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CRITERIA OF LOCAL IN TIME REGULARITY OF THE NAVIER-STOKES EQUATIONS BEYOND SERRIN'S CONDITION

REINHARD FARWIG
Fachbereich Mathematik, Technische Universität Darmstadt
64289 Darmstadt, Germany
E-mail: farwig@mathematik.tu-darmstadt.de

HIDEO KOZONO
Mathematical Institute, Tôhoku University
Sendai, 980-8578 Japan
E-mail: kozono@math.tohoku.ac.jp

HERMANN SOHR
Fakultät für Elektrotechnik, Informatik und Mathematik, Universität Paderborn
33098 Paderborn, Germany
E-mail: hsohr@math.uni-paderborn.de

Abstract. Let u be a weak solution of the Navier-Stokes equations in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ and a time interval $[0,T)$, $0 < T \leq \infty$, with initial value u_0, external force $f = \operatorname{div} F$, and viscosity $\nu > 0$. As is well known, global regularity of u for general u_0 and f is an unsolved problem unless we pose additional assumptions on u_0 or on the solution u itself such as *Serrin's condition* $\|u\|_{L^s(0,T;L^q(\Omega))} < \infty$ where $2/s + 3/q = 1$. In the present paper we prove several local and global regularity properties by using assumptions beyond Serrin's condition e.g. as follows: If the norm $\|u\|_{L^r(0,T;L^q(\Omega))}$ and a certain norm of F satisfy a ν-dependent smallness condition, where Serrin's number $2/r + 3/q > 1$, or if u satisfies a *local leftward* L^s-L^q-condition for every $t \in (0,T)$, then u is regular in $(0,T)$.

1. Introduction and main results. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$ in the sense that $\partial\Omega$ is uniform of class $C^{2,1}$, and let $[0,T)$ be a time interval

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with $0 < T \leq \infty$. We consider the Navier-Stokes system

$$u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0, \qquad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0, \tag{1.1}$$

with external force $f = \operatorname{div} F$, $F \in L^2(\Omega \times (0,T))$, initial value $u_0 \in L^2_\sigma(\Omega)$ and viscosity $\nu > 0$. Then we are interested in weak solutions u of this system defined as follows.

DEFINITION 1.1. A vector field

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); W^{1,2}_0(\Omega)) \tag{1.2}$$

is called a *weak solution* of the system (1.1) with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$, $F = (F_{i,j})^3_{i,j=1} \in L^2(\Omega \times (0,T))$, if the relation

$$-\langle u, v_t\rangle_{\Omega,T} + \nu\langle \nabla u, \nabla v\rangle_{\Omega,T} - \langle uu, \nabla v\rangle_{\Omega,T} = \langle u_0, v(0)\rangle_\Omega - \langle F, \nabla v\rangle_{\Omega,T} \tag{1.3}$$

is satisfied for all test functions $v \in C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$.

Here we use the following notations: $\langle\cdot,\cdot\rangle_\Omega$ means the usual pairing of functions on Ω, $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $\Omega \times [0,T)$, $L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$ with $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega); \operatorname{div} v = 0\}$ and $W^{1,2}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Moreover, $uu = (u_iu_j)^3_{i,j=1}$ for $u = (u_1, u_2, u_3)$.

We know, see [13, V, (3.6.3)], that there exists a weak solution u as in Definition 1.1 which additionally satisfies the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|^2_2 + \nu\int_\sigma^t \|\nabla u\|^2_2\, d\tau \leq \frac{1}{2}\|u(\sigma)\|^2_2 - \int_\sigma^t \langle F, \nabla u\rangle_\Omega\, d\tau \tag{1.4}$$

for almost all $\sigma \in [0,T)$, including $\sigma = 0$, and all $t \in [\sigma, T)$. This energy inequality is needed for the local in time identification of u with strong solutions.

Each weak solution u satisfies the condition

$$u \in L^r(0,T; L^q(\Omega)), \quad 2 \leq q,\ r < \infty, \quad \frac{2}{r} + \frac{3}{q} = \frac{3}{2}. \tag{1.5}$$

Without loss of generality we may assume in the following that

$$u : [0,T) \to L^2_\sigma(\Omega) \quad \text{is weakly continuous,} \tag{1.6}$$

with $u(0) = u_0$. Further, there exists a distribution p, called an *associated pressure*, such that

$$u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = f \tag{1.7}$$

holds in the sense of distributions, see [13, Chapter V.1]. Conversely, if u satisfies (1.2), (1.6), $u(0) = u_0$, and if (1.7) holds with some p in the sense of distributions, then u is a weak solution in the sense of Definition 1.1.

