in L^{2,n-2+2\alpha}(\Omega^{(0)}_{R_0}(x^0))$ for a fixed $\alpha\in(0,1)$. There exist positive numbers θ and $R\le R_0$ such that the assumption*

$$\|u_x\|_{L^{2,n}(Q_R(z^0);\delta)}<\theta \tag{26}$$

guarantees the estimate

$$\langle u\rangle^{(\beta,\frac{\beta}{2})}_{Q_{\frac{R}{2}}(z^0)} \le c_{10}(1+\|u_x\|_{2,Q_R(z^0)}). \tag{27}$$

Parameters θ and R in (26) depend only on the data of the problem. The exponent β is an arbitrary number in $(0,1)$ provided that $\Omega^{(0)}_R(x^0)=\emptyset$, and $\beta\le\alpha$ in the case $\Omega^{(0)}_R(x^0)\ne\emptyset$. The constant c_{10} depends on $R^{-1},\nu,\mu,b_0,\theta,\beta,n$. Moreover, c_{10} may also depend on $\|\phi_x\|_{L^{2,n-2+2\alpha}(\Omega^{(0)}_R(x^0))}$ and $\mathcal{C}^1$-characteristics of $\gamma_R(x^0)$.

4. Model setting of the problem. The most interesting cases as regards the location of $z^0\in Q\cup\partial' Q$ are the following: $\Gamma_R(z^0)=\mathbb{P}_R(z^0)\cap\Gamma\ne\emptyset$, and $\Omega^{(0)}_R(x^0)=\mathbb{P}_R(z^0)\cap\{t=0\}\ne\emptyset$ (z^0 is close to the parabolic boundary of Q). We consider this case below.

Let y^0 be the nearest point to x^0 at $\partial\Omega$. We introduce $\mathcal{C}^1$-diffeomorphism $y=y(x)$ of some neighborhood $V(y^0)$ so that $V(y^0)\cap\partial\Omega\subset\gamma_{R_0}(x^0)$, $x^0\in V(y^0)\cap\overline{\Omega}$ and $y(V(y^0)\cap\Omega)=B_1^+(0)$, $y(V(y^0)\cap\partial\Omega)=\gamma_1(0)$. Here and below $\gamma_r(0)=B_r(0)\cap\{y_n=0\}$. The function $\tilde u(y,t)=u(x(y),t)$ is a solution of the problem

$$\begin{gathered}\tilde u^k_t-(\mathcal{A}^{\alpha\beta}_{kl}(\xi,\tilde u)\tilde u^l_{y_\beta})_{y_\alpha}+D^k(\xi,\tilde u,\tilde u_y)=0,\\ \tilde u|_{\Gamma_1}=0,\quad \Gamma_1=\gamma_1(0)\times(0,T),\quad \xi=(y,t)\in B_1^+(0)\times(0,T),\\ \tilde u|_{t=0}=\phi(x(y)),\end{gathered} \tag{28}$$

where the functions $\mathcal{A}^{\alpha\beta}_{kl}$ and D^k satisfy conditions of the form **a)**, **b)**, and **c)** for $y\in B_1^+(0)$ but with the other parameters depending on the $\mathcal{C}^1$-norm of the diffeomorphism $y(x)$; $\psi(y)=\phi(x(y))\in L_{2,n-2+2\alpha}(B_1^+)$. The solution $\tilde u$ of (28) satisfies the inequality (24) with the other constants.

Later on, we use the initial notation of the variables and functions, and prove our Theorem in the following local setting:

$$\begin{gathered}u^k_t-(A^{\alpha\beta}_{kl}(z,u)u^l_{x_\beta})_{x_\alpha}+b^k(z,u,u_x)=0,\quad z\in Q^+=B_1^+(0)\times(0,T),\\ u|_{\Gamma_1}=0,\qquad u|_{B_1^+(0)\times\{0\}}=\phi(x),\end{gathered} \tag{29}$$

and condition (26) holds in a cylinder $Q_R(z^0)\subset Q^+$. We suppose that $\Gamma_R(z^0)=\mathbb{P}_R(z^0)\cap\Gamma_1\ne\emptyset$ and $\mathbb{P}_R(z^0)\cap\{t=0\}\ne\emptyset$.

There is no loss of generality in assuming that

$$R^\alpha \leq \theta. \tag{30}$$

Here θ is the parameter in condition (26), and $\theta \leq \theta_0$, where θ_0 was fixed by the data to guarantee the validity of (21). We will sharp the choice of R and θ later.

