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ON THE EXISTENCE AND REGULARITY OF THE SOLUTIONS TO THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN PRESENCE OF MASS DIFFUSION

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Abstract. This paper is devoted to the study of the incompressible Navier-Stokes equations with mass diffusion in a bounded domain in R^3 with C^3 boundary. We prove the existence of weak solutions, in the large, and the behavior of the solutions as the diffusion parameter $\lambda \to 0$. Moreover, the existence of L^2-strong solution, in the small, and in the large for small data, is proved. Asymptotic regularity (the regularity after a finite period) of a weak solution is studied. Finally, using the Dore-Venni theory, the problem of the L^q-maximal regularity is investigated.

1. Introduction. Presented in this paper is a general existence and regularity theory of a nonhomogeneous Navier-Stokes model with mass diffusion. Observe that the model includes as particular cases the classical Navier-Stokes equations and the density-dependent Navier-Stokes equations. The Navier-Stokes equations are largely studied while the literature concerning the model considered in this paper is not very extensive. So far, we focus our attention mainly on the $L^2(\Omega), L^q(\Omega)$-theory in bounded domains. But the procedure used here can be adapted to unbounded domains. Moreover, we do not insist on the precise critical descriptions of the spaces of the initial data.

1.1. *The model.* We now derive the equations of our physical model (see [9]).

Let Ω be a bounded domain in R^3 with boundary Γ. We consider in Ω the motion of a continuous medium consisting of two components, for example, water and dissolved salt. Let $\rho_i^0 = cost > 0, i = 1, 2$ be the characteristic densities of the components of the mixture and α and c be the volume and mass concentrations of one of the components, say water.

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We introduce the concept of mean densities $\rho_1 = \alpha\rho_1^0, \rho_2 = (1-\alpha)\rho_2^0$ of the components, $\rho = \rho_1 + \rho_2$ of the solution, and also the velocities v_1 and v_2. The mean-volume and mass-velocities of the mixture are defined in the usual way: $v = \alpha v_1 + (1-\alpha)v_2$ and $w = cv_1 + (1-c)v_2$. The equations of the motion in $Q_T = \Omega \times (0,T)$ are given by

$$\begin{cases} \rho(\partial_t w + w\cdot\nabla w - f) - \mu'\Delta w - (\mu+\mu')\nabla\nabla\cdot w + \nabla p = 0, \\ \nabla\cdot u = 0, \quad \partial_t\rho + \nabla\cdot(w\rho) = 0; \quad \nabla\cdot v = 0. \end{cases} \tag{1}$$

Here p is the pressure, f the external force and μ, μ' the viscosity constants such that $\mu > 0$ and $3\mu' + \mu > 0$.

Making use of the Fick diffusion law

$$w = v - \frac{\lambda}{\rho}\nabla\rho,$$

($\lambda > 0$ is the constant diffusion coefficient) we get

$$\begin{cases} \rho(\partial_t v + v\cdot\nabla v - f) - \mu\Delta v - \lambda((v\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)v) \\ -\frac{\lambda^2}{\rho}((\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho) + \nabla\pi = 0, \\ \partial_t\rho + v\cdot\nabla\rho - \lambda\Delta\rho = 0; \quad \nabla\cdot v = 0. \end{cases} \tag{2}$$

Here $\pi = p + \lambda v\cdot\nabla\rho - \lambda^2\Delta\rho + \lambda(2\mu+\mu')\Delta\log\rho$ is the modified pressure.

We complete the system with the following initial-boundary conditions (n is the unit outward normal to Γ)

$$v = 0,\ \partial_n\rho = 0,\ \text{on } \Gamma, \forall t > 0,$$
$$v(0) = v_0,\ \rho(0) = \rho_0 \text{ in } \Omega.$$

We notice that

$$\nabla\cdot\left(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho\right) = \frac{1}{\rho}\left[(\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho\right].$$

1.2. *Known results.* The system (2) includes as particular cases the classical Navier-Stokes equations ($\rho = const$), the density-dependent Navier-Stokes equations ($\lambda = 0$), the reduced model (λ^2-terms are omitted). The classical Navier-Stokes system is largely studied (see the classical books by Ladyzhenskaya and Temam). The other models are less known and the literature on the complete model (2) is not very extensive.

Density-dependent Navier-Stokes equations. A. Kazhikhov proved in [8], via a Galerkin-type approximation (semi Galerkin method), the existence, in the large, of a weak solution (see Definition 1 below) and established the existence of a local strong solution. The uniqueness of strong solution at this time is an open problem. In [10] Ladyzhenskaya and Solonnikov proved the uniqueness for a smoother solution of the model with $\lambda = 0$. The above results are proved in the case that the initial density ρ_0 does not vanish. In [13] the author solved the open problem of the existence of weak solution of the variational inequality associated to the model in arbitrary L^2-convex sets. Moreover, in [16] and [17], error estimates bounds for the Galerkin approximations and regularity are investigated.

Reduced model. Kazhikhov and Smagulov [9] established the global existence of weak solution and local strong solution under the assumption $\lambda < 2\mu/(M-m)$ ($M = \sup\rho$,

$m = \inf \rho$ in Q_T) via the semi Galerkin approximation and an estimate of a sort of time fractional derivative:

$$\int_0^T |v(t+h) - v(t)|_2^2 dt \le c\sqrt{h},$$

in space dimension $n = 3, 4$ and assuming the initial density bounded from below by a positive constant.

The author in [15], among other things, extended the Kazhikhov and Smagulov results to arbitrary space dimension and the initial density can vanish, making use of the estimate of the time derivative of ρv. For further development of this method see [11].

Complete model. Beirão da Veiga [2], [3], Secchi [19] established the local existence of strong solutions using fixed point arguments. Moreover, behavior as $\lambda \to 0$ and $t \to \infty$ is considered.

Damázio, Guillén, and Rojas-Modar [5] study an approximation of solution by means of an iterative method, establish some convergence rates for the approximation of the exact solution for strong and more regular solutions.

The paper is organized as follows. In section 2 we introduce notation and functional spaces. In section 3 the main results of the paper are presented. Section 4 is devoted to the proof of Theorem 1. In section 5 the existence of L^2 strong solution is proved. In section 6 the problem of the existence of periodic solution is discussed. In section 7 the asymptotics and $C^\infty((0,\bar{T}) \times \bar{\Omega})$ regularity for a weak solution is investigated and we prove Theorem 3. Finally, in section 8 Theorem 4, concerning the maximal regularity, is proved.

2. Notation and functional spaces. In the sequel we will assume that Ω denotes an open set in R^m which is generally assumed to be bounded hence $\bar{\Omega}$ is compact. Γ denotes the boundary of Ω. Moreover, it is assumed that Ω is a smooth domain of class C^k with k a positive integer. Furthermore, we assume that the unit normal vector field $n(x)$ with $x \in \Gamma$ is outward to Γ. If it is necessary we consider also an extension of n in a neighborhood of $\bar{\Omega}$.

