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L^q-APPROACH TO WEAK SOLUTIONS OF THE OSEEN FLOW AROUND A ROTATING BODY

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Abstract. We consider the time-periodic Oseen flow around a rotating body in $\mathbb{R}^3$. We prove *a priori* estimates in L^q-spaces of weak solutions for the whole space problem under the assumption that the right-hand side has the divergence form. After a time-dependent change of coordinates the problem is reduced to a stationary Oseen equation with the additional term $-(\omega \wedge x) \cdot \nabla u + \omega \wedge u$ in the equation of momentum where ω denotes the angular velocity. We prove the existence of generalized weak solutions in L^q-space using Littlewood-Paley decomposition and maximal operators.

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1. Introduction. Many physical phenomena involve moving or deformable structures interacting with fluids and are of great concern for aerospace, mechanical, biomedical applications, sedimentation. From the mathematical point of view, they have been studied extensively over the last few years. When the domain depends on time, we refer to [21], [14], [22], [4]. In this paper we consider the case when we fix a rotation of a body and we investigate the flow around the body. In recent years the analysis of the Navier-Stokes equations describing the flow around or past a rotating body has attracted much attention, see [18], [12], [10], [11], [29]–[31], [19], [26], [28], [7], [8], [9], [16], [17], [32], [34], [13]. Further references on moving bodies in fluids are given in [16].

We study the stationary Oseen system in the whole three-dimensional space:

$$\begin{aligned} -\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f \quad \text{in } \mathbb{R}^3 \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3 \\ u &\to 0 \quad \text{as } |x|\to\infty. \end{aligned} \tag{1.1}$$

Here, $\wedge$ denotes the usual exterior product of three-dimensional vectors. Note that the second and the third terms are linearized convective terms, and that in unbounded domains they are not subordinated to the Laplacian. Let us also note that

$$\nabla\cdot[-(\omega\wedge x)\cdot\nabla u + \omega\wedge u] = 0. \tag{1.2}$$

As a consequence $\Delta p = \nabla\cdot f$ and we can write the reduced equation

$$-\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u = g \;\text{ in }\; \mathbb{R}^3, \tag{1.3}$$

where $g = f - \nabla p$.

The linear system (1.1) has been analyzed in L^q-spaces, $1 < q < \infty$, in [10] proving the *a priori* estimates

$$\begin{aligned} \|\nu\nabla^2 u\|_q + \|\nabla p\|_q &\le c\|f\|_q, \\ \|k\partial_3 u\|_q + \|-(\omega\wedge x)\cdot\nabla u + \omega\wedge u\|_q &\le c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q \end{aligned} \tag{1.4}$$

with the constant $c > 0$ independent of ω, ν, k. Further the results were improved in [7] in weighted spaces and the authors have obtained the following *a priori* estimates

$$\begin{aligned} \|\nu\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} &\le c\|f\|_{q,w}, \\ \|k\partial_3 u\|_{q,w} + \|-(\omega\wedge x)\cdot\nabla u + \omega\wedge u\|_{q,w} &\le c\left(1 + \frac{k^5}{\nu^2|\omega|^{5/2}}\right)\|f\|_{q,w}, \end{aligned}$$

where the weights (denoted by w) belong to the more general Muckenhoupt class $\tilde{A}_q^-$, see [7], with the constant $c > 0$ independent of ν, ω, k.

Let us recall in two steps the natural introduction of the previous Oseen system starting with a viscous flow either past a threedimensional rigid body, rotating with an angular velocity $\omega = |\omega|\,(0,0,1)^T$, $|\omega| \ne 0$, or around a rotating body which is moving in the direction of its axis of rotation. We assume this viscous flow modelled by the incompressible Navier-Stokes equations with the velocity $u_\infty = ke_3 \ne 0$ at infinity. Then, given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{f} = \tilde{f}(y,t)$, the velocity

$v = v(y,t)$ and the pressure $q = q(y,t)$ solve the well known nonlinear system:

$$\begin{aligned} \partial_t v - \nu\Delta v + (v\cdot\nabla)\, v + \nabla q &= \tilde{f} && \text{in } (0,+\infty)\times\Omega(t),\\ \operatorname{div} v &= 0 && \text{in } (0,+\infty)\times\Omega(t),\\ v(y,t) &= \omega\wedge y && \text{on } (0,+\infty)\times\partial\Omega(t),\\ v(y,t) &\to u_\infty \neq 0 && \text{as } |y|\to\infty. \end{aligned} \tag{1.5}$$

Due to the rotation with angular velocity ω, the time-dependent exterior domain $\Omega(t)$ is given by

$$\Omega(t) = O_\omega(t)\Omega,$$

where $\Omega\subset\mathbb{R}^3$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|\,t & -\sin|\omega|\,t & 0\\ \sin|\omega|\,t & \cos|\omega|\,t & 0\\ 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$

Introducing the change of variables

$$x = O_\omega(t)^T y \tag{1.7}$$

and the new functions

$$u(x,t) = O_\omega^T(t)(v(y,t)-u_\infty),\;\; p(x,t) = q(y,t),\;\; f(x,t) = O_\omega(t)^T\tilde{f}(y,t) \tag{1.8}$$

we arrive at the modified Navier-Stokes system, this is the first step:

$$\begin{aligned} \partial_t u - \nu\Delta u \;&+\; (u\cdot\nabla)\, u + (O_\omega^T(t)u_\infty)\cdot\nabla u\\ &-\; (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p \;=\; f && \text{in } (0,+\infty)\times\Omega,\\ \operatorname{div} u \;&=\; 0 && \text{in } (0,+\infty)\times\Omega,\\ u(x,t) \;&=\; \omega\wedge x - O_\omega^T(t)u_\infty && \text{on } (0,+\infty)\times\partial\Omega,\\ u(x,t) \;&\to\; 0 && \text{as } |x|\to\infty. \end{aligned} \tag{1.9}$$

Due to the new coordinate system attached to the rotating body, equation in (1.9) contains two new linear terms, the classical Coriolis force term $\omega\wedge u$ (up to a multiplicative constant) and the additional term $(\omega\wedge x)\cdot\nabla u$.

The second step consists of the linearization of equation (1.9) at $u=0$, assuming the case $u_\infty||\omega$ and then $O_\omega^T(t)u_\infty = ke_3$, for all $t>0$, and assuming $\Omega=\mathbb{R}^3$. Thus we get the modified Oseen system (1.1).