5. Proof of the main Theorem. To prove Theorem in the local setting (29), we fix a point $\hat{z} \in \tilde{Q}_{\frac{3R}{4}}(z^0) = \overline{\Omega_{\frac{3R}{4}}(x^0)} \times [0, t^0 + (\frac{3R}{4})^2] \supseteq Q_{\frac{3R}{4}}(z^0)$ and $r \leq \frac{R}{4a}$ (the constant $a \geq 2$ is fixed by (21)). We denote

$$Q_r(\hat{z}) = Q_R(z^0) \cap \mathbb{P}_r(\hat{z}), \quad u_{r,\hat{z}} = \fint_{Q_r(\hat{z})} u(z)\, dz, \quad \hat{A}^{\alpha\beta}_{kl} = A^{\alpha\beta}_{kl}(\hat{z}, u_{r,\hat{z}})$$

and consider the following problem:

$$\begin{aligned} v^k_t - \hat{A}^{\alpha\beta}_{kl} v^l_{x_\beta x_\alpha} = 0, \quad z \in Q_r(\hat{z}), \\ v|_{\partial' Q_r(\hat{z})} = u(z). \end{aligned} \tag{31}$$

The problem has a unique solution, it is smooth up to $\Gamma'_r(\hat{z}) \cup \Omega^{(0)}_r(\hat{x})$, where $\Gamma'_r(\hat{z}) = \partial' Q_r(\hat{z}) \cap \Gamma_R(z^0)$ and $\Omega^{(0)}_r(\hat{x}) = \partial' Q_r(\hat{z}) \cap \{t = 0\}$. (The sets $\Gamma'_r(\hat{z})$ or $\Omega^{(0)}_r(\hat{x})$ may be empty.)

First, we consider the case $\Omega^{(0)}_r(\hat{z}) \neq \emptyset$. In this case, we introduce the function $\tilde{v}(z) = v(z) - \phi(x)$, it solves the problem

$$\tilde{v}^k_t - \hat{A}^{\alpha\beta}_{kl} \tilde{v}^l_{x_\beta x_\alpha} = (\hat{A}^{\alpha\beta}_{kl} \phi^l_{x_\beta}(x))_{x_\alpha}, \quad z \in Q_r(\hat{z}), \tilde{v}|_{\Gamma'_r(\hat{z}) \cup \Omega^{(0)}_r(\hat{x})} = 0. \tag{32}$$

The following Campanato estimates are valid for $\tilde{v}$ [16]:

$$\int_{Q_\varrho(\xi)} |\tilde{v} - \tilde{v}_{\varrho,\xi}|^2\, dz \leq c_{11} \left[\left(\frac{\varrho}{\hat{r}} \right)^{n+4} \int_{Q_{\hat{r}}(\xi)} |\tilde{v} - \tilde{v}_{\hat{r},\xi}|^2\, dz + c_\phi \hat{r}^{n+2+2\alpha} \right], \tag{33}$$

and

$$\int_{Q_\varrho(\xi)} |\tilde{v}_x|^2\, dz \leq c_{12} \left[\left(\frac{\varrho}{\hat{r}} \right)^{n+2} \int_{Q_{\hat{r}}(\xi)} |\tilde{v}_x|^2\, dz + c_\phi \hat{r}^{n+2\alpha} \right], \quad \varrho \leq \hat{r} \leq \frac{r}{2}, \tag{34}$$

where $\xi \in \overline{\Omega_{\frac{r}{2}}(\hat{x})} \times [0, \hat{t} + (\frac{r}{2})^2]$ and $c_\phi = \|\phi_x\|_{L^{2,n-2+2\alpha}(\Omega^{(0)}_R(x^0))}$.

It is easy to see that estimates (33) and (34) imply the corresponding inequalities for v. Putting

$$\Phi(\varrho, \xi) = \int_{Q_\varrho(\xi)} |v - v_{\varrho,\xi}|^2\, dz,$$

we rewrite estimate (33) in the form

$$\Phi(\varrho, \xi) \leq c_{13} \left[\left(\frac{\varrho}{\hat{r}} \right)^{n+4} \Phi(\hat{r}, \xi) + c_\phi r^{2\alpha} \hat{r}^{n+2} \right], \quad \varrho \leq \hat{r} \leq \frac{r}{2}. \tag{35}$$

Due to the well-known Lemma by S. Campanato [19], the inequality

$$\Phi(\varrho, \xi) \leq c_{14} \varrho^{n+2} \left(\frac{\Phi(\hat{r}, \xi)}{\hat{r}^{n+2}} + c_\phi r^{2\alpha} \right), \tag{36}$$

$\varrho \leq \hat{r} \leq \frac{r}{2}$, $c_{14} = c_{14}(c_{13}, n)$, follows from (35).

Now we put $\hat{r} = \frac{r}{2}$ in (36) and obtain that

$$\frac{\Phi(\varrho,\xi)}{\varrho^{n+2}} \le c_{14}\Big\{\frac{\Phi(\frac{r}{2},\xi)}{(\frac{r}{2})^{n+2}} + c_\phi r^{2\alpha}\Big\} \le c_{15}\Big\{\frac{\Phi(r,\hat{z})}{r^{n+2}} + c_\phi r^{2\alpha}\Big\}. \tag{37}$$

We take the supremum over all admissible ξ and $\varrho \le \frac{r}{2}$ to conclude that

$$[v]^2_{\mathcal{L}^{2,n+2}(\tilde{Q}_{\frac{r}{2}}(\hat{z});\delta)} \le c_{15}\Big\{\fint_{Q_r(\hat{z})} |v - v_{r,\hat{z}}|^2\,dz + c_\phi r^{2\alpha}\Big\} \tag{38}$$

in the case $\Omega_r^{(0)}(\hat{x}) \ne \emptyset$.