To simplify the discussion, we do not distinguish in our notations whether the functions are R- or R^m-valued, and c denotes a constant. We define $C_0^\infty(\Omega)$ to be the linear space of infinitely many times differentiable functions with compact supports in Ω. Now let $(C_0^\infty(\Omega))'$ denote the dual space of $C_0^\infty(\Omega)$, the space of distributions on Ω. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $(C_0^\infty(\Omega))'$ and $C_0^\infty(\Omega)$.

Let $\alpha = (\alpha_1, ..., \alpha_n) \in N^n$ and set $|\alpha| = \sum_{i=1}^n \alpha_i$. We set

$$\frac{\partial}{\partial x_i} = \partial_{x_i}, \quad D_x^\alpha = \partial^{|\alpha|}_{x_1^{\alpha_1}, ..., x_n^{\alpha_n}},$$

$\nabla = (\partial_{x_1}, ..., \partial_{x_m})$ the gradient operator and $\nabla\cdot$ the divergence operator.

We denote by $\mathcal{C}_0^\infty$ the linear subspace of divergence free functions of C_0^∞. For any $s, q,\ s \ge 0, q \ge 1$, $H_q^s(\Omega)$ denotes the usual Slobodeckii-Sobolev space of order s on $L^q(\Omega)$. Further, the norm (defined intrinsically involving first order differences of the highest-order derivatives) on $H_q^s(\Omega)$ is denoted by $\|\phi\|_{s,q}$.

We do not consider in this paper Besov space $B^s_q(\Omega)$ (the norm involves the second rather than the first differences). Notice that B^s_q and H^s_q coincide if $q = 2$, $\forall s$ but not for non integral s. When $q = 2$, $H^s_2(\Omega)$ is usually denoted by $H^s(\Omega)$ and we drop the subscript $q = 2$ when referring to its norm. $H^s(\Omega)$ $(s \in N)$ is a Hilbert space for the scalar product

$$((u,v))_s = \sum_{|\alpha|\le s} \int_\Omega D^\alpha u D^\alpha v dx.$$

In particular, in $L^q(\Omega)$, we write the L^q-duality pairing $(u,v)_q = \int_\Omega uvdx$ with $u \in L^q$ and $v \in L^{q'}$ with $q' = q/(q-1)$ and the norm $|v|_q$.

Further, we define $H^s_{q,0}(\Omega)$ the closure of $C^\infty_0(\Omega)$ for the norm $\|\cdot\|_{s,q}$.

We denote $H^{-s}_{q'}(\Omega)$ the dual space of $H^s_{q,0}(\Omega)$ and $\|\cdot\|_{-s,q'}$ denotes its norm where q' satisfies $1/q + 1/q' = 1$.

Let us introduce the following spaces of divergence-free functions. We denote by

$$V^s = \{v | v \in H^s_0(\Omega), \nabla \cdot v = 0\}.$$

V^s is the closure of $\mathcal{C}^\infty_0(\Omega)$ for the norm $\|\cdot\|_s$, and it is a closed subspace of $H^s(\Omega)$.

We set $V^1 = V$ and $V^0 = H$. In similar manner we define the spaces V^s_q. Moreover, we introduce the projection operator P_q $(P_2 \equiv P) : L^q \to V^0_q$. It is well known that the operator P_q is continuous on L^q and the subspace V^0_q is complemented. Thus, the following decomposition of L^q

$$L^q = Range P_q \oplus Ker P_q$$

holds true.

It is interesting to observe that $Ker P_q = \{\phi \in L^q | \phi = \nabla p_1 + \nabla p_2\}$ where p_1, p_2 are generalized solutions of the problems

$$\Delta p_1 = 0,\ \partial_n p_1 = f \text{ on } \Gamma,$$

and

$$\Delta p_2 = \nabla \cdot g,\ p_2 = 0 \text{ on } \Gamma,$$

respectively. Here $g \in L^q$ and $f \in H^{-1/q}_q(\Gamma)$ with $\langle f, 1\rangle_\Gamma = 0$. We recall that analogous decomposition of L^q holds working with the subspace $\bar{V} = \{\phi | \phi \in L^q, \nabla \cdot \phi = 0\}$.

We further define the Stokes operator on L^q

$$A_q = -P_q \Delta,$$

with domain $D(A_q) = H^2_q(\Omega) \cap V^1_q$.

For any Banach space X and for any $T > 0$ we denote by $L^r(0,T;X)$ the set of X-valued functions defined a.e. in $[0,T]$ and L^r summable in the sense of Bochner. Frequently, we consider $X = H^s_q(\Omega)$. In such cases, for any $\phi \in L^r(0,T;H^s_q(\Omega))$, ϕ stands for the function $\phi(t)$ or $\phi(\cdot,t)$.

Throughout the paper we denote $Q_t = (0,t) \times \Omega$ and the parabolic Slobodeckii-Sobolev space $H^{s,r}_q(Q_T)$ of order s in space variable and of order r in time variable on L^q. We will denote by $\|\cdot\|^{s,r}_q$ the norm in this space. In the following we make use of the inequality, for $q > 3$,

$$\sup_{(x,t)\in Q_T} |v| \le \|v\|_{H_q^{2,1}(Q_T)}.$$

In addition, let us consider the affine space

$$\tilde{H}^k(\Omega) = \left\{\phi \in H^k(\Omega), \partial_n\phi = 0 \text{ on } \Gamma, \int_\Omega \rho dx = \int_\Omega \rho_0 dx\right\}.$$

In this manner, the functions in $\tilde{H}^k$ are uniquely fixed and we can not distinguish the norms $\|\phi\|_{H^2}$ and $|\Delta\phi|_2$ in $\tilde{H}^2$, $\|\phi\|_{H^3}$ and $|\nabla\Delta\phi|_2$ in $\tilde{H}^3$.

Throughout the paper we shall use the following propositions.

PROPOSITION 1 (Gagliardo-Nirenberg inequality). *Let $\Omega \subset R^n$ bounded and sufficiently regular. The multiplicative inequality*

$$\sum_{|\alpha|=r} |D^\alpha\phi|_q \le c|\phi|_{q_1}^{1-\theta}\Big(\sum_{|\alpha|=l} |D^\alpha\phi|_{q_2}\Big)^\theta, \tag{3}$$

for $1 \le q_1, q_2 \le \infty$, $0 \le r \le l$,

$$\frac{n}{q} - r = (1-\theta)\frac{n}{q_1} + \theta\Big(\frac{n}{q_2} - l\Big), \quad \frac{r}{l} \le \theta \le 1,$$

holds with the following exceptions:

a) if $r = 0$, $l < \frac{n}{q_2}$, and $q_1 = \infty$ and Ω unbounded, we assume in addition that either $\phi \to 0$ as $x \to \infty$ or $\phi \in L^p$ for some $p > 0$;

b) if $1 < q_1 < \infty$ and $l - r - \frac{n}{q_2}$ is a non-negative integer, then (3) does not hold for $\theta = 1$.

The following propositions are commonly used in the theory of ordinary differential equations. We will give a brief proof of Proposition 4 only.

We assume that $\phi(t)$, $\psi(t)$, $h(t)$, $f(t)$ are smooth non-negative functions defined for all $t > 0$.

