PARABOLIC AND NAVIER–STOKES EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 81
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

ON THE STEFAN PROBLEM WITH A SMALL PARAMETER

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Abstract. We consider the multidimensional two-phase Stefan problem with a small parameter κ in the Stefan condition, due to which the problem becomes singularly perturbed. We prove unique solvability and a coercive uniform (with respect to κ) estimate of the solution of the Stefan problem for $t \le T_0$, T_0 independent of κ, and the existence and estimate of the solution of the Florin problem (Stefan problem with $\kappa = 0$) in Hölder spaces.

1. Introduction. Statement of the problem. Classical solution of the multidimensional Stefan problem was studied by A. Friedman and D. Kinderlehrer [15], L. A. Caffarelli [11], [12], D. Kinderlehrer and L. Nirenberg [17], A. M. Meirmanov [19], E. I. Hanzawa [16], B. V. Bazaliy [1], E. V. Radkevich [20], B. V. Bazaliy and S. P. Degtyarev [2], M. A. Borodin [10], G. I. Bizhanova [5], [6], G. I. Bizhanova and V. A. Solonnikov [9].

J. F. Rodrigues, V. A. Solonnikov and F. Yi have investigated multidimensional one-phase Stefan problem with a small parameter [21]. There was obtained the existence of the solution of the corresponding Florin problem in the Hölder space $C^{2+\beta,1+\beta/2}$, $0<\beta<\alpha$ with the help of the imbedding theorem applied to the solution of Stefan problem from the space $C^{2+\alpha,1+\alpha/2}$, $\alpha \in (0,1)$.

A. Fasano, M. Primicerio and E. V. Radkevich [13] proved existence of the solution of the multidimensional one-phase Florin problem [14] in Hölder spaces. In [5], [6] G. I. Bizhanova established existence, uniqueness and estimates of the solution of the two-phase Florin problem in weighted Hölder spaces with time power weights [3] in the cases when the free boundary is the graph of a function on the plane $x_n = 0$ and on the unit sphere.

We consider the multidimensional two-phase Stefan problem in bounded domains of arbitrary configuration with a small parameter κ in the Stefan condition at the principal

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2000 *Mathematics Subject Classification*: Primary 35R35; Secondary 35B25, 35K60, 80A22.

Key words and phrases: free boundary problem, parabolic equation, small parameter, Hölder space, existence, estimate of solution.

The paper is in final form and no version of it will be published elsewhere.

term, velocity of a free boundary. Letting κ tend to zero we obtain the Florin problem [14] (degenerate Stefan problem with $\kappa = 0$). We note that the classes of the free boundaries in the Stefan and Florin problems are different, that is, the Stefan problem with a small parameter is singularly perturbed one. In Ch. 2 we prove existence, uniqueness and a uniform with respect to κ estimate of the solution of the Stefan problem for $t \le T_0$, T_0 independent of κ (Theorems 2.2′, 2.3), then we prove the existence and estimate of the solution of the two-phase Florin problem without loss of smoothness of the solution (Theorems 2.1), in Appendix A the existence of the inverse Jacobian matrix is proved, in Appendix B the linear model problem is considered with a small parameter κ corresponding to the Stefan problem.

Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded domain with a boundary Σ, κ a small parameter. Assume there is a closed surface $\gamma_\kappa(t) \subset \Omega$, $t \in [0,T]$, dividing Ω into two sub-domains $\Omega_1^{(\kappa)}(t)$ and $\Omega_2^{(\kappa)}(t)$ with the boundaries $\partial\Omega_1^{(\kappa)}(t) = \Sigma \cup \gamma_\kappa(t)$, $\partial\Omega_2^{(\kappa)}(t) = \gamma_\kappa(t)$. At the initial moment $t = 0$, $\gamma_\kappa(0) := \Gamma$ and $\Omega_j^{(\kappa)}(0) := \Omega_j$, $j = 1,2$. Let $\operatorname{dist}(\Gamma, \Sigma) \ge d_0 = \text{const} > 0$, $\operatorname{diam}\Omega_2 \ge d_0$. These conditions guarantee that the boundary $\gamma_\kappa(t)$ will not touch Σ and the domain $\Omega_2^{(\kappa)}(t)$ will not degenerate at least for small t.

Let $\Gamma \in C^{2+\alpha}$, $\alpha \in (0,1)$. Then $\gamma_\kappa(t)$ may be represented by the equation [19], [16], [9]

$$x = \xi + \rho_\kappa(\xi,t)\,N(\xi),\ \xi = \xi(x) \in \Gamma,\ t \in [0,t_0], \tag{1.1}$$

where $\rho_\kappa|_{t=0} = 0$, $N(\xi) = (N_1,\dots,N_n) \in C^{2+\alpha}(\Gamma;\mathbb{R}^n)$ is a unit vector field determined on Γ and such that $\nu_0(\xi)\,N^T(\xi) \ge d_1 = \text{const} > 0$, $\nu_0(\xi)$ is a unit normal to Γ directed into Ω_2, N^T a column-vector.

Let

$$Q_{jT}^{(\kappa)} = \{(x,t) :\, x \in \Omega_j^{(\kappa)}(t),\, t \in\, (0,T)\},\ \ \Omega_{jT} = \Omega_j \times (0,T),\ j = 1,2,$$
$$\Omega_T = \Omega \times (0,T),\ \Sigma_T = \Sigma \times [0,T],\ \Gamma_T = \Gamma \times [0,T].$$

We study two-phase Stefan problem with a small parameter κ. It is required to find the functions $u_{j\kappa}(x,t)$, $j = 1,2$, and $\rho_\kappa(\xi,t)$ satisfying the parabolic equations, initial and boundary conditions

$$\partial_t u_{j\kappa} - a_j\,\Delta u_{j\kappa} = 0 \text{ in } Q_{jT}^{(\kappa)},\ j = 1,2, \tag{1.2}$$

$$\gamma_\kappa(t)|_{t=0} = \Gamma,\ u_{j\kappa}|_{t=0} = u_{0j}(x) \quad \text{in} \ \ \Omega_j, \ \ j = 1,2, \tag{1.3}$$

$$u_{1\kappa}|_\Sigma = p(x,t),\ t \in (0,T), \tag{1.4}$$

and conditions on the free boundary $\gamma_\kappa(t)$, $t \in (0,T)$,

$$u_{1\kappa} = u_{2\kappa} = 0, \tag{1.5}$$

$$\lambda_1\,\partial_{\nu_\kappa} u_{1\kappa} - \lambda_2\,\partial_{\nu_\kappa} u_{2\kappa} = -\kappa\, V_{\nu_\kappa}, \tag{1.6}$$

where a_j, λ_j, $j = 1,2$, are positive constants, κ small parameter, $\nu_\kappa(x,t)$ is a unit normal to $\gamma_\kappa(t)$ directed into $\Omega_2^{(\kappa)}(t)$, $\partial_t = \partial/\partial t$, $\partial_{\nu_\kappa} = \partial/\partial\nu_\kappa$ the normal derivative, V_{ν_κ} velocity of the free boundary in the direction of ν_κ. Due to the equation of $\gamma_\kappa(t)$ (1.1) $V_{\nu_\kappa} = \nu_\kappa N^T \partial_t \rho_\kappa$.

We consider (1.2)–(1.6) as a problem with a small parameter κ, therefore we have ascribed an index κ to all unknowns. This problem is singularly perturbed because κ is in the principal term in a Stefan condition (1.6).

Putting $\kappa = 0$ in the problem (1.2)–(1.6) we obtain the Florin [14] or degenerate Stefan problem with unknown functions u_j, $j = 1, 2$, ρ, which satisfy parabolic equations

$$\partial_t u_j - a_j \Delta u_j = 0 \text{ in } Q_{jT}, \quad j = 1, 2, \tag{1.7}$$

with initial and boundary conditions

$$\gamma(t)|_{t=0} = \Gamma, \; u_j|_{t=0} = u_{0j}(x) \text{ in } \Omega_j, \quad j = 1, 2, \tag{1.8}$$

$$u_1|_\Sigma = p(x,t), \; t \in (0,T), \tag{1.9}$$

and conditions on a free boundary $\gamma(t)$, $t \in (0,T)$,

$$u_1 = u_2 = 0, \tag{1.10}$$

$$\lambda_1 \partial_\nu u_1 - \lambda_2 \partial_\nu u_2 = 0, \tag{1.11}$$

where as above $\gamma(t) \subset \Omega$, $t \in [0,T]$, is a closed surface, $\partial\Omega_1(t) = \Sigma \cup \gamma(t)$, $\partial\Omega_2(t) = \gamma(t)$, $\Omega_j := \Omega_j(0)$, Q_{jT}, Ω_{jT} are defined as $Q^{(\kappa)}_{jT}$, $\Omega^{(\kappa)}_{jT}$.

We study the problems in the Hölder spaces $C^{l,l/2}_{x\ \ t}(\bar\Omega_T)$, l positive non-integer, of functions $u(x,t)$ with the norm [18]

$$|u|^{(l)}_{\Omega_T} := \sum_{2k+|m|<l} |\partial^k_t \partial^m_x u|_{\Omega_T} + \sum_{2k+|m|=[l]} [\partial^k_t \partial^m_x u]^{(l-[l])}_{\Omega_T} + \sum_{2k+|m|=[l]-1} [\partial^k_t \partial^m_x u]^{(\frac{1+l-[l]}{2})}_{\Omega_T},$$

where the last term is omitted if $[l] = 0$,

$$|\,v|_{\Omega_T} = \max_{(x,t)\in\bar\Omega_T} |v|,$$

$$[v]^{(\alpha)}_{x,\Omega_T} = \max_{(x,t),\,(z,t)\in\bar\Omega_T} |v(x,t)-v(z,t)|\,|x-z|^{-\alpha},$$

$$[v]^{(\alpha)}_{t,\Omega_T} = \max_{(x,t),\,(x,t_1)\in\bar\Omega_T} |v(x,t)-v(x,t_1)|\,|t-t_1|^{-\alpha}, \; \alpha \in (0,1).$$

$\overset{\circ}{C}{}^{l,l/2}_{x\ \ t}(\bar\Omega_T)$ is the set of $u(x,t) \in C^{l,l/2}_{x\ \ t}(\bar\Omega_T)$ satisfying $\partial^k_t u|_{t=0} = 0$, $k \le [l/2]$.

To study solutions in Hölder spaces it is necessary to require compatibility conditions of initial and boundary data. The compatibility conditions of zero and first order for the problem (1.2)–(1.6) are as follows

$$u_{01}|_\Sigma = p\,(x,0), \;\; u_{01}|_\Gamma = u_{02}|_\Gamma = 0, \tag{1.12}$$

$$a_1 \Delta u_{01}|_\Sigma = \partial_t p(x,0), \;\; \frac{a_1 \Delta u_{01}|_\Gamma}{\partial_{\nu_0} u_{01}|_\Gamma} = \frac{a_2 \Delta u_{02}|_\Gamma}{\partial_{\nu_0} u_{02}|_\Gamma}, \tag{1.13}$$

$$(\lambda_1 \partial_{\nu_0} u_{01} - \lambda_2 \partial_{\nu_0} u_{02})|_\Gamma - \kappa \frac{a_j \Delta u_{0j}|_\Gamma}{\partial_{\nu_0} u_{0j}|_\Gamma} = 0, \; j = 1, 2. \tag{1.14}$$

For the Florin problem (1.7)–(1.11) the compatibility conditions have the form (1.12), (1.13) and

$$(\lambda_1\,\partial_{\nu_0}u_{01}-\lambda_2\,\partial_{\nu_0}u_{02})|_{\Gamma}\;=\;0 \tag{1.15}$$

((1.15) is the condition (1.14) with $\kappa=0$).