We repeat the above considerations to deduce for v_x from (34) the estimate

$$\|v_x\|^2_{L^{2,n}(\tilde{Q}_{\frac{r}{2}}(\hat{z}))} \le c_{16}\Big\{\fint_{Q_r(\hat{z})} |v_x|^2\,dz + c_\phi r^{2\alpha}\Big\}, \tag{39}$$

when $\Omega_r^{(0)}(\hat{x}) \ne \emptyset$.

If $\Omega_r^{(0)}(\hat{x}) = \emptyset$, we do not transform problem (31) to (32), and obtain estimates of $[v]^2_{\mathcal{L}^{2,n+2}}(Q_{\frac{r}{2}}(\hat{z};\delta))$ and $\|v_x\|^2_{L^{2,n}(Q_{\frac{r}{2}}(\hat{z};\delta))}$ similar to (38) and (39) where the terms with c_ϕ are absent.

Further, we consider function $w = u - v$, $w|_{\partial' Q_r(\hat{z})} = 0$. It satisfies the identity

$$\int_{Q_r(\hat{z})} (w_t^k h^k + \hat{A}^{\alpha\beta}_{kl} w^l_{x_\beta} h^k_{x_\alpha})\,dz = \int_{Q_r(\hat{z})} (F^k_\alpha(z) h^k_{x_\alpha} + f^k(z) h^k)\,dz,$$
$$h \in L_2(\Lambda_r(\hat{t}); \overset{0}{W}{}^1_2(\Omega_r(\hat{x}))). \tag{40}$$

Here $F^k_\alpha(z) = \hat{A}^{\alpha\beta}_{kl} u^l_{x_\beta}(z)$, $f^k(z) = u^k_t(z)$. From (40) with $h = w$, it follows that

$$\sup_{\Lambda_r(\hat{t})} \|w(\cdot,t)\|^2_{2,\Omega(\hat{x})} + \|w_x\|^2_{2,Q_r(\hat{z})} \le c_{17}(\|u_x\|^2_{2,Q_r(\hat{z})} + r^2\|u_t\|^2_{2,Q_r(\hat{z})}). \tag{41}$$

From (41) and (24), the estimate

$$\sup_{\Lambda_r(\hat{t})} \|w(\cdot,t)\|^2_{2,\Omega_r(\hat{x})} + \|w_x\|^2_{2,Q_r(\hat{z})} \le c_{18}(\|(1+|u_x|)\|^2_{2,Q_{2r}(\hat{z})} + r^2\|\phi_x\|^2_{2,\Omega_{2r}(\hat{x})}) \tag{42}$$

follows.

As a consequence of (38), (25), (42), and (26), we obtain the estimate

$$[w]^2_{\mathcal{L}^{2,n+2}(\tilde{Q}_{\frac{r}{2}}(\hat{z}))} \le c_{19}(\theta^2 + r^{2\alpha}) \le_{(30)} 2c_{19}\theta^2, \tag{43}$$

for the case $\Omega^{(0)}(\hat{x}) \ne \emptyset$. (The estimate of $[w]^2_{\mathcal{L}^{2,n+2}(Q_{\frac{r}{2}}(\hat{z}))}$ we derive in the other case.)

The inequality

$$\|w_x\|^2_{L^{2,n}(\tilde{Q}_{\frac{r}{2}}(\hat{z}))} \le c_{20}\theta^2 \tag{44}$$

follows from (39), (42), and (26), provided that $\Omega_R^{(0)}(\hat{x}) \ne \emptyset$. If $\Omega_r^{(0)}(\hat{x}) = \emptyset$, then the same estimate can be derived for $\|w_x\|^2_{L^{2,n}(Q_{\frac{r}{2}}(\hat{z}))}$.

Besides (42), we need a global L_m-estimate of $|w_x|$ in $Q_r(\hat{z})$ for an exponent $m > 2$. Using identity (40) and the condition $w|_{\partial' Q_r(\hat{z})} = 0$, it is easy to deduce the reverse

Hölder inequalities in all cylinders $Q_\varrho \subset Q_r(\hat z)$ (we admit $\partial Q_\varrho \cap \partial' Q_r(\hat z) \neq \emptyset$). By the parabolic version of the Gehring Lemma, there exists a number $m = m(\nu, \mu, n) > 2$ such that

$$\Big(\fint_{Q_r(\hat z)} |w_x|^m\,dz\Big)^{\frac{2}{m}} \le c_{21}\Big\{\fint_{Q_r(\hat z)} |w_x|^2\,dz + \Big(\fint_{Q_r(\hat z)} |F|^m\,dz\Big)^{\frac{2}{m}} + r^2\Big(\fint_{Q_r(\hat z)} |f|^{\frac{lm}{2}}\,dz\Big)^{\frac{4}{lm}}\Big\}, \quad l = \frac{2(n+2)}{n+4}. \tag{45}$$

We may consider (45) with $m \le \min\{p, \frac{2(n+4)}{n+2}\}$, where $p > 2$ is the exponent from (21).