Remark 1. The study of the whole space problem is of interest because we need the results about existence, uniqueness and boundedness of a solution in order to get respective results also in the case of exterior domains. This complete study will be the object of a forthcoming paper [27], we will use the so called localization procedure, see [25].

Remark 2. We would like to mention that there exists another type of transformation (a local transformation) which was introduced by Inou and Wakimoto [24]. The transformation is applied by several authors, see e.g. [38].

We introduce notation. The class $C_0^\infty(\mathbb{R}^3)$ consists of C^∞ functions with compact supports contained in $\mathbb{R}^3$. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$.

We define the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q} = \{v \in L^q_{\text{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\}/\mathbb{R}. \tag{1.10}$$

Remark 3. Another possibility of the definition of the homogeneous Sobolev spaces can be found in the work of Galdi [15]. He defines the homogeneous Sobolev spaces in the following way

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q}$$

and from Theorem II.6.3, and Remark II.6.2 [15] he gives the following characterisation

$$\begin{aligned}\widehat{W}^{1,q}(\mathbb{R}^3) &= \{v \in L^1_{\text{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\},\ q \geq 3,\\ &= \{v \in L^1_{\text{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3, v \in L^{\frac{3q}{3-q}}(\mathbb{R}^3)\},\ q < 3.\end{aligned} \tag{1.11}$$

We mention [25], Proposition 2.4 for characterisation of the spaces $\widehat{W}^{1,q}(\mathbb{R}^3)$.

Lemma 1.1.

- *For $1 < r < n$ we have $\widehat{W}^{1,r}(\mathbb{R}^3) = \{u \in L^s(\mathbb{R}^3) : \nabla u \in L^r(\mathbb{R}^3)\}$ where $s = \frac{3r}{3-r}$.*
- *Let $r \geq n$. Suppose $u_k \in C_0^\infty(\mathbb{R}^3)$, $k = 1, 2, \dots$ is a Cauchy sequence in $\widehat{W}^{1,r}(\mathbb{R}^3)$. Then there is a Cauchy sequence $w_k \in C_0^\infty$ with $\nabla u \in L^r(\mathbb{R}^3)$ satisfying*

$$\begin{aligned}&\|\nabla u_k - \nabla w_k\|_{L^r(\mathbb{R}^3)} \to 0,\\ &w_k \to u \text{ in } L^r_{loc}(\mathbb{R}^3),\\ &\nabla w_k \to \nabla u \text{ in } L^r(\mathbb{R}^3) \text{ as } k \to \infty.\end{aligned} \tag{1.12}$$

Such a u is unique up to additive constants. In this case, we have the inclusion $\widehat{W}_0^{1,r}(\mathbb{R}^3) \subset \{[u] \in L^r_{loc}(\mathbb{R}^3)/\mathbb{R} : \nabla u \in L^r(\mathbb{R}^3)\}$ where $[u] = \{w \in L^r_{loc}(\mathbb{R}^3) : w - u \in \mathbb{R}\}$.

Remark 4. We would like to mention that definitions (1.10) and (1.11) are equivalent in the following sense. In definition (1.10) the elements of space are classes of functions since we factorized the homogeneous spaces $\widehat{W}^{1,r}$ by constants. In definition (1.11) we divide into two cases:

- the case $1 < r < n$ where Sobolev imbedding is valid
- the case of $r \geq n$ where limits of Cauchy sequences are unique up to constant, see previous Lemma.

Remark 5. We would like to mention that a different approach was given by Girault and collaborators. They introduce Sobolev spaces with weights where the density of weighted Sobolev spaces in C_0^∞ is satisfied automatically from the definition, see [1].

Their dual space is defined in the following way

$$\widehat{W}^{-1,q}(\mathbb{R}^3) = (\widehat{W}^{1,q/(q-1)}(\mathbb{R}^3))^*,\ \text{with norm } \|\cdot\|_{-1,q}. \tag{1.13}$$

A characterisation of the normed dual spaces can be found in [15] page 72–74.

Remark 6. The definition of dual spaces is important for extension of Bogovskii operator to negative homogeneous spaces; for more details see [5, 20].

We will use systematic notations ∂_j for partial derivatives in Cartesian coordinates and ∂_r or ∂_θ in cylindrical coordinates. We are now interested in the weak solution to (1.1).

DEFINITION 1.1. Let $1 < q < \infty$. Given $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we call $\{u, p\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ a *weak solution* to (1.1) if

$$\begin{aligned} &(1) \qquad \nabla \cdot u = 0 \quad \text{in} \quad L^q(\mathbb{R}^3), \\ &(2) \qquad (\omega \wedge x) \cdot \nabla u - \omega \wedge u \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3, \end{aligned} \tag{1.14}$$

$\{u, p\}$ satisfies $(1.1)_2$ in the sense of distributions, that is,

$$\begin{aligned} &\nu\langle \nabla u, \nabla \varphi\rangle - \langle (\omega \wedge x) \cdot \nabla u - \omega \wedge u, \varphi\rangle \\ &+k\left\langle \frac{\partial u}{\partial x_3}, \varphi \right\rangle - \langle p, \nabla \cdot \varphi\rangle = \langle f, \varphi\rangle, \\ &\qquad \varphi \in C_0^\infty(\mathbb{R}^3)^3. \end{aligned} \tag{1.15}$$

In fact, as usual, equation (1.15) holds by density for all $\varphi \in \widehat{W}^{1,q/(q-1)}(\mathbb{R}^3)^3$.

In Definition 1.1 we use that functions from $\varphi \in \widehat{W}^{1,q/(q-1)}(\mathbb{R}^3)^3$ can be approximated by functions from C_0^∞, for more details see [31].

In the work of Galdi [15], the author defines q-generalized solutions (see page 189) which are similar to our Definition 1.1.

The main results are the following

THEOREM 1.1. *Let $1 < q < \infty$ and suppose $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, then the problem* (1.1) *possesses a weak solution $(u, p) \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ satisfying the estimate*

$$\|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q} \ \le\ C\|f\|_{-1,q}, \tag{1.16}$$

with some $C > 0$, which depends on q.

THEOREM 1.2. *The solution $\{u, p\}$ given by Theorem 1.1 is unique up to a constant multiple of ω for u.*

COROLLARY 1.1. *Let $1 < q < 4$, $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ and let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be the unique weak solution to problem* (1.1)*. Then there exists $\alpha \in \mathbb{R}$ such that*

$$u - \alpha e_3 \in L^s(\mathbb{R}^3)^3 \textit{ for all } s > 1, 1/s \in 1/q - [1/4, 1/3].$$

Moreover

$$\|u - \alpha e_3\|_s \le C\|f\|_{-1,q}$$

with a constant $C = C(\nu, k, \omega, s) > 0$.