We will use Definition 1.1 with obvious modifications if the interval $[0,T)$ is replaced by any other interval $[t_0, T)$ with $0 < t_0 < T$, and with $u|_{t=t_0} = u_0$.

A weak solution u in Definition 1.1 is uniquely determined by u_0 and f if *Serrin's condition*

$$u \in L^s(0,T; L^q(\Omega)), \quad 2 < s < \infty, \quad 3 < q < \infty, \quad \frac{2}{s} + \frac{3}{q} = 1, \tag{1.8}$$

is satisfied, see [12], [13], i.e., if

$$\|u\|_{L^s(0,T;L^q(\Omega))} = \|u\|_{q,s} = \left(\int_0^T \|u\|_q^s \, d\tau \right)^{\frac{1}{s}} < \infty,$$

where $\|u\|_q = \|u(t)\|_{L^q(\Omega)} = (\int_\Omega |u(x,t)|^q \, dx)^{1/q}$. More precisely, u is unique within the class defined by (1.8). The same result holds in the limit case $s = \infty$, $q = 3$, see [10].

Moreover, if u satisfies (1.8), then u is regular in the sense that

$$u \in C^\infty(\overline{\Omega} \times (0,T)), \ p \in C^\infty(\overline{\Omega} \times (0,T)), \tag{1.9}$$

provided $\partial\Omega$ and f are of class C^∞, see [13, Theorem V.1.8.2]. Hence a weak solution u satisfying (1.8) is called a *strong solution*. A similar result was proved for the limit case $s = \infty$, $q = 3$ in a series of papers, see e.g. [11].

A point $t \in (0,T)$ is called a *regular point* of a weak solution u if there exists a subinterval $(t-\delta, t+\delta) \subset (0,T)$, $\delta > 0$, such that $u \in L^s(t-\delta, t+\delta; L^q(\Omega))$ with s, q as in (1.8). Otherwise t is called a *singular point* of u.

Now our first main result reads as follows:

THEOREM 1.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and let $0 < T \leq \infty$. Furthermore let*[1] *$4 < s < \infty$, $3 < q < 6$, $1 < q^* < q$ and $1 \leq r \leq s$ be given with $\frac{2}{s} + \frac{3}{q} = 1$ and $\frac{1}{3} + \frac{1}{q} = \frac{1}{q^*}$. For $u_0 \in L^2_\sigma(\Omega)$ and $f = \operatorname{div} F$, $F \in L^2(0,T; L^2(\Omega)) \cap L^s(0,T; L^{q^*}(\Omega))$, consider a weak solution u of the Navier-Stokes system* (1.1) *satisfying the strong energy inequality* (1.4).

(i) Assume $0 \neq u_0 \in L^q_\sigma(\Omega)$. Then there exist constants $\varepsilon_* = \varepsilon_*(q,\Omega) > 0$ and $c_0 = c_0(q,\Omega,r) > 0$ with the following property: If

$$\int_0^T \|F\|_{q^*}^s \, d\tau \leq \nu^{2s-1} \varepsilon_* \quad \text{and} \quad \int_0^T \|u\|_q^r \, d\tau \leq c_0 \frac{\nu^{s+r-1} \varepsilon_*}{\|u_0\|_q^s}, \tag{1.10}$$

then u is regular in the sense that $u \in L^s(0,T; L^q(\Omega))$.

(ii) Suppose for each $T_1 \in (0,T)$ there is some $0 < \delta = \delta(T_1) < T_1$ such that u satisfies the leftward L^s-L^q-condition

$$u \in L^s(T_1 - \delta, T_1; L^q(\Omega)).$$

Then u is regular in the sense that $u \in L^s_{\mathrm{loc}}((0,T); L^q(\Omega))$.

We remark that the constant $c_0 = c_0(q,\Omega,r) > 0$ in (1.10) mainly depends on the boundedness of the Stokes semigroup $\{e^{-tA_q} : t > 0\}$, see §2 below, but that $\varepsilon_* = \varepsilon_*(q,\Omega) > 0$ is related to the nonlinearity of the Navier-Stokes system. Note that if $r < s$ and consequently $\frac{2}{r} + \frac{3}{q} > 1$, then Theorem 1.2 (i) yields the regularity of the weak solution u beyond Serrin's barrier $\frac{2}{s} + \frac{3}{q} = 1$. The proof is based on the following theorem yielding a local in time regularity result.