Such an estimate for $|w_x|$ but with the last term in the form

$$\Big(\fint_{Q_r(\hat z)} |f|^{l+m-2}\,dz\Big)^{\frac{2}{l+m-2}}$$

was obtained in [12]. Here we need the inequality (45). One can deduce it, following the idea of proving Theorem 2.2 of Chapter 4 [19] for the standard euclidian metric. Now we estimate all terms in the right-hand side of (45) as follows:

$$\fint_{Q_r(\hat z)} |w_x|^2\,dz \le_{(42)} c_{22}\Big(\fint_{Q_r(\hat z)} (1+|u_x|)^2\,dz + \fint_{\Omega_r^{(0)}(\hat z)} |\phi_x|^2\,dx\Big),$$

$$\Big(\fint_{Q_r(\hat z)} |F|^m\,dz\Big)^{\frac{2}{m}} \le c_{23}\Big(\fint_{Q_r(\hat z)} |u_x|^m\,dz\Big)^{\frac{2}{m}} dz$$

$$\le_{(21)} c_{24}\fint_{Q_{ar}(\hat z)} (1+|u_x|)^2\,dz + \fint_{\Omega_{ar}^{(0)}(\hat x)} |\phi_x|^2\,dx,$$

$$r^2\Big(\fint_{Q_r(\hat z)} |f|^{\frac{lm}{2}}\,dz\Big)^{\frac{4}{lm}} \le r^2\fint_{Q_r(\hat z)} |u_t|^2\,dz.$$

Now from (45) and (24), it follows that

$$\Big(\fint_{Q_r(\hat z)} |w_x|^m\,dz\Big)^{\frac{2}{m}} \le c_{25}\Big[\fint_{Q_{ar}(\hat z)} (1+|u_x|)^2\,dz + \fint_{\Omega_{ar}^{(0)}(\hat x)} |\phi_x|^2\,dx\Big]. \tag{46}$$

Moreover, the function w satisfies the identity

$$\int_{Q_r(\hat z)} [w_t^k\eta^k + \hat A_{kl}^{\alpha\beta} w_{x_\beta}^l \eta_{x_\alpha}^k + b^k(z,u,u_x)\eta^k + \Delta A_{kl}^{\alpha\beta} u_{x_\beta}^l \eta_{x_\alpha}^k]\,dz = 0,$$

$$\eta|_{\Gamma_r(\hat z)} = 0, \tag{47}$$

$\Delta A_{kl}^{\alpha\beta} = A_{kl}^{\alpha\beta}(z,u) - A_{kl}^{\alpha\beta}(\hat z, u_{r,\hat z})$.

Further, we consider in detail the case $\Omega_r^{(0)}(\hat x) \neq \emptyset$ and put in (47) $\eta(z) = w(z)((2T)^s - |w(z)|^s)_+$, the parameters $T > 1$ and $s \in (0,1)$ will be chosen later. The function η is bounded in $Q_r(\hat z) = \Omega_r \times (0, \hat t + r^2)$, ($|\eta| \le (2T)^{s+1}$), and $\eta|_{\partial' Q_r(\hat z)} = 0$.

We obtain the inequality

$$\int_{Q_r} [w_t w((2T)^s - |w|^s)_+ + \hat{A}^{\alpha\beta}_{kl} w^l_{x_\beta} w^k_{x_\alpha} ((2T)^s - |w|^s)_+ \\ + \hat{A}^{\alpha\beta}_{kl} w^l_{x_\beta} w^k (-s|w|^{s-2}(w \cdot w_{x_\alpha})\chi_+)]\, dz \\ \le \int_{Q_r} |\Delta A||u_x|(|w_x|((2T)^s - |w|^s)_+ + s|w|^s|w_x|\chi_+)\, dz \\ + b_0 \int_{Q_r} |u_x|^2|w|((2T)^s - |w|^s)_+\, dz + \mu \int_{Q_r} |w|((2T)^s - |w|^s)_+\, dz, \quad Q_r = Q_r(\hat{z}),$$

where $\chi_+(z)$ is the characteristic function of the set $\{z \in Q_r(\hat{z}) : |w| < 2T\}$.

