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LINEAR FLOW PROBLEMS IN 2D EXTERIOR DOMAINS FOR 2D INCOMPRESSIBLE FLUID FLOWS

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Abstract. The paper analyzes the issue of existence of solutions to linear problems in two dimensional exterior domains, linearizations of the Navier-Stokes equations. The systems are studied with a slip boundary condition. The main results prove the existence of distributional solutions for arbitrary data.

1. Introduction. Linear problems play the crucial role in investigations of asymptotic structure of solutions to the stationary Navier-Stokes equations in exterior domains. Qualitative properties of these solutions follow from the fact that the nonlinear term $v \cdot \nabla v$ does not determine the behaviour of the system in a neighbourhood of infinity. Thus analysis of a suitable linearization can give us appropriate information about the analyzed equations.

We investigate two linear problems. Both come from linearization of the Navier-Stokes equations and are expressed in terms of the vorticity α of the fluid. We consider these problems in the two-dimensional exterior domain

$$\Omega = \mathbb{R}^2 \setminus B,$$

where $B \subset \mathbb{R}^2$ is a simply connected bounded domain with smooth boundary.

Slip boundary conditions

$$\vec{n} \cdot \mathbb{T}(v,p) \cdot \vec{\tau} + f(v \cdot \vec{\tau}) = 0 \tag{1.1}$$

(see below for notation) are assumed in both cases, which allows us to show existence of solutions without assumptions on smallness of the data (see [11]).

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What is worth noticing is that from (1.1) one cannot directly get full information about the velocity v (only its tangential part), however in terms of the vorticity α this condition rewrites as Dirichlet condition:

$$\alpha = (2\chi - f/\nu)(v \cdot \vec{\tau}), \tag{1.2}$$

where χ is the curvature of $\partial\Omega$ (see [14]).

The first problem we consider, which is in fact the Stokes problem expressed in terms of the vorticity of the fluid, is as follows:

$$\begin{aligned} -\nu\Delta\alpha &= \operatorname{rot} F && \text{in } \Omega,\\ \operatorname{rot} v &= \alpha && \text{in } \Omega,\\ \operatorname{div} v &= 0 && \text{in } \Omega,\\ v \cdot \vec{n} &= 0 && \text{on } \partial\Omega,\\ \alpha &= (2\chi - f/\nu)(v \cdot \vec{\tau}) && \text{on } \partial\Omega,\\ v &\to v_\infty && \text{as } |x| \to \infty. \end{aligned} \tag{1.3}$$

For this system we prove the following theorem:

THEOREM 1. *Let $\nu > 0$, $f \in L^\infty(\partial\Omega)$ and $\operatorname{rot} F \in (H^2_0(\Omega))^*$. Then there exists a weak solution $v \in D^1(\Omega)$ to problem (1.3) in the sense of Definition 3.1 such that*

$$\|\nabla u\|_{L^2(\Omega)} \leq DATA. \tag{1.4}$$

What is substantial in this theorem is the fact that the solution v is constructed together with full information about the kernel of the *rot-div* operator (see Section 2).

One of the most important and difficult questions arising in this problem is if condition (1.3_6) is fulfilled for a weak solution v, since there are examples of solenoidal vector fields which satisfy (1.4) and are unbounded at infinity. In [6] the author assumes extra conditions on integrability of the gradient ∇v for $p \in (1, 2)$, which imply existence of solutions satisfying condition (1.3_6). In [7], [8], [1] and [2] one can find similar results for a restricted version of our problem.

The second system we study is the linearization of the Navier-Stokes problem around a given vector field $\widetilde{v_0}$. We introduce it as follows:

$$\begin{aligned} -\nu\Delta\alpha + \widetilde{v_0} \cdot \nabla\alpha &= \operatorname{rot} F && \text{in } \Omega,\\ \operatorname{rot} v &= \alpha && \text{in } \Omega,\\ \operatorname{div} v &= 0 && \text{in } \Omega,\\ \alpha &= (2\chi - f/\nu)(v \cdot \vec{\tau}) && \text{on } \partial\Omega,\\ v \cdot \vec{n} &= 0 && \text{on } \partial\Omega,\\ v &\to v_\infty && \text{as } |x| \to \infty, \end{aligned} \tag{1.5}$$

where $\widetilde{v_0}$ is a given divergence free vector field, which we define later, satisfying $\widetilde{v_0} = 0$ on $\partial\Omega$ and $\widetilde{v_0} \to v_\infty$ as $|x| \to \infty$. This is a modification of the Oseen system, with $\widetilde{v_0}$ in place of v_∞.

Our second theorem states the existence of solutions to (1.5):

THEOREM 2. *Given $\nu > 0$, $f \in L^\infty(\partial\Omega)$ and $\operatorname{rot} F \in (H^2_0(\Omega))^*$. There exists a weak solution $v \in D^1(\Omega)$ to problem (1.5) in the sense of Definition 4.2 such that*

$$\|\nabla u\|_{L^2(\Omega)} \leq DATA. \tag{1.6}$$

In this theorem we show existence of solutions in the space being the projection on the range of the *rot-div* operator (we omit the kernel part). This is due to the additional term $\widetilde{v_0} \cdot \nabla\alpha$, which presence does not allow the methods used in the Stokes problem to recover information about the kernel part of the *rot-div* operator, to be applied directly.

NOTATION. In the above we use the following notation: v is a velocity vector field, p the corresponding pressure, ν the viscous positive constant coefficient, f the nonnegative friction coefficient, $\mathbb{T}(v,p)$ is the Cauchy stress tensor, i.e. $\mathbb{T}(v,p) = \nu\mathbb{D}(v) + p\mathbb{I}$, where $\mathbb{D}(v) = \{v_{i,j} + v_{j,i}\}_{i,j=1}^2$ is the symetric part of the gradient ∇v, and $\mathbb{I}$ is the identity matrix. Moreover $\vec{n}, \vec{\tau}$ are respectively normal and tangential vector to boundary $\partial\Omega$.