[1]In the meantime the restriction $4 < s < \infty$, $3 < q < 6$, see also Lemma 2.1 below, has been removed by the authors. For the more general result when $2 < s < \infty$, $3 < q < \infty$ see the forthcoming paper *Very weak, weak and strong solutions to the instationary Navier-Stokes system*, Nečas Center for Mathematical Modeling, Lecture Notes, Vol. 1, Prague, 2007, 15–68.

THEOREM 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and let $0 < T \le \infty$. Consider a weak solution u of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$ and $f = \operatorname{div} F$, $F \in L^2(\Omega \times (0,T))$, satisfying the strong energy inequality* (1.4)*. Moreover let*[1] $4 < s < \infty$, $3 < q < 6$, $1 < q^* < q$, $1 \le r \le s$, *and* $0 \le \beta \le \frac{r}{s}$ *with* $\frac{2}{s} + \frac{3}{q} = 1$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{q^*}$.

Then there is a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If $0 < T_0 < T_1 < T' < T$, and if

$$\int_{T_0}^{T'} \|F(\tau)\|_{q^*}^s \, d\tau \le \varepsilon_* \nu^{2s-1}, \qquad \frac{1}{T_1 - T_0} \int_{T_0}^{T_1} (T' - \tau)^\beta \|u(\tau)\|_q^r \, d\tau \le \varepsilon_* \nu^{r-\beta}, \tag{1.11}$$

then u is regular in some interval $(T_1 - \delta, T') \subset (0,T)$, $\delta > 0$, in the sense that Serrin's condition

$$u \in L^s(T_1 - \delta, T'; L^q(\Omega))$$

is satisfied. In particular, T_1 is a regular point of u. If $\beta = 0$, then $T' = T \le \infty$ is allowed.

COROLLARY 1.4. *Let u be a weak solution in $\Omega \times [0,T)$ and let r, s, q, q^* be exponents as in Theorem 1.3.*

(i) Let $T = \infty$ and assume that

$$\int_0^\infty \|F\|_{q^*}^s \, d\tau \le \varepsilon_* \nu^{2s-1} \quad \text{and} \quad T_1 > \frac{1}{\varepsilon_* \nu^r} \|u\|^r_{L^r(0,\infty; L^q(\Omega))} \tag{1.12}$$

with ε_* as in (1.11)*. Then u is regular for $t \ge T_1$, i.e., $u \in L^s(T_1, \infty; L^q(\Omega))$.*

(ii) Let $0 < T_1 < T \le \infty$ and assume that

$$\liminf_{\delta \to 0} \frac{1}{\delta^{1-\beta}} \int_{T_1 - \delta}^{T_1} \|u(\tau)\|_q^r \, d\tau = 0, \quad 0 \le \beta \le \frac{r}{s}. \tag{1.13}$$

Then there exist T' and $\delta_0 > 0$, $0 < T_1 - \delta_0 < T_1 < T' \le T$, such that u is regular in $(T_1 - \delta_0, T')$ in the sense $u \in L^s(T_1 - \delta_0, T'; L^q(\Omega))$. In particular, T_1 is a regular point.

We note that the condition (1.13) may be replaced by the slightly weaker smallness condition

$$\liminf_{\delta \to 0} \frac{1}{\delta^{1-\beta}} \int_{T_1 - \delta}^{T_1} \|u(\tau)\|_q^r \, d\tau < \varepsilon_* \nu^{r-\beta} 2^{-\beta} \tag{1.14}$$

with ε_* as in Theorem 1.3.

If $r = s$, then the local leftward Serrin condition $\int_{T_1 - \delta_0}^{T_1} \|u(\tau)\|_q^s \, d\tau < \infty$ with some fixed $\delta_0 > 0$ is sufficient for (1.13) when $\beta = \frac{r}{s} = 1$ and implies that T_1 is a regular point. Furthermore, (1.13) is satisfied with $0 < \beta \le \frac{r}{s} \le 1$ if $T_1 \in (0,T)$ is a Lebesgue point of $t \mapsto \|u(t)\|_q^r$, $t \in (0,T)$, in the sense that

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{T_1 - \delta}^{T_1} \|u(\tau)\|_q^r \, d\tau = \|u(T_1)\|_q^r. \tag{1.15}$$

Conversely, if $T_1 \in (0,T)$ is a singular point of u in the sense that there is no $T' > T_1$ such that u is contained in $L^s(T_1, T'; L^q(\Omega))$, then for all $\beta \in \left[0, \frac{r}{s}\right]$

$$\liminf_{\delta \to 0} \frac{1}{\delta^{1-\beta}} \int_{T_1 - \delta}^{T_1} \|u(\tau)\|_q^r \, d\tau \ge \varepsilon_* \nu^{r-\beta} 2^{-\beta}. \tag{1.16}$$

The set of such points T_1 (is empty or) has Lebesgue measure zero.