From this inequality it follows that

$$\int_{Q_r} \left(\frac{|w|^2}{2}\right)'_t ((2T)^s - |w|^s)_+\, dz + \frac{\nu}{2} \int_{Q_r} |w_x|^2((2T)^s - |w|^s)_+\, dz \\ \le c_{26} T^s \Big\{ s \int_{Q_r} |w_x|^2\, dz + \int_{Q_r} \omega^2(r^2; |u - u_{r,\hat{z}}|^2)|u_x|^2\, dz \\ + b_0 \int_{Q_r} |u_x|^2 |w| \chi_+\, dz + \int_{Q_r} |w|\chi_+\, dz \Big\}. \tag{48}$$

The first integral in the left-hand side of (48) is nonnegative. Indeed,

$$J = \int_{Q_r} \left(\frac{|w|^2}{2}\right)'_t ((2T)^s - |w|^s)_+\, dz = \int_{\Omega_r(\hat{x})} \left(\int_0^{|w(x,t)|} \xi((2T)^s - \xi^s)_+\, d\xi \right) dx \Big|_{t=0}^{t=\hat{t}+r^2} \\ = \int_{\Omega_r(\hat{x})} \left(\int_0^{|w(x,\hat{t}+r^2)|} \xi((2T)^s - \xi^s)_+\, d\xi \right) dx \ge 0. \tag{49}$$

Now we'll estimate from below the second integral in the left-hand side of (48). First, we fix the number

$$k_* = \frac{T}{2^{\frac{1}{s}}}, \tag{50}$$

and put

$$q = \frac{1}{2\sqrt{n}}, \quad \tilde{Q}_\varrho(\hat{z}) = \Omega_\varrho(\hat{x}) \times (0, \hat{t} + \varrho^2) \supseteq Q_\varrho(\hat{z}) = \mathbb{P}_\varrho(\hat{z}) \cap Q^+, \\ \tilde{Q}^+_{qr}(\hat{z}) = \{z \in \tilde{Q}_{qr}(\hat{z}) : |w(z) - w_{qr,\hat{z}}| > k_*\} \\ \tilde{Q}^-_{qr}(\hat{z}) = \{z \in \tilde{Q}_{qr}(\hat{z}) : |w(z) - w_{qr,\hat{z}}| \le k_*\}.$$

Note that

$$|w_{qr,\hat{z}}| \le \frac{|Q_r|}{|Q_{qr}|} \fint_{Q_r(\hat{z})} |w|\, dz = c_{27} \fint_{Q_r(\hat{z})} |w|\, dz \le c_{28} \left(\fint_{Q_r(\hat{z})} |w_x|^2\, dz \right)^{\frac{1}{2}} \\ \le_{(42)} c_{29} \left(\fint_{Q_{2r}(\hat{z})} (1 + |u_x|)^2\, dz + \fint_{\Omega^{(0)}_{2r}(\hat{x})} |\phi_x|^2\, dx \right)^{\frac{1}{2}} \le_{(26)} c_{30}(\theta + r^\alpha) \le_{(30)} c_{31}\theta.$$

We assume that the parameters $\theta \le 1$, and get the inequality

$$|w_{qr,\hat{z}}| \le c_{31}. \tag{51}$$

Let the numbers T and s be fixed to satisfy the condition

$$k_* = \frac{T}{2^{\frac{1}{s}}} > c_{31}. \tag{52}$$

It yields the estimate

$$|w(z)| \le |w(z) - w_{qr,\hat z}| + |w_{qr,\hat z}| \le 2k_*, \quad z \in \tilde Q^-_{qr}(\hat z).$$

Consequently,

$$(2T)^s - |w(z)|^s \ge (2T)^s - (2k_*)^s = \frac{(2T)^s}{2} \tag{53}$$

on the set $\tilde Q^-_{qr}(\hat z)$.

From the above it follows that

$$\mathcal{L} = \int_{Q_r} |w_x|^2((2T)^s - |w|^s)_+ \, dz \ge \int_{\tilde Q^-_{qr}(\hat z)} |w_x|^2((2T)^s - |w|^s)_+ \, dz$$
$$\ge \frac{(2T)^s}{2} \int_{\tilde Q^-_{qr}(\hat z)} |w_x|^2 \, dz. \tag{54}$$

Taking into account the estimates for J and $\mathcal{L}$ and dividing inequality (48) by $(2T)^s$, we derive the relation

$$\int_{\tilde Q^-_{qr}(\hat z)} |w_x|^2 \, dz \le c_{32} \Big\{ s \int_{Q_r(\hat z)} |w_x|^2 \, dz + \int_{Q_r(\hat z)} \omega^2 |u_x|^2 \, dz$$
$$+ \int_{Q_r(\hat z)} |u_x|^2 |w| \chi_+ \, dz + \int_{Q_r(\hat z)} |w| \, dz \Big\}. \tag{55}$$

Now we explain the estimating of every integral in the right-hand side of (55).