COROLLARY 1.2. *Let $1 < q < 3$, $\nu > 0$, $k > 0, f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, and let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be the unique weak solution to problem* (1.1)*. Then*

$$\|u/|x|\|_q \le \frac{c}{\nu}\|f\|_{-1,q}$$

with $c = c(q, \omega) > 0$.

In Theorem 1.1 and Theorem 1.2, due to our choice of the right-hand side, we improve Farwig's a priori estimates (1.4) . We extend to the Oseen problem the analysis done by Hishida [31] for the Stokes problem.

2. Mathematical preliminaries

2.1. *Definition of Littlewood-Paley decomposition*

Definition 2.1. Let $\chi \in \mathcal{D}(\mathbb{R}^d)$, $d \in \mathbb{N}$, be such that $|\xi| \le 1/2$ implies $\chi(\xi) = 1$ and $|\xi| \ge 1$ implies $\chi(\xi) = 0$. Let ψ be defined as $\psi(\xi) = \chi(\xi/2) - \chi(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \chi(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi(\xi/2^j)\mathcal{F}f$. Then for all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^d)$ we have $f = S_N f + \sum_{j \ge N} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^d)$, this equality is called the Littlewood-Paley decomposition of the distribution f.

Theorem 2.1 (Littlewood-Paley decomposition of $L^p(\mathbb{R}^d)$). *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $1 < p < \infty$. Then the following assertions are equivalent:*

(i) $f \in L^p(\mathbb{R}^d)$,
(ii) $S_0 \in L^p(\mathbb{R}^d)$ *and* $(\sum_{j\in\mathbb{N}} |\Delta_j f|^2)^{1/2} \in L^p(\mathbb{R}^d)$,
(iii) $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ *and* $(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2)^{1/2} \in L^p(\mathbb{R}^d)$.

Moreover, the following norms are equivalent on L^p :

$$\|f\|_p, \qquad \|S_0 f\|_p + \Big\|\Big(\sum_{j\in\mathbb{N}} |\Delta_j f|^2\Big)^{1/2}\Big\|_p \quad \textit{and} \quad \Big\|\Big(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2\Big)^{1/2}\Big\|_p.$$

Proof. See [33].

2.2. *Bogovskii operator*

Definition 2.2. Let $\mathcal{D}(\Delta_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ denote the usual domain of definition of the Laplace operator $\Delta = \Delta_q$ in L^q space with zero Dirichlet boundary condition. We set

$$L_0^q(\Omega) = \left\{ u \in L_q(\Omega) : \int_\Omega u \, dx = 0 \right\}.$$

We introduce the Bogovskii operator and we recall its properties. For a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary in $C^{0,1}$ Bogovskii [2], [3] constructed a bounded linear operator $\mathcal{R} : L_0^q(\Omega) \to W_0^{1,q}(\Omega)^n$ such that $u = \mathcal{R}g$ is a solution of

$$\begin{aligned} \operatorname{div} u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

satisfying $\|\mathcal{R}g\|_{W^{1,q}(\Omega)^n} \le c\|g\|_q$. Additionally $\mathcal{R}$ maps $W_0^{1,q}(\Omega) \cap L_0^q(\Omega)$ into $W_0^{2,q}(\Omega)$, see [2].

The Bogovskii operator was studied in a more general class of domains, see e.g. [5].

Assumptions I. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain with boundary $\partial\Omega \in C^{1,1}$, and suppose one of the following cases

(i) Ω is bounded,
(ii) Ω is an exterior domain, i.e., a domain having a compact nonempty complement.

(iii) Ω is a perturbed half space, i.e., there exists some open ball B such that $\Omega \setminus B = \mathbb{R}^n_+ \setminus B$.

LEMMA 2.1 (Farwig, Sohr). *Let $\Omega = \mathbb{R}^n$ or let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain satisfying Assumption I, further let $1 < q < \infty$. Then there exists a linear bounded operator $\mathcal{R} : W^{1,q}(\Omega) \cap \widehat{W}^{-1,q}(\Omega) \to \mathcal{D}(\Delta_q)^n$ if Ω is unbounded or $\mathcal{R} : W^{1,q}(\Omega) \cap L^q_0(\Omega) \to \mathcal{D}(\Delta_q)^n$ if Ω is bounded such that $u = \mathcal{R}g$ is a solution of (2.1) for all $g \in W^{1,q}(\Omega) \cap \widehat{W}^{-1,q}(\Omega)$ or $g \in W^{1,q}(\Omega) \cap L^q_0(\Omega)$ respectively; $u = \mathcal{R}g$ satisfies the estimates*

$$\|u\|_q \leq c\|g\|_{-1,q} \quad \text{and} \quad \|u\|_{W^{2,q}(\Omega)} \leq c(\|\nabla g\|_q + \|g\|_{-1,q}),$$

where $c = c(\Omega, q) > 0$ is a constant.

Proof. See [5].

2.3. *Maximal operator.* For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^n)$ let

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transform of u. Its inverse is denoted by $\mathcal{F}^{-1}$. Moreover, we define the centered Hardy-Littlewood maximal operator

$$\mathcal{M}u(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u(y)|\, dy, \quad x \in \mathbb{R}^n,$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where again Q runs through the set of all cubes centered at x.

3. Computation of ∇u. Using the fact that the space $\{g \mid g = \nabla \cdot G,\ G \in C_0^\infty(\mathbb{R}^3)^{3\times 3}\}$ is dense in $\widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we can write either f in the divergence form in the Oseen system (1.1) or $g = \nabla \cdot G$ in the reduced Oseen system 1.3, assuming firstly $G \in C_0^\infty\left(\mathbb{R}^3\right)^{3\times 3}$. We will work in the space of tempered distributions because we have in mind to apply the Fourier transform. We will derive the following formal expressions of $\widehat{u}$, u, and ∇u:

$$\widehat{u}(\xi) = \int_0^\infty e^{-(\nu|\xi|^2 + ik\xi_3)t} O^T_\omega(t)\, \widehat{g}(O_\omega(t)\xi)\, dt,$$

yielding $u(\cdot)$ in the form

$$u(x) = \int_0^\infty E_t * O^T_\omega(t)\, g(O_\omega(t). - kte_3)(x)\, dt,$$

where

$$E_t(x) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{|x|^2}{4\nu t}}.$$

Observing the previous integral the solution can be rewritten as

$$u(x) = \int_{\mathbb{R}^3} \Gamma\left(x, y\right)\, \nabla \cdot G\left(y\right)\, dy,$$

where

$$\Gamma(x, y) = \int_0^\infty E_t\left(O_\omega(t)x - y - kte_3\right)\, O^T_\omega(t)\, dt.$$

Therefore we can compute the gradient of u,

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma\left(x, y\right) : G\left(y\right)\, dy,$$

and come back to its Fourier transform.