2. Preliminaries. Given a bounded smooth domain $\Omega \subseteq \mathbb{R}^3$ as in Section 1 we use the well-known spaces $L^q(\Omega)$, $1 < q < \infty$, with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and pairing $\langle v, w\rangle = \langle v, w\rangle_\Omega = \int_\Omega v \cdot w\,dx$ for $v \in L^q(\Omega)$, $w \in L^{q'}(\Omega)$, $q' = \frac{q}{q-1}$. Moreover, given $0 < T \le \infty$, we need the Bochner spaces $L^s(0,T;L^q(\Omega))$, $1 \le s < \infty$, with norm $\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s} = (\int_0^T \|\cdot\|_q^s\,dt)^{1/s}$ and the corresponding pairing $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\Omega,T}$ on $L^s(0,T;L^q(\Omega)) \times L^{s'}(0,T;L^{q'}(\Omega))$, $s' = \frac{s}{s-1}$. Furthermore, we will use the smooth function spaces $C_0^\infty(\Omega)$, $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega); \operatorname{div} v = 0\}$ and the space $L^q_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$.

Concerning the Stokes operator $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$, $D(A_q) \subseteq L^q_\sigma(\Omega)$, and the Helmholtz projection $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ in L^q-spaces we refer to [1], [3] – [7]. In particular we need the following estimates, see [4]:

$$\|v\|_\gamma \le C\|A_q^\alpha v\|_q \text{ for all } v \in D(A_q^\alpha),\ 1 < q \le \gamma,\ 0 \le \alpha \le 1, \quad \text{where } 2\alpha + \frac{3}{\gamma} = \frac{3}{q}, \tag{2.1}$$

$$\|A_q^\alpha e^{-\nu t A_q} v\|_q \le C\nu^{-\alpha} e^{-\nu\delta t} t^{-\alpha}\|v\|_q \text{ for all } v \in L^q_\sigma(\Omega),\ t > 0, \quad \text{where } \delta = \delta(\Omega, q) > 0 \text{ and } 0 \le \alpha \le 1, \tag{2.2}$$

$$\|A_q^{-\frac{1}{2}} P_q \operatorname{div} v\|_q \le C\|v\|_q \text{ for all } v = (v_{ij})_{i,j=1}^3 \in L^q(\Omega), \tag{2.3}$$

$$\|v\|_{L^s(0,T;L^q(\Omega))} \le C\frac{1}{\nu}\|f\|_{L^s(0,T;L^q(\Omega))} \text{ for all } f \in L^s(0,T;L^q_\sigma(\Omega)), \quad \text{where } v(t) = A_q\int_0^t e^{-\nu(t-\tau)A_q} f(\tau)\,d\tau. \tag{2.4}$$

The constants C in (2.1)–(2.4) depend on Ω and q, s, α, but are independent of v and ν. Note that the norms $\|A_q^{1/2}v\|_q$ and $\|\nabla v\|_q$ are equivalent for $v \in D(A_q^{1/2})$.

To prove our main results we have to identify the given weak solution u locally in time with strong solutions, i.e. with weak solutions satisfying Serrin's regularity condition. There are many results on the existence of such solutions for some given interval $[0, T)$, $0 < T \le \infty$, if the initial value u_0 satisfies a certain smallness condition, see, e.g., [8]–[10], [14]. However, we need some particular weak assumption on u_0 and will apply Theorem 1 in [4] for bounded domains.