The function spaces $\dot{H}_0^2(\Omega)$ and $D^1(\Omega)$ are introduced in Section 2.

Our paper is organized as follows. In Section 2 we deal with a *rot-div* problem, which is fundamental in our considerations, introduce basic definitions and auxiliary lemmas used in next sections. In Section 3 we give a weak formulation of the problem (1.3) and prove Theorem 1. Similar arrangement is for system (1.5) in Section 4.

2. rot-div problem. In this section we focus on properties of the *rot-div* problem in two types of domain, bounded and exterior, however both non-simply connected. Namely, we consider the following system:

$$\begin{aligned} \operatorname{rot} v &= \alpha \qquad && \text{in } \Omega, \\ \operatorname{div} v &= 0 && \text{in } \Omega, \\ v \cdot \vec{n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Our goal is to get precise information about the kernel of this operator. We will need to consider a proper function space for solutions.

First let us notice that from $(2.1_{2,3})$ and Poincaré Lemma we conclude existence of a scalar stream function Φ such that

$$v = \nabla^\perp \Phi, \tag{2.2}$$

where $\nabla^\perp\Phi = (-\partial_{x_2}\Phi, \partial_{x_1}\Phi)$. From (2.1_3) we see that

$$\nabla\Phi \cdot \vec{\tau} = \nabla^\perp\Phi \cdot \vec{n} = v \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega, \tag{2.3}$$

so

$$\Phi \equiv \text{const} \tag{2.4}$$

on each connected component of $\partial\Omega$.

We decompose Φ as

$$\Phi = \varphi + \psi \tag{2.5}$$

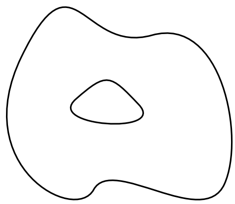
where ψ is the kernel part of the operator (2.1).

From (2.1_1) we easily see that φ and ψ fulfil the following equations in Ω:

$$\Delta\varphi = \alpha \quad \text{and} \quad \Delta\psi = 0 \qquad \text{in } \Omega. \tag{2.6}$$

Further requirements on φ and ψ, e.g. value on the boundary, will differ in the case of bounded and unbounded domain; that is why we state them in separate subsections.

2.1. *Case of a bounded non-simply connected domain.* In this subsection we study system (2.1) in a bounded non-simply connected domain Ω, as in the picture:



Domain

Let us assume that the boundary $\partial\Omega$ decomposes into Γ_0 and Γ_1, where Γ_0 is the inner boundary and Γ_1 is the outer boundary.

Since we are only interested in the gradient of φ and ψ we assume that

$$\varphi \equiv \psi \equiv 0 \qquad \text{on } \Gamma_0, \tag{2.7}$$

since $\psi \equiv \text{const}$ on Γ_1 and $\Pi^1(\Omega) = \mathbb{Z}$ we may set

$$\psi \equiv 1 \qquad \text{on } \Gamma_1 \tag{2.8}$$

and then, since $\Delta\psi = 0$ in Ω, we conclude that the kernel of operator (2.1) has one dimension, i.e. every vector field $\nabla^\perp \widetilde{\psi}$ from the kernel can be represented as

$$\nabla^\perp \widetilde{\psi} = C_\psi \nabla^\perp \psi \tag{2.9}$$

for a proper constant C_ψ.

For a non-kernel function φ we may set

$$\varphi \equiv 0 \qquad \text{on } \Gamma_1. \tag{2.10}$$

Let us introduce the following function space:

$$\widetilde{H}_0^2(\Omega) = \overline{\{f \in C^\infty(\Omega) : f_{|\Gamma_0} \equiv 0, \nabla^2 f \in L^2(\Omega)\}}^{\|\nabla^2 \cdot\|_{L^2(\Omega)}} \tag{2.11}$$

which is a Banach space with respect to the norm

$$\|f\|_{\widetilde{H}_0^2(\Omega)} = \|\nabla^2 f\|_{L^2(\Omega)}. \tag{2.12}$$

Since we are interested in the velocity vector v we introduce the following function space:

$$\widetilde{D}^1(\Omega) = \{\nabla^\perp f : f \in \widetilde{H}_0^2(\Omega)\} \tag{2.13}$$

which is also a Banach space with respect to the norm $\|\nabla \cdot \|_{L^2(\Omega)}$.

It is easily seen that the class $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω is not dense in this space. To be precise: no function from the kernel of the operator (2.1) can be approximated in $\widetilde{H}_0^2(\Omega)$ with functions from $C_0^\infty(\Omega)$.

2.2. *Case of an unbounded non-simply connected domain.* In this subsection we will state some results in an unbounded domain, needed for further calculations.

From the case of a bounded domain we see that we may assume that φ and ψ fulfil the following system:

$$\begin{aligned} \Delta\varphi &= \alpha, & \Delta\psi &= 0 & &\text{in } \Omega,\\ \varphi &= 0, & \psi &= 0 & &\text{on } \partial\Omega, \end{aligned} \tag{2.14}$$

From the physical point of view (we recall that $v = \nabla^\perp\varphi + \nabla^\perp\psi$) we may assume that $\nabla\varphi + \nabla\psi \to 0$ as $|x| \to \infty$, however later we approximate φ with smooth functions of compact support in Ω that is why we assume that $\nabla\psi \to 0$ as $|x| \to \infty$. Finally we write:

$$\nabla\varphi \to 0 \quad \text{and} \quad \nabla\psi \to 0 \qquad \text{as } |x| \to \infty. \tag{2.15}$$

Basic function spaces. Similarly to $\widetilde{H}^2_0(\Omega)$ for a bounded domain we introduce

$$\dot{H}^2_0(\Omega) = \overline{\{f \in C^\infty_0(\Omega) : f_{|\partial\Omega} \equiv 0, \nabla^2 f \in L^2(\Omega)\}}^{\|\nabla^2 \cdot\|_{L^2(\Omega)}} \tag{2.16}$$

which is a Banach space with respect to the norm $\|\nabla^2 \cdot \|_{L^2(\Omega)}$. Similarly we introduce a proper space for te velocity vector v:

$$D^1(\Omega) = \{\nabla^\perp f : f \in \dot{H}^2_0(\Omega)\} \tag{2.17}$$

which is a Banach space with respect to the norm $\|\nabla \cdot \|_{L^2(\Omega)}$.