The first term will be estimated by (42). The second integral is estimated in the standard way with the help of (21):

$$\int_{Q_r(\hat z)} \omega^2(r^2; |u - u_{r,\hat z}|^2)|u_x|^2 \, dz \le \Big(\fint_{Q_r(\hat z)} |u_x|^p \, dz \Big)^{\frac{2}{p}} \Big(\fint_{Q_r(\hat z)} \omega^{\frac{2p}{p-2}} \, dz \Big)^{\frac{p-2}{p}} |Q_r|$$
$$\le_{(21)} c_{33} \Big(\int_{Q_{ar}(\hat z)} (1 + |u_x|)^2 \, dz + r^2 \int_{\Omega^{(0)}_{ar}(\hat x)} |\phi_x|^2 \, dx \Big) \omega^{\frac{p-2}{p}} \Big(r^2; \fint_{Q_r} |u - u_{r,\hat z}|^2 \, dz \Big)$$
$$\le_{(25)} c_{33} \Big(\int_{Q_{ar}(\hat z)} (1 + |u_x|)^2 \, dz + r^2 \int_{\Omega^{(0)}_{ar}(\hat x)} |\phi_x|^2 \, dx \Big)$$
$$\times \omega^{\frac{p-2}{p}} \Big(r^2; c_9 \Big(\fint_{Q_{2r}} (1 + |u_x|)^2 \, dz + \fint_{\Omega^{(0)}_{2r}(\hat x)} |\phi_x|^2 \, dx \Big) \Big). \tag{56}$$

Further,

$$\int_{Q_r(\hat z)} |u_x|^2 |w| \chi_+ \, dz \le \Big(\fint_{Q_r(\hat z)} |u_x|^p \, dz \Big)^{\frac{2}{p}} \Big(\fint_{Q_r(\hat z)} |w|^{\frac{p}{p-2}} \chi_+ \, dz \Big)^{1 - \frac{2}{p}} |Q_r|$$
$$\le_{(21)} c_{34} \Big(\int_{Q_{ar}(\hat z)} (1 + |u_x|)^2 \, dz + r^2 \int_{\Omega^{(0)}_{ar}(\hat x)} |\phi_x|^2 \, dx \Big) T^{\frac{4-p}{p}} \Big(\fint_{Q_r(\hat z)} |w|^2 \, dz \Big)^{\frac{p-2}{p}}$$
$$\le_{(42),(26)} c_{35} T^{\frac{4-p}{p}} (\theta^2 + r^2)^{\frac{p-2}{p}} \Big(\int_{Q_{ar}(\hat z)} (1 + |u_x|)^2 \, dz + c_\phi r^{n+2\alpha} \Big). \tag{57}$$

At last,

$$\int_{Q_r(\hat z)} |w|\,dz \le c(n)(r^2 \int_{Q_r(\hat z)} |w_x|^2\,dz + r^{n+2})$$
$$\le_{(42)} c_{36} r^2 \int_{Q_{2r}(\hat z)} (1+|u_x|)^2\,dz + c_{37} r^{n+2}. \tag{58}$$

Using (56)–(58), we derive from (55) the inequality

$$\int_{\tilde Q_{qr}(\hat z)} |w_x|^2\,dz \le \int_{\tilde Q^+_{qr}(\hat z)} |w_x|^2\,dz + c_{38}\{s + \omega^{\frac{p-2}{p}}(r^2; c_{39}\theta^2)$$
$$+ T^{\frac{4-p}{p}} \theta^{\frac{2(p-2)}{p}} + r^2\}\Big(\int_{Q_{ar}(\hat z)} (1+|u_x|)^2\,dz + c_\phi r^{n+2\alpha}\Big). \tag{59}$$

Now we estimate the integral $\mathcal{M} = \int_{\tilde Q^+_{qr}(\hat z)} |w_x|^2\,dz$ in (59):

$$\mathcal{M} \le \Big(⨍_{\tilde Q_{qr}(\hat z)} |w_x|^m\,dz\Big)^{\frac{2}{m}} |\tilde Q^+_{qr}|^{1-\frac{2}{m}} |\tilde Q_{qr}|^{\frac{2}{m}}$$
$$\le_{(46),(42)} c_{39}\Big(\int_{Q_{ar}(\hat z)} (1+|u_x|)^2\,dz + c_\phi r^{n+2\alpha}\Big)\Big(\frac{|\tilde Q^+_{qr}|}{|\tilde Q_{qr}|}\Big)^{1-\frac{2}{m}}. \tag{60}$$

The next step is to explain that the ratio $\frac{|\tilde Q^+_{qr}|}{|\tilde Q_{qr}|}$ is a decreasing function with respect to the parameter k_* (see (50)).