PROPOSITION 2. *Suppose $\phi(0) = \phi_0$ and $\frac{d}{dt}\phi(t) + \psi(t) \le g(\phi(t)) + f(t)$ for $t \ge 0$ where g is a non-negative Lipschitz continuous function defined for $\phi \ge 0$. Then $\phi(t) \le F(t;\phi_0)$ for $t \in [0, T(\phi_0))$ where F is the solution of the initial value problem $\frac{d}{dt}F(t;\cdot) = g(F(t)) + f(t)$, $F(0;\cdot) = \phi_0$ and $[0, T(\phi_0))$ is the largest interval to which it can be continued. Also, if g is non-decreasing then*

$$\int_0^t \psi d\tau \le \tilde{F}(t,\phi_0),$$

with

$$\tilde{F}(t;\phi_0) = \phi_0 + \int_0^t (g(F(\tau;\phi_0)) + f(\tau))d\tau.$$

PROPOSITION 3. *Suppose $\phi(0) = \phi_0$ and $\frac{d}{dt}\phi(t) + \psi(t) \le h(t)\phi(t) + f(t)$ for $t \ge 0$. Then*

$$\phi(t) \le F(t;\phi_0), \quad \int_0^t \psi d\tau \le \tilde{F}(t,\phi_0),$$

with

$$F(t,\phi_0) = \Big(\phi_0 + \int_0^t f(\tau)e^{\int_0^\tau -h(\sigma)d\sigma}d\tau\Big)e^{\int_0^t h(\tau)d\tau},$$

$$\tilde{F}(t;\phi_0) = \phi_0 + \int_0^t (h(\tau)(F(\tau;\phi_0)) + f(\tau))d\tau.$$

Thus, the estimates for ϕ and $\int_0^t \psi(\tau)d\tau$ are obtained from estimates for ϕ_0, $\int_0^t f(\tau)d\tau$ and $\int_0^t h(\tau)d\tau$.

Proposition 4. *In Proposition 2 assume $f \equiv 0$ and suppose $g(\phi) \leq c_1\phi^2$ for $\phi \leq c_2$ where c_1, c_2 are given positive numbers. Suppose also $E = \int_0^\infty \phi dt < \infty$. Then, for $t > (E/c_2)\exp(c_1 E)$, we have*

$$\begin{cases} \phi(t) \leq \dfrac{\exp(c_1E) - 1}{c_1 t}, \\ \displaystyle\int_t^\infty \psi(\tau)d\tau \leq \dfrac{\exp(2c_1E) - \exp(c_1E)}{c_1 t}. \end{cases} \tag{4}$$

Proof. Consider an arbitrary instant $t^* \geq c_2^{-1}Ee^{c_1E}$. It is $\phi^* = \phi(t^*) \leq c_2$. If on the contrary $\phi^* > c_2$, then, thanks to the comparison theorem for differential inequalities, $\phi(t) > \eta(t)$ for $0 \leq t \leq t^*$, where $\eta(t)$ is the solution of the equation $\eta'(t) = c_1\eta^2(t)$ satisfying $\eta(t^*) = c_2$. Furthermore,

$$\eta(t) = \eta(t^*)e^{-c_1\int_t^{t^*}\eta(s)ds},$$

then

$$E \geq \int_0^{t^*}\phi(s)ds \geq \int_0^{t^*}\eta(s)ds \geq c_2\int_0^{t^*} e^{-c_1\int_t^{t^*}\eta(s)ds}dt \geq t^*c_2e^{-Ec_1}. \tag{**}$$

This contradicts the assumption $t^* > c_2^{-1}Ee^{c_1E}$. So we have $\phi(t^*) \leq c_2$. From $(**)$ we get

$$\phi(t^*) \leq \frac{Ee^{c_1E}}{t^*}.$$

The precise estimate $(4)_1$ is obtained considering the explicit expression of η, i.e. $\eta(t) = c_2[1 - c_2c_1(t - t^*)]^{-1}$. $(4)_2$ follows by integration.

3. Statements. First, we give the definitions of weak and strong solution of system (2).

Definition 1. (v, ρ) is called a weak solution to problem (2) if

i) $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$,

$$\rho \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\ 0 < m \leq \rho \leq M;$$

ii) the diffusion equation is satisfied a. e. in Q_T;

iii) the following integral identity holds for all smooth solenoidal $\phi, \phi(T) = 0$,

$$\int_0^T ((\rho v, \partial_t\phi) + (\rho v, v\cdot\nabla\phi) - \mu(\nabla v, \nabla\phi) - \lambda(((v\cdot\nabla)\phi, \nabla\rho) + ((\nabla\rho\cdot\nabla)\phi, v)) - \left(\frac{\lambda^2}{\rho}\nabla\rho\otimes\nabla\rho, \nabla\phi\right) + (\rho f, \phi))dt = -(\rho_0 v_0, \phi(0)).$$

Definition 2. The functions v, ρ, π are a strong solution of the problem (2) if their derivatives occurring in (2) are summable and they satisfy the system and the boundary conditions almost everywhere in the corresponding measure.

Definition 3. The functions v, ρ, π are an L^q-solution ($q > 1$) of the problem (2) if their derivatives occurring in (2) are L^q functions and they satisfy the system and the boundary conditions almost everywhere in the corresponding measure.

The following theorems are the basic results of the paper.

Theorem 1. *Let $v_0 \in H$, $\rho_0 \in \tilde{H}^1(\Omega)$ with $0 < m \leq \rho_0 \leq M$, and $f \in L^2(Q_T)$ or $L^2(0, T; H^{-1}(\Omega))$. Then there exists at least one weak solution if the parameters, M, m, μ, λ satisfy the relation (21) below. Moreover, as $\lambda \to 0$, $(v^\lambda, \rho^\lambda)$ converges to a weak solution of the density dependent Navier-Stokes equations.*

Theorem 2. *If, in addition to the assumptions of Theorem 1, $v_0 \in V, \rho_0 \in \tilde{H}^2(\Omega), f \in L^2(Q_T)$ then there exists a $\bar{T}$ such that there exists a unique strong solution (v, ρ).*

In the next theorem we assume $f \equiv 0$ for simplicity of exposition. In the case of nonhomogeneous forces the assumptions follow from the estimates that we are proving in Theorem 3 below.

Theorem 3. *Let $v_0 \in H$, $\rho_0 \in \tilde{H}^1(\Omega)$ with $0 < m \leq \rho_0 \leq M$ and Ω be smooth. Then there exists at least one weak solution of problem (2) such that there exist $\bar{T}$ (small enough) and T^* (sufficiently big) such that $D_x^k D_t^l v, D_x^k D_t^l \rho$ depend continuously on $t \in (0, \bar{T}) \cup (T^*, \infty)$ in $L^2(\Omega)$ for all integers $k \geq 0$ and $l \geq 0$. Consequently, $(v, \rho) \in C^\infty((0, \bar{T}) \cup (T^*, \infty) \times \bar{\Omega})$.*

Theorem 4. *If, in addition to the assumptions of Theorem 1, $v_0 \in H_q^{2-\frac{2}{q}} \cap V_q, \rho_0 \in H_r^{2-\frac{2}{r}}(\Omega) \cap \tilde{H}^{3/2}(\Omega), f \in L^q(Q_T)$, $q > 3$, $r \geq 5$, then there exists a $\bar{T}$ such that there exists a unique strong L^q solution (v, ρ, π).*

Since we are going to prove Theorems $1-4$ using fixed point arguments we follow the sequent scheme: first, we assign the velocity and consider the diffusion equation proving the existence and a priori estimates. Next, we pass to consider the existence of a linearized momentum equation and, finally, we conclude the scheme proving the existence of a fixed point.