2.3. *Kernel function ψ.* Now we find some information about the kernel of the operator (2.1) for an unbounded domain. Let us recall the system for ψ:

$$\begin{aligned} \Delta\psi &= 0 && \text{in } \Omega, \\ \psi &= 0 && \text{on } \partial\Omega, \\ \nabla\psi &\to 0 && \text{as } |x| \to \infty. \end{aligned} \tag{2.18}$$

Notice that we have no explicit information about the behaviour of ψ at infinity.

We show now that in the class of tempered distributions the above system has one-parameter family of solutions.

LEMMA 2.1. *Given the system*

$$\begin{aligned} \Delta\psi &= 0 && \textit{in } \Omega, \\ \psi &= 0 && \textit{on } \partial\Omega, \\ \nabla\psi &\to 0 && \textit{as } |x| \to \infty. \end{aligned} \tag{2.19}$$

There exists a one-dimensional space of solutions in the class of tempered distributions.

Proof. Existence of a solution is obvious. Let ψ be a solution to (2.19). Introducing a smooth cut-off function $\eta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\eta \equiv 0 \text{ in } B_{R_1}, \qquad \eta \equiv 1 \text{ in } \mathbb{R}^2, \backslash B_{R_2}, \tag{2.20}$$

where $B_{R_1} \subset \mathbb{R}^2$ is a ball containing the hole of Ω and $R_1 < R_2$.

Then we see that $\varphi\eta$ satisfies the following conditions:

$$\begin{aligned} \Delta(\varphi\eta) &= 2\nabla\varphi \cdot \nabla\eta + \varphi\Delta\eta \qquad \text{in } \mathbb{R}^2, \\ \nabla(\varphi\eta) &\to 0 \qquad \text{as } |x| \to \infty. \end{aligned} \tag{2.21}$$

Denoting $2\nabla\varphi \cdot \nabla\eta + \varphi\Delta\eta =: F$ we find that F has compact support in $\mathbb{R}^2$ and we may take

$$\widetilde{\varphi} = E * F, \tag{2.22}$$

where E is the fundamental solution for the Laplace operator, as a solution to (2.21). Properties of $\widetilde{\varphi}$ are well known since support of F is compact. Thus – $\widetilde{\varphi}$ has logarithmic growth and its gradient tends to 0 like $1/|x|$ as $|x| \to \infty$.

Further we get a system for $\varphi\eta - \widetilde{\varphi}$:

$$\begin{aligned} \Delta(\varphi\eta - \widetilde{\varphi}) &= 0 && \text{in } \mathbb{R}^2, \\ \nabla(\varphi\eta - \widetilde{\varphi}) &\to 0 && \text{as } |x| \to \infty. \end{aligned} \tag{2.23}$$

Taking the Fourier transform $(\hat{\ })$ of the first equation we get:

$$|\xi|^2 (\varphi\eta - \widetilde{\varphi})\hat{} = 0, \tag{2.24}$$

hence

$$\text{supp } (\varphi\eta - \widetilde{\varphi})\hat{} \subset \{0\}, \tag{2.25}$$

which implies

$$(\varphi\eta - \widetilde{\varphi}) = p(x), \tag{2.26}$$

where $p(x)$ is a polynomial. Since we have assumption (2.23) we conclude that $p(x) = c$, for some constant c.

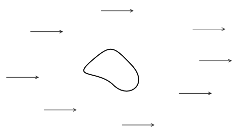
This shows that every solution in the class of tempered distributions to problem (2.19) has logarithmic growth at infinity and its gradient behaves like $1/|x|$.

To finish the proof we need to show that two solutions ψ_1, ψ_2 to (2.19) such that

$$\lim_{|x|\to\infty} \frac{\psi_1}{\ln|x|} = \lim_{|x|\to\infty} \frac{\psi_2}{\ln|x|} = c \tag{2.27}$$

for some constant c, are equal. It is however simple, since this property (in $\mathbb{R}^2$) implies analyticity of $\psi_1 - \psi_2$ at infinity and thus $\psi_1 - \psi_2 = 0$, since $\psi_1 - \psi_2 = 0$ on $\partial\Omega$. ∎

3. Stokes problem. In this section we investigate the Stokes problem gathered by the system (1.3). First we give its weak formulation. The following picture shows the situation being considered.



3.1. *Construction of* v_0. For the sake of further considerations (this section together with Section 4) we need to construct a vector field $v_0 \in H^1(\Omega)$ which for given $\epsilon > 0$ fulfils the following requirements:

$$\begin{aligned} &\text{div } v_0 = 0 && \text{in } \Omega, \\ &v_0 \cdot \vec{n} = -v_\infty \cdot \vec{n} && \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

and for every $\varphi \in \dot{H}^2_0(\Omega)$ the following inequality holds:

$$\left| \int_\Omega (v_0 \cdot \nabla\varphi)^2 \, dx \right| \le \epsilon \|\Delta\varphi\|^2_{L^2(\Omega)}. \tag{3.2}$$

Moreover, the following estimate is valid

$$\|v_0\|_{H^1(\Omega)} \le C \|v_\infty \cdot \vec{n}\|_{H^{1/2}(\partial\Omega)}. \tag{3.3}$$

This construction is the same as in the case of Lemma 2.1 in [12]. Here we give only a sketch of its structure.