Let us compute explicitly $\widehat{u}$. First of all, due to the geometry of the problem it is reasonable to introduce cylindrical coordinates $(r,\theta,x_3)\in(0,\infty)\times[0,2\pi)\times\mathbb{R}$. Then $(\omega\wedge x)\cdot\nabla u=|\omega|\,(-x_2\partial_1 u+x_1\partial_2 u)$ may be rewritten in the form

$$(\omega\wedge x)\cdot\nabla u=|\omega|\,\partial_\theta u$$

using the angular derivative ∂_θ applied to $u(r,x_3,\theta)$.

With the Fourier variable, $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$ we get from (1.1)

$$(\nu|\xi|^2+ik\xi_3)\widehat{u}-|\omega|\partial_\varphi\widehat{u}+|\omega|e_3\wedge\widehat{u}+i\xi\widehat{p}=\widehat{f},\quad i\xi\cdot\widehat{u}=0. \tag{3.1}$$

It is clear that $(e_3\wedge\xi)\cdot\nabla_\xi=-\xi_2\partial/\partial\xi_1+\xi_1\partial/\partial\xi_2=\partial_\varphi$ is the angular derivative in Fourier space when using cylindrical coordinates. Since (1.2) we have $i\xi\cdot(\partial_\varphi\widehat{u}-e_3\wedge\widehat{u})=0$, the unknown pressure p is explicitly given by $-|\xi|^2\widehat{p}=i\xi\cdot\widehat{f}$. Denoting $g=f-\nabla p$ then we get

$$-\partial_\varphi\widehat{u}+\frac{1}{|\omega|}(\nu|\xi|^2+ik\xi_3)\widehat{u}+e_3\wedge\hat{u}=\frac{1}{|\omega|}\widehat{g}, \tag{3.1$'$}$$

a first order differential equation with respect to φ for $\widehat{u}:=\widehat{u}(\sqrt{\xi_1^2+\xi_2^2},\varphi,\xi_3)$.

To deal with the term $\omega\wedge u$ note that $\partial_\phi O(\varphi)=\omega\wedge O(\varphi)$ in the sense of linear maps. Applying the $O(\varphi)$ to (3.1)$'$ the unknown $\widehat{v}(\varphi)=O(\varphi)^T\widehat{u}(\varphi)$ solves the problem

$$-\partial_\varphi\widehat{v}+\frac{1}{|\omega|}(\nu|\xi|^2+ik\xi_3)\widehat{v}=\frac{1}{|\omega|}\widehat{g}.$$

This inhomogeneous, linear ordinary differential equation of first order with respect to φ has a unique 2π-periodic solution

$$\widehat{v}(\varphi)=\frac{1/|\omega|}{1-e^{-2\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}}\int_0^{2\pi}e^{(-\nu|\xi|^2+ik\xi_3)t}O_{e_3}^T(\varphi+t)\widehat{g}(O_{e_3}(t)\xi)dt.$$

Then

$$\widehat{u}=\frac{1}{1-e^{-2\pi(\nu|\xi|^2+ik\xi_3)/|\omega|}}\int_0^{2\pi/|\omega|}e^{-(\nu|\xi|^2+ik\xi_3)t}O_{|\omega|}^T(t)\widehat{g}(O_{|\omega|})(t)\xi)dt.$$

Applying the geometric series and the $2\pi/|\omega|$-periodicity of the map $t\mapsto O_{|\omega}^T(t)\widehat{g}(O_{|\omega|}(t)\xi)$ we get the unique $2\pi/|\omega|$-periodic solution

$$\widehat{u}(\xi)=\int_0^\infty e^{-(\nu|\xi|^2+ik\xi_3)t}O_\omega^T(t)\,\widehat{g}(O_\omega(t)\xi)\,dt. \tag{3.2}$$

$\widehat{u}$ solves the reduced equation (3.1)$'$ in the Fourier space, we have

$$u(x)=\int_0^\infty\frac{1}{(4\pi\nu t)^{3/2}}\exp\left(-\frac{|\,.\,|^2}{4\nu t}\right)*O_\omega^T(t)\,g(O_\omega(t).-kte_3)(x)\,dt, \tag{3.3}$$

$$\nabla u(x)=\int_0^\infty\nabla\frac{1}{(4\pi\nu t)^{3/2}}\exp\left(-\frac{|\,.\,|^2}{4\nu t}\right)*O_\omega^T(t)\,g(O_\omega(t).-kte_3)(x)\,dt, \tag{3.4}$$

yielding

$$\nabla u(x)=\int_{\mathbb{R}^3}\nabla_x\Gamma(x,y)\,g(y)\,dy, \tag{3.5}$$

with $g=\nabla\cdot G$ and

$$\Gamma(x,y)=\int_0^\infty\frac{1}{(4\pi\nu t)^{3/2}}\exp\left(-\frac{|O_\omega(t)\,x-y-k\,t\,e_3|^2}{4\nu t}\right)O_\omega^T(t)\,dt.$$

Remark 7. Taking into account the expression $g = \nabla \cdot G$, $G \in C_0^\infty\left(\mathbb{R}^3\right)^9$, we can integrate by parts:

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma(x,y) : G(y)\, dy$$

Observing that $|y - O_\omega(t)x + kte_3| = |x - O_\omega^T(t)(y - kte_3)| = |x - O_\omega^T(t)y - kte_3|$ we get $\nabla_x \nabla_y \Gamma(x,y) = \nabla_x^2 \Gamma(x,y)$. So we have

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x^2 \Gamma(x,y) : G(y)\, dy = -\nabla^2 \int_{\mathbb{R}^3} \Gamma(x,y) : G(y)\, dy.$$

4. Proof of the main theorem. In this section we estimate the L^q-norm of each component of $TG(x) := \triangle \int_{\mathbb{R}^3} \Gamma(x,y) : G(y)\, dy$, say $TG_{i,k}(\cdot)$ by the L^q-norm of $G_{i,\,k}(\cdot)$, and then we apply these results for L^q-estimate of ∇u.