We consider the parabolic cube $D_r(\hat z) = \{(x,t) : |x_i - \hat x_i| < r,\ i \le n,\ |t-\hat t| < r^2\}$ and put

$$w_0(z) = \begin{cases} w(z), & z \in Q_r(\hat z); \\ 0, & z \in D_r(\hat z) \setminus Q_r(\hat z). \end{cases}$$

We assert that

$$\tilde Q^+_{qr}(\hat z) \subset \{z \in D_{qr}(\hat z) : |w^0(z) - w^0_{qr,\hat z}| > k_*/2\} \equiv D^{(k_*)}_{qr}(\hat z),$$
$$w^0_{qr,\hat z} = \frac{1}{|D_{qr}|} \int_{D_{qr}(\hat z)} w^0\,dz. \tag{61}$$

Indeed, let $z \in \tilde Q_{qr}(\hat z)$ and $|w(z) - w_{qr,\hat z}| > k_*$ then

$$|w^0(z) - w^0_{qr,\hat z}| = |w(z) - w^0_{qr,\hat z}| \ge \big||w(z) - w_{qr,\hat z}| - |w_{qr,\hat z} - w^0_{qr,\hat z}|\big| \equiv l.$$

Looking at the derivation of (50), one can see that

$$|w^0_{qr,\hat z}| \le \frac{|Q_r|}{|D_{qr}|} ⨍_{Q_r(\hat z)} |w|\,dz \le c_{27} ⨍_{Q_r(\hat z)} |w|\,dz \le \dots \le c_{31}\theta \le_{(52)} k_*.$$

We make stronger condition (52) and assume that

$$k_* > 4c_{31}. \tag{62}$$

Then $|w_{qr,\hat z} - w^0_{qr,\hat z}| \le 2c_{31} \le \frac{k_*}{2}$ and $l > k_* - \frac{k_*}{2} = \frac{k_*}{2}$. Imbedding (61) follows.

Moreover, it is easy to check that

$$[w^0]_{\mathcal{L}^{2,n+2}(D_{qr}(\hat z);\delta)} \le c_{40}([w]_{\mathcal{L}^{2,n+2}(\tilde Q_{\frac{r}{2}}(\hat z;\delta))}$$
$$+\|w_x\|_{L^{2,n}(\tilde Q_{\frac{r}{2}};\delta)}) \le_{(43),(44)} c_{41}\theta \le c_{41}. \tag{63}$$

Due to the parabolic version of the John-Nirenberg theorem, we can assert that there exist positive numbers H and β such that

$$\frac{|\{z \in D_R(\hat z) : |g(z) - g_{R,\hat z}| > \lambda\}|}{|D_R|} \le H \exp\Big(\frac{-\beta\lambda}{[g]_{\mathcal{L}^{2,n+2}(D_R;\delta)}}\Big), \tag{64}$$

for a function $g \in \mathcal{L}^{2,n+2}(D_R;\delta)$, $\lambda > 0$. The numbers H and β in (64) depend on the dimension n only.

This fact was proved for the standard euclidian metric (see, for example, [24]) and can be generalized for the parabolic metric in the same way.

We put $g(z) = w^0(z)$, $R = qr, \lambda = \frac{k_*}{2}$, and obtain the estimate

$$\frac{|\tilde Q^+_{qr}(\hat z)|}{|\tilde Q_{qr}|} \le_{(61)} c(n)\frac{|D^{(k_*)}_{qr}(\hat z)|}{|D_{qr}|} \le_{(64)} c(n)H\exp\Big(\frac{-\beta k_*}{2[w^0]_{\mathcal{L}^{2,n+2}(D_{qr};\delta)}}\Big). \tag{65}$$

Using (63) and (65), we derive from (60) the inequality

$$\mathcal{M} \le c_{42}\exp(-c_{43}k_*)\Big(\int_{Q_{ar}(\hat z)} (1+|u_x|)^2\,dz + c_\phi r^{n+2\alpha}\Big). \tag{66}$$

Now it follows from (59) and (66) that

$$\int_{\tilde Q_{qr}(\hat z)} |w_x|^2\,dz \le c_{44}\{\exp(-c_{43}k_*) + s + \omega^{\frac{p-2}{p}}(r^2; c_{39}\theta^2)$$
$$+T^{\frac{4-p}{p}}\theta^{\frac{2(p-2)}{p}} + r^2\}\Big(\int_{Q_{ar}(\hat z)} (1+|u_x|)^2\,dz + c_\phi r^{n+2\alpha}\Big). \tag{67}$$

We put $\psi(\varrho,\hat z) = \int_{Q_\varrho(\hat z)}(1+|u_x|)^2\,dz$ and consider (34) for v (with $\xi = \hat z$, $\hat r = qr$) and (67) to obtain the estimate

$$\psi(\varrho,\hat z) \le c_{45}\Big\{\Big(\frac{\varrho}{r}\Big)^{n+2} + \Big[s + \exp\Big(-c_{43}\frac{T}{2^{1}/s}\Big) + \omega^{\frac{p-2}{p}}(r^2; c_{39}\theta^2)$$
$$+T^{\frac{4-p}{p}}\theta^{\frac{2(p-2)}{p}} + r^2\Big]\Big\}\psi(ar,\hat z) + c_{46}(1 + T^{\frac{4-p}{p}}\theta^{\frac{2(p-2)}{p}})r^{n+2\alpha},$$
$$\varrho \le qr. \tag{68}$$

Obviously, inequality (68) is valid also for $\varrho \in (qr, ar]$. Further we change the notation ar by r, and consider $r \le \frac{R}{4}$. It gives the inequality

$$\psi(\varrho,\hat z) \le c_{47}\Big\{\Big(\frac{\varrho}{r}\Big)^{n+2} + [\cdots]\Big\}\psi(r,\hat z) + c_{48}(1 + T^{\frac{4-p}{p}}\theta^{\frac{2(p-2)}{p}})r^{n+2\alpha}, \varrho \le r. \tag{69}$$

In (69) the expression $[\cdots]$ coincides with the square brackets of (68).