First we introduce (t_1, t_2)-*coordinates* as follows. Let $s : [0, L] \to \mathbb{R}^2$ be a normal parameterization of boundary $\partial\Omega$, i.e.

$$s([0, L]) = \partial\Omega, \quad s(0) = s(L) = x_0 \in \partial\Omega, \quad \text{and} \quad |s'(t)| = 1 \tag{3.4}$$

for a fixed point x_0, and L the length of $\partial\Omega$. Next we introduce the following mapping $p : [0, L] \times [0, \zeta] \to \mathbb{R}^2$ such that

$$p(t_1, t_2) = s(t_1) + t_2\vec{n}(s(t_1)), \tag{3.5}$$

where $\vec{n}$ is the inner normal vector to $\partial\Omega$. If ζ is small enough (compared to the curvature χ of $\partial\Omega$), then the map is one-to-one and $p \in C^1$. Moreover

$$dist(p(t_1, t_2), \partial\Omega) = t_2. \tag{3.6}$$

Using the definition we compute the gradient of map p as follows

$$p_{,t_1} = (1 + t_2\chi)\vec{\tau}(s(t_1)), \qquad p_{,t_2} = \vec{n}(s(t_1)). \tag{3.7}$$

Then we see that

$$p_{,1} \perp p_{,2} \quad \text{and} \quad (\nabla p)^{-1} = \left(\frac{1}{1 + t_2\chi}\vec{\tau}, \vec{n}\right)^T. \tag{3.8}$$

By (t_1, t_2) we denote coordinates obtained using p.

Now, since $\int_{\partial\Omega} v_\infty \cdot \vec{n}\, d\sigma = 0$ we can define

$$D(t) = \frac{L}{2\pi} \int_0^t v_\infty \cdot \vec{n}(p(s, 0))\, ds.$$

and its extension (in (t_1, t_2) coordinates) $E : \mathcal{S}^1 \times \mathbb{R}^+ \to \mathbb{R}$, where $\mathcal{S}^1$ is the circle of circumference L, such that $E \in H^2_{\mathrm{loc}}(\mathcal{S}^1 \times \mathbb{R}^+)$. Now we take $\xi : [0, L] \times [0, \zeta] \to \mathbb{R}$ defined as follows:

$$\xi(t_1, t_2) = E(t_1, t_2)\eta_\epsilon(t_2), \tag{3.9}$$

where a smooth function $\eta_\epsilon(t)$ is defined as follows:

$$\eta_\epsilon(t) = \begin{cases} 1 & \text{for } t < \gamma^2(\epsilon), \\ \epsilon \ln \frac{\gamma(\epsilon)}{t} & \text{for } \gamma^2(\epsilon) \le t < \gamma(\epsilon), \\ 0 & \text{for } t \ge \gamma^2(\epsilon), \end{cases} \tag{3.10}$$

where $\gamma(\epsilon) = \exp\left(-\frac{1}{\epsilon}\right)$. Then $\xi \in H^2([0, L] \times \mathbb{R}^+)$.

For $\epsilon < \zeta$ we use the mapping $p : [0, L] \times [0, \zeta] \to \mathbb{R}^2$ to define ξ on Ω:

$$\xi(x) = \xi(p^{-1}(x)) \text{ for } x \in p([0, L] \times [0, \zeta]) \text{ and } \xi(x) = 0 \text{ otherwise.} \tag{3.11}$$

To avoid misunderstanding we will write ∇_t for the gradient in (t_1, t_2) coordinates, and ∇_x for the gradient in (x_1, x_2) coordinates. Then we have:

$$\nabla_x\xi \cdot \vec{\tau} = d \qquad \text{on } \partial\Omega, \tag{3.12}$$

since $\nabla_x\xi \cdot \vec{\tau} = \nabla_t\xi \cdot (\nabla p)^{-1} \cdot \vec{\tau} = \nabla_t\xi \cdot [1, 0] = \xi_{,t_1} = D'(t)r'(t) = d(p(t, 0))$.

Our desired field will be given as follows

$$v_0 = \nabla_x^\perp \xi \qquad \text{in } \Omega. \tag{3.13}$$

By the construction conditions (3.1) are satisfied.

For a more detailed view of this construction (in particular for the proof of inequality (3.3)) we refer the reader to [12].

3.2. *Reformulation.* To show existence first we introduce a decomposition of v as:

$$v = v_0^\infty + u, \tag{3.14}$$

where $v_0^\infty = v_0 + v_\infty$. Then for u we get the following system:

$$\begin{aligned} \operatorname{rot} u &= \alpha - \operatorname{rot} v_0^\infty && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u \cdot \vec{n} &= 0 && \text{on } \partial\Omega, \\ u &\to 0 && \text{as } |x| \to \infty. \end{aligned} \tag{3.15}$$

For u satisfying these conditions we may assume existence of φ and ψ such that $\psi, \varphi \in \dot{H}_0^2(\Omega)$ and:

$$u = \nabla^\perp \varphi + C_\psi \nabla^\perp \psi. \tag{3.16}$$

Here ψ is taken as a kernel part of u of the *rot-div* operator.

3.3. *Weak formulation and existence.* Let us multiply (1.3_1) by a function θ such that $\theta = 0$ on $\partial\Omega$, and integrate by parts. We get the weak formulation of our problem:

DEFINITION 3.1. We say that a pair $\varphi, C_\psi \psi \in \dot{H}_0^2(\Omega)$ is a *weak solution* to problem (1.3) iff the following identity holds for all $\theta \in \dot{H}_0^2(\Omega)$:

$$\begin{aligned} -\nu \int_\Omega \Delta\varphi \Delta\theta + \nu \int_{\partial\Omega} [(2\chi - f/\nu)((\nabla^\perp \varphi + C_\psi \nabla^\perp \psi) \cdot \vec{\tau}) - 2d_{,s}] \frac{\partial\theta}{\partial\vec{n}} \\ = \nu \int_\Omega \operatorname{rot} v_0 \Delta\theta - \int_\Omega F \cdot \nabla^\perp \theta - \nu \int_{\partial\Omega} (2\chi - f/\nu)(v_0 \cdot \vec{\tau}) \frac{\partial\theta}{\partial\vec{n}}. \end{aligned} \tag{3.17}$$

Proof of Theorem 1. To show existence of weak solutions first we introduce a scalar product in $\dot{H}_0^2(\Omega)$ as follows:

$$(u, v)_{\dot{H}_0^2(\Omega)} = \int_\Omega \mathbb{D}(u) \cdot \mathbb{D}(v) + \int_{\partial\Omega} f(u \cdot \vec{\tau})(v \cdot \vec{\tau}). \tag{3.18}$$

To show that this is indeed a scalar product one can use the Korn inequality, whose proof can be found in [14].