2. Stefan problem with a small parameter

THEOREM 2.1. *Let* $\Sigma,\ \Gamma\in C^{2+\alpha}$, $\alpha\in(0,1)$. *For any functions* $u_{0j}\in C^{2+\alpha}(\bar\Omega_j)$, $j=1,2$, $p\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Sigma_T)$ *satisfying the compatibility conditions* (1.12), (1.13), (1.15) *and the condition* $\partial_{\nu_0}u_{0j}|_\Gamma\le -d_2$ *or* $\partial_{\nu_0}u_{0j}|_\Gamma\ge d_2$, $j=1,2$, $d_2=\mathrm{const}>0$ *there exists* $T_0>0$ *such that the Florin problem* (1.7)–(1.11) *has a solution* $u_j\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\bar Q_{jT_0})$, $j=1,2$, $\rho\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Gamma_{T_0})$ *and the following estimate holds:*

$$\sum_{j=1}^{2}|u_j|_{Q_{jt}}^{(2+\alpha)}+|\rho|_{\Gamma_t}^{(2+\alpha)}\le C_1\Big(\sum_{j=1}^{2}|u_{0j}|_{\Omega_j}^{(2+\alpha)}+|p|_{\Sigma_t}^{(2+\alpha)}\Big) \tag{2.1}$$

for $0<t\le T_0$.

In [9] unique solvability of the Stefan problem (1.2)–(1.6) was proved with $\kappa=1/c_0$, c_0 arbitrary positive value, in the weighted Hölder spaces $C_s^{2+\alpha}(\Omega_T)$, $1<s\le 2+\alpha$ with time power weight [3]. From this result the following theorem follows.

THEOREM 2.2 ([9]). *Let* $\Sigma,\ \Gamma\in C^{2+\alpha}$, $\alpha\in(0,1)$. *For any functions* $u_{0j}\in C^{2+\alpha}(\bar\Omega_j)$, $j=1,2$, $p\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Sigma_T)$ *satisfying the compatibility conditions* (1.12)–(1.14) *and the conditions*

$$0<\kappa\le\kappa_0,\ \partial_{\nu_0}u_{0j}|_\Gamma\le -d_3\ \ or\ -\kappa_0\le\kappa<0,\ \partial_{\nu_0}u_{0j}|_\Gamma\ge d_3, \tag{2.2}$$

$j=1,2$, $d_3=\mathrm{const}>0$, *there exists* $T_1>0$ *such that the Stefan problem* (1.2)–(1.6) *has a unique solution* $u_{j\kappa}\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\bar Q_{jT_1}^{(\kappa)})$, $j=1,2$, $\rho_\kappa\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Gamma_{T_1})$, $\partial_t\rho_\kappa\in C_{x\ \ t}^{1+\alpha,1+\alpha/2}(\Gamma_{T_1})$ *and the following estimate holds:*

$$\sum_{j=1}^{2}|u_{j\kappa}|_{Q_{jt}^{(\kappa)}}^{(2+\alpha)}+|\rho_\kappa|_{\Gamma_t}^{(2+\alpha)}+|\partial_t\rho_\kappa|_{\Gamma_t}^{(1+\alpha)}\le C\Big(\sum_{j=1}^{2}|u_{0j}|_{\Omega_j}^{(2+\alpha)}+|p|_{\Sigma_t}^{(2+\alpha)}\Big),\ t\le T_1. \tag{2.3}$$

This theorem does not permit us to obtain the solvability of the Florin problem (1.7)–(1.11) putting κ to zero in the solution of Stefan problem (1.2)–(1.6), because the constant C in (2.3) and T_1 depend on κ. So we have to prove

THEOREM 2.2′. *Let the conditions of Theorem 2.2 be fulfilled. Then there exists* $T_0>0$ *such that the Stefan problem* (1.2)–(1.6) *has a unique solution* $u_{j\kappa}\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\bar Q_{jT_0}^{(\kappa)})$, $j=1,2$, $\rho_\kappa\in C_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Gamma_{T_0})$, $\kappa\partial_t\rho_\kappa\in C_{x\ \ t}^{1+\alpha,1+\alpha/2}(\Gamma_{T_0})$ *and the following estimate holds:*

$$\sum_{j=1}^{2}|u_{j\kappa}|_{Q_{jt}^{(\kappa)}}^{(2+\alpha)}+|\rho_\kappa|_{\Gamma_t}^{(2+\alpha)}+|\kappa\partial_t\rho_\kappa|_{\Gamma_t}^{(1+\alpha)}\le C_2\Big(\sum_{j=1}^{2}|u_{0j}|_{\Omega_j}^{(2+\alpha)}+|p|_{\Sigma_t}^{(2+\alpha)}\Big),\ t\le T_0, \tag{2.4}$$

where T_0 *and the constant* C_2 *do not depend on* κ.

We reduce (1.2)–(1.6) to the problem in the fixed domain $\Omega_1 \cup \Omega_2$ with the help of the coordinate transformation [16, 9, 8]

$$x = y + \chi(\lambda(y))\,\rho_\kappa(\xi,\tau)\,N(\xi),\ y \in \mathcal{O},\quad \xi = \xi(y) \in \Gamma,$$
$$x = y,\ y \in \overline{\Omega}\backslash\mathcal{O},\quad t = \tau, \tag{2.5}$$

where $\mathcal{O}$ is a $2\lambda_0$-neighborhood of Γ, $\lambda_0 > 0$ sufficiently small value depending on Γ and such that $\gamma_\kappa(t) \subset \mathcal{O}$ for $t \in [0, t_0]$, $\lambda(y)$ is the distance between $\xi = \xi(y) \in \Gamma$ and $y \in \mathcal{O}$ lying on a vector $N(\xi)$ or its continuation (see [9]), $\chi(\lambda)$ is a smooth cut-off function: $\chi = 1$, $|\lambda| < \lambda_0$, $\chi = 0$, $|\lambda| \geq 2\lambda_0$. The mapping (2.5) transforms Γ into $\gamma_\kappa(t)$ and the domains Ω_j into the unknown ones $\Omega_j^{(\kappa)}(t)$, $j = 1, 2$.

We note that the points $y \in \bar{\Omega}\backslash\mathcal{O}$ (or $|\lambda| \geq 2\lambda_0$) remain fixed ($x = y$). We keep the variable t instead of a new one τ.

We construct auxiliary functions $\rho_0(\xi, t)$ on Γ_T under the conditions

$$\rho_0|_{t=0} = 0,\ \partial_t\rho_0|_{t=0} \equiv \partial_t\rho|_{t=0} = -\frac{a_1\Delta u_{01}|_\Gamma}{\nu_0 N^T\partial_{\nu_0}u_{01}|_\Gamma} \tag{2.6}$$

and $V_j(y, t)$, $j = 1, 2$, as the solutions of the Cauchy problems

$$\partial_t V_j - a_j\,\Delta V_j - \chi\,\partial_t\rho_0 N\,\nabla^T V_j = 0 \ \text{ in } R^n_T, \tag{2.7}$$

$$V_j|_{t=0} = \widetilde{u}_{0j}(y) \text{ in } \mathbb{R}^n, \tag{2.8}$$

where $j = 1, 2$ and the tilde denotes the smooth extension of a function into $\mathbb{R}^n$, $R^n_T = \mathbb{R}^n \times (0, T)$.

Lemma 2.1 ([18, 22, 9, 8]). *For arbitrary functions* $u_{0j} \in C^{2+\alpha}(\overline{\Omega}_j)$, $j = 1, 2$, *each one of the problems* (2.6), (2.7)–(2.8) *has a unique solution* $\rho_0 \in C_y^{3+\alpha,}{}_t^{\frac{3+\alpha}{2}}(\Gamma_T)$, $V_j \in C_y^{2+\alpha,1}{}_t^{+\alpha/2}(R^n_T)$, $j = 1, 2$, *and the following estimates are valid*

$$|\rho_0|^{(3+\alpha)}_{\Gamma_T} \leq C_3|u_{01}|^{(2+\alpha)}_{\Omega_1}, \tag{2.9}$$

$$|V_j|^{(2+\alpha)}_{R^n_T} \leq C_4|u_{0j}|^{(2+\alpha)}_{\Omega_j},\ j = 1, 2. \tag{2.10}$$

In the problem (1.2)–(1.6) we make the following substitutions

$$\rho_\kappa(\xi, t) = \rho_0(\xi, t) + \psi_\kappa(\xi, t),$$
$$u_{j\kappa}(y + \chi\,\rho_\kappa N, t) = v_{j\kappa}(y, t) + V_j(y, t), \tag{2.11}$$

where ψ_κ, $v_{j\kappa}$ are the new unknown functions satisfying zero initial conditions $\partial_t^k v_{j\kappa}|_{t=0} = 0$, $\partial_t^k\psi_\kappa|_{t=0} = 0$, $k = 0, 1$; $j = 1, 2$.

With the help of the expansion formulas (A.5) and (A.4) of the inverse Jacobian matrix J^{-1} of the transformation (2.5) and J_0^{-1} respectively [8] ($J_0 = J(\rho_0 + \psi)|_{\psi=0}$, see (A.2)) we extract linear principal terms with respect to unknown functions, known functions and remainder terms containing the rests after separating linear terms and known functions. Then we obtain the parabolic problem for the unknown functions $v_{j\kappa}$, $j = 1, 2$, ψ_κ

$$\partial_t v_{j\kappa} - a_j\,\Delta v_{j\kappa} - (\partial_t\psi_\kappa - a_j\,\Delta\psi_\kappa)\,\chi\,N J_0^{-T}\,\nabla^T V_j$$
$$= f_j(y, t) + F_j(v_{j\kappa}, \psi_\kappa) \text{ in } \Omega_{jT},\ j = 1, 2, \tag{2.12}$$

with boundary and zero initial conditions

$$v_{1\kappa}|_{\Sigma} = p_1(y,t),\ t \in (0,T), \tag{2.13}$$

and the transmission conditions on Γ, $t \in (0,T)$,

$$v_{j\kappa}|_{\Gamma} = \eta_j(y,t),\ j = 1,2, \tag{2.14}$$

$$\begin{aligned}(\lambda_1\,\partial_{\nu_0} v_{1\kappa} - \lambda_2\,\partial_{\nu_0} v_{2\kappa} + \kappa\,\nu_0\,N^T\,\partial_t\psi_\kappa \\ -\nu_0\,N^T[(\lambda_1\,\nabla V_1 - \lambda_2\,\nabla V_2)J_0^{-1}J_0^{-T} + \kappa\,N J_0^{-T}\,\partial_t\rho_0]\,\nabla^T\psi_\kappa)\,|_{\Gamma} \\ = \varphi(y,t;\kappa) + \Phi(v_{1\kappa}, v_{2\kappa}, \psi_\kappa;\kappa)|_{\Gamma},\end{aligned} \tag{2.15}$$

where "T" means transposed matrix and column-vector, $\nu_0\,N^T \geq d_1 > 0$,

$$f_j = \chi\,\partial_t\rho_0 N J_0^{-T}\,\nabla^T V_j - \partial_t V_j + a_j(J_0^{-T}\,\nabla^T)^T J_0^{-T}\,\nabla^T V_j, \tag{2.16}$$

$$\begin{aligned}F_j = \chi\,\partial_t(\rho_0 + \psi_\kappa)\,N J^{-T}(\nabla^T v_{j\kappa} - J_1^T J_0^{-T}\,\nabla^T V_j) \\ +a_j[\nabla\,B^T + (B^T\,J^{-T}\,\nabla^T)^T\,J^{-T} J_{11}^T \\ -(J_0^{-T} J_1^T J^{-T}\,\nabla^T)^T + (J^{-T}\,\nabla^T)^T J^{-T} J_{12}^T]\,J_0^{-T}\,\nabla^T V_j \\ -a_j[\nabla\,B^T + (B^T\,J^{-T}\nabla^T)^T]J^{-T}\nabla^T v_{j\kappa} - a_j(\nabla\psi)\nabla^T(\chi N J_0^{-T}\nabla^T V_j),\end{aligned} \tag{2.17}$$