By the well-known Campanato Lemma (see, for example, [19], Chapter 2, Lemma 2.1) there exists a number $\delta_0 = \delta_0(c_{47}, n, \alpha) > 0$ such that if

$$[\cdots] < \delta_0 \tag{70}$$

in inequality (69), then

$$\psi(\varrho, \hat{z}) \le c_{49}\left(\frac{\varrho}{r}\right)^{n+2\alpha}\{\psi(r, \hat{z}) + (1 + T^{\frac{4-p}{p}}\theta^{\frac{2(p-2)}{p}})r^{n+2\alpha}\}, \quad \varrho \le r. \tag{71}$$

Here c_{49} depends on c_{47}, n, and α.

At last, we fix the parameters s, T, θ and R. First, let s be fixed in $(0, 1)$ to satisfy

$$s < \frac{\delta_0}{4}. \tag{72}$$

Then, we fix $T > 1$ to obtain (62) and the inequality

$$\exp\left(-c_{43}\frac{T}{2^{1/s}}\right) < \frac{\delta_0}{4}. \tag{73}$$

At last, we choose numbers θ_1 and R_1 to obtain the inequality

$$T^{\frac{4-p}{p}}\theta_1^{\frac{2(p-2)}{p}} + \omega^{\frac{p-2}{p}}(R_1; c_{39}\theta_1^2) + R_1^2 < \frac{\delta_0}{2} \tag{74}$$

Now we may fix $R \le 4R_1$ and $\theta \le \min\{\theta_0, \theta_1\}$ in assumption (26). (Parameter θ_0 was fixed by the data to ensure inequality (21).)

Conditions (72)–(74) guarantee the validity of (70). Under restrictions $r \le \frac{R}{4} \le R_1$ and $\theta \le \min\{\theta_0, \theta_1\}$, inequality (71) follows. We obtain the relation

$$\frac{1}{\varrho^{n+2\alpha}}\int_{\tilde{Q}_\varrho(\hat{z})}|u_x|^2\,dz \le c_{50}\left\{R^{-(n+2\alpha)}\int_{Q_R(z^0)}|u_x|^2\,dz + 1\right\}, \tag{75}$$

for all $\hat{z} \in \tilde{Q}_{\frac{3R}{4}}(z^0)$, $\varrho \le \frac{R}{4}$.

We have considered the case $\Omega_r^{(0)}(\hat{x}) \ne \emptyset$. Now we address the situation $\Omega_r^{(0)}(\hat{x}) = \emptyset$. It should be noted only that we do not transform problem (31) to (32) in this case, and apply estimates (33) and (34) with the function v. Moreover, now estimates (35) – (36) and others do not include the terms with c_ϕ. One can repeat all considerations and assert that estimate (75) is valid for all $\hat{z} \in Q_{\frac{3R}{4}}(z^0)$, $\varrho \le \frac{R}{4}$, provided that $\Omega^{(0)}(\hat{x}) = \emptyset$.

Inequalities (25) and (75) ensure the estimate

$$[u]^2_{\mathcal{L}^{,n+2+2\alpha}(Q_{\frac{3R}{4}}(z^0);\delta)} \le c_{51}\{R^{-(n+2\alpha)}\|u_x\|^2_{2,Q_R(z^0)} + 1\}. \tag{76}$$

Due to the isomorphism of $\mathcal{L}^{2,n+2+2\alpha}(Q;\delta)$ and the corresponding Hölder space, estimate (76) yields the estimate of the seminorm in the Hölder space

$$\langle u\rangle^{(\alpha,\alpha/2)}_{Q_{\frac{3R}{4}}(z^0)} \le c_{52}\{R^{-(n+2\alpha)}\|u_x\|^2_{2,Q_R(z^0)} + 1\}. \tag{77}$$

We have proved the Theorem for the situation $\Omega_R^{(0)}(x^0) \ne \emptyset$ and $\Gamma_R(z^0) \ne \emptyset$. In the other cases, all steps of the proof are only simplified. ■

Remark 3. It was assumed in the Theorem that a solution u of (1), (2) has the derivative $u_t \in L_2(Q_{R_0}(z^0))$. It will be desirable to remove this restrictive assumption (more exactly, do not apply inequalities (24)).

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