Next we introduce an orthonormal basis with respect to this scalar product:

$$\operatorname{span}(\{\psi, \varphi_1, \varphi_2, \ldots\}) = \dot{H}_0^2(\Omega). \tag{3.19}$$

We look for a solution u in the form (3.16), where Ψ is the kernel function, and $\varphi \in \operatorname{span}\{\varphi_1, \varphi_2, \ldots\}$.

Further we need the following identity:

$$\int_\Omega \alpha^2 = \int_\Omega |\mathbb{D}(v)|^2 + \int_{\partial\Omega} 2\chi (v \cdot \tau)^2. \tag{3.20}$$

Using it we find that (3.17) is equivalent to the following:

$$\nu \int_\Omega \mathbb{D}(v) \cdot \mathbb{D}(\nabla^\perp \theta) + \int_{\partial\Omega} f(v \cdot \vec{\tau}) \left(\frac{\partial\theta}{\partial\vec{n}} \right) = - \int_\Omega F \cdot \nabla^\perp \theta. \tag{3.21}$$

Setting $\theta = \psi$ and remembering that $\varphi \in \operatorname{span}\{\varphi_1, \ldots\}$, i.e. $\varphi \perp \psi$ in the sense of (3.18), we find:

$$C_\psi = - \frac{\int_\Omega F \cdot \nabla^\perp \psi + (v_0, \nabla^\perp \psi)_{\dot{H}_0^2(\Omega)}}{(\nabla^\perp \psi, \nabla^\perp \psi)_{\dot{H}_0^2(\Omega)}}. \tag{3.22}$$

Having v_0 and $C_\psi \psi$ we show existence of φ using the Galerkin method.

We introduce the following approximation space $V^N(\Omega) = \mathrm{span}\{\varphi_1, \dots, \varphi_N\}$. Then we search for approximate solution $\varphi^N \in V^N(\Omega)$ in the form:

$$\varphi^N = \sum_{i=1}^{N} c_i^N \varphi_i. \tag{3.23}$$

Identity (3.17) must be fulfilled for $\theta = \psi, \varphi_1, \dots, \varphi_N$. Since $\varphi^N \perp \psi$ and (3.22) we see again that it is valid for $\theta = \psi$. To show existence of coefficients c_i^N such that (3.17) is valid for $\theta = \varphi_1, \dots, \varphi_N$ we use the following Lemma:

LEMMA 3.2. *Suppose that a continuous mapping $P : V^N(\Omega) \to V^N(\Omega)$ satisfies $(P(\varphi^N), \varphi^N)_{V^N(\Omega)} > 0$ for all $\varphi^N \in V^N$ such that $\|\varphi^N\|_{V^N(\Omega)} = M$. Then there exists $\varphi_0^N \in V^N(\Omega)$ such that $P(\varphi_0^N) = 0$ and $\|\varphi_0^N\|_{V^N} \leq M$.*

To use this lemma we define a mapping $P : V^N(\Omega) \to V^N(\Omega)$ as follows: first we define

$$u^N = v_0 + C_\psi \nabla^\perp \psi + \sum_{i=1}^{N} c_i^N \nabla^\perp \varphi_i, \tag{3.24}$$

next

$$\begin{aligned} P(\varphi^N) := \Bigg(\sum_{i=1}^{N} \nu \int_\Omega \mathbb{D}(u^N) \cdot \mathbb{D}(\nabla^\perp \varphi_i) \\ + \int_{\partial\Omega} f(u^N \cdot \vec{\tau}) \left(\frac{\partial \varphi_i}{\partial \vec{n}} \right) + \int_\Omega F \cdot \nabla^\perp \varphi_i \Bigg) \cdot \varphi_i. \end{aligned} \tag{3.25}$$

Since $\{\varphi_i\}$ is an orthonormal basis we easily find that:

$$\begin{aligned} (P(\varphi^N), \varphi^N)_{V^N(\Omega)} &= \nu \int_\Omega \mathbb{D}(u^N) \cdot \mathbb{D}(\nabla^\perp \varphi^N) \\ &\quad + \int_{\partial\Omega} f(u^N \cdot \vec{\tau}) \left(\frac{\partial \varphi^N}{\partial \vec{n}} \right) + \int_\Omega F \cdot \nabla^\perp \varphi^N \\ &= (u^N, \varphi^N)_{V^N(\Omega)} + \int_\Omega F \cdot \nabla^\perp \varphi^N \\ &= (\varphi^N, \varphi^N)_{V^N(\Omega)} + (v_0, \varphi^N)_{V^N(\Omega)} + \int_\Omega F \cdot \nabla^\perp \varphi^N \\ &\geq \|\varphi^N\|^2_{V^N(\Omega)} - C(v_0, F) \|\varphi^N\|_{V^N(\Omega)} > 0 \end{aligned}$$

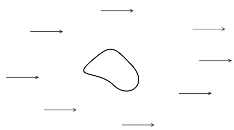
for all φ^N such that $\|\varphi^N\|_{V^N(\Omega)} > C(v_0, F)$, where $C(v_0, F)$ does not depend on N.

Using Lemma 3.2 we get that there exists φ_0^N (i.e. coefficients c_i^N) such that φ_0^N is a solution to (3.17) and $\|\varphi_0^N\|_{V^N(\Omega)} \leq C(v_0, F)$.

Since all these solutions are bounded we are able to choose a subsequence $\{\varphi_0^i\}_{i=1}^\infty$ weakly convergent in $\dot{H}_0^2$ (for simplicity we do not add another index) for which the limit function $\varphi_0 = \text{w-}\lim_{i\to\infty} \varphi_0^i \in \dot{H}_0^2(\Omega)$ satisfies (3.17) in the sense of distributions.