$$p_1 = (p(y,t) - V_1(y,t))|_{\Sigma},\quad \eta_j = -V_j(y,t)|_{\Gamma},\ j = 1,2, \tag{2.18}$$

$$\varphi = -\nu_0\,J_0^{-1}[J_0^{-T}\nabla^T(\lambda_1 V_1 - \lambda_2 V_2)|_{\Gamma} + \kappa N^T\partial_t\rho_0], \tag{2.19}$$

$$\begin{aligned}\Phi = \nu_0(B^T + J^{-1}B)\,J^{-T}\nabla^T(\lambda_1\,v_{1\kappa} - \lambda_2\,v_{2\kappa}) \\ -\nu_0\,\mathcal{M}\nabla^T(\lambda_1\,V_1 - \lambda_2\,V_2) \\ +\kappa\,\nu_0 J^{-1}\,(B\,N^T\partial_t\psi_\kappa + (J_{12} - B\,J_{11})\,J_0^{-1}N^T\partial_t\rho_0),\end{aligned} \tag{2.20}$$

$$B = J_{01} + J_1,$$

$$\mathcal{M} = J^{-1}[B\,J_{11}^T + J_{01}^T J_0^{-T} J_{11}^T - J_0^{-T} J_{12}^T]\,J^{-T} + J^{-1}(B\,J_{11} - J_{12})\,J_0^{-1}J_0^{-T},$$

where the matrices $J = I + J_{01} + J_1$, $J_0 = I + J_{01}$, $J_1 = J_{11} + J_{12}$ are determined by formulae (A.1)–(A.3). Here we omit the index κ at the matrices J^{-1}, $J_1 = J_{11} + J_{12}$, for convenience.

THEOREM 2.3. *Let the assumptions of Theorem 2.2 be fulfilled. Then there exists $T_0 > 0$ such that the Stefan problem (2.12)–(2.15) has a unique solution $v_{j\kappa} \in \overset{\circ}{C}{}_{y}^{2+\alpha,}{}_{t}^{1+\alpha/2}(\bar\Omega_{jT_0})$, $j = 1,2$, $\psi_\kappa \in \overset{\circ}{C}{}_{y}^{2+\alpha,}{}_{t}^{1+\alpha/2}(\Gamma_{T_0})$, $\kappa\partial_t\psi_\kappa \in \overset{\circ}{C}{}_{y}^{1+\alpha,}{}_{t}^{\frac{1+\alpha}{2}}(\Gamma_{T_0})$ and this solution satisfies the following estimate for $t \leq T_0$*

$$\sum_{j=1}^{2} |v_{j\kappa}|_{\Omega_{jt}}^{(2+\alpha)} + |\psi_\kappa|_{\Gamma_t}^{(2+\alpha)} + |\kappa\partial_t\psi_\kappa|_{\Gamma_t}^{(1+\alpha)} \leq C_5\Big(\sum_{j=1}^{2} |u_{0j}|_{\Omega_j}^{(2+\alpha)} + |p\,|_{\Sigma_t}^{(2+\alpha)}\Big), \tag{2.21}$$

where T_0 and the constant C_5 do not depend on κ.

From the formulae (2.11) with $x = y + \chi\,\rho_\kappa\,N$ due to this theorem and Lemma 2.1 we shall have Theorem 2.2′ and estimate (2.4).

We consider the given functions $h(y,t) = (f_1,\, f_2,\, p_1,\, \eta_1,\, \eta_2,\, \varphi)$, $t \le t_1$ in the problem (2.12)–(2.15). They are expressed via the inverse matrix J_0^{-1} which exists for $t \le t_1$ (see (A.6) and [8]).

Lemma 2.2. *Let* $\Sigma,\ \Gamma \in C^{2+\alpha}$, $\alpha \in (0,1)$. *For any functions* $u_{0j} \in C^{2+\alpha}(\bar\Omega_j)$, $j = 1,2$, $p \in C^{2+\alpha,1+\alpha/2}_{y\quad t}(\Sigma_T)$ *satisfying the compatibility conditions* (1.12)–(1.14), $f_j \in \mathring{C}^{\alpha,\alpha/2}_{y\ \ t}(\bar\Omega_{jt_1})$, $\eta_j \in \mathring{C}^{2+\alpha,1+\alpha/2}_{y\quad t}(\Gamma_{t_1})$, $j = 1,2$, $p_1 \in \mathring{C}^{2+\alpha,1+\alpha/2}_{y\quad t}(\Sigma_{t_1})$, $\varphi \in \mathring{C}^{1+\alpha,\frac{1+\alpha}{2}}_{y\quad t}(\Gamma_{t_1})$ *and the following estimate holds*

$$\sum_{j=1}^{2}(|f_j|^{(\alpha)}_{\Omega_{jt}} + |\eta_j|^{(2+\alpha)}_{\Gamma_t}) + |p_1|^{(2+\alpha)}_{\Sigma_t} + |\varphi|^{(1+\alpha)}_{\Gamma_t} \le C_6\Big(\sum_{j=1}^{2}|u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t}\Big),\ t \le t_1, \tag{2.22}$$

where the constant C_6 *does not depend on* κ.

Proof. Estimate (2.22) is derived by direct evaluation of the functions (2.16), (2.18), (2.19) with the help of the estimates (2.9), (2.10) for the functions ρ_0 and V_j, $j = 1,2$, and (A.6) for the matrix J_0^{-1}.

Functions $f_j|_{t=0}$, $j = 1,2$, are zero by the equation (2.7) and condition $J_0^{-1}|_{t=0} = I$, I the identity matrix. For the functions $p_1,\ \eta_j, \varphi$ we have

$$p_1|_{t=0} = p(x,0) - u_{01}|_\Sigma = 0,\ \partial_t p_1|_{t=0} = p_t(x,0) - a_j\,\Delta u_{01}|_\Sigma = 0$$

by compatibility conditions (1.12), (1.13) and $\chi|_\Sigma = 0$;

$$\begin{aligned} \eta_j|_{t=0} &= -V_j|_{t=0,\Gamma} = -u_{0j}|_\Gamma = 0,\\ \partial_t\eta_j|_{t=0} &= -a_j\,\Delta u_{0j}|_\Gamma - N\,\nabla^T u_{0j}|_\Gamma\partial_t\rho_0|_{t=0}\\ &= -a_j\,\Delta\, u_{0j}|_\Gamma + \frac{a_1\Delta u_{01}|\,\Gamma}{\nu_0\,N^T\partial_{\nu_0}u_{01}|\,\Gamma}\,\partial_N u_{0j}|\,\Gamma = 0,\ j = 1,2, \end{aligned}$$

by the conditions (2.8), (2.6), (1.12), (1.13), $\chi|_\Gamma = 1$ and the identity $\partial_N u_{0j}|\,\Gamma = \nu_0\,N^T\partial_{\nu_0}u_{0j}|\,\Gamma$, $j = 1,2$;

$$\begin{aligned} \varphi|_{t=0} &= -[(\lambda_1\partial_{\nu_0}u_{0j} - \lambda_1\partial_{\nu_0}u_{0j})|\,\Gamma + \kappa\nu_0\,N^T\,\partial\rho_0|_{t=0}]\\ &= -\left[(\lambda_1\partial_{\nu_0}u_{01} - \lambda_1\partial_{\nu_0}u_{02})|\,\Gamma - \kappa\frac{a_1\Delta u_{01}|_\Gamma}{\partial_{\nu_0}u_{01}}\right] = 0 \end{aligned}$$

by (2.6) and compatibility condition (1.14). ■

To prove Theorem 2.3 we consider a linear problem with the unknown functions $Z_{j\kappa}$, $j = 1,2$, Ψ_κ satisfying zero initial conditions

$$\partial_t Z_{j\kappa} - a_j\,\Delta Z_{j\kappa} - \alpha_j(x,t)(\partial_t\Psi_\kappa - a_j\,\Delta\Psi_\kappa) = f_j(x,t) \text{ in } \Omega_{jT},\ j = 1,2, \tag{2.23}$$

$$Z_{1\kappa}|_\Sigma = p_1(x,t),\ t \in (0,T), \tag{2.24}$$

$$Z_{j\kappa}|_\Gamma = \eta_j(x,t),\ j = 1,2, \tag{2.25}$$

$$(\lambda_1\,\partial_{\nu_0}Z_{1\kappa} - \lambda_2\,\partial_{\nu_0}Z_{2\kappa})|_\Gamma + \kappa\,\partial_t\Psi_\kappa + d(x,t)\,\nabla^T\Psi_\kappa = \varphi(x,t),\ t \in (0,T), \tag{2.26}$$

where λ_j are positive constants, $j = 1,2$, $d = (d_1,\dots,d_n)$.

THEOREM 2.4. *Let* $\Sigma,\ \Gamma \in C^{2+\alpha}$, $\alpha \in (0,1)$, $\alpha_j(x,t) \in C_{x\ t}^{\alpha,\alpha/2}(\bar\Omega_{jT})$, $d_i(x,t) \in C_{x\ \ t}^{1+\alpha,1+\alpha/2}(\Gamma_T)$, $j=1,2$, $i=1,\dots,n$, *and*

$$0<|\kappa|\le\kappa_0,\quad -\kappa\,\alpha_j(x,0)|_\Gamma \ge d_4 = const>0,\ j=1,2. \tag{2.27}$$

Then for any $f_j \in \overset{\circ}{C}{}_{x\ t}^{\alpha,\alpha/2}(\bar\Omega_{jT})$, $p_1 \in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Sigma_T)$, $\eta_j \in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Gamma_T)$, $j=1,2$, $\varphi \in \overset{\circ}{C}{}_{x\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma_T)$ *the problem* (2.23)–(2.26) *has a unique solution* $Z_{j\kappa} \in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\alpha/2}(\bar\Omega_{jT})$, $j=1,2$, $\Psi_\kappa \in \overset{\circ}{C}{}_{x\ \ t}^{2+\alpha,1+\alpha/2}(\Gamma_T)$, $\kappa\partial_t\Psi_\kappa \in \overset{\circ}{C}{}_{x\ \ t}^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma_T)$ *and it satisfies the estimate*

$$\sum_{j=1}^{2}|Z_{j\kappa}|^{(2+\alpha)}_{\Omega_{jt}}+|\Psi_\kappa|^{(2+\alpha)}_{\Gamma_t}+|\kappa\partial_t\Psi_\kappa|^{(1+\alpha)}_{\Gamma_t} \tag{2.28}$$
$$\le C_7\Big(\sum_{j=1}^{2}(|f_j|^{(\alpha)}_{\Omega_{jt}}+|\eta_j|^{(2+\alpha)}_{\Gamma_t})+|p_1|^{(2+\alpha)}_{\Sigma_t}+|\varphi|^{(1+\alpha)}_{\Gamma_t}\Big),\ t\le T,$$

where the constant C_7 *does not depend on* κ.