4. Proof of Theorem 1. Following the proof's scheme mentioned above, first we investigate the diffusion equation $(2)_2$.

4.1. *Existence of the diffusion equation and a priori estimates.* We deduce a priori estimates of the solution of the problem

$$\begin{cases} \partial_t \rho + \psi \cdot \nabla \rho - \lambda \Delta \rho = 0, \\ \rho(0) = \rho_0, 0 < m \leq \rho_0 \leq M, \ \partial_n \rho = 0 \text{ on } \Gamma. \end{cases} \tag{5}$$

Here ψ is a smooth divergence free function with normal component vanishing on Γ and m, M are fixed constants.

The existence of the diffusion equation (5) can be obtained using Galerkin method. This procedure is well known in literature so we omit details. We now prove a priori estimates largely based on the multiplicative inequality

$$|\nabla \rho|_4^2 \leq c |\rho|_\infty |\Delta \rho|_2, \tag{6}$$

that holds for all space dimensions.

First, we notice that the maximum principle holds and $m \leq \rho \leq M$.

Now, we prove three levels of regularity for ρ.

Multiplying (5) by ρ and integrating by parts in Ω (the boundary terms vanish) we get

$$d_t|\rho|_2^2 + \lambda|\nabla\rho|_2^2 = 0,$$

then

$$|\rho(t)|_2^2 + \lambda\int_0^t |\nabla\rho|_2^2 d\tau \leq |\rho_0|_2^2. \tag{7}$$

Now, multiplying (5) by $-\lambda\Delta\rho$ and after integration by parts on Ω (the boundary terms vanish), we obtain

$$\frac{\lambda}{2}d_t|\nabla\rho|_2^2 + \lambda^2|\Delta\rho|_2^2 = -\lambda(\nabla\psi\cdot\nabla\rho, \nabla\rho). \tag{8}$$

In virtue of (6) we have $\lambda|(\nabla\psi\cdot\nabla\rho,\nabla\rho)| \leq c\lambda|\nabla\psi|_2|\nabla\rho|_4^2 \leq c|\nabla\psi|_2^2 + \frac{\lambda^2}{2}|\Delta\rho|_2^2$ thus we get

$$\lambda|\nabla\rho(t)|_2^2 + \lambda^2\int_0^t |\Delta\rho|_2^2 d\tau \leq \lambda|\nabla\rho_0|_2^2 + c\int_0^t \|\psi\|^2 d\tau. \tag{9}$$

Notice that the above estimate requires that $\psi \in L^2(0,T;H^1(\Omega))$, only. We conclude this section with the H^3 estimate of ρ. First, we apply the ∇ operator to (5) and then multiply the result by $-\nabla\Delta\rho$, after integration by parts (bearing in mind that the boundary terms vanish), we deduce

$$\frac{1}{2}d_t|\Delta\rho|_2^2 + \lambda|\Delta\nabla\rho|_2^2 = (\nabla(\psi\cdot\nabla\rho), \Delta\nabla\rho).$$

In view of Proposition 1, $|\nabla(\psi\cdot\nabla\rho)|_2^2 \leq c|\nabla\psi|_2^2\|\rho\|_2\|\rho\|_3$, thus we obtain

$$d_t|\Delta\rho|_2^2 + \lambda|\nabla\Delta\rho|_2^2 \leq \frac{c}{\lambda^3}|\nabla\psi|_2^4|\Delta\rho|_2^2, \tag{10}$$

consequently

$$|\Delta\rho(t)|_2^2 + \lambda\int_0^t |\nabla\Delta\rho|_2^2 d\tau \leq c|\Delta\rho_0|_2^2 \exp\int_0^t \frac{|\nabla\psi|_2^4}{\lambda^3}d\tau. \tag{11}$$

4.2. *Auxiliary problem.* In this section we solve the linear problem:

Given $f \in L^2(Q_T)$ and $\rho^\epsilon \in L^2(0,T;\tilde{H}^3(\Omega)) \cap L^\infty(0,T;\tilde{H}^2(\Omega))$, find a solution $v^\epsilon \in L^2(0,T;H^2(\Omega)\cap V) \cap H^1(0,T;L^2(\Omega))$ of the problem

$$\begin{cases} \rho^\epsilon(\partial_t v^\epsilon + \bar{u}\cdot\nabla v^\epsilon - f) - \mu\Delta v^\epsilon - \lambda((\bar{u}\cdot\nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon\cdot\nabla)v^\epsilon) \\ -\nabla\cdot(\frac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon\otimes\nabla\rho^\epsilon) + \nabla\pi^\epsilon = 0, \\ \nabla\cdot v^\epsilon = 0. \end{cases} \tag{12}$$

Here $\bar{u}$ is the composition of regularization (with parameter ϵ) by convolution of $u \in L^2(0,T;V)$ with respect x-variable and of the projection P. Moreover, ρ^ϵ is the solution of (5) with $\psi \equiv \bar{u}$.

We introduce the following functional space

$$\mathcal{F} = \{\phi|\phi \in L^2(0,T;H^2(\Omega)\cap V) \cap H^1(0,T;L^2(\Omega))\}.$$

We denote $\|\cdot\|_{\mathcal{F}}$ the natural norm on $\mathcal{F}$.

We let

$$
(13)\qquad \begin{cases} E(v^\epsilon,\phi) = \displaystyle\int_0^T (\rho^\epsilon(\partial_t v^\epsilon + \bar{u}\cdot\nabla v^\epsilon) + \mu A v^\epsilon - \lambda(\nabla\rho^\epsilon\cdot\nabla)v^\epsilon, \partial_t\phi \\ \qquad\qquad + \bar{\mu}A\phi + \alpha\phi)dt + (v^\epsilon(0), \alpha\rho(0)\phi(0) + \mu A\phi(0)); \\ L(f,\phi) = \displaystyle\int_0^T \left(\lambda(\bar{u}\cdot\nabla)\nabla\rho^\epsilon + \nabla\cdot\left(\frac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon\otimes\nabla\rho^\epsilon\right) + \rho^\epsilon f,\right. \\ \qquad\qquad \left.\partial_t\phi + \bar{\mu}A\phi + \alpha\phi\right)dt + (v_0, \alpha\rho(0)\phi(0) + \mu A\phi(0)). \end{cases}
$$

Here ϕ is a solenoidal smooth function and $\bar{\mu} = \frac{\mu}{M}$.

First, $E(v^\epsilon,\phi)$ and $L(f,\phi)$ are a continuous bilinear form and a continuous linear form on $\mathcal{F}$, respectively.