This completes the proof of Theorem 1. ■

4. Linearization of the N-S problem. In this section we investigate problem (1.5) where $\widetilde{v}_0 = v_0 + v_\infty$, v_0 has compact support and the system (3.1)-(3.2) is fulfilled. The system comes from a linearization of the Navier-Stokes system on a rotation level. The domain in this case is the same as before, i.e.



4.1. *Weak formulation.* Weak formulation of system (1.5) can be obtained similarly as for the Stokes problem. First we recall that the desired solution v can be stated in the following form:

$$v = v_0 + \nabla^{\perp}\varphi + \nabla^{\perp}\psi, \tag{4.1}$$

where v_0 is the vector field constructed in Section 3. We make a formal projection and search for a solution in the form

$$v = v_0 + \nabla^{\perp}\varphi, \tag{4.2}$$

where $\varphi \in \dot{H}^2_0(\Omega)$.

The first approach to defining weak solutions might be as follows: we say that $\varphi \in \dot{H}^2_0(\Omega)$ is a weak solution to problem (1.5) iff the following identity holds for all $\theta \in \dot{H}^2_0(\Omega)$:

$$\begin{aligned} -\nu\int_{\Omega}\Delta\varphi\Delta\theta + \nu\int_{\partial\Omega}(2\chi - f/\nu)(\nabla^{\perp}\varphi\cdot\vec{\tau})\frac{\partial\theta}{\partial\vec{n}} - \int_{\Omega}\widetilde{v}_0\cdot\nabla\theta\alpha \\ = \nu\int_{\Omega}\operatorname{rot} v_0\Delta\theta - \int_{\Omega}F\cdot\nabla^{\perp}\theta - \nu\int_{\partial\Omega}(2\chi - f/\nu)(v_0\cdot\vec{\tau})\frac{\partial\theta}{\partial\vec{n}}, \end{aligned} \tag{4.3}$$

however we encounter difficulties with defining the meaning of $\int_{\Omega}\widetilde{v}_0\cdot\nabla\theta\alpha$, in particular $v_{\infty}\int_{\Omega}\theta_{,1}\Delta\varphi$. As we shall see we can replace it by $-\int_{\Omega}\Delta\theta\varphi_{,1} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}\,d\sigma$. In the next lemma we show this and the fact that $\varphi_{,1} \in L^2(\Omega)$.

Lemma 4.1. *For $\theta \in \dot{H}^2_0(\Omega)$ and $\varphi \in \dot{H}^2_0(\Omega)$ - a solution to (4.12), the following term is well defined:*

$$\int_{\Omega}\theta_{,1}\Delta\varphi \tag{4.4}$$

Proof. First let us assume that $\theta \in C_0^{\infty}(\Omega)$. Then the following calculations are valid:

$$\begin{aligned} \int_{\Omega}\theta_{,1}\Delta\varphi &= -\int_{\Omega}\nabla\theta_{,1}\nabla\varphi + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n} \\ &= \int_{\Omega}\Delta\theta_{,1}\varphi + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n} - \int_{\partial\Omega}(\nabla\theta_{,1}\cdot\vec{n})\varphi \\ &= -\int_{\Omega}\Delta\theta\varphi_{,1} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n} + \int_{\partial\Omega}\Delta\theta\varphi\vec{n}^{(1)} \\ &= -\int_{\Omega}\Delta\theta\varphi_{,1} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}. \end{aligned}$$

We can rewrite the above as follows:

$$\int_{\Omega}\Delta\theta\varphi_{,1} = -\int_{\Omega}\theta_{,1}\Delta\varphi + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}. \tag{4.5}$$

Now since φ is a solution (in the sense of distributions) we find:

$$\int_\Omega \Delta\theta\varphi_{,1} = \nu\int_\Omega (\Delta\varphi + \operatorname{rot} v_0)\Delta\theta + \int_\Omega v_0\cdot\nabla\theta(\Delta\varphi + \operatorname{rot} v_0)$$
$$-\nu\int_{\partial\Omega}(2\chi - f/\nu)(v_0 + \nabla^\perp\varphi)\cdot\vec\tau\frac{\partial\theta}{\partial\vec n} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec n. \quad (4.6)$$

In this form we are able to get estimates on $L^2(\Omega)$ norm of $\varphi_{,1}$, that is using the following definition of a norm:

$$\|\varphi_{,1}\|_{L^2(\Omega)} = \sup_{f\in C_0^\infty(\Omega),\|f\|_{L^2(\Omega)}\le 1}(\varphi_{,1}, f)_{L^2(\Omega)}. \quad (4.7)$$

We use it together with (4.5). First for arbitrary $f\in C_0^\infty(\Omega)$ we solve the following system:

$$\Delta\theta = f \qquad \text{in } \Omega, \quad (4.8)$$
$$\theta = 0 \qquad \text{on } \partial\Omega. \quad (4.9)$$

It is well known that there exists a solution θ to this problem in the class $\dot H_0^2(\Omega)$ (we do not need uniqueness) which, since $\|f\|_{L^2(\Omega)}\le 1$, satisfies the following inequality:

$$\|\theta\|_{\dot H_0^2(\Omega)}\le C. \quad (4.10)$$

It is then easily seen that in (4.5) the right hand side is well defined and can be estimated by a constant C, independent of f. Thus

$$\varphi_{,1}\in L^2(\Omega) \quad (4.11)$$

and the term $\int_\Omega\theta_{,1}\Delta\varphi$ is well defined. ∎

We are now ready to give a proper definition of weak solution to problem (1.5):