Proof. We derive (2.28) with the help of the Schauder method. Let $\xi_0\in\Gamma$ be an arbitrary point. In (2.23), (2.25), (2.26) we make the substitution

$$Z_{j\kappa}=z_{j\kappa}+\alpha_j(\xi_0,0)\Psi_\kappa,\ j=1,2,$$

where $z_{j\kappa}$ are new unknown functions, then we obtain the problem for $z_{j\kappa}$, $j=1,2$, Ψ_κ

$$\partial_t z_{j\kappa}-a_j\,\Delta z_{j\kappa}=f_j(x,t) \tag{2.29}$$
$$+(\alpha_j(x,t)-\alpha_j(\xi_0,0))(\partial_t\Psi_\kappa-a_j\,\Delta\Psi_\kappa)\ \text{in}\ \ \Omega_{jT},\ j=1,2,$$

$$z_{j\kappa}|_\Gamma+\alpha_j(\xi_0,0)\Psi_\kappa=\eta_j(x,t),\ j=1,2, \tag{2.30}$$

$$(\lambda_1\,\partial_{\nu_0}z_{1\kappa}-\lambda_2\,\partial_{\nu_0}z_{2\kappa})|_\Gamma+\kappa\,\partial_t\Psi_\kappa+d(x,t)\,\nabla^T\Psi_\kappa=\varphi(x,t),\ t\in(0,T). \tag{2.31}$$

Up to translation and rotation, we can assume that the origin of coordinates is at ξ_0 and x_n-axis coincides with the normal ν_0 to Γ directed into Ω_2 (that is $\xi_0=0$). Let $B_{2\delta_0}=\{x: |x-\xi_0|<2\delta_0,\ x\in\Omega\}$, $\delta_0>0$. Choosing δ_0 sufficiently small we can represent the surface $\Gamma\cap B_{2\delta_0}$ by an equation $x_n=q(x')$, where $q\in C^{2+\alpha}(\bar B_{2\delta_0})$, $q(0)=0$, $\partial_{x_\mu}q(0)=0$, $\mu=1,\dots,n-1$.

Let $\zeta(x)$ be a smooth cut-off function such, that $\zeta(x)=1$ if $|x|\le\delta_0$, $\zeta(x)=0$ if $|x|\ge 2\delta_0$.

We multiply (2.29) by $\zeta(x)$ and the conditions (2.30), (2.31) by $\zeta(x)|_\Gamma$, extend $q(x')$ into an entire space $\mathbb{R}^{n-1}$ preserving smoothness and notation and make a change of coordinates $y=Y(x): y'=x',\ y_n=x_n-q(x')$.

Let $D_1=\mathbb{R}^n_-$, $D_2=\mathbb{R}^n_+$, $D_{jT}=D_j\times(0,T)$, R be the plane $y_n=0$, $R_T=R\times[0,T]$.

We denote

$$\hat z_j(y,t)=\zeta(x)\,z_{j\kappa}(x,t)|_{x=Y^{-1}(y)},\ \hat\psi(y',t)=(\zeta(x)|_\Gamma\Psi_\kappa)|_{x=Y^{-1}(y)},$$
$$\hat f_j(y,t)=\zeta(x)\,f_j(x,t)|_{x=Y^{-1}(y)},\ \hat\eta_j(y',t)=(\zeta(x)|_\Gamma\,\eta_j(x,t))|_{x=Y^{-1}(y)},$$
$$\hat\varphi(y',t)=(\zeta(x)|_\Gamma\varphi(x,t))|_{x=Y^{-1}(y)},\ \ j=1,2,$$

and extend by zero the functions $\hat{z}_j$, $\hat{f}_j$ into D_j, $j = 1, 2$, and $\hat{\psi}$, $\hat{\eta}_j$, $\hat{\varphi}$ into $\mathbb{R}^{n-1}$. Then for the functions $\hat{z}_j$, $j = 1, 2$, and $\hat{\psi}$ we obtain the model conjunction problem

$$\partial_t \hat{z}_j - a_j \, \Delta \hat{z}_j = \hat{f}_j(y,t) + Q_j(z_{j\kappa}, \Psi_\kappa; \zeta) \text{ in } D_{jT},\ j = 1, 2, \tag{2.32}$$

$$\hat{z}_j|_{y_n=0} + \alpha_j(\xi_0, 0)\hat{\psi} = \hat{\eta}_j(y', t),\ j = 1, 2, \tag{2.33}$$

$$\begin{aligned}(\lambda_1 \, \partial_{y_n} \hat{z}_1 - \lambda_2 \, \partial_{y_n} \hat{z}_2)|_{y_n=0} + \kappa \, \partial_t \hat{\psi} + d'(\xi_0, 0)) \, \nabla^T \hat{\psi}_\kappa \\ = \hat{\varphi}(y', t) + P(z_{1\kappa}, z_{2\kappa}, \Psi_\kappa; \zeta)|_{y_n=0}, \quad t \in (0, T),\end{aligned} \tag{2.34}$$

where

$$\begin{aligned} Q_j = & -a_j \, (2\nabla\zeta \, \nabla^T + \Delta\zeta)(z_{j\kappa} - \alpha_j(\xi_0, 0)\Psi_\kappa)|_{x=Y^{-1}(y)} \\ & + \zeta(x)(\alpha_j(x,t) - \alpha_j(\xi_0, 0))(\partial_t \Psi_\kappa - a_j \, \Delta\Psi_\kappa)|_{x=Y^{-1}(y)} \\ & - a_j \Big(\Delta'_{y'} \, q \partial_{y_n} + 2 \sum_{\mu=1}^{n-1} \partial_{y_\mu} q \, \partial^2_{y_\mu y_n} - \nabla' q \nabla'^T q \partial^2_{y_n} \Big) \\ & \times (\hat{z}_j - \alpha_j(\xi_0, 0)\zeta(x)\Psi_\kappa|_{x=Y^{-1}(y)}), \end{aligned}$$

$$\begin{aligned} P = & -(\zeta(x)(\nu_0(x) - \nu_0(\xi_0))\nabla^T_x(\lambda_1 \, z_{1\kappa} - \lambda_2 \, z_{2\kappa}) \\ & + \zeta(x)(d(x,t) - d(\xi_0, 0))\nabla^T_x \Psi_\kappa)|_{\Gamma,\ x=Y^{-1}(y)} \\ & + (\Psi_\kappa d(x,t) + (\lambda_1 \, z_{1\kappa} - \lambda_2 \, z_{2\kappa})\nu_0(x))\nabla^T_x \zeta(x)|_{\Gamma,\ x=Y^{-1}(y)}, \end{aligned}$$

$\nabla' = (\partial_{y_1}, \ldots, \partial_{y_{n-1}})$, $\Delta' = \partial^2_{y_1} + \ldots + \partial^2_{y_{n-1}}$, $d' = (d_1, \ldots, d_{n-1})$.

(2.32)–(2.34) is the problem (B.1)–(B.4), for the solution of which we have an estimate (B.6) under the conditions (B.5). These conditions for the problem (2.32)–(2.34): $-\kappa\alpha_j(\xi_0, 0)\lambda_j > 0$, $j = 1, 2$, are fulfilled due to (2.27) and $\lambda_j > 0$. We apply (B.6) to the solution of the problem (2.32)–(2.34)

$$\begin{aligned} ||\hat{w}||_t := & \sum_{j=1}^{2} |\hat{z}_j|^{(2+\alpha)}_{D_{jt}} + |\hat{\psi}|^{(2+\alpha)}_{R_t} + |\kappa\partial_t\hat{\psi}|^{(1+\alpha)}_{R_t} \\ & \le C_8 \Big(\sum_{j=1}^{2} (|\hat{f}_j + Q_j|^{(\alpha)}_{D_{jt}} + |\hat{\eta}_j|^{(2+\alpha)}_{R_t}) + |\hat{\varphi} + P|^{(1+\alpha)}_{R_t} \Big), \end{aligned}$$

where $\hat{w} = (\hat{z}_1,\, \hat{z}_2,\, \hat{\psi})$, then we estimate the norms $|Q_j|^{(\alpha)}_{D_{jt}}$, $|P|^{(1+\alpha)}_{R_t}$ and choosing δ_0 and T_3 sufficiently small we find

$$||\hat{w}||_t \le q||\hat{w}||_t + C_8 \Big(\sum_{j=1}^{2} (|\hat{f}_j|^{(\alpha)}_{D_{jt}} + |\hat{\eta}_j|^{(2+\alpha)}_{R_t}) + |\hat{\varphi}|^{(1+\alpha)}_{R_t} \Big),$$

$q \in (0, 1)$, $t \le T_3$, where T_3 and the constant C_8 do not depend on κ. From this we derive an estimate for $||\hat{w}||_t$. Returning to the original coordinates $\{x\}$ and remembering that $\zeta(x)=1$, if $|x|\le\delta_0$ we obtain an estimate for Ψ_κ

$$(2.35)\quad |\Psi_\kappa|^{(2+\alpha)}_{\Gamma_{\delta_0,t}}+|\kappa\partial_t\Psi_\kappa|^{(1+\alpha)}_{\Gamma_{\delta_0,t}}\le C_9\Big(\sum_{j=1}^2(|\hat f|_j|^{(\alpha)}_{B^{(j)}_{2\delta_0,t}}+|\hat\eta_j|^{(2+\alpha)}_{\Gamma_{2\delta_0,t}})+|\hat\varphi|^{(1+\alpha)}_{\Gamma_{2\delta_0,t}}\Big)$$
$$\le C_{10}\Big(\sum_{j=1}^2(|f_j|^{(\alpha)}_{D_{jt}}+|\eta_j|^{(2+\alpha)}_{\Gamma_t})+|\varphi|^{(1+\alpha)}_{\Gamma_t}\Big),\ t\le T_3,$$

where $B^{(j)}_{2\delta_0,t}=(\Omega_j\cap B_{2\delta_0})\times(0,t),\ \Gamma_{2\delta_0,t}=(\Gamma\cap B_{2\delta_0})\times(0,t)$.

With the help of (2.35) due to the arbitrariness of a point $\xi_0\in\Gamma$, center of a ball B_{δ_0} we obtain an estimate

$$(2.36)\quad |\Psi_\kappa|^{(2+\alpha)}_{\Gamma_t}+|\kappa\partial_t\Psi_\kappa|^{(1+\alpha)}_{\Gamma_t}\le C_{11}\Big(\sum_{j=1}^2(|f_j|^{(\alpha)}_{D_{jt}}+|\eta_j|^{(2+\alpha)}_{\Gamma_t})+|\varphi|^{(1+\alpha)}_{\Gamma_t}\Big),\ t\le T_3,$$

where the constant C_{11} does not depend on κ.

In the equations of the problem (2.23)–(2.26) we move all the terms containing Ψ_κ to the right-hand sides, then we obtain the first boundary value problems (2.23), (2.24), (2.25) for $Z_{1\kappa}$, $j=1$, and (2.23), (2.25) for $Z_{2\kappa}$, $j=2$. Every one of these problems has a unique solution satisfying an estimate [18]

$$(2.37)\quad \begin{aligned}|Z_{1\kappa}|^{(2+\alpha)}_{\Omega_{1t}}&\le C_{12}(|f_1|^{(\alpha)}_{\Omega_{1t}}+|p_1|^{(2+\alpha)}_{\Sigma_t}+|\eta_1|^{(2+\alpha)}_{\Gamma_t}+|\Psi_\kappa|^{(2+\alpha)}_{\Gamma_t}),\\ |Z_{2\kappa}|^{(2+\alpha)}_{\Omega_{2t}}&\le C_{13}(|f_2|^{(\alpha)}_{\Omega_{2t}}+|\eta_2|^{(2+\alpha)}_{\Gamma_t}+|\Psi_\kappa|^{(2+\alpha)}_{\Gamma_t}),\end{aligned}$$

where the constants C_{12}, C_{13} do not depend on κ.

Combining the estimates (2.36), (2.37) we obtain the required estimate (2.28) for $t\le T_3$.

The existence of the solution of the problem (2.23)–(2.26) is proved by constructing a regularizer [18] and applying Theorem B.1.