Moreover, taking in account the diffusion equation and recalling that $|\nabla\phi|_3^2 \leq c|\nabla\phi|_2|A\phi|_2$, we get

$$
\begin{aligned}
(14)\qquad E(\phi,\phi) &= \int_0^T (\rho^\epsilon(\partial_t\phi + \bar{u}\cdot\nabla\phi) + \mu A\phi - \lambda(\nabla\rho^\epsilon\cdot\nabla)\phi, \partial_t\phi + \bar{\mu}A\phi + \alpha\phi)dt \\
&\quad + (v^\epsilon(0), \alpha\rho(0)\phi(0) + \mu A\phi(0)) \\
&\geq \int_0^T \left(\frac{1}{2}|\sqrt{\rho^\epsilon}\partial_t\phi|_2^2 + \bar{\mu}\frac{\mu}{2}|A\phi|_2^2 + \alpha\mu|\nabla\phi|_2^2 + \alpha(\rho^\epsilon\bar{u}\cdot\nabla\phi,\phi)\right. \\
&\quad + \frac{\alpha\lambda}{2}(\Delta\rho^\epsilon\phi,\phi) + (\rho^\epsilon\bar{u}\cdot\nabla\phi - \lambda(\nabla\rho^\epsilon\cdot\nabla)\phi, \partial_t\phi + \bar{\mu}A\phi) \\
&\quad \left. + \alpha(\rho^\epsilon\partial_t\phi,\phi) + \frac{\mu}{2}d_t|\nabla\phi|_2^2\right)dt + (\rho(0)\phi(0), \alpha\phi(0)) + \mu|\nabla\phi(0)|_2^2 \\
&\geq \int_0^T \left(\frac{1}{2}|\sqrt{\rho^\epsilon}\partial_t\phi|_2^2 + \bar{\mu}\frac{\mu}{2}|A\phi|_2^2 + \alpha\mu|\nabla\phi|_2^2\right. \\
&\quad \left. {}_c|\bar{u}|_\infty^2|\nabla\phi|_2^2 - c\lambda^2|\nabla\rho^\epsilon|_6^2|\nabla\phi|_3^2 - \frac{1}{8}(|\sqrt{\rho^\epsilon}\partial_t\phi|_2^2 + \bar{\mu}\mu|A\phi|_2^2)\right)dt \\
&\quad + \frac{\alpha}{2}(|\sqrt{\rho^\epsilon(T)}\phi(T)|_2^2 + |\sqrt{\rho(0)}\phi(0)|_2^2) + \frac{\mu}{2}|\nabla\phi(T)|_2^2 + \frac{\mu}{2}|\nabla\phi(0)|_2^2 \\
&\geq c\|\phi\|_{\mathcal{F}},
\end{aligned}
$$

for suitable α.

Thanks to the Lax-Milgram theorem there exists a solution $v^\epsilon \in \mathcal{F}$ of the problem

$$
(15)\qquad E(v^\epsilon,\phi) = L(f,\phi).
$$

Now, let $\tilde{\phi}$ be a solution of the problem

$$
(16)\qquad \begin{cases} \partial_t\tilde{\phi} + \bar{\mu}A\tilde{\phi} + \alpha\tilde{\phi} = g, \\ \nabla\cdot\tilde{\phi} = 0,\ \tilde{\phi}(0) = 0,\ \tilde{\phi} = 0 \text{ on } \Gamma. \end{cases}
$$

Here g is a smooth divergence free function.

Replacing in (15) ϕ by $\tilde{\phi}$, we obtain

$$\int_0^T \Big(\rho^\epsilon(\partial_t v^\epsilon + \bar{u}\cdot\nabla v^\epsilon) + \mu A v^\epsilon - \lambda((\bar{u}\cdot\nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon\cdot\nabla)v^\epsilon) \\ - \nabla\cdot\left(\frac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon\otimes\nabla\rho^\epsilon\right) - \rho^\epsilon f, g\Big)dt = 0. \tag{17}$$

This implies that v^ϵ satisfies a.e. in Q_T

$$\rho^\epsilon(\partial_t v^\epsilon + \bar{u}\cdot\nabla v^\epsilon) + \mu\Delta v^\epsilon - \lambda((\bar{u}\cdot\nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon\cdot\nabla)v^\epsilon) \\ - \nabla\cdot\left(\frac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon\otimes\nabla\rho^\epsilon\right) + \nabla\pi^\epsilon - \rho^\epsilon f = 0. \tag{18}$$

Now, let $\tilde{\phi}$ be the solution of the problem

$$\begin{cases} \partial_t\tilde{\phi} + \bar{\mu}A\tilde{\phi} + \alpha\tilde{\phi} = 0, \\ \nabla\cdot\tilde{\phi} = 0,\ \tilde{\phi}(0) = h(x),\ \tilde{\phi} = 0 \text{ on } \Gamma. \end{cases} \tag{19}$$

Here $h(x)$ is a smooth divergence free function.

Replacing ϕ by $\tilde{\phi}$ in (15) we obtain

$$(v(0) - v_0, \alpha\rho(0)h + \mu Ah) = 0,$$

consequently $v(0) = v_0$.

The existence of the auxiliary problem is proved.

4.3. *Approximate problem.* We denote by ϕ_ϵ the regularization of ϕ using convolution (with respect to the x-variable) and then applying the projection operator P. We set $P\phi_\epsilon \equiv \bar{\phi}$. We notice that $\bar{\phi}$ is a regular function with the normal component vanishing on Γ. We now consider the following approximate problem:

Find a solution

$$v^\epsilon \in L^2(0,T;H^2(\Omega)\cap V)\cap H^1(0,T;L^2(\Omega)),$$
$$\rho^\epsilon \in L^\infty(Q_T)\cap L^\infty(0,T;\tilde{H}^2(\Omega))\cap L^2(0,T;\tilde{H}^3(\Omega))$$

of the problem

$$\begin{cases} \rho^\epsilon(\partial_t v^\epsilon + \bar{v}^\epsilon\cdot\nabla v^\epsilon - f) - \mu\Delta v^\epsilon - \lambda((\bar{v}^\epsilon\cdot\nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon\cdot\nabla)v^\epsilon) \\ -\nabla\cdot\left(\dfrac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon\otimes\nabla\rho^\epsilon\right) + \nabla\pi^\epsilon = 0, \\ \partial_t\rho^\epsilon + \bar{v^\epsilon}\cdot\nabla\rho^\epsilon - \lambda\Delta\rho^\epsilon = 0, \\ \nabla\cdot v^\epsilon = 0, \end{cases} \tag{20}$$

with the usual initial-boundary conditions.

We are going to show the existence of the approximate problem by fixed point argument. Let B be a convex set in $L^2(0,T;V)$ defined by $\|\phi\|_{L^2(0,T;V)} \le R$ with R a positive real number and $u \in B$.

Now we define the map $v^\epsilon = Gu$ given by the composition of $r : u \to \bar{u}$, $g : \bar{u} \to \rho^\epsilon$ and $h : (\bar{u}, \rho^\epsilon) \to v^\epsilon$. The fixed point of G is the solution of the approximate problem (20).

The existence and a priori estimates of the solution of the diffusion equation are established in section 4.1 with $\psi = \bar{u}$.