To this end, we follow the way used by Farwig, Hishida, Müller [12] and by Hishida [31] because till now we do not have a more direct analysis. By means of the Fourier transform we have

$$\widehat{TG}\,(\xi) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty |\xi|^2 \exp(-\nu\, |\xi|^2\, t) O_\omega^T(t)\, \widehat{G}\,((O_\omega(t) \cdot - k\, t\, e_3)\, \xi)\; dt.$$

Which we can rewrite as

$$\widehat{TG}\,(\xi) = \frac{1}{\nu(2\pi)^{3/2}} \int_0^\infty |\xi|^2 \exp(-\, |\xi|^2\, t) O_{\omega/\nu}^T(t)\, \widehat{G}\left(\left(O_{\omega/\nu}(t) \cdot - k\, \frac{t}{\nu}\, e_3\right) \xi\right)\; dt.$$

Let us temporarily denote $TG_{i,\,k}(x)$ by $F(x)$. A deep tool from harmonic analysis requires us to define an appropriate function $\varphi(\cdot) \in C_0^\infty((0,\infty); \mathcal{S}(\mathbb{R}^3))$ such that with the so called square operator

$$S(F)(x) = \int_0^\infty |\varphi(t,\cdot) * F(x)|^2\, \frac{dt}{t}$$

we obtain the equivalence of L^q-norms given by the theorem of E. M. Stein, Chapter I, Section 8.23 [37],

$$c_1 \|F\|_q \le \|S(F)^{1/2}\|_q \le c_2 \|F\|_q.$$

The necessary properties of $\varphi(t,\cdot)$ for $t > 0$ are

$$\operatorname{supp} \widehat{\varphi}(t,\cdot) \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2\sqrt{t}} < |\xi| < \frac{2}{2\sqrt{t}} \right\},$$

$$\int_0^\infty \widehat{\varphi}(t,\xi)^2 \frac{dt}{t} = 1, \qquad \int_{\mathbb{R}^3} \varphi(t,x)\, dx = 0.$$

We start with Littlewood-Paley decomposition. We define $\psi \in \mathcal{S}(\mathbb{R}^3)$ by its Fourier transform

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}} |\xi|^2 e^{-|\xi|^2} \quad \text{and} \quad \widehat{\psi}_t(\xi) = \widehat{\psi}(\sqrt{t}\xi) \text{ for } t > 0, \tag{4.1}$$

and so for all $t > 0$

$$\psi_t(x) = t^{-3/2} \psi\left(\frac{x}{\sqrt{t}}\right), \qquad \widehat{\psi}_t(\xi) = \frac{1}{(2\pi)^{3/2}} t |\xi|^2 e^{-\nu\, t\, |\xi|^2}. \tag{4.2}$$

So we get

$$\widehat{TG}(\xi) = \frac{1}{\nu}\int_0^\infty \widehat{\psi}_t(\xi)\, O^T_{\omega/\nu}(t)\, \widehat{G}\left(\left(O_{\omega/\nu}(t). - k\frac{t}{\nu}e_3\right)\xi\right)\frac{dt}{t}\,. \tag{4.3}$$

We define the multiplier operator Δ_j such that

$$\widehat{\Delta_j f(\xi)} := \widehat{\chi}^j(\xi)\widehat{f}(\xi) \tag{4.4}$$

where

$$\widehat{\chi}^j(\xi) = \widehat{\chi}_0\left(\frac{\xi}{2^{j+1}}\right) - \widehat{\chi}_0\left(\frac{|\xi|}{2^j}\right) \tag{4.5}$$

with

$$\widehat{\chi}_0(.) : |\xi| \to \mathbb{R}, \chi \in C^\infty, \widehat{\chi}_0|_{\{|\xi|\le\frac{1}{2}\}} = 1, \widehat{\chi}_0|_{\{|\xi|\ge 1\}} = 0. \tag{4.6}$$

Note that

$$\sum_{j\in\mathbb{Z}} \widehat{\chi}^j(\xi) = 1. \tag{4.7}$$

and

$$f(.) = \sum_{j\in\mathbb{Z}} \Delta_j f(.). \tag{4.8}$$

We define χ^j for $\xi \in \mathbb{R}^3$ and $j \in \mathbb{Z}$ by its Fourier transform

$$\widehat{\chi}^j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^3,$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^3 \setminus \{0\}$ and

$$\operatorname{supp} \widehat{\chi}^j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^3 : 2^{j-1} < |\xi| < 2^{j+1}\}. \tag{4.9}$$

Using χ^j we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{n/2}}\chi_j * \psi_t, \qquad \widehat{\psi^j} = \widehat{\Delta_j\psi(.)} = \widehat{\chi}^j \cdot \widehat{\psi}_t. \tag{4.10}$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^3$. We start with the procedure of the Littlewood-Paley decomposition of $F = TG_{i,k}$. For $G_{i,k} \in \mathcal{S}'(\mathbb{R}^3)$ the property $TG_{i,k} \in L^p(\mathbb{R}^3)$ is equivalent to the property

$$TG_{i,k} = \sum_{j=-\infty}^{+\infty} \Delta_j TG_{i,k} \quad \text{and} \quad \left(\sum_{j=-\infty}^{+\infty} |\Delta_j TG_{i,k}|^2\right)^{1/2} \in L^p\left(\mathbb{R}^3\right).$$

where

$$\Delta^j = \mathcal{F}^{-1}\widehat{\psi^j}\left(\frac{\xi}{2^j}\right)\mathcal{F}, \qquad \Delta^j_t = \mathcal{F}^{-1}\widehat{\psi}^j_t\left(\frac{\xi}{2^j}\right)\mathcal{F}.$$

We define

$$\Delta\Gamma = \sum_{j\in Z} \Delta_j \Delta\Gamma \tag{4.11}$$

leading to

$$\Delta\int_{\mathbb{R}^3} \Gamma(x,y) : G(y)dy, \quad G \in C_0^\infty(\mathbb{R}^3)^9, (G_{ik})_{1\le i\le 3, 1\le k\le 3}. \tag{4.12}$$

We define the linear operator

$$TG_{ik}(x) = \int_{\mathbb{R}^3} \Delta\Gamma(x,y)_{ki} G_{ik}(y)\,dy. \tag{4.13}$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we have to prove that this infinite series converges even in the operator norm on L^q.

For later use we cite the following lemma, see [12].

LEMMA 4.1. *The functions Δ^j, Δ^j_t, $j \in \mathbb{Z}$, $t > 0$, have the following properties:*

(i) $\operatorname{supp} \widehat{\Delta}^j_t \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$.