LEMMA 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and let $4 < s < \infty$, $3 < q < 6$, $1 < q^* < q$ satisfy $\frac{1}{3} + \frac{1}{q} = \frac{1}{q^*}$ and $\frac{2}{s} + \frac{3}{q} = 1$. Moreover, let $u_0 \in L^q_\sigma(\Omega)$ and $f = \operatorname{div} F$, $F \in L^s(0,T;L^{q^*}(\Omega))$, $0 < T \le \infty$. Then there is a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If*

$$\int_0^T \|F\|_{q^*}^s\,d\tau \le \varepsilon_*\nu^{2s-1} \quad \textit{and} \quad \int_0^T \|e^{-\nu\tau A_q}u_0\|_q^s\,d\tau \le \varepsilon_*\nu^{s-1}, \tag{2.5}$$

then there exists a unique weak solution u in $\Omega \times [0,T)$ of the Navier-Stokes system (1.1) *satisfying Serrin's condition*

$$u \in L^s(0,T;L^q(\Omega)) \tag{2.6}$$

and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega \, d\tau \ , \ 0 \le t < T. \tag{2.7}$$

Proof. In the case $\nu = 1$ the existence result of [4, Theorem 1] yields – under the smallness condition (2.5), see [4, (4.23)] – a unique solution u in the following so-called *very weak sense*: It satisfies (2.6) and the relation

$$-\langle u, v_t\rangle_{\Omega,T} - \nu\langle u, \Delta v\rangle_{\Omega,T} - \langle uu, \nabla v\rangle_{\Omega,T} = \langle u_0, v(0)\rangle_\Omega - \langle F, \nabla v\rangle_{\Omega,T} \tag{2.8}$$

for all $v \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$. It is straightforward to generalize this result to $\nu \neq 1$ and to check that (2.5) is the corresponding smallness condition with constant $\varepsilon_* = \varepsilon_*(q,\Omega) > 0$; for details on the dependence on ν see [5] concerning the theory of very weak solutions in three-dimensional exterior domains and in particular the condition [5, (5.12)].

In order to prove that u is a weak solution satisfying (2.6) we have to show several regularity properties. We start with the case that $4 < s \le 8$ and hence $4 \le q < 6$. Due to the proof in [4, (4.19)] we know that u satisfies the relation

$$\tilde{u}(t) \equiv u(t) - E(t) = -\int_0^t A_q^{\frac{1}{2}} e^{-\nu(t-\tau)A_q} A_q^{-\frac{1}{2}} P_q \operatorname{div}(uu)\, d\tau, \ 0 \le t < T, \tag{2.9}$$

with

$$E(t) = e^{-\nu t A_q} u_0 + \int_0^t e^{-\nu(t-\tau)A_q} f(\tau)\, d\tau.$$

Using (2.3) and Hölder's inequality we obtain that

$$\|A_{q/2}^{-\frac{1}{2}} P_{q/2} \operatorname{div}(uu)\|_{q/2} \le C\|uu\|_{q/2} \le C\|u\|_q^2 \tag{2.10}$$

where here and in the following C is a generic positive constant depending only on q and Ω. By (2.9)

$$A_q^{\frac{1}{2}} \tilde{u}(t) = -A_q \int_0^t e^{-\nu(t-\tau)A_q} A_q^{-\frac{1}{2}} P_q \operatorname{div}(uu)\, d\tau, \ 0 \le t < T, \tag{2.11}$$

and using (2.4) we get the estimate

$$\|\nabla \tilde{u}\|_{\frac{q}{2},\frac{s}{2}} \le C\|A_{q/2}^{\frac{1}{2}} \tilde{u}\|_{\frac{q}{2},\frac{s}{2}} \le C\frac{1}{\nu}\|uu\|_{\frac{q}{2},\frac{s}{2}} \le C\frac{1}{\nu}\|u\|_{q,s}^2 < \infty. \tag{2.12}$$

This shows that

$$\nabla \tilde{u} \in L^{s/2}(0,T; L^{q/2}(\Omega)) \tag{2.13}$$

and, since $4 \le q < 6$, $4 < s \le 8$, that

$$\nabla \tilde{u} \in L^2_{\text{loc}}([0,T); L^2(\Omega)), \ \tilde{u} \in L^2_{\text{loc}}([0,T); W_0^{1,2}(\Omega)). \tag{2.14}$$

By virtue of (2.2) and (2.3), Hölder's inequality and the properties of q and s we obtain from (2.9) the estimate

$$\begin{aligned}
(2.15)\qquad \|\tilde u(t)\|_2 &\le \frac{C}{\nu^{\frac12}}\int_0^t \frac{1}{(t-\tau)^{\frac12}}\, e^{-\nu\delta(t-\tau)}\|uu\|_2\,d\tau\\
&\le \frac{C}{\nu^{\frac12}}\int_0^t \frac{1}{(t-\tau)^{\frac12}}\, e^{-\nu\delta(t-\tau)}\|uu\|_{\frac q2}\,d\tau\\
&\le C\nu^{-1+\frac2s}\|uu\|_{\frac q2,\frac s2} \le C\nu^{-1+\frac2s}\|u\|_{q,s}^2.
\end{aligned}$$