DEFINITION 4.2. We say that $\varphi\in\dot H_0^2(\Omega)$ is a *weak solution* to problem (1.5) iff the following identity holds for all $\theta\in\dot H_0^2(\Omega)$:

$$-\nu\int_\Omega\Delta\varphi\Delta\theta + \nu\int_{\partial\Omega}(2\chi - f/\nu)(\nabla^\perp\varphi\cdot\vec\tau)\frac{\partial\theta}{\partial\vec n} - \int_\Omega(\widetilde v_0\cdot\nabla\theta(\operatorname{rot} v_0) + v_0\cdot\nabla\theta\Delta\varphi)$$
$$+\int_\Omega\Delta\theta\varphi_{,1} - \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec n$$
$$= \nu\int_\Omega\operatorname{rot} v_0\Delta\theta - \int_\Omega F\cdot\nabla^\perp\theta - \nu\int_{\partial\Omega}(2\chi - f/\nu)(v_0\cdot\vec\tau)\frac{\partial\theta}{\partial\vec n}. \quad (4.12)$$

Proof of Theorem 2. To show existence of this solution we proceed as earlier, i.e. we must compute proper estimates for $\|\varphi\|_{\dot H_0^2(\Omega)}$, however in this case we have the additional term $\int_\Omega\widetilde v_0\cdot\nabla\theta\alpha$. We thus need the following lemma:

LEMMA 4.3. *For every $\epsilon > 0$ there exists $\widetilde v_0$ such that*

$$\left|\int_\Omega\widetilde v_0\cdot\nabla\varphi\Delta\varphi\right|\le\epsilon\|\varphi\|^2_{\dot H_0^2(\Omega)} \quad (4.13)$$

for all $\varphi\in C_0^\infty(\Omega)$.

Proof. We recall that $\widetilde v_0 = v_0 + v_\infty$, where v_0 has compact support and since $\widetilde v_0 = 0$ on $\partial\Omega$ we have also $v_0 = -v_\infty$ on $\partial\Omega$. Our first step is to remove v_∞ from (4.13). We do this with a few integrations by parts. Let us split our integral as follows:

$$\int_\Omega \widetilde{v}_0 \cdot \nabla\varphi\Delta\varphi = \int_\Omega \widetilde{v}_0 \cdot \nabla\varphi(\varphi_{,11} + \varphi_{,22}) = I_1 + I_2, \tag{4.14}$$

and since $v_\infty = (v_\infty, 0)$:

$$\begin{aligned} I_1 = \int_\Omega \widetilde{v}_0 \cdot \nabla\varphi\varphi_{,11} &= \int_\Omega (v_0 + v_\infty) \cdot \nabla\varphi\varphi_{,11} \\ &= \int_\Omega (v_0^{(1)} + v_\infty)\varphi_{,1}\varphi_{,11} + \int_\Omega v_0^{(2)}\varphi_{,2}\varphi_{,11} = I_{11} + I_{12}, \end{aligned} \tag{4.15}$$

similarly $I_2 = I_{21} + I_{22}$. Next we integrate:

$$\begin{aligned} I_{11} = \int_\Omega (v_0^{(1)} + v_\infty)\varphi_{,1}\varphi_{,11} &= \frac{1}{2}\int_\Omega (v_0^{(1)} + v_\infty)(\varphi_{,1}^2)_{,1} \\ &= -\frac{1}{2}\int_\Omega v_{0,1}^{(1)}(\varphi_{,1}^2) + \frac{1}{2}\int_{\partial\Omega} (v_0 + v_\infty)\varphi_{,1}^2 n_1 \\ &= -\frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,1}^2 \end{aligned}$$

since $v_0 + v_\infty = 0$ on $\partial\Omega$. Similarly

$$\begin{aligned} I_{12} = \int_\Omega v_0^{(2)}\varphi_{,2}\varphi_{,11} &= -\int_\Omega v_0^{(2)}\varphi_{,21}\varphi_{,1} - \int_\Omega v_{0,1}^{(2)}\varphi_{,2}\varphi_{,1} \\ &= -\frac{1}{2}\int_\Omega v_0^{(2)}(\varphi_{,1})_{,2}^2 - \int_\Omega v_{0,1}^{(2)}\varphi_{,2}\varphi_{,1} \\ &= \frac{1}{2}\int_\Omega v_{0,2}^{(2)}(\varphi_{,1})^2 - \int_\Omega v_{0,1}^{(2)}\varphi_{,2}\varphi_{,1}. \end{aligned}$$

In the same way we transform I_{21} and I_{22} to finally get:

$$\begin{aligned} I = -\frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,1}^2 + \frac{1}{2}\int_\Omega v_{0,2}^{(2)}\varphi_{,1}^2 - \int_\Omega v_{0,1}^{(2)}\varphi_{,2}\varphi_{,1} \\ - \int_\Omega v_{0,2}^{(1)}\varphi_{,1}\varphi_{,2} + \frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,2}^2 - \frac{1}{2}\int_\Omega v_{0,2}^{(2)}\varphi_{,2}^2 \end{aligned} \tag{4.16}$$

Recalling that div $v = 0$ we find:

$$\begin{aligned} I &= -\int_\Omega v_{0,1}^{(1)}\varphi_{,1}^2 - \int_\Omega v_{0,1}^{(2)}\varphi_{,2}\varphi_{,1} - \int_\Omega v_{0,2}^{(1)}\varphi_{,1}\varphi_{,2} - \int_\Omega v_{0,2}^{(2)}\varphi_{,2}^2 \\ &= -\int_\Omega \nabla\varphi \cdot \nabla v_0 \cdot \nabla\varphi. \end{aligned}$$

In this form we are able to get the desired estimate. to appear.

First, let us introduce a vector field V in (t_1, t_2) coordinates:

$$V(p(t_1, t_2)) := \begin{pmatrix} \frac{v_\infty}{2}[1 + cos((\pi/\zeta)t_2)] \\ 0 \end{pmatrix} \tag{4.17}$$

for which the following conditions are valid:

$$V(p(t_1, 0)) = \begin{pmatrix} v_\infty \\ 0 \end{pmatrix}, \qquad V(p(t_1, \zeta)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.18}$$

where $\zeta = \zeta(\Omega)$ is the constant from the construction of the mapping $p(t_1, t_2)$.