The solution of the problem (2.23)–(2.26) obtained for $t\le T_3$ (T_3, independent of κ), may be extended on $(0,T)$ as in [9], [4]. ■

Proof of Theorem 2.3. We introduce the Hölder spaces. Let $\overset{\circ}{\mathcal{D}}{}^{2+\alpha}(\Gamma_T)$ be the space of functions $\psi_\kappa(\xi,t)$ such that $\psi_\kappa(\xi,t)\in\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\Gamma_T)$, $\kappa\partial_t\psi_\kappa\in\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{y\ \ t}(\Gamma_T)$. Let

$$\mathcal{B}(\Omega_T):=\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\bar\Omega_{1T})\times\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\bar\Omega_{2T})\times\overset{\circ}{\mathcal{D}}{}^{2+\alpha}(\Gamma_T),$$
$$\mathcal{H}(\Omega_T):=\overset{\circ}{C}{}^{\alpha,\alpha/2}_{y\ t}(\bar\Omega_{1T})\times\overset{\circ}{C}{}^{\alpha,\alpha/2}_{y\ t}(\bar\Omega_{2T})\times\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\Sigma_T)$$
$$\times\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\Gamma_T)\times\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\Gamma_T)\times\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{y\ \ t}(\Gamma_T)$$

be the spaces of the functions $w_\kappa=(v_{1\kappa},v_{2\kappa},\psi_\kappa)$ and $h=(f_1,f_2,p_1,\eta_1,\eta_2,\varphi)$ respectively with the norms

$$(2.38)\quad \|w_\kappa\|_{\mathcal{B}(\Omega_T)}:=\sum_{j=1}^2|v_{j\kappa}|^{(2+\alpha)}_{\Omega_{jT}}+|\psi_\kappa|^{(2+\alpha)}_{\Gamma_T}+|\kappa\partial_t\psi_\kappa|^{(1+\alpha)}_{\Gamma_T}.$$

$$(2.39)\quad \|h\|_{\mathcal{H}(\Omega_T)}:=\sum_{j=1}^2|f_j|^{(\alpha)}_{\Omega_{jT}}+|p_1|^{(2+\alpha)}_{\Sigma_T}+\sum_{j=1}^2|\eta_j|^{(2+\alpha)}_{\Gamma_T}+|\varphi|^{(1+\alpha)}_{\Gamma_T}.$$

We have reduced free boundary problem (1.2)–(1.6) to the nonlinear one (2.12)–(2.15) in the given domain $\Omega_1 \cup \Omega_2$. We write this problem in the operator form

$$\mathcal{A}[w_\kappa] = h + \mathcal{N}[w_\kappa], \tag{2.40}$$

where $w_\kappa = (v_{1\kappa}, v_{2\kappa}, \psi_\kappa)$ is an unknown vector, $h = (f_1, f_2, p_1, \eta_1, \eta_2, \varphi)$ a given one, $\mathcal{A}$ is the linear operator determined by all the terms in the left-hand sides of the equations and conditions of the problem (2.12)–(2.15), $\mathcal{N} = (F_1,\ F_2,\ 0,\ 0,\ 0,\ \Phi)$ a nonlinear operator, moreover $\mathcal{A}\colon \mathcal{B}(\Omega_T) \to \mathcal{H}(\Omega_T)$, $\mathcal{N}\colon \mathcal{B}(\Omega_T) \to \mathcal{H}(\Omega_T)$.

In the left-hand sides of the equations and conditions of the problem (2.12)–(2.15) there are the same linear terms as in the problem (2.23)–(2.26). The condition (2.27): $-\kappa\alpha_j(x,0)|_\Gamma \geq d_4 > 0$ with $\alpha_j(x,0)|_\Gamma = \chi N J_0^{-T} \nabla^T V_j|_{\Gamma,\, t=0} = \partial_N u_{0j}|_\Gamma = \nu_0 N^T \partial_{\nu_0} u_{0j}|_\Gamma$ is fulfilled by $\nu_0 N^T \geq d_1 > 0$ and (2.2). So we can apply Theorem 2.4 to the problem (2.40), represent it in the form

$$w_\kappa = \mathcal{A}^{-1}[h + \mathcal{N}[w_\kappa]] \tag{2.41}$$

where $\mathcal{A}^{-1}$ is the inverse operator, and by (2.28) obtain

$$\begin{aligned} ||w_\kappa||_{\mathcal{B}(\Omega_T)} &\equiv ||\mathcal{A}^{-1}[h + \mathcal{N}[w_\kappa]]||_{\mathcal{B}(\Omega_T)} \\ &\leq C_7\Big(\|h\|_{\mathcal{H}(\Omega_T)} + \sum_{j=1}^{2} |F_j(v_{j\kappa}, \psi_\kappa)|^{(\alpha)}_{\Omega_{jT}} + |\Phi(v_{1\kappa}, v_{2\kappa}, \psi_\kappa; \kappa)|^{(1+\alpha)}_{\Gamma_T}\Big). \end{aligned} \tag{2.42}$$

Let $B(M) \subset \mathcal{B}(\Omega_{T_0}$ be a closed ball with center at zero: $B(M) := \{\, w_\kappa \mid v_{j\kappa} \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{\;y\quad\ t}(\bar\Omega_{jT_0}),\ j = 1,2,\ \psi_\kappa \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{\;y\quad\ t}(\Gamma_{T_0}),\ \kappa\partial_t\psi_\kappa \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{\;y\quad\ t}(\Gamma_{T_0}),\ ||w_\kappa||_{\mathcal{B}(\Omega_{T_0})} \leq M,\ t \leq T_0\}$, $M = C_7||h||_{\mathcal{H}(\Omega_{T_0})}(1-q)^{-1}$, $q \in (0,1)$, where $||w_\kappa||_{\mathcal{B}(\Omega_T)}$, $||h||_{\mathcal{H}(\Omega_T)}$ are the norms of the vectors $w_\kappa = (v_{1\kappa},\ v_{2\kappa},\ \psi_\kappa)$ and $h = (f_1,\ f_2,\ p_1,\ \eta_1,\ \eta_2,\ \varphi)$ determined by (2.38) and (2.39).

We prove that the operator $\mathcal{A}^{-1}[h+\mathcal{N}[w_\kappa]]$ acts from the closed ball $B(M)$ into itself and is contractive. For this we estimate the norms (2.42) and

$$\begin{aligned} \|\mathcal{A}^{-1}[h + \mathcal{N}[w_\kappa]] - \mathcal{A}^{-1}[h + \mathcal{N}[\widetilde{w}_\kappa]]\|_{\mathcal{B}(\Omega_t)} &\equiv \|\mathcal{A}^{-1}[\mathcal{N}[w_\kappa] - \mathcal{N}[\widetilde{w}_\kappa]]\|_{\mathcal{B}(\Omega_t)} \\ &\leq C_7\Big(\sum_{j=1}^{2} |F_j(v_{j\kappa}, \psi_\kappa) - F_j(\widetilde{v}_{j\kappa}, \widetilde{\psi}_\kappa)|^{(\alpha)}_{\Omega_{jt}} \\ &\quad + |\Phi(v_{1\kappa}, v_{2\kappa}, \psi_\kappa; \kappa) - \Phi(\widetilde{v}_{1\kappa}, \widetilde{v}_{2\kappa}, \widetilde{\psi}_\kappa; \kappa)|^{(1+\alpha)}_{\Gamma_t}\Big) \end{aligned} \tag{2.43}$$

for $w, \widetilde{w} \in B(M)$.

We evaluate the norms of the functions (2.17) F_j, $j = 1,2$, (2.20) Φ in (2.42) applying the estimates (A.17) of the inverse Jacobian matrix J^{-1}; (A.9), (A.7), (A.8) for $J_1 = J_{11} + J_{12}$; (A.6), (A.13) for J_0^{-1}, J_{01} and (A.11), (A.12), then we obtain

$$||\mathcal{A}^{-1}[h + \mathcal{N}[w_\kappa]]||_{\mathcal{B}(\Omega_t)} \leq C_7\, ||h||_{\mathcal{H}(\Omega_t)} + r_1(t, |\psi_\kappa|^{(2+\alpha)}_{\Gamma_t})\, ||w_\kappa||_{\mathcal{B}(\Omega_t)}, \tag{2.44}$$

where

$$r_1 = C_{14}\, t^{\frac{\alpha}{2}}(t^{\frac{2-\alpha}{2}} + t^{\frac12}|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t})(1 + t^{\frac12}|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t})(1+t^{\frac12})(1+|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t}) + C_{15}\, t^{\frac12}(1+t^{\frac12}|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t})(1+|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t} + t^{\frac12}|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t}) + C_{16}\,(t^{\frac{1+\alpha}{2}}|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t} + t^{\frac{1-\alpha}{2}} + t^{\frac12} + t)(1+|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t})^2. \tag{2.45}$$

In the same manner we estimate the norms in (2.43)

$$\|\mathcal{A}^{-1}[h+\mathcal{N}[w_\kappa]] - \mathcal{A}^{-1}[h+\mathcal{N}[\widetilde{w}_\kappa]]\|_{\mathcal{B}(\Omega_t)} \le r_2(t, |v_{1\kappa}|^{(2+\alpha)}_{\Omega_{1t}}, |v_{2\kappa}|^{(2+\alpha)}_{\Omega_{2t}}, |\psi_\kappa|^{(2+\alpha)}_{\Gamma_t})\, ||w_\kappa - \widetilde{w}||_{\mathcal{B}(\Omega_t)}, \tag{2.46}$$

where r_2 is similar to (2.45) and $r_2(0, M, M, M) = 0$.

We find T_4 from the inequalities

$$r_1(t, M) \le q, \quad r_2(t, M, M, M) \le q, \quad q \in (0, 1),$$

then from (2.44) and (2.46) we have

$$||\mathcal{A}^{-1}[h+\mathcal{N}[w_\kappa]]||_{\mathcal{B}(\Omega_t)} \le C_7\, ||h||_{\mathcal{H}(\Omega_t)} + q\, ||w_\kappa||_{\mathcal{B}(\Omega_t)} \le C_7\, ||h||_{\mathcal{H}(\Omega_t)} + q\, M \le M \equiv C_7\, ||h||_{\mathcal{H}(\Omega_{T_0})}(1-q)^{-1}, \tag{2.47}$$

$$\|\mathcal{A}^{-1}[h+\mathcal{N}[w_\kappa]] - \mathcal{A}^{-1}[h+\mathcal{N}[\widetilde{w}_\kappa]]\|_{\mathcal{B}(\Omega_t)} \le q\, ||w_\kappa - \widetilde{w}||_{\mathcal{B}(\Omega_t)}, \tag{2.48}$$

for all $w, \widetilde{w} \in B(M)$, $t \le T_0 = \min(t_0, t_1, t_2, T_4,)$ (the parametrization of a free boundary (1.1) is valid for $t \le t_0$; for $t \le t_1$ and $t \le t_2$ the inverse matrices J_0^{-1} and J^{-1} exist).

From (2.47) and (2.48) by the contraction mapping principle it follows that the problem (2.41) or (2.12)–(2.15) has a unique solution $w_\kappa = (v_{1\kappa}, v_{2\kappa}, \psi_\kappa) \in \mathcal{B}(\Omega_{T_0})$.

We can see that T_0 and the constant $C_7(1-q)^{-1}$ do not depend on κ.