We now build v^ϵ. The existence of a solution of the linear problem

$$
\begin{cases} \rho^\epsilon(\partial_t v^\epsilon + \bar{u}\cdot\nabla v^\epsilon - f) - \mu\Delta v^\epsilon - \lambda((\bar{u}\cdot\nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon\cdot\nabla)v^\epsilon) \\ -\nabla\cdot\left(\dfrac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon\otimes\nabla\rho^\epsilon\right) + \nabla\pi^\epsilon = 0, \quad \nabla\cdot v^\epsilon = 0, \\ \partial_t\rho^\epsilon + \bar{u}\cdot\nabla\rho^\epsilon - \lambda\Delta\rho^\epsilon = 0, \end{cases} \tag{21}
$$

with the usual initial-boundary conditions, is proved in section 4.2. Moreover, if $u \in B$ then v^ϵ belongs to a bounded set in $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Therefore, v^ϵ belongs to a compact set in $L^2(0,T;V)$. To conclude the existence of a fixed point, we have only to choose λ such that $\|v^\epsilon\|_{L^2(0,T;V)} \le R$.

To do this, we multiply the first equation in (21) by v^ϵ and, after integration by parts, we get

$$
\begin{aligned} &\frac{1}{2}d_t|\sqrt{\rho^\epsilon}v^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2 \\ &\quad\le \lambda|(\rho^\epsilon D_i\bar{u}_j, D_j v_i^\epsilon)| + \frac{\lambda^3}{2m}|\Delta\rho^\epsilon|_2^2 + c^2\frac{\lambda M^2}{2m}|\nabla v^\epsilon|_2^2 + \frac{6M^2}{\mu}|f|_2^2 \\ &\quad\le \frac{1}{2}\left(\frac{\lambda(M-m)}{2}|\nabla\bar{u}|_2^2 + \frac{\lambda^3}{m}|\Delta\rho^\epsilon|_2^2 + \left(\frac{\lambda(M-m)}{2} + \frac{c^2\lambda M^2}{m}\right)|\nabla v^\epsilon|_2^2\right) + C|f|_2^2. \end{aligned} \tag{22}
$$

Here c is the constant in (6). Assuming

$$
\frac{\lambda(M-m)}{2} + \frac{c^2\lambda M^2}{m} \le \frac{\mu}{2} \tag{23}
$$

and setting $R^2 \ge C(\frac{\lambda^2}{m}|\nabla\rho(0)|_2^2 + |\sqrt{\rho}(0)v(0)|_2^2 + \|f\|_{L^2(Q_T)})$ (C is a constant big enough) we obtain $\|v^\epsilon\|_{L^2(0,T;V)} \le R$. The proof of the existence of a solution of the approximate problem (21) is completed.

4.3. *Existence of weak solution of problem (2).* In this section we prove the existence of a weak solution of (2).

We deduce a priori estimates starting from approximate problem (20).

Energy estimate. Multiplying by v^ϵ (20) and after integrations by parts we obtain

$$
\frac{1}{3}(d_t|\sqrt{\rho^\epsilon}v^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2) \le \frac{1}{2}\left(\lambda(M-m) + \frac{\lambda c^2 M^2}{m}\right)|\nabla v^\epsilon|_2^2 + \frac{\lambda^3}{2m}|\Delta\rho^\epsilon|_2^2 + \frac{6M^2}{\mu}|f|_2^2. \tag{24}
$$

Consequently, we obtain, in view of (23),

$$
\|v^\epsilon\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \le c. \tag{25}
$$

Moreover, the diffusion equation gives

$$
m \le \rho^\epsilon \le M, \quad \|\rho^\epsilon\|_{L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} \le c. \tag{26}
$$

Thus, there exists a subsequence (denoted again by) $\{v^\epsilon, \rho^\epsilon\}$ such that, for $\epsilon \to 0$,

$$
\begin{cases} v^\epsilon \to v \text{ weak* in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;V); \\ \rho^\epsilon \to \rho \text{ weak* in } L^\infty(Q_T) \text{ and weakly in } L^2(0,T;H^2(\Omega)); \\ v^\epsilon\rho^\epsilon \to \gamma, \quad v_i^\epsilon \bar{v}_j^\epsilon \rho^\epsilon \to \alpha_{ij}, \text{ weakly in } L^p,\ p > 1. \end{cases} \tag{27}
$$

To complete the existence proof of a weak solution we have to show $\gamma = v\rho$ and $\alpha_{ij} = v_i v_j \rho$. For this we estimate the time derivative of ρ^ϵ and v^ϵ.

Time derivative estimates and compactness result. The estimates (25), (26) and the diffusion equation imply, for a $q > 1$,

$$\|\partial_t \rho^\epsilon\|_{L^q(0,T;L^2(\Omega))} \leq c.$$

Consequently, $\rho^\epsilon \to \rho$ strongly in $L^p(Q_T)$ for any $p > 1$ and one deduces that $\gamma = \rho v$.

The estimate (25), (26) shows that $\partial_t(P\rho^\epsilon v^\epsilon)$ is bounded in $L^2(0,T;H^{-2}(\Omega))$, uniformly with respect to ϵ while $\rho^\epsilon v^\epsilon$ and thus $P\rho^\epsilon v^\epsilon$ are bounded in $L^\infty(0,T;L^2(\Omega))$, uniformly with respect to ϵ. Hence, by classical compactness theorems, $\{P\rho^\epsilon v^\epsilon\}$ is compact in $L^2(0,T;H^{-1}(\Omega))$. In particular, since (subsequence) $\{\rho^\epsilon v^\epsilon\}$ converges weakly to ρv, $\{P\rho^\epsilon v^\epsilon\}$ converges to $P\rho v$ in $L^2(0,T;H^{-1}(\Omega))$. Hence, we have

$$\begin{aligned}
(28)\qquad & \int_0^T \int_\Omega \rho^\epsilon |v^\epsilon|^2 dx dt = \int_0^T (\rho^\epsilon v^\epsilon, v^\epsilon) dt \\
& = \int_0^T (P\rho^\epsilon v^\epsilon, v^\epsilon) dt = \int_0^T \langle P\rho^\epsilon v^\epsilon, v^\epsilon\rangle_{H^{-1}\times H^1} dt \\
& \to \int_0^T \langle P\rho v, v\rangle_{H^{-1}\times H^1} = \int_0^T (P\rho v, v) dt = \int_0^T \int_\Omega \rho |v|^2 dx dt.
\end{aligned}$$

Now, we observe that $\sqrt{\rho^\epsilon} v^\epsilon$ tends to $\sqrt{\rho^\epsilon} v$ weakly in $L^2(Q_T)$. This weak convergence combined with (28) yields the strong convergence in $L^2(Q_T)$ of $\sqrt{\rho^\epsilon} v^\epsilon$ to $\sqrt{\rho} v$. This convergence implies that in (27) $\alpha_{ij} = \rho v_i v_j$.

Passing to the limit $\epsilon \to 0$. In virtue of the above estimates, the limit of the diffusion equation in (5) as $\epsilon \to 0$ gives $\partial_t \rho + v \cdot \nabla \rho - \lambda \Delta \rho = 0$ a.e. in Q_T.