(ii) *For $m > \frac{n}{2}$ let $h(x) = (1+|x|^2)^{-m}$ and $h_t(x) = t^{-n/2} h(\frac{x}{\sqrt{t}})$, $t > 0$. Then there exists a constant $c > 0$ independent of $j \in \mathbb{Z}$ such that*

$$|\Delta^j(x)| \le c 2^{-2|j|}\, h_{2^{-2j}}(x), \quad x \in \mathbb{R}^n,$$
$$\|\Delta^j\|_1 \le c 2^{-2|j|}\,.$$

Proof. See [12].

From the general definition of a Littlewood-Paley decomposition of L^q choose $\widetilde{\varphi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \le \widetilde{\varphi} \le 1$ and

$$\int_0^\infty \widetilde{\varphi}(s)^2 \,\frac{ds}{s} = \frac{1}{2}.$$

Then define $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s > 0$

$$\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \qquad \operatorname{supp} \widehat{\varphi}_s \subset A\left(\frac{1}{2\sqrt{s}}, \frac{2}{\sqrt{s}}\right), \tag{4.14}$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2 \,\frac{ds}{s} = 1$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$.

THEOREM 4.1. *Let $1 < q < \infty$. Then there are constants $c_1, c_2 > 0$ depending on q and φ such that for all $f \in L^q$*

$$c_1 \|f\|_q \le \left\| \left(\int_0^\infty \int_0^\infty |\varphi_s * f(.)|^2 \frac{ds}{s} \right) \right\|_{q/2} \le c_2 \|f\|_q$$

where $\varphi_s \in \mathcal{S}(\mathbb{R}^n)$ is defined by (4.14).

Proof. See [37], Chapter I, Section 8.23.

We apply Theorem 4.1 to the operator $T_j G_{ik}$:

$$c_1 \|T_j G_{ik}\|_q \le \left\| \int_0^\infty |(\varphi(t,.) * T_j G_{ik})(x)|^2 \,\frac{dt}{t} \right\|_{q/2} \le c_2 \|T_j G_{ik}\|_q. \tag{4.15}$$

5. Proofs. As a preliminary version of Theorem 1. 1 we prove the following proposition.

PROPOSITION 5.1. *Let $j \in \mathcal{Z}$. The linear operator T defined by* (4.3) *satisfies the estimate*

$$\|T_j G_{ik}\|_q \le c \|G_{ik}\|_q \quad \textit{for all} \quad G \in L^q,\ q \in (2, \infty)$$

with a constant $c = c(q, w) > 0$ independent of f.

Proof. We define the sublinear operator $\mathcal{M}^j$, a modified maximal operator, by

$$\mathcal{M}^j\varphi(x) = \sup_{s>0}\int_{A_s}(|\Delta_t^j| * |\varphi|)\left(O_{\omega/\nu}(t)^T x + \frac{k}{\nu}te_3\right)\frac{dt}{t}\,, \tag{5.1}$$

where $A_s = [\frac{s}{16}, 16s]$.

First step. We will prove the preliminary estimate

$$\|T_j G_{ik}\|_q \le c\|\Delta^j\|_1^{1/2}\|\mathcal{M}^j\|_{L^{(q/2)'}}^{1/2}\|G_{ik}\|_q\,, \quad j\in\mathbb{Z}. \tag{5.2}$$

To prove (5.2) we use the Littlewood-Paley decomposition of L^q,

$$c_1^2\|f\|_q^2 \le \left\|\int_0^\infty |\varphi_s * f(\cdot)|^2\,\frac{ds}{s}\right\|_{q/2} \le c_2^2\|f\|_q^2. \tag{5.3}$$

By a duality argument we find some function $0\le g\in L^{(q/2)'}$ with $\|g\|_{(q/2)'}=1$ such that

$$\left\|\int_0^\infty |\varphi_s * T_jG_{ik}(\cdot)|^2\,\frac{ds}{s}\right\|_{q/2} = \int_0^\infty\int_{\mathbb{R}^n}|\varphi_s * T_jG_{ik}(x)|^2 g(x)\,dx\,\frac{ds}{s}. \tag{5.4}$$

To estimate the right-hand side of (5.4) note that

$$\varphi_s * T_jG(x) = \int_0^\infty O(t)^T_{\omega/\nu}(t)(\varphi_s * \Delta_t^j * G_{ik})\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\Big)\,\frac{dt}{t}\,,$$

where $\varphi_s * \Delta_t^j = 0$ unless $t\in A(s,j) := [2^{2j-4}s, 2^{2j+4}s]$. Since $\int_{t\in A(s,j)}\frac{dt}{t} = \log 2^8$ for every $j\in\mathbb{Z}$, $s>0$, we get by the inequality of Cauchy-Schwarz and the associativity of convolutions that

$$\begin{aligned}|\varphi_s * T_jG_{ik}(x)|^2 &\le c\int_{A(s,j)}\left|(\Delta_t^j*(\varphi_s * G_{ik}))\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\Big)\right|^2\frac{dt}{t}\\ &\le c\|\Delta^j\|_1\int_{A(s,j)}(|\Delta_t^j| * |\varphi_s * G_{ik}|^2)\Big(O(t)_{\omega/\nu}x + \frac{k}{\nu}te_3\Big)\,\frac{dt}{t};\end{aligned}$$

here we used the estimate $|(\Delta_t^j * (\varphi_s * G_{ik}))(y)|^2 \le \|\Delta_t^j\|_1(|\Delta_t^j| * |\varphi_s * G_{ik}|^2)(y)$ and the identity $\|\Delta_t^j\|_1 = \|\Delta^j\|_1$, see Lemma 4.1. Thus

$$\begin{aligned}&\|T_jG_{ik}\|_q^2\\ &\le c\int_0^\infty\int_{A(s,j)}\int_{\mathbb{R}^n}(|\Delta_t^j| * |\varphi_s * G_{ik}|^2)(x)\Big(O(t)^T_{\omega/\nu}x - \frac{k}{\nu}te_3\Big)g(x)\,dx\frac{dt}{t}\frac{ds}{s}\end{aligned} \tag{5.5}$$

since Δ_t^j is radially symmetric. By definition of $\mathcal{M}^j$ the innermost integral is bounded by $\mathcal{M}^jg(x)$ uniformly in $s>0$. Hence we may proceed in (5.5) using Hölder's inequality as follows:

$$\|T_jG\|_q^2 \le c\|\Delta^j\|_1\int_{\mathbb{R}^n}\left(\int_0^\infty|\varphi_s * G|^2(x)\,\frac{ds}{s}\right)\mathcal{M}^jg(x)\,dx. \tag{5.6}$$

Now (5.3) and the normalization $\|g\|_{(q/2)'}=1$ complete the proof of (5.2).