Applying (2.47) and an estimate (2.22) for the vector h in (2.42) we find an estimate

$$||w_\kappa||_{\mathcal{B}(\Omega_t)} := \sum_{j=1}^{2} |v_{j\kappa}|^{(2+\alpha)}_{\Omega_{jt}} + |\psi_\kappa|^{(2+\alpha)}_{\Gamma_t} + |\kappa \partial_t \psi_\kappa|^{(1+\alpha)}_{\Gamma_t} \le C_7\,(1-q)^{-1}\, ||h||_{\mathcal{H}(\Omega_{T_0})} \le C_5\Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t}\Big), \tag{2.49}$$

$t \le T_0$, with $C_5 = C_6\, C_7(1-q)^{-1}$ independent of κ (C_6 is from (2.22) for the vector h). ■

Proof of Theorem 2.1. Due to Theorem 2.3 Stefan problem (2.12)–(2.15) has a unique solution $v_{j\kappa} \in \overset{\circ}{C}{}^{2+\alpha, 1+\alpha/2}_{y \quad t}(\bar{\Omega}_{jT_0})$, $j = 1, 2$, $\psi_\kappa \in \overset{\circ}{C}{}^{2+\alpha, 1+\alpha/2}_{y \quad t}(\Gamma_{T_0})$, $\kappa \partial_t \psi_\kappa \in \overset{\circ}{C}{}^{1+\alpha, \frac{1+\alpha}{2}}_{y \quad t}(\Gamma_{T_0})$ and it satisfies a uniform (with respect to κ) estimate (2.49) ((2.21)) for $t \le T_0$, that is, the sequences $\{v_{j\kappa}\}$, $j = 1, 2$, $\{\psi_\kappa\}$ and $\{\kappa \partial_t \psi_\kappa\}$, as $\kappa \to 0$ are compact in $\overset{\circ}{C}{}^{2,1}_{y\,t}(\bar{\Omega}_{jT_0})$, $\overset{\circ}{C}{}^{2,1}_{y\,t}(\Gamma_{T_0})$ and $\overset{\circ}{C}{}^{1,1/2}_{y \quad t}(\Gamma_{T_0})$ respectively. We choose converging subsequences

$$\{v_{j\kappa_n}\}, \quad j = 1, 2, \quad \{\psi_{\kappa_n}\} \text{ and } \{\kappa_n \partial_t \psi_{\kappa_n}\} \tag{2.50}$$

and denote

$$\lim_{\kappa_n \to 0} v_{j\kappa_n} = v_j, \ \lim_{\kappa_n \to 0} \psi_{\kappa_n} = \psi. \tag{2.51}$$

Here $v_j \in \overset{\circ}{C}{}^{2,1}_{y\,t}(\bar{\Omega}_{jT_0})$, $\psi \in \overset{\circ}{C}{}^{2,1}_{y\,t}(\Gamma_{T_0})$.

We rewrite the problem (2.12)–(2.15) for the functions of the subsequences (2.50) and with κ_n instead of κ in a Stefan condition (2.15), in the problem we let $\kappa_n \to 0$, then we obtain that functions v_j, $j = 1, 2$, ψ are the solution of the Florin problem

$$(2.52)\qquad \begin{gathered} \partial_t v_j - a_j \,\Delta v_j - (\partial_t \psi - a_j \,\Delta \psi)\, \chi \, N J_0^{-T}\, \nabla^T V_j \\ = f_j(y,t) + F_j(v_j, \psi) \text{ in } \Omega_{jT},\ j = 1, 2, \\ v_1|_\Sigma = p_1(y,t), \quad t \in (0,T),\ v_j|_\Gamma = \eta_j(y,t),\ j = 1, 2, \\ (\lambda_1 \,\partial_{\nu_0} v_1 - \lambda_2 \,\partial_{\nu_0} v_2 - \nu_0 \, N^T (\lambda_1 \nabla V_1 - \lambda_2 \nabla V_2)\, J_0^{-1} J_0^{-T}\, \nabla^T \psi)\,|_\Gamma \\ = \varphi(y,t;0) + \Phi(v_1, v_2, \psi; 0)|_\Gamma,\ t \in (0,T), \end{gathered}$$

where functions f_j, F_j, p_1, η_j, φ, Φ are determined by formulae (2.16)–(2.20).

From (2.49) we have the following estimate

$$(2.53)\qquad \begin{gathered} \sum_{j=1}^{2} |v_{j\kappa_n}|_{C^{2,1}_{y\ t}(\bar\Omega_{jt})} + |\psi_{\kappa_n}|_{C^{2,1}_{y\ t}(\Gamma_t)} + |\kappa_n \partial_t \psi_{\kappa_n}|_{C^{1,1/2}_{y\ \ t}(\Gamma_t)} \\ \le C_5 \Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t} \Big),\ t \le T_0, \end{gathered}$$

we let $\kappa_n \to 0$ in (2.53), then due to (2.51) we obtain an estimate for the functions v_j, $j = 1, 2$, ψ

$$(2.54)\qquad \sum_{j=1}^{2} |v_j|_{C^{2,1}_{y\ t}(\bar\Omega_{jt})} + |\psi|_{C^{2,1}_{y\ t}(\Gamma_t)} \le C_5 \Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t} \Big),\ t \le T_0.$$

Now we show that the functions v_j, $j = 1, 2$, ψ possess higher smoothness. For that we should estimate the Hölder constants

$$(2.55)\qquad [\partial_y^2 v_j]^{(\alpha)}_{\Omega_{jT_0}},\ [\partial_t v_j]^{(\alpha)}_{\Omega_{jT_0}},\ [\partial_y v_j]^{(\frac{1+\alpha}{2})}_{t, \Omega_{jT_0}}, [\partial_y^2 \psi]^{(\alpha)}_{\Gamma_{T_0}},\ [\partial_t \psi]^{(\alpha)}_{\Gamma_{T_0}},\ [\partial_y \psi]^{(\frac{1+\alpha}{2})}_{t, \Gamma_{T_0}}.$$

Consider, for instance, $[\partial_t v_j]^{(\alpha)}_{y,\, \Omega_{jT_0}}$. We represent the difference as $\partial_t v_j(y,t) - \partial_t v_j(z,t) = \partial_t v_j(y,t) - \partial_t v_{j\kappa_n}(y,t) + \partial_t v_{j\kappa_n}(y,t) + \partial_t v_{j\kappa_n}(z,t) - \partial_t v_{j\kappa_n}(z,t) - \partial_t v_j(z,t)$, (y,t), $(z,t) \in \bar\Omega_{jT_0}$, then

$$(2.56)\qquad \begin{gathered} |\partial_t v_j(y,t) - \partial_t v_j(z,t)| \le |\partial_t v_j(y,t) - \partial_t v_{j\kappa_n}(y,t)| \\ + |\partial_t v_j(z,t) - \partial_t v_{j\kappa_n}(z,t)| + |\partial_t v_{j\kappa_n}(y,t) - \partial_t v_{j\kappa_n}(z,t)|. \end{gathered}$$

We apply (2.49) to the function $v_{j\kappa_n}$

$$\begin{gathered} |\partial_t v_{j\kappa_n}(y,t) - \partial_t v_{j\kappa_n}(z,t)| \le [\partial_t v_{j\kappa}]^{(\alpha)}_{y,\, \Omega_{jT_0}} |y - z|^\alpha \\ \le C_5 \Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_{T_0}} \Big) |y - z|^\alpha \end{gathered}$$

and let $\kappa_n \to 0$ in (2.56) taking into account the convergence of subsequence $\{v_{j\kappa_n}\}$ in $\mathring{C}^{2,1}_{y\ t}(\bar\Omega_{jT_0})$ to v_j (see (2.51)), then we obtain an inequality

$$|\partial_t v_j(y,t) - \partial_t v_j(z,t)| \le C_5 \Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t} \Big) |y - z|^\alpha,$$

$t \le T_0$, which leads to the estimate of the Hölder constant

$$[\partial_t v_j]^{(\alpha)}_{y,\,\Omega_{jT_0}} \le C_5\Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_{T_0}}\Big). \tag{2.57}$$

In the same manner we derive such estimates for all other Hölder constants in (2.55). By (2.54) and estimates (2.57) of the Hölder constants we have $v_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{\;y\quad\;\; t}(\bar\Omega_{jT_0})$, $j=1,2$, $\psi \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{\;y\quad\;\; t}(\Gamma_{T_0})$ and

$$\sum_{j=1}^{2} |v_j|^{(2+\alpha)}_{\Omega_{jt}} + |\psi|^{(2+\alpha)}_{\Gamma_t} \le C_{17}\Big(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t}\Big),\ t \le T_0. \tag{2.58}$$

We rewrite the substitutions (2.11) and coordinate transformation (2.5) with κ_n instead of κ

$$\begin{gathered} \rho_{\kappa_n} = \rho_0 + \psi_{\kappa_n},\quad u_{j\kappa_n}(y + \chi\, N\rho_{\kappa_n}, t) = v_{j\kappa_n}(y,t) + V_j(y,t),\\ x = y + \chi(\lambda)\,\rho_{\kappa_n}(\xi,\tau)\,N(\xi),\ y \in \mathcal{O},\quad \xi \in \Gamma,\ x = y,\ y \in \bar\Omega\backslash\mathcal{O}, \end{gathered} \tag{2.59}$$

then we find

$$\rho_{\kappa_n} = \rho_0 + \psi_{\kappa_n},\ u_{j\kappa_n}(x,t) = v_{j\kappa_n}(x - \chi\, N\rho_{\kappa_n}, t) + V_j(x - \chi\, N\rho_{\kappa_n}, t), \tag{2.60}$$

$j = 1, 2$.

In (2.60) we let $\kappa_n \to 0$, take into account (2.51) and denote by ρ and u_j the functions in the right–hand sides

$$\rho := \rho_0 + \psi,\quad u_j(x,t) := v_j(x - \chi\, N\rho, t) + V_j(x - \chi\, N\rho, t),\ j = 1, 2. \tag{2.61}$$

In the coordinate transformation (2.59) we let κ_n tend to zero making use of (2.51) and $\rho := \rho_0 + \psi$,

$$x = y + \chi(\lambda)\,\rho(\xi,\tau)\,N(\xi),\ y \in \mathcal{O},\quad \xi \in \Gamma,\ x = y,\ y \in \bar\Omega\backslash\mathcal{O}. \tag{2.62}$$

From (2.61) we obtain that $\rho \in C^{2+\alpha,1+\alpha/2}_{\;x\quad\;\; t}(\Gamma_{T_0})$, $u_j \in C^{2+\alpha,1+\alpha/2}_{\;x\quad\;\; t}(\bar Q_{jT_0})$, $j = 1, 2$, where $Q_{jT_0} = \{(x,t) : x \in \Omega_j(t),\ t \in (0,T)\}$, $\partial\Omega_1(t) = \Sigma \cup \gamma(t)$, $\partial\Omega_2(t) = \gamma(t)$, $\gamma(t)$ is a surface: $x = \xi + \rho(\xi,t)\,N(\xi)$, $\xi = \xi(x) \in \Gamma$, $t \in [0,t_0]$, this equation is derived from (1.1) written for $k_n \to 0$ and by (2.51). In (2.61) we make use of the estimates (2.9), (2.10) for the functions ρ_0, V_j; (2.58) for v_j, ψ, then we obtain an estimate (2.1) for the functions $u_j(x,t)$ and ρ.

We show that the functions (2.61) $u_j(x,t)$, $j = 1, 2$, and ρ are the solution of the problem (1.7)–(1.11). For that we substitute obtained functions $u_j(x,t)$, $j = 1, 2$, and ρ in the problem (1.7)–(1.11), apply a transform (2.62) and making the change of the functions

$$\rho = \rho_0 + \psi,\quad u_j(y + \chi\, N\rho, t) = v_j(y,t) + V_j(y,t),\ j = 1, 2,$$

we get the problem (2.52), the solution of which are the functions v_j, $j = 1, 2$, and ψ. That means that the functions $u_j(x,t)$, $j = 1, 2$, and ρ are the solution of the Florin problem (1.7)–(1.11). ∎

A. Estimates of the Jacobian matrix J**.** Consider the Jacobian matrix J of the coordinate transformation (2.5) leaving an index κ at $\rho_\kappa = \rho_0 + \psi_\kappa$

(A.1) $J = \{\delta_{ij} + \partial_{y_j}(N_i\,\chi(\rho_0+\psi))\}_{1\le i,\,j\le n} = I + (\nabla^T N\chi\,(\rho_0+\psi))^T := I + J_{01} + J_1,$

(A.2) $J_0 = I + J_{01},\ \ J_{01} = (\nabla^T N\chi\,\rho_0)^T,$

(A.3) $J_1 = (\nabla^T N\chi\,\psi)^T = N^T\chi\nabla\psi + \psi(\nabla^T(N\chi))^T := J_{11} + J_{12},$

where δ_{ij} is the Kronecker delta, $N = (N_1,\dots,N_n) \in C^{2+\alpha}(\Gamma;\mathbb{R}^n)$ is a unit vector in the equation of a free boundary (1.1), I identity matrix, and "^{T}" means transposed matrix and column vector.