Now, let ϕ be a smooth function divergence free such that $\phi(T) = 0$ vanishing on Γ. Multiplying (20) by ϕ and after integration by parts we get

$$\begin{aligned}
(29)\qquad & \int_0^T \Big((\rho^\epsilon v^\epsilon, \partial_t \phi) + (\rho^\epsilon \bar{v}^\epsilon, v^\epsilon \cdot \nabla \phi) - \mu(\nabla v^\epsilon, \nabla \phi) \\
& - \lambda(((\bar{v}^\epsilon \cdot \nabla)\phi, \nabla \rho^\epsilon) + ((\nabla \rho^\epsilon \cdot \nabla)\phi, v^\epsilon)) \\
& - \Big(\frac{\lambda^2}{\rho^\epsilon} \nabla \rho^\epsilon \otimes \nabla \rho^\epsilon, \nabla \phi\Big) + (\rho^\epsilon f, \phi)\Big) dt = -(\rho_0 v_0, \phi(0)).
\end{aligned}$$

The estimates (25), (26) guarantee the convergence of all terms in (29), hence passing to the limit $\epsilon \to 0$ we have proved that (ρ, v) is a weak solution of (2). To complete the proof of Theorem 1 we study the behavior of the weak solution as $\lambda \to 0$.

Behavior of the weak solution as $\lambda \to 0$. In this section we consider the λ-parameter dependent weak solution $(v^\lambda, \rho^\lambda)$ of the problem (2) and prove that it converges as $\lambda \to 0$, in a certain sense, to a weak solution of the nonhomogeneous Navier-Stokes equations

$$(30)\qquad \begin{cases} \rho(\partial_t v + v \cdot \nabla v - f) - \mu \Delta v + \nabla \pi = 0, \\ \partial_t \rho + v \cdot \nabla \rho = 0, \\ \nabla \cdot v = 0,\ v(0) = v_0,\ \rho(0) = \rho_0. \end{cases}$$

The definition of weak solution of (30), mutatis mutandis, is Definition 1. Giving evidence on the role of the parameter λ, we have proved in the above steps that

$$
\begin{cases}
m \le \rho^\lambda \le M, \ \left(\sqrt{\lambda} \sup\limits_{0\le t\le T} |\nabla\rho^\lambda|_2, \lambda\displaystyle\int_0^T |\nabla\rho^\lambda|_2^2 dt\right) \le c, \\
\lambda^2 \displaystyle\int_0^T |\Delta\rho^\lambda|_2^2 dt \le c, \ \|\partial_t\rho^\lambda\|_{L^2(0,T;H^{-1}(\Omega))} \le c, \\
\left(\|\sqrt{\rho^\lambda}v^\lambda\|_{L^\infty(0,T;L^2(\Omega))}, \displaystyle\int_0^T |\nabla v^\lambda|_2^2 dt\right) \le c.
\end{cases} \tag{31}
$$

In (31) c is a constant independent of λ with λ small enough.

(31) yields $\rho^\lambda \to \rho$ weakly in $L^p(Q_T)$ for any $p>1$ and strongly in $L^2(0,T;H^{-1}(\Omega))$, and $v^\lambda \to v$ in $L^\infty(0,T;L^2(\Omega))$ weak* and weakly in $L^2(0,T;H^1(\Omega))$, consequently $\rho^\lambda v^\lambda \to \rho v$ weakly in $L^2(Q_T)$ and for any smooth function $\phi \in C_0^1(Q_T)$,

$$
\lambda^2\left|\int_0^T \left(\nabla\cdot\left(\frac{1}{\rho^\lambda}\nabla\rho^\lambda\otimes\nabla\rho^\lambda\right),\phi\right)dt\right| \le \lambda^2 m^{-1}\int_0^T |\nabla\rho^\lambda|_2^2 dt\|\nabla\phi\|_{L^\infty(Q_T)} \to 0
$$

as $\lambda \to 0$. Moreover,

$$
\begin{aligned}
&\int_0^T (\partial_t\rho^\lambda + v^\lambda\cdot\nabla\rho^\lambda - \lambda\Delta\rho^\lambda, \phi)dt \\
&\quad = -\int_0^T ((\rho^\lambda, \partial_t\phi + v^\lambda\cdot\nabla\phi) - \lambda(\nabla\rho^\lambda,\nabla\phi))dt \to -\int_0^T (\rho, \partial_t\phi + v\cdot\nabla\rho)dt
\end{aligned} \tag{32}
$$

as $\lambda \to 0$ for all $\phi \in C_0^1(Q_T)$. The equation of the conservation of mass $(30)_2$ is satisfied in weak sense by the limit (v,ρ).

We now consider the convergence of the momentum equation. As usual in equations type Navier-Stokes the crucial term in passing to the limit in the momentum equation is $\rho^\lambda v^\lambda \otimes v^\lambda$. Thanks to the estimates (25) and (28) the convergence is guaranteed if ρ^λ converges strongly in $L^2(Q_T)$, for example. This seems not to be true. We indicate the argument that can be used to overcome this difficulty but it does not give better results than the method introduced above. Instead of $\partial_t P\rho^\lambda v^\lambda$ we make use of the estimate type fractional derivative

$$
\int_0^T |v^\lambda(t+h) - v^\lambda(t)|_2^2 dt \le c\sqrt{h},
$$

that can be proved as in [14] (for some details see section 6). This argument requires that $v^\lambda \in L^2(0,T;V)$ and the space dimension ≤ 4. So the Ascoli-Arzelà-Kolmogorov-Riesz compactness theorem yields $v^\lambda \to v$ as $\lambda \to 0$ strongly in $L^2(Q_T)$.

Now, it is a routine matter to prove that (v,ρ) satisfies

$$
\int_0^T ((\rho v, \partial_t\phi) + (\rho v, v\cdot\nabla\phi) - \mu(\nabla v,\nabla\phi) + (\rho f,\phi))dt = -(\rho_0 v_0, \phi(0)),
$$

for every $\phi \in C^1(Q_T), \nabla\cdot\phi = 0, \phi(T) = 0$.

Theorem 1 is completely proved.

5. Proof of Theorem 2. We prove Theorem 2 by a fixed point argument following the scheme of section 4.

Let $B(R) \subset \mathcal{F}$ be a ball with center the origin and radius R ($\mathcal{F}$ has been introduced in section 4.2). We fix a function $u \in B(R)$ and section 4.1 gives the solution of diffusion equation (5) with $\psi \equiv u$ and relative estimates on ρ up to the 3-spatial derivatives. Then we consider the linear problem

$$\begin{aligned}(33)\qquad & \rho\partial_t v - \mu\Delta v + \rho u\cdot\nabla u - \lambda((u\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)u) \\ & \quad - \frac{\lambda^2}{\rho}\Big((\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho\Big) + \nabla\pi - \rho f = 0,\end{aligned}$$

complemented with the usual initial-boundary conditions. The existence of a solution of system (33) has been established in the section 4.2.