Hence (2.14) and (2.15) imply that

$$(2.16)\qquad \tilde u \in L^\infty([0,T);L^2_\sigma(\Omega))\cap L^2_{\mathrm{loc}}([0,T);W^{1,2}_0(\Omega)).$$

Concerning $E(t)$ standard energy estimates, see e.g. [13, Theorem V.1.4.1], yield the inequalities

$$(2.17)\qquad \|E\|^2_{2,\infty}+\nu\|\nabla E\|^2_{2,2}\le \|u_0\|_2^2+\frac1\nu\|F\|^2_{2,2}.$$

With the help of (2.16) and (2.17) we conclude that

$$(2.18)\qquad u \in L^\infty([0,T);L^2_\sigma(\Omega))\cap L^2_{\mathrm{loc}}([0,T);W^{1,2}_0(\Omega)).$$

Since $u\in L^s(0,T';L^q(\Omega))$ for all $0<T'<T$, Hölder's inequality yields

$$(2.19)\qquad uu\in L^2_{\mathrm{loc}}([0,T);L^2(\Omega)).$$

Using (2.18) and (2.19), a calculation shows that (2.8) implies (1.3), and that the energy inequality (2.7) is satisfied; see also [13, Theorem V.1.4.1] concerning the last property. Consequently u is a weak solution of (1.1) satisfying (2.6) and (2.7). Hence it is also a strong solution. The uniqueness of u with these properties follows from Serrin's uniqueness argument, see [12], [13]. This completes the proof in the case that $4<s\le 8$.

In the second case we assume that $8<s<\infty$ and $3<q<4$. Now we need several steps. First let $s_1=s$, $q_1=q$. Then we get as in (2.9)–(2.13) that $\nabla\tilde u\in L^{s_1/2}(0,T;L^{q_1/2}(\Omega))$. Defining $s_2=\frac{s_1}{2}$ and $q_2>q_1$ such that $\frac13+\frac1{q_2}=\frac1{q_1/2}$, $\frac2{s_2}+\frac3{q_2}=1$, we obtain by Sobolev's embedding theorem that $\tilde u\in L^{s_2}(0,T;L^{q_2}(\Omega))$. Moreover, using (2.1), (2.2) we see that $E\in L^{s_2}(0,T;L^{q_2}(\Omega))$ which leads to $u\in L^{s_2}(0,T;L^{q_2}(\Omega))$. Proceeding in the same way, let $s_k=\frac{s_{k-1}}{2}$ and $q_k>q_{k-1}$ such that $\frac13+\frac1{q_k}=\frac1{q_{k-1}/2}$, $\frac2{s_k}+\frac3{q_k}=1$, for $k\in\mathbb N$. Since $\frac13-\frac1{q_k}=2^{k-1}(\frac13-\frac1{q_1})$, we choose $k\in\mathbb N$ such that $\frac13-\frac1{q_{k-1}}<\frac1{12}\le\frac13-\frac1{q_k}$, leading to $4\le q_k<6$, $4<s_k\le 8$. Now $q_k/2\ge 2$, and using (2.12), (2.15) with q,s replaced by q_k,s_k, we obtain the properties (2.14), (2.16). This yields the result in the same way as in the first case. Now the proof of the lemma is complete. ■

3. Proof of the theorems. First we have to prove Theorem 1.3.

Proof of Theorem 1.3. Given the bounded domain $\Omega\subseteq\mathbb R^3$, $0<T_0<T_1<T'<T$ and u,q,r,s,β as in this theorem, we have to prove the existence of some constant $\varepsilon_*=\varepsilon_*(\Omega,q)>0$ yielding regularity of u on $(T_1-\delta,T')$ if (1.11) is satisfied. Note that if $\beta=0$ then the subsequent proof will also work for $T'=T$.