In [8] expansion formulae of the inverse matrices J_0^{-1}, J^{-1} were obtained

$$J_0^{-1} \equiv (I + J_{01})^{-1} = I - J_{01}J_0^{-1}, \tag{A.4}$$

$$J^{-1} \equiv (I + B)^{-1} = I - BJ^{-1},\ B = J_{01} + J_1, \tag{A.5}$$

existence of the matrices J_0^{-1}, J^{-1} was proved for small $t \le t_1$ and their estimates found in the weighted Hölder spaces with time power weights [3]. From these results and estimates in the classical Hölder spaces it follows that

$$\|J_0^{-1}\|_{\Gamma_t}^{(\alpha+\nu)} \le \frac{1}{1-q},\quad \nu = 0,1,\quad q \in (0,1),\quad t \le t_1, \tag{A.6}$$

under the condition $\rho_0(\xi(y),t) \in C_y^{3+\alpha,}{}_t^{\frac{3+\alpha}{2}}(\Gamma_T)$, $\alpha \in (0,1)$, $\rho_0|_{t=0} = 0$, where

$$||\{a_{ij}\}_{1\le i,j\le n}||_{\Gamma_T}^{(l)} := n\max_{i,j}|a_{ij}|_{\Gamma_T}^{(l)}.$$

The existence and estimate of the inverse matrix J^{-1} were proved under the assumptions $\psi(\xi(y),t) \in \mathring{C}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Gamma_T)$, $\partial_t\psi \in \mathring{C}_y^{1+\alpha,}{}_t^{\frac{1+\alpha}{2}}(\Gamma_T)$. We should obtain similar results, if $\psi(\xi,t) \in \mathring{C}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Gamma_T)$.

Lemma A.1. *Let $\psi(\xi(y),t) \in \mathring{C}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Gamma_T)$, $\alpha \in (0,1)$. Then for the matrix $J_1 := J_{11} + J_{12}$ the following estimates hold for $t \le T$*

$$\|J_{11}\|_{\Gamma_t}^{(\alpha+\nu)} = n\max_{i,j}|N_i\,\chi\,\partial_{y_j}\psi|_{\Gamma_t}^{(\alpha+\nu)} \le C_1\,t^{\frac{1-\nu}{2}}\,|\psi|_{\Gamma_t}^{(2+\alpha)}, \tag{A.7}$$

$$\|J_{12}\|_{\Gamma_t}^{(\alpha+\nu)} = n\max_{i,j}|\psi\,\partial_{y_j}(N_i\,\chi)|_{\Gamma_t}^{(\alpha+\nu)} \le C_2\,t^{\frac{2-\nu}{2}}\,|\psi|_{\Gamma_t}^{(2+\alpha)}, \tag{A.8}$$

$$\|J_1\|_{\Gamma_t}^{(\alpha+\nu)} \le C_3\,t^{\frac{1-\nu}{2}}\,|\psi|_{\Gamma_t}^{(2+\alpha)}, \tag{A.9}$$

$$\|J_1^2\|_{\Gamma_t}^{(\alpha+\nu)} \le C_4\,t^{\frac{2+\alpha-\nu}{2}}\,|\psi|_{\Gamma_t}^{(2+\alpha)},\ \nu = 0,1. \tag{A.10}$$

Proof. The estimates (A.7)–(A.10) are derived by direct evaluation of the norms with the help of the estimates

$$|f_1|_{\Omega_t}^{(l)} \le C_5 t^{\frac{r}{2}}|f_1|_{\Omega_t}^{(l+r)}, \tag{A.11}$$

$$|f_1\,f_2|_{\Omega_t}^{(l)} \le C_6 t^{\frac{l+r}{2}}|f_1|_{\Omega_t}^{(l+r)}|f_2|_{\Omega_t}^{(l)},\ t \le T, \tag{A.12}$$

for the functions $f_1 \in \mathring{C}_y^{l+r,}{}_t^{\frac{l+r}{2}}(\bar\Omega_T)$, $f_2 \in \mathring{C}_y^{l,}{}_t^{\frac{l}{2}}(\bar\Omega_T)$, l, $l+r$ positive non-integers, $r \ge 0$. ■

Lemma A.2. *Let* $\rho_0(\xi(y),t) \in C_{y}^{3+\alpha,\frac{3+\alpha}{2}}{}_{t}(\Gamma_T)$, $\alpha \in (0,1)$, $\rho_0|_{t=0} = 0$, $\psi(\xi(y),t) \in \overset{\circ}{C}{}_{y}^{2+\alpha,1+\alpha/2}{}_{t}(\Gamma_T)$. *Then*

$$\|J_{01}\|_{\Gamma_t}^{(\alpha+\nu)} = n\max_{i,j}|\partial_{y_j}(N_i\,\chi\,\rho_0)|_{\Gamma_t}^{(\alpha+\nu)} \le C_7\, t^{\frac{2+\alpha-\nu}{2}}\,|\rho_0|_{\Gamma_t}^{(3+\alpha)}, \tag{A.13}$$

$$\|J_{01}^2\|_{\Gamma_t}^{(\alpha+\nu)} \le C_8\, t^{\frac{4-\alpha-2\nu}{2}}\,|\rho_0|_{\Gamma_t}^{(3+\alpha)}, \tag{A.14}$$

$$\|J_{01}J_1\|_{\Gamma_t}^{(\alpha+\nu)} \le C_9\, t^{\frac{3+\alpha-\nu}{2}}\,|\rho_0|_{\Gamma_t}^{(3+\alpha)}|\psi|_{\Gamma_t}^{(2+\alpha)}, \tag{A.15}$$

$$\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(\alpha+\nu)} \le C_{10}\, t^{\frac{2+\alpha-\nu}{2}}\,(1+t^{1-\alpha}+t^{\frac{1-\alpha}{2}})(|\rho_0|_{\Gamma_t}^{(3+\alpha)}+|\psi|_{\Gamma_t}^{(2+\alpha)})^2, \tag{A.16}$$

$\nu = 0,1$, $t \le T$.

Proof. The estimates (A.13), (A.14) follow from the estimates of the matrices J_{01}, J_{01}^2 in the weighted Hölder spaces obtained in Lemma 5 and Corollary A.2 in [8].

To derive (A.15) we make use of the formulae (A.12) (because $\partial_{y_j}\rho_0$ may be considered as a function from $\overset{\circ}{C}{}_{y}^{\alpha+\nu,\frac{\alpha+\nu}{2}}{}_{t}(\Gamma_T)$), and (A.13), (A.9), and get

$$\|J_{01}J_1\|_{\Gamma_t}^{(\alpha+\nu)} \le C_{11}\, t^{\frac{1+\alpha}{2}}\,\|J_{01}\|_{\Gamma_t}^{(\alpha+\nu)}\|J_1\|_{\Gamma_t}^{(1+\alpha)} \le C_{12}\, t^{\frac{3+\alpha-\nu}{2}}\,|\rho_0|_{\Gamma_t}^{(3+\alpha)}|\psi|_{\Gamma_t}^{(2+\alpha)}.$$

Applying the estimates (A.12), (A.10), (A.14), (A.15) in an inequality

$$\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(\alpha+\nu)} \le \|J_{01}^2\|_{\Gamma_t}^{(\alpha+\nu)} + \|J_{01}J_1\|_{\Gamma_t}^{(\alpha+\nu)} + \|J_1J_{01}\|_{\Gamma_t}^{(\alpha+\nu)} + \|J_1^2\|_{\Gamma_t}^{(\alpha+\nu)}$$

we obtain (A.16). ∎

Theorem A.1. *Let* $\psi(\xi(y),t) \in \overset{\circ}{C}{}_{y}^{2+\alpha,1+\alpha/2}{}_{t}(\Gamma_T)$, $\alpha \in (0,1)$, $\rho_0(\xi(y),t) \in \overset{\circ}{C}{}_{y}^{3+\alpha,\frac{3+\alpha}{2}}{}_{t}(\Gamma_T)$, $\rho_0|_{t=0} = 0$, *and* $|\psi|_{\Gamma_T}^{(2+\alpha)} \le M$, $|\rho_0|_{\Gamma_T}^{(3+\alpha)} \le M_1$, $M > 0$, $M_1 > 0$. *Then there is* $t_2 \in (0,T]$ *such that the inverse Jacobian matrix* J^{-1} *exists, can be represented in the form*

$$J^{-1} = \sum_{k=0}^{\infty}(J_{01}+J_1)^{2k}\,(I-(J_{01}+J_1))$$

and satisfies an estimate

$$\|J^{-1}\|_{\Gamma_t}^{(\alpha+\nu)} \le C_{13}(1+t^{\frac{1-\nu}{2}}|\psi|_{\Gamma_t}^{(2+\alpha)}),\ \nu=0,1,\ t\le t_2. \tag{A.17}$$

Proof. Such theorem was proved in [8] under the assumption $\psi(\xi,t) \in \overset{\circ}{C}{}_{y}^{2+\alpha,1+\alpha/2}{}_{t}(\Gamma_T)$ and $\partial_t\psi(\xi,t) \in \overset{\circ}{C}{}_{y}^{1+\alpha,\frac{1+\alpha}{2}}{}_{t}(\Gamma_T)$.

First, we shall prove the existence of the inverse matrix $(I-(J_{01}+J_1)^2)^{-1}$. With the help of (A.12) we derive

$$\begin{aligned}\|((J_{01}+J_1)^2)^2\|_{\Gamma_t}^{(\alpha+\nu)} &\le C_{14}\, t^{\frac{\alpha+\nu}{2}}\,\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(\alpha+\nu)}\,\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(\alpha+\nu)}\\ &\le C_{14}C_{15}t^{\frac{1+\alpha}{2}}\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(\alpha+\nu)}\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(1+\alpha)},\end{aligned}$$

$$\|((J_{01}+J_1)^2)^k\|_{\Gamma_t}^{(\alpha+\nu)} \le \|(J_{01}+J_1)^2\|_{\Gamma_t}^{(\alpha+\nu)}(C_{14}C_{15}\, t^{\frac{1+\alpha}{2}}\,\|(J_{01}+J_1)^2\|_{\Gamma_t}^{(1+\alpha)})^{k-1}, \tag{A.18}$$

$\nu = 0, 1,\ k = 2, 3, \ldots$. Formula (A.18) is proved by induction. In (A.18) we apply (A.16) for $(J_{01} + J_1)^2$

$$\|((J_{01} + J_1)^2)^k\|_{\Gamma_t}^{(\alpha+\nu)} \le \mu_{1+\nu}(t)(C_{14}C_{15}\, t^{\frac{1+\alpha}{2}}\, \mu_2(t))^{k-1},$$

where

$$\mu_{1+\nu}(t) = C_{10}\, t^{\frac{2+\alpha-\nu}{2}}\, (1 + t^{1-\alpha} + t^{\frac{1-\alpha}{2}})(M + M_1)^2,\ \nu = 0, 1,$$

and choose $t_2 > 0$ from the inequalities

$$\mu_{1+\nu}(t) \le q, \quad C_{14}C_{15}\, t^{\frac{1+\alpha}{2}}\, \mu_2(t) \le q, \quad \nu = 0, 1, \quad q \in (0, 1),$$

then we have

$$\|((J_{01} + J_1)^2)^k\|_{\Gamma_t}^{(\alpha+\nu)} \le q^k, \quad \nu = 0, 1, \quad k = 2, 3, \ldots, \quad t \le t_2, \tag{A.19}$$

and

$$\sum_{k=0}^{\infty} \|((J_{01} + J_1)^2)^k\|_{\Gamma_t}^{(\alpha+\nu)} \le \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad \nu = 0, 1, \quad t \le t_2. \tag{A.20}$$