Second step. We investigate the estimate $\|\mathcal{M}^jg\|_{(q/2)'}$. Since $\frac{q}{2}\in(1,\infty)$ is arbitrary, we have to consider $\|\mathcal{M}^j\|_{L^p}$ for arbitrary $p\in(1,\infty)$. For this reason we define the classical Hardy-Littlewood maximal operator $\mathcal{M}$ on $L^p(\mathbb{R}^3)$ by

$$\mathcal{M}g(x) := \sup_{s>0}\frac{1}{|B_s(x)|}\int_{B_s(x)}|g(x)|\,dy$$

and a "helical" maximal operator

$$\mathcal{M}_{\text{hel}}g(\theta, x_3) := \sup_{s>0} \frac{1}{s} \int_{A_s} |g| \left| \theta - \frac{\omega}{\nu} t, x_3 + \frac{k}{\nu} t \right|,$$

for functions g depending on (θ, x_3), which are 2π-periodic in θ. Since $0 \le h \in L^1(\mathbb{R}^3)$ is radially symmetric and strictly decreasing,

$$\sup_{r>0} h_r * u(x) \le c\mathcal{M}u(x).$$

Then

$$\mathcal{M}_j g(x) \le c2^{-2|j|} \mathcal{M}(\mathcal{M}_{hel} g_r(.,.))(x),$$

where $g_r(\theta, x_3) = g(r, \theta, x_3) = g(x)$ is considered as a function of θ, x_3 and

$$\|\mathcal{M}_j g\|_p \le C2^{-2j} \|\mathcal{M}_{\text{hel}} g_r(\cdot, \cdot)\|_{L^p(\mathbb{R}^3)}.$$

due to L^p continuity of $\mathcal{M}$.

To estimate $\mathcal{M}_{\text{hel}} g_r(\cdot, \cdot)$ in $L^p(\mathbb{R}^3)$, fix $r > 0$ and use the 2π-periodicity of g_r with respect to θ to get that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^{2\pi} |\mathcal{M}_{\text{hel}} g_r(\theta, x_3)|^p d\theta dx_3 \\
&\le \int_{\mathbb{R}} \int_0^{2\pi} \left| \sup_{s>0} \frac{1}{s} \int_{-16s}^{16s} |g_r| \left(\theta - \frac{\omega}{k} \left(x_3 + \frac{k}{\nu} t \right), x_3 + \frac{k}{\nu} t \right) dt \right|^p d\theta dx_3 \\
&= \int_{\mathbb{R}} \int_0^{2\pi} \left| \sup_{s>0} \frac{1}{s} \int_{-16s}^{16s} \gamma_{r,\theta} \left(x_3 + \frac{k}{\gamma} t \right) dt \right|^p d\theta dx_3,
\end{aligned}$$

where

$$\gamma_{r,\theta}(y_3) = |g_r| \left(\theta - \frac{\omega}{k} y_3, y_3 \right).$$

Thus we get applying Hardy-Littlewood maximal operator on $\mathbb{R}^1$ that

$$\|\mathcal{M}_{\text{hel}} g_r(\cdot, \cdot)\|_{L^p(\mathbb{R}^3)} \le c\|g\|_{L^p(\mathbb{R}^3)}. \tag{5.7}$$

From Lemma 4.1 and Proposition 5.1 the operator T_j satisfies the estimates

$$\|T_j G_{ik}\|_q \le C2^{-2|j|} \|G_{ik}\|_q, \ j \in \mathcal{Z}, \ q \in (2, \infty), c = c(q) > 0.$$

Then $T = \sum_{j=-\infty}^{\infty} T_j$ converges in the operator norm on $L^q(\mathbb{R}^3)^3$ and $\|TG\|_q \le c\|G\|_q$, for every $G \in S(\mathbb{R}^3)^3$.

Third step. For $1 < q < 2$ we use the adjoint operator T^* given by

$$T^* G(x) = \int_0^{\infty} (\Delta_t * O_{\omega/\nu}(t) G) \left(O^T_{\omega/\nu}(t) x + \frac{k}{\nu} t e_3 \right) \frac{dt}{t}, \tag{5.8}$$

with $G \in \mathcal{S}(\mathbb{R}^3)^3$ and then by same argument we get that T^* is bounded in $L^{\frac{q}{q-1}}(\mathbb{R}^3)^9$, so T is L^q bounded for $1 < q < 2$. This implies the following estimate

$$\|\nabla u\|_q \le \|G\|_q. \tag{5.9}$$

Fourth step. Now, using Farwig-Sohr Lemma 2.1 we know that there is $G \in L^q(\mathbb{R}^3)^9$ such that

$$\nabla \cdot G = f, \ \|G\|_{q,\mathbb{R}^3} \le C\|f\|_{-1,q,\mathbb{R}^3}.$$

Let $G_k \in C_0^\infty(\mathbb{R}^3)^9$ such that $\|G_k - G\|_{q,\mathbb{R}^3} \to 0$ as $k \to \infty$. Let u_k be solution of our fundamental solution (3.3) with $f = \nabla \cdot G_k$. For each k and $m \in \mathbb{N}$, we choose a constant vector $b_k^m \in \mathbb{R}^3$ satisfying

$$\int_{B_m} (u_k(x) + b_k^m)dx = 0$$

so that

$$\|u_k + B_k^m\|_{q,B_m} \le C_m \|\nabla u_k\|_{q,B_m} \le C_m \|\nabla u_k\|_{q,\mathbb{R}^3} \le c_m \|G_k\|_{q,\mathbb{R}^3}$$

by Poincaré inequality and by (5.9).