Using the weak continuity of the weak solution $u : [0,T) \to L^2_\sigma(\Omega)$, see (1.6), we know that $u(t_0) \in L^2_\sigma(\Omega)$ is well defined for all $t_0 \in [0,T)$. Furthermore, since $\nabla u \in L^2(0,T;L^2(\Omega))$, see (1.4) for $\sigma = 0$, and since $3 < q < 6$, the embedding inequality $\|u(t)\|_q \le C_1\|u(t)\|_6 \le C_2\|\nabla u(t)\|_2$ with $C_j = C_j(\Omega,q) > 0$, $j = 1,2$, implies that $u \in L^2(0,T;L^q_\sigma(\Omega))$. Then the Lebesgue point argument shows that there is a null set $N \subseteq (0,T)$ such that $\|u(t_0)\|_q$ is well defined by the property

$$\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \|u(\tau)\|_q^2 \, d\tau = \|u(t_0)\|_q^2 \tag{3.1}$$

for all $t_0 \in (0,T)\backslash N$. Moreover, since the energy inequality (1.4) holds for a.a. $\sigma \in [0,T)$, we may assume in the following that the null set $N \subseteq (0,T)$ is chosen in such a way that both (3.1) and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_{t_0}^t \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(t_0)\|_2^2 - \int_{t_0}^t \langle F, \nabla u\rangle_\Omega \, d\tau, \quad t_0 \le t < T, \tag{3.2}$$

hold for all $t_0 \in (0,T)\backslash N$.

Let $t_0 \in (T_0,T_1)\backslash N$. Then $u(t_0) \in L^q_\sigma(\Omega)$, and we are able to apply the local existence results of Lemma 2.1, replacing the existence interval $[0,T)$ by the interval $[t_0,T')$, and using $u(t_0)$ as initial value. Hence, if the smallness condition

$$\int_{t_0}^{T'} \|F\|_{q^*}^s \, d\tau \le \varepsilon_* \nu^{2s-1}, \quad \int_0^{T'-t_0} \|e^{-\nu\tau A_q} u(t_0)\|_q^s \, d\tau \le \varepsilon_* \nu^{s-1} \tag{3.3}$$

is satisfied with ε_* as in Lemma 2.1, then we obtain a unique weak solution $\tilde u$ on the interval $[t_0,T')$, corresponding to Definition 1.1, of the Navier-Stokes system

$$\begin{aligned} \tilde u_t - \nu\Delta\tilde u + \tilde u\cdot\nabla\tilde u + \nabla\tilde p = f, &\quad \operatorname{div}\tilde u = 0, \\ \tilde u_{|\partial\Omega} = 0, &\quad \tilde u_{|t=t_0} = u(t_0), \end{aligned} \tag{3.4}$$

satisfying

$$\tilde u \in L^\infty(t_0,T';L^2_\sigma(\Omega)) \cap L^2_{\rm loc}([t_0,T');W_0^{1,2}(\Omega)), \quad \tilde u \in L^s(t_0,T';L^q(\Omega)), \tag{3.5}$$

and the energy inequality

$$\frac{1}{2}\|\tilde u(t)\|_2^2 + \nu \int_{t_0}^t \|\nabla\tilde u\|_2^2 \, d\tau \le \frac{1}{2}\|u(t_0)\|_2^2 - \int_{t_0}^t \langle F, \nabla\tilde u\rangle_\Omega \, d\tau\,, \ t_0 \le t < T'.$$

By Serrin's uniqueness argument, see [12], [13, V, Theorem 1.5.1], we conclude that $u = \tilde u$ on $[t_0,T')$. This yields the properties (3.5) with $\tilde u$ replaced by u, and we get the desired result of Theorem 1.3.

Thus it remains to prove the existence of some $t_0 \in (T_0,T_1)\backslash N$ as above such that (3.3) is satisfied. First assume that the conditions

$$\int_{T_0}^{T'} \|F\|_{q^*}^s \, d\tau \le \varepsilon_*' \nu^{2s-1}, \quad \frac{1}{T_1-T_0} \int_{T_0}^{T_1} (T'-\tau)^\beta \, \|u(\tau)\|_q^r \, d\tau \le \varepsilon_*' \nu^{r-\beta} \tag{3.6}$$

are satisfied with some constant $\varepsilon_*' > 0$ to be determined below. Then we find at least one $t_0 \in (T_0,T_1)\backslash N$ such that

$$(T'-t_0)^\beta \|u(t_0)\|_q^r \le \frac{1}{T_1-T_0}\int_{T_0}^{T_1} (T'-\tau)^\beta \|u(\tau)\|_q^r\, d\tau \le \varepsilon'_* \nu^{r-\beta}. \tag{3.7}$$