The similar conditions are fulfilled by the vector field

$$V_\epsilon(p(t_1,t_2)) := \begin{cases} V(p(t_1,t_2/\epsilon)) & \text{for } t_2 \le \zeta\epsilon, \\ 0 & \text{for } \zeta\epsilon \le t_2 \le \zeta. \end{cases} \tag{4.19}$$

From the construction of V_ϵ it is easily seen that

$$\|\nabla V_\epsilon\|_{L^2(\Omega)} \le C(\Omega, V)\frac{1}{\epsilon^{1/2}}. \tag{4.20}$$

We may now estimate the integral I:

$$I = -\int_\Omega \nabla\varphi \cdot \nabla(v_0 + V_\epsilon) \cdot \nabla\varphi + \int_\Omega \nabla\varphi \cdot \nabla V_\epsilon \cdot \nabla\varphi =: I_1 + I_2. \tag{4.21}$$

Since $v_0 + V_\epsilon = 0$ on $\partial\Omega$ we may integrate I_1 by parts and get:

$$I_1 = \int_\Omega \nabla\varphi(v_0 + V_\epsilon)\Delta\varphi + \frac{1}{2}\int_\Omega ((v_0 + V_\epsilon)\cdot\nabla)(|\nabla\varphi|^2) = I_{11} + I_{12}. \tag{4.22}$$

Now, from the Gauss theorem and since div $v_0 = 0$ in Ω

$$I_{12} = -\frac{1}{2}\int_\Omega (\text{div } V_\epsilon)(|\nabla\varphi|^2). \tag{4.23}$$

Gathering all these calculations we get:

$$I = \int_\Omega \nabla\varphi v_0 \Delta\varphi + \int_\Omega \nabla\varphi V_\epsilon \Delta\varphi - \frac{1}{2}\int_\Omega (\text{div } V_\epsilon)(|\nabla\varphi|^2) + \int_\Omega \nabla\varphi\cdot\nabla V_\epsilon\cdot\nabla\varphi$$
$$=: I_1 + I_2 + I_3 + I_4. \tag{4.24}$$

Before we estimate these integrals let us introduce the following notation:

$$\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \epsilon\zeta\}. \tag{4.25}$$

Integrals I_3 and I_4 are similar and we estimate them first. Since supp $V_\epsilon \subset \Omega_\epsilon$:

$$I_3 + I_4 \le C\left(\int_{\Omega_\epsilon} |\nabla V_\epsilon|^2\right)^{1/2}\left(\int_{\Omega_\epsilon} |\nabla\varphi|^4\right)^{1/2}. \tag{4.26}$$

We must use the following inequality:

$$\|\nabla\varphi\|_{L^4(\Omega_\epsilon)} \le \|\varphi\|^{1/4}_{L^2(\Omega_\epsilon)}\|\nabla^2\varphi\|^{3/4}_{L^2(\Omega_\epsilon)}. \tag{4.27}$$

Since $\varphi \equiv 0$ on $\partial\Omega$ and $\varphi \in H^2(\Omega_\epsilon)$ we may use embedding theorem $H^2(\Omega_\epsilon) \subset C^\alpha(\Omega_\epsilon)$, where $\alpha < 1$, to conclude that

$$\|\varphi\|_{L^2(\Omega_\epsilon)} \le C(\Omega_\epsilon)\|\nabla^2\varphi\|_{L^2(\Omega_\epsilon)} \cdot \epsilon^{\frac{1+2\alpha}{2}}. \tag{4.28}$$

Indeed:

$$\|\varphi\|^2_{L^2(\Omega_\epsilon)} \le \int_0^L \int_0^{\zeta\epsilon} |\varphi(t_1,t_2)|^2 |Jp| dt_1 dt_2 \tag{4.29}$$

$$\le C(\Omega)\int_0^L \int_0^{\zeta\epsilon} t_2^{2\alpha}\|\nabla^2\varphi\|^2_{L^2(\Omega_\epsilon)} dt_1 dt_2 \tag{4.30}$$

$$\le C(\Omega)\|\nabla^2\varphi\|^2_{L^2(\Omega_\epsilon)}\epsilon^{1+2\alpha}. \tag{4.31}$$

Inserting this inequality to (4.27) we get:

$$\|\nabla\varphi\|^2_{L^4(\Omega_\epsilon)} \le C(\Omega)\|\nabla^2\varphi\|^2_{L_2(\Omega_\epsilon)} \cdot \epsilon^{\frac{1+2\alpha}{4}}. \tag{4.32}$$

Now from (4.20), (4.26) and (4.32) we conclude:

$$I_3 + I_4 \leq C(\Omega, V)\epsilon^{\frac{2\alpha-1}{4}} \|\nabla^2\varphi\|^2_{L_2(\Omega)}. \tag{4.33}$$

Since $\alpha < 1$ we may choose ϵ small enough to get

$$I_3 + I_4 \leq \epsilon\|\nabla^2\varphi\|^2_{L_2(\Omega)}. \tag{4.34}$$

The estimate of the integral I_2 is similar:

$$I_2 = \int_\Omega \nabla\varphi V_\epsilon \Delta\varphi \tag{4.35}$$

$$\leq \left(\int_{\Omega_\epsilon} |V_\epsilon|^4\, dx\right)^{1/4} \left(\int_{\Omega_\epsilon} |\nabla\varphi|^4\, dx\right)^{1/4} \left(\int_\Omega |\nabla^2\varphi|^2\right)^{1/2} \tag{4.36}$$

and since

$$\left(\int_{\Omega_\epsilon} |V_\epsilon|^4\, dx\right)^{1/4} \leq \left(\int_\Omega |V|^4\, dx\right)^{1/4} = C(V) < \infty \tag{4.37}$$

we get the desired estimate:

$$I_2 \leq \epsilon\|\nabla^2\varphi\|^2_{L^2(\Omega)}. \tag{4.38}$$

To finish the proof we need to estimate I_1, but this is the subject of Lemma 2.1 from [12]. ∎

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