Therefore, there exist $u^{(m)} \in W^{1,q}(B_m)^3$ and $V \in L^q(\mathbb{R}^3)^9$ such that

$$\|u_k + b_k^{(m)} - u^{(m)}\|_{q,B_m} \to 0, \quad \|\nabla u_k - V\|_{q,\mathbb{R}^3} \to 0, \quad k \to \infty,$$

with

$$\nabla u^{(m)}(x) = V(x) \text{ (a.a. } x \in B_m).$$

We first set

$$\tilde{u} = u^{(1)} \text{ on } B_1; \; b_k = b_k^{(1)}.$$

Consider next the case $m = 2$; since $\nabla u^{(2)}(x) = V(x) = \nabla u^{(1)}(x) = \nabla \tilde{u}(x)$ for a.a. $x \in B_1 \subset B_2$, the difference $u^{(2)}(x) - \tilde{u}(x) =: a$ is a constant vector and

$$\begin{aligned} &|B_1|^{1/q}|b_k^{(2)} - b_k - a| = \|b_k^{(2)} - b_k - a\|_{q,B_1} \le \\ &\|u_k + b_k - \tilde{u}\|_{q,B_1} + \|u_k + b_k^{(2)} - u^{(2)}\|_{q,B_2} \to 0, \; k \to \infty \end{aligned} \tag{5.10}$$

One extends $\tilde{u}$ by

$$\tilde{u} = u^{(2)} - a \text{ on } B_2.$$

Then (5.10) implies

$$\|u_k + b_k - \tilde{u}\|_{q,B_2} \le \|u_k + b_k^{(2)} - u^{(2)}\|_{q,B_2} + |B_2|^{1/q}|b_k^{(2)} - b_k - a| \to 0 \tag{5.11}$$

as $k \to \infty$. By induction there exists a function $\tilde{u} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ so that

$$\|u_k + b_k - \tilde{u}\|_{q,B_m} + |\nabla u_k - \nabla \tilde{u}\|_{q,\mathbb{R}^3} \to 0, \quad k \to \infty, \tag{5.12}$$

for all $m \in \mathbb{N}$. We define

$$L = -\Delta - \frac{\partial}{\partial x_3} - (\omega \wedge x) \cdot \nabla + \omega \wedge .$$

From definition of L together with $Lu_k = \nabla \cdot G_k$ we have

$$Lb_k = \omega \wedge b_k = L(u_k + b_k) - \nabla \cdot G_k \to L\tilde{u} - \nabla \cdot G \text{ in } \mathcal{D}'(\mathbb{R}^3)^3 \text{ as } k \to \infty.$$

Since there is a constant vector $b \in \mathbb{R}^3$ such that

$$\omega \wedge b_k \to \omega \wedge b = Lb$$

as $k \to \infty$. Consequently, we get

$$L(\tilde{u} - b) = \nabla \cdot G \text{ in } \mathcal{D}'(\mathbb{R}^3)^3$$

and $u = \tilde{u} - b$ is the desired solution. By (5.12) we have $\|u_k - \nabla u\|_{q,\mathbb{R}^3} \to 0$ and, therefore, the estimate (1.14) holds.

Fifth step. It remains to prove the uniqueness. We use the duality method. We consider the adjoint equation

$$L^*v \equiv -\Delta v + (\omega \wedge x) \cdot \nabla v - \omega \wedge v + \frac{\partial u}{\partial x_3} = \nabla \cdot G \tag{5.13}$$

with $G \in C_0^\infty(\mathbb{R}^3)^9$. It has the solution

$$\widehat{v}(\xi) = \int_0^\infty e^{-\nu|\xi|^2 t} O_\omega(t)(\mathcal{G}f(O_\omega^T(t). - kte_3))(\xi)dt. \tag{5.14}$$

Applying the same argument we get

$$\|\nabla v\|_{r,\mathbb{R}^3} \le C\|G\|_{r,\mathbb{R}^3}, \text{for all } v \in \widehat{W}^{1,r}(\mathbb{R}^3),\ r \in (1,\infty). \tag{5.15}$$

Let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be a weak solution of $Lu = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$. One can take as a test function to get

$$\langle Lu, v\rangle = 0.$$

Similarly, one takes u as a test function for (5.13) in $\widehat{W}^{-1,q/(q-1)}(\mathbb{R}^3)^3$ to obtain

$$\langle u, L^*v\rangle = \langle u, \nabla \cdot G\rangle.$$

Therefore,

$$\langle u, \nabla \cdot G\rangle = 0.$$

Since $G \in (C_0^\infty)^9$ is arbitrary, we obtain $u = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ by Theorem 1.2. u is a constant vector, but it is a constant multiple of ω because $\omega \wedge u = 0$.

To complete the proof of Theorem 1.1, we have to show the following lemma

LEMMA 5.1. *Let $v \in \mathcal{S}(\mathbb{R}^3)$ be the solution of*

$$-\Delta v + \frac{\partial v}{\partial x_3} - (\omega \wedge x) \cdot \nabla v = 0 \quad in\ \mathbb{R}^3.$$

Then $\operatorname{supp} \widehat{v} \subset \{0\}$.

Proof. This was proved in [10].

Proof of Theorem 1.1. As we explained before, the term $-(\omega\wedge x)\cdot\nabla u+\omega\wedge u$ is divergence free. The pressure is formally obtained from the problem

$$p = -\nabla \cdot (-\Delta)^{-1} f.$$

Since $(-\Delta)^{-1}$ can be justified as a bounded operator from $\widehat{W}^{-1,q}(\mathbb{R}^3)$ to $\widehat{W}^{1,q}(\mathbb{R}^3)$ we get

$$\|\nabla p\|_q \le c\|f\|_{1,q},$$

which implies that

$$\|f - \nabla p\|_{-1,q} \le c\|f\|_{-1,q}.$$

This completes the proof of Theorem 1.1.

Proof of Corollary 1.1. From [6] there exists $\alpha \in \mathbb{R}^3$ such that

$$v = u - \alpha \in L^s(\mathbb{R}^3),\ \text{for all } s > 1,\ 1/s \in 1/q - [1/4, 1/3].$$

Let

$$\tilde{L}u' := -\partial_\theta u' + u^\perp.$$

Since $\tilde{L}v' = -\partial_\theta v' + v^\perp = \tilde{L}u' - a^\perp$ and applying integration with respect to θ we get

$$2\pi a' = \int_0^{2\pi} \tilde{L}u' d\theta - \int_0^{2\pi} v' d\theta \in L^q + L^s,$$

which implies $a' = 0$.

Proof of Corollary 1.2. From Theorem II5.1 of [15] yields the estimate

$$\left(\int_{\mathbb{R}^3} \frac{|u(x) - u_\infty|}{|x|} dx \right)^{1/q} \leq \frac{q}{3-q} \left(\int_{\mathbb{R}^3} |\nabla u(x)|^q dx \right)^{1/q}.$$

Moreover, by lemma 5.2 of [15]

$$\int_{y=1} |u(Ry)|^q do(y) = o(R^{q-3})$$

as $R \to \infty$. Since $u \in L^s(\mathbb{R}^3)$ that u_∞ vanishes.

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