Hence, by virtue of (2.2) with $\alpha = 0$ and of the condition (3.7),

$$\begin{aligned}\int_0^{T'-t_0} \|e^{-\nu\tau A_q} u(t_0)\|_q^s\, d\tau \ &\le\ c_0 \int_0^{T'-t_0} e^{-\nu\delta s\tau}\, d\tau\, \|u(t_0)\|_q^s \\ &\le c_0 (T'-t_0)^{\frac{\beta s}{r}} \left(\int_0^{T'-t_0} e^{-\nu\delta s\tau}\, d\tau\right)^{1-\frac{\beta s}{r}} \|u(t_0)\|_q^s \\ &\le c_0(\varepsilon'_*\nu^{r-\beta})^{\frac{s}{r}} (\nu^{\frac{\beta s}{r}-1}) = c_0(\varepsilon'_*)^{\frac{s}{r}} \nu^{s-1},\end{aligned}$$

where $c_0 = c_0(q,\Omega,\beta,r) > 0$ is a generic constant. This estimate shows how to choose the smallness constant ε'_* in (3.6) depending on ε_* in (3.3), in order to prove Theorem 1.3. For simplicity we denote the constant ε'_* finally in Theorem 1.3 again by ε_*. The proof is complete. ∎

Proof of Theorem 1.2. (i) By Lemma 2.1 there exists some $\delta = \delta(u_0,\nu,\Omega,q,\varepsilon_*) \in (0,T)$ such that $u \in L^s(0,\delta;L^q(\Omega))$. Actually, the second part of condition (2.5) shows that we may choose $\delta = c_0\varepsilon_*\nu^{s-1}\|u_0\|_q^{-s}$ with ε_* as in (2.5) and $c_0 = c_0(\Omega,q) > 0$. Next let $T_0 = \frac{\delta}{2}$, $T_1 = \delta$, and denoting the constant ε_* from (1.11) here by ε'_*, we assume that

$$\int_0^T \|u\|_q^r\, d\tau \le \frac{\delta}{2}\varepsilon'_*\nu^r = \frac{c_0}{2}\varepsilon_*\varepsilon'_*\nu^{s+r-1}\|u_0\|_q^{-s}$$

is satisfied. Using Theorem 1.3 with $\beta=0$ and $T'=T$ we conclude that $u\in L^s(T_1,T;L^q(\Omega))$ and even $u \in L^s(0,T;L^q(\Omega))$. This proves (i).

(ii) In this case we use Theorem 1.3 with $r = s$ and $\beta = \frac{r}{s} = 1$. Let $T_1 \in (0,T)$ and choose $0 < \delta < T_1$ such that $u \in L^s(T_1-\delta,T_1;L^q(\Omega))$ satisfies the estimate $2\|u\|^s_{L^s(T_1-\delta,T_1;L^q(\Omega))} \le \varepsilon_*\nu^{r-1}$ with ε_* from (1.11). Moreover, we can reach with $T' = T_1+\delta$ and $T_0 = T_1-\delta$, that

$$\int_{T_0}^{T'} \|F\|_{q*}^s\, d\tau = \int_{T_1-\delta}^{T+\delta} \|F\|_{q*}^s\, d\tau \le \varepsilon_*\nu^{2s-1} \tag{3.8}$$

and

$$\frac{1}{T_1-T_0}\int_{T_0}^{T_1} (T'-t)\|u(t)\|_q^s\, dt \le 2\int_{T_0}^{T_1} \|u(t)\|_q^s\, dt \le \varepsilon_*\nu^{r-1}.$$

Then Theorem 1.3 implies that $u \in L^s(T_1-\delta,T_1+\delta;L^q(\Omega))$. We can find such a $\delta > 0$ for each $T_1 \in (0,T)$ and get the result. ∎

Proof of Corollary 1.4. (i) Condition (1.12) implies (1.11) with $\beta = 0$ for some sufficiently small $T_0 > 0$.

(ii) Assume that (1.13) (or only (1.14)) holds. Then we find a sufficiently small $\delta > 0$ such that with $T' = T_1+\delta$ and $T_0 = T_1-\delta$ the estimates

$$\frac{1}{T_1-T_0}\int_{T_0}^{T_1} (T'-\tau)^\beta\|u(t)\|_q^r\, d\tau \le 2^\beta\delta^{\beta-1}\int_{T_1-\delta}^{T_1} \|u(t)\|_q^r\, d\tau \le \varepsilon_*\nu^{r-\beta}$$

and (1.11) are satisfied. ∎

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