From this estimate it follows that the inverse matrix $(I - (J_{01} + J_1)^2)^{-1}$ exists, is expressed in the form

$$(I - (J_{01} + J_1)^2)^{-1} = \sum_{k=0}^{\infty}((J_{01} + J_1)^2)^k \tag{A.21}$$

and satisfies the estimate

$$\|(I - (J_{01} + J_1)^2)^{-1}\|_{\Gamma_t}^{(\alpha+\nu)} \le \frac{1}{1-q}, \quad \nu = 0, 1, \quad t \le t_2, \quad q \in (0, 1).$$

Using (A.5) and (A.21) we can obtain formally the identity

$$\begin{aligned} J^{-1} &= (I - (J_{01} + J_1)^2)^{-1}\, (I - (J_{01} + J_1)) \\ &\equiv \sum_{k=0}^{\infty}((J_{01} + J_1)^2)^k (I - (J_{01} + J_1)). \end{aligned} \tag{A.22}$$

On the basis of (A.19) we can show as in [8] that the matrix in the right hand side of (A.22) is the left and right inverse matrix to the Jacobian matrix $J = I + J_{01} + J_1$, that is, (A.22) is valid. With the help of the estimates (A.20), (A.13), (A.9) we obtain (A.17):

$$\begin{aligned} \|J^{-1}\|_{\Gamma_t}^{(\alpha+\nu)} &\le C_{16}\, \frac{1}{1-q}(1 + C_{17}\, t^{\frac{2-\alpha-\nu}{2}}\, |\rho_0|_{\Gamma_t}^{(3+\alpha)} + C_{18} t^{\frac{1-\nu}{2}}\, |\psi|_{\Gamma_t}^{(2+\alpha)}) \\ &\le C_{13}(1 + t^{\frac{1-\nu}{2}}\, |\psi|_{\Gamma_t}^{(2+\alpha)}),\ \nu = 0, 1,\ t \le t_2. \ \blacksquare \end{aligned}$$

B. Model problem with a small parameter. Let $D_1 = \mathbb{R}^n_-$, $D_2 = \mathbb{R}^n_+$, $D_{jT} = D_j \times (0, T)$, R be the plane $x_n = 0$, $R_T = R \times [0, T]$.

In the proof of Theorem 2.4 for a linear problem we have reduced it to the linear model conjunction problem (2.32)–(2.34) with a small parameter. We consider this problem. It is required to find functions $z_j,\ j = 1, 2$, and $\psi(x', t)$ under the conditions

$$\partial_t z_j - a_j\, \Delta z_j = f_j(x, t) \ \text{ in } \ D_{jT}, \quad j = 1, 2, \tag{B.1}$$

$$\psi|_{t=0} = 0,\ z_j|_{t=0} = 0 \ \text{ in } D_j,\ j = 1, 2, \tag{B.2}$$

(B.3) $$z_1 - \beta_1\psi = \eta_1(x',t),\ z_2 - \beta_2\psi = \eta_2(x',t) \ \text{ on } \ R_T,$$

(B.4) $$b\nabla^T z_1 - c\nabla^T z_2 + h'\nabla'\psi + \kappa\,\partial_t\psi = \varphi(x',t) \text{ on } R_T,$$

where all coefficients are constant, $a_j > 0$, $b = (b', b_n)$, $b' = (b_1, \dots, b_{n-1})$, $c = (c', c_n)$, $c' = (c_1, \dots, c_{n-1})$, $h' = (h_1, \dots, h_{n-1})$, κ a small parameter.

THEOREM B.1. *Let*

(B.5) $$0 < |\kappa| \le \kappa_0,\ b_n\,\beta_1\,\kappa > 0,\ c_n\,\beta_2\,\kappa > 0.$$

For any $f_j \in \overset{\circ}{C}{}^{\alpha,\alpha/2}_{x\ t}(D_{jT})$, $\alpha \in (0,1)$, $\eta_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x'\ \ \ t}(R_T)$, $j = 1,2$, $\varphi \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ t}(R_T)$ *the problem (B.1)-(B.4) has a unique solution* $z_j(x,t) \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x\ \ \ t}(D_{jT})$, $j = 1,2$, $\psi(x',t) \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x'\ \ \ t}(R_T)$, $\kappa\,\partial_t\psi(x',t) \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ t}(R_T)$, *and it satisfies the estimate*

(B.6) $$\sum_{j=1}^{2} |z_j|^{(2+\alpha)}_{D_{jT}} + |\psi|^{(2+\alpha)}_{R_T} + |\kappa\partial_t\psi|^{(1+\alpha)}_{R_T} \le C_1\Big(\sum_{j=1}^{2}(|f_j|^{(\alpha)}_{D_{jT}} + |\eta_j|^{(2+\alpha)}_{R_T}) + |\varphi|^{(1+\alpha)}_{R_T}\Big),$$

where the constant C_1 does not depend on κ.

Proof. We construct auxiliary functions $V_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x\ \ \ t}(D_{jT})$, $j = 1,2$, as the solutions of the following first boundary value problems

$$\partial_t V_j - a_j\,\Delta V_j = f_j(x,t) \ \text{ in } \ D_{jT},\ \ V_j|_{x_n=0} = \eta_j(x',t),\ \ j = 1,2.$$

For the solutions of these problems the following estimates are valid [18]

(B.7) $$|V_j|^{(2+\alpha)}_{D_{jT}} \le C_{1+j}(|f_j|^{(\alpha)}_{D_{jT}} + |\eta_j|^{(2+\alpha)}_{R_T}),\ j = 1,2.$$

After substitutions in (B.1)–(B.4)

(B.8) $$z_j = u_j + V_j,\ j = 1,2,$$

we obtain the problem for the functions u_j, $j = 1,2$, ψ

$$\partial_t u_j - a_j\,\Delta u_j = 0 \ \text{ in } \ D_{jT},\ \ j = 1,2,$$

(B.9) $$u_1 - \beta_1\psi = 0,\ u_2 - \beta_2\psi = 0 \ \text{ on } \ R_T,$$

$$b\nabla u_1 - c\nabla u_2 + h'\nabla'\psi + \kappa\,\partial_t\psi = g(x',t) \text{ on } R_T,$$

where $g = \varphi - (b\nabla^T V_1 - c\nabla^T V_2)|_{x_n=0} \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ t}(R_T)$ and

(B.10) $$|g|^{(1+\alpha)}_{R_T} \le C_4\sum_{j=1}^{2}|V_j|^{(2+\alpha)}_{D_{jt}} + |\varphi|^{(1+\alpha)}_{R_t}.$$

Applying Laplace (L) transform on t and Fourier (F) transform on x' to the problem (B.9) we find

(B.11) $$FL[u_j(x,t)] := \tilde{u}_j(s',x_n,p) = \frac{\beta_j}{\kappa\,\zeta}\,\tilde{g}(s',p)\,e^{-r_j|x_n|},\ \ \tilde{\psi} = \frac{1}{\kappa\,\zeta}\tilde{g},$$

where $j = 1,2$,

$$\zeta = \ p + \frac{\beta_1 b_n}{\kappa}r_1 + \frac{\beta_2 c_n}{\kappa}r_2 + i\frac{d's'}{\kappa},\ r_j = \frac{\sqrt{p + a_j s'^2}}{\sqrt{a_j}},$$

$d' = \beta_1 b' - \beta_2 c' + h'$, $\operatorname{Re}\zeta \geq a_0 = \text{const} > 0$ due to (B.5). With the help of the inverse Laplace transform on p and Fourier transform on s' applied to the functions (B.11) we obtain the solution to the problem (B.9) in the explicit form [4], [7]

$$u_j(x,t) = \frac{\beta_j}{\kappa}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} g(y',\tau) G_j(x'-y', x_n, t-\tau)\, dy', \tag{B.12}$$

$$\psi(x',t) = \frac{1}{\beta_j} u_j(x',0,t), \quad j = 1,2, \tag{B.13}$$

where

$$G_j(x,t) = \int_0^t \partial_{x_n} g_j(x' - d'\sigma/\kappa, (-1)^j x_n, \sigma/\kappa, t-\sigma)\, d\sigma,$$

$$g_1(x' - d'\sigma/\kappa, -x_n, \sigma/\kappa, t) = 4a_1 a_2 \int_0^t d\tau_1 \int_{\mathbb{R}^{n-1}} \Gamma_1(x' - \eta' - d'\sigma/\kappa, \beta_1 b_n \sigma/\kappa - x_n, t - \tau_1)$$
$$\times \partial_{\eta_n} \Gamma_2(\eta', \beta_2 c_n \sigma/\kappa - \eta_n, \tau_1)|_{\eta_n = 0}\, d\eta'$$
$$\equiv 2a_1 \int_0^t d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{1}{(2\sqrt{\pi a_1 (t-\tau_1)})^n} \frac{\beta_2 c_n\,\sigma/\kappa}{(2\sqrt{\pi a_2 \tau_1})^n \tau_1}$$
$$\times e^{-\frac{(x' - \eta' - d'\sigma/\kappa)^2 + (\beta_1 b_n\,\sigma/\kappa - x_n)^2}{4a_1(t-\tau_1)}} e^{-\frac{\eta'^2 + (\beta_2 c_n \sigma/\kappa)^2}{4a_2\tau_1}}\, d\,\eta', \ x_n < 0,$$

$$g_2(x - d'\sigma/\kappa, x_n, \sigma/\kappa, t) = 4a_1 a_2 \int_0^t d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{\eta_n} \Gamma_1(\eta', \beta_1 b_n \sigma/\kappa + \eta_n, \tau_1)$$
$$\times \Gamma_2(x' - \eta' - d'\sigma/\kappa, \beta_2 c_n \sigma/\kappa + x_n, t - \tau_1)|_{\eta_n = 0}\, d\eta'$$
$$\equiv -2a_2 \int_0^t d\,\tau_1 \int_{\mathbb{R}^{n-1}} \frac{\beta_1 b_n\,\sigma/\kappa}{(2\sqrt{\pi a_1 \tau_1})^n \tau_1} \frac{1}{(2\sqrt{\pi a_2 (t-\tau_1)})^n}$$
$$\times e^{-\frac{\eta'^2 + (\beta_1 b_n\,\sigma/\kappa)^2}{4a_1\tau_1}} e^{-\frac{(x' - \eta' - d'\sigma/\kappa)^2 + (\beta_2 c_n \sigma/\kappa + x_n)^2}{4a_2(t-\tau_1)}}\, d\,\eta', \ x_n > 0,$$

$\Gamma_j(x,t) = \frac{1}{(2\sqrt{a_j \pi t})^n} e^{-\frac{x^2}{4a_j t}}$ is a fundamental solution to the heat equation (B.1). In [7] the problem (B.9) was studied with a small parameter κ. For its solution (B.12), (B.13) by direct evaluation the following uniform (with respect to κ) estimate was derived for every given $T > 0$

$$\sum_{j=1}^2 |u_j|_{D_{jT}}^{(2+\alpha)} + |\psi|_{R_T}^{(2+\alpha)} + |\kappa \partial_t \psi|_{R_T}^{(1+\alpha)} \leq C_5 |g|_{R_T}^{(1+\alpha)}, \tag{B.14}$$

where the constant C_5 does not depend on κ.

Applying estimates (B.14), (B.7) for u_j, V_j in formula (B.8) and gathering estimates for z_j, ψ and (B.10) for g we obtain an estimate (B.6) and Theorem B.1. ∎

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