5.1. *A priori estimates.* Now, we multiply by v (33) and after integration by parts we obtain

$$\begin{aligned}(34)\qquad & d_t|\sqrt{\rho}v|_2^2 + \mu|\nabla v|_2^2 \le cM^2|u|_3^2|\nabla u|_2^2 + c\frac{\lambda^2}{\mu}|\nabla u|_2^2 \\ & \quad + c\lambda^2\frac{M}{\mu}|\nabla\rho|_3|\nabla u|_2^2 + \frac{\lambda^4c^2M^2}{\mu m^2}|\Delta\rho|_2^2 + \frac{M^2}{\mu}|f|_2^2.\end{aligned}$$

Now, we multiply (33) by $\partial_t v$, integrate over Ω, and obtain

$$\begin{aligned}(35)\qquad & |\sqrt{\rho}\partial_t v|_2^2 + \frac{\mu}{2}d_t|\nabla v|_2^2 \\ & = \Big(-\rho u\cdot\nabla u + \rho f + \lambda((u\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)u) + \frac{\lambda^2}{\rho}\Big((\nabla\rho\cdot\nabla)\nabla\rho \\ & \quad - \frac{1}{\rho}|\nabla\rho|^2\nabla\rho + \Delta\rho\nabla\rho\Big), \partial_t v\Big) \\ & \le c(|\rho u|_\infty^2|\nabla u|_2^2 + \lambda^2(|u|_\infty^2|\Delta\rho|_2^2 + |\nabla\rho|_6^2|\nabla u|_3^2) + \frac{\lambda^4}{m^2}|\Delta\rho|_2^3\|\rho\|_3 \\ & \quad + M|f|_2^2) + \frac{1}{2}|\sqrt{\rho}\partial_t v|_2^2 \le c(M^2\|u\|^3|Au|_2 + \lambda^2\|u\||Au|_2|\Delta\rho|_2^2 \\ & \quad + \frac{\lambda^4}{m^2}M^2|\Delta\rho|_2^3\|\rho\|_3 + M|f|_2^2) + \frac{1}{2}|\sqrt{\rho}\partial_t v|_2^2.\end{aligned}$$

Then, integrating (35) with respect to t, we get

$$\begin{aligned}(36)\qquad & \int_0^t |\sqrt{\rho}\partial_t v|_2^2 d\tau + \mu\sup_{0\le\tau\le t}|\nabla v(t)|_2^2 \le \mu|\nabla v(0)|_2^2 \\ & \quad + c\int_0^t\Big(M^2\|u\|^3|Au|_2 + \lambda^2\|u\||Au|_2|\Delta\rho|_2^2 + \frac{\lambda^4}{m^2}M^2|\Delta\rho|_2^3\|\rho\|_3 + M|f|_2^2\Big)d\tau.\end{aligned}$$

Now, we consider the Stokes problem

$$(37)\qquad \mu Av = -P\Big(\rho\partial_t v + \rho u\cdot\nabla u - \lambda((u\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)u) - \nabla\cdot\Big(\frac{\lambda^2}{\rho}\nabla\rho\otimes\nabla\rho\Big) - \rho f\Big).$$

From the theory of the Stokes problem there exists a constant c such that

$$\int_0^t |\mu\Delta v|_2^2 d\tau \le c\biggl(\mu|\nabla v(0)|_2^2 + \int_0^t \biggl(M^2\|u\|^3|Au|_2 + \lambda^2\|u\||Au|_2|\Delta\rho|_2^2 + \frac{\lambda^4}{m^2}M^2|\Delta\rho|_2^3\|\rho\|_3 + M|f|_2^2\biggr)d\tau\biggr). \tag{38}$$

Choosing $R^2 = C(|\nabla v_0|_2^2 + |\Delta\rho_0|_2^2 + M\|f\|_{Q_T}^2)$ with C big enough, (10), (36), (37) yield

$$\begin{aligned}\int_0^t (|\sqrt{\rho}\partial_t v|_2^2 + \mu|Av|_2^2)d\tau + \mu \sup_{0\le\tau\le t}|\nabla v(t)|_2^2 &\le c\biggl(|\nabla v(0)|_2^2 + \sqrt{t}R^3\biggl(\int_0^t |Au|_2^2 d\tau\biggr)^{1/2} + \sqrt{t}R \sup_{0\le\tau\le t}|\Delta\rho|_2^2\biggl(\int_0^t |Av|_2^2 d\tau\biggr)^{1/2} \\ &\quad + \sqrt{t}\sup_{0\le\tau\le t}|\Delta\rho|_2^3\biggl(\int_0^t \|\rho\|_3^2 d\tau\biggr)^{1/2} + M\|f\|_{Q_T}^2\biggr) \\ &\le c(|\nabla v(0)|_2^2 + \sqrt{t}(R^4 + R^8 + R^{12}) + M\|f\|_{Q_T}^2) \le R^2,\end{aligned} \tag{39}$$

for $t = \bar{T}$ small enough. Hence (39) implies

$$GB \subseteq B.$$

We now prove the continuity of G. Let $\{u^n\} \subset B$ be a sequence such that $u^n \to u$ in $L^2(Q_{\bar{T}})$. We notice that $u \in B$. Now, let ρ^n, ρ be solutions of

$$\begin{cases} \partial_t\rho^n + u^n\cdot\nabla\rho^n - \lambda\Delta\rho^n = 0;\ \rho^n(0) = \rho_0,\ \partial_n\rho^n = 0 \text{ on } \Gamma, \\ \partial_t\rho + u\cdot\nabla\rho - \lambda\Delta\rho = 0;\ \rho(0) = \rho_0,\ \partial_n\rho = 0 \text{ on } \Gamma, \end{cases} \tag{40}$$

respectively. Then $\tau^n = \rho^n - \rho$ satisfies

$$\partial_t\tau^n + u^n\cdot\nabla\tau^n - \lambda\Delta\tau^n = -U^n\cdot\nabla\rho;\ \tau^n(0) = 0,\ \partial_n\tau^n = 0 \text{ on } \Gamma. \tag{41}$$

Here $U^n = u^n - u$. The estimate (11), Gronwall's lemma and Proposition 1 yield that $\tau^n \to 0$ in $L^2(0,\bar{T};H^2(\Omega)) \cap L^\infty(0,\bar{T};H^1(\Omega))$. Now, let v^n, v be the solutions of

$$\begin{cases} \rho^n\partial_t v^n - \mu\Delta v^n + \rho^n u^n\cdot\nabla u^n - \lambda((u^n\cdot\nabla)\nabla\rho^n + (\nabla\rho^n\cdot\nabla)u^n) \\ \quad - \nabla\cdot\left(\dfrac{\lambda^2}{\rho^n}\nabla\rho^n\otimes\nabla\rho^n\right) + \nabla\pi^n - \rho^n f = 0, \\ \rho\partial_t v - \mu\Delta v + \rho u\cdot\nabla u - \lambda((u\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)u) \\ \quad - \nabla\cdot\left(\dfrac{\lambda^2}{\rho}\nabla\rho\otimes\nabla\rho\right) + \nabla\pi - \rho f = 0, \end{cases} \tag{42}$$

with $v^n(0) = v(0) = v_0$, respectively. Then $V^n = v^n - v$ satisfies

$$\rho\partial_t V^n - \mu\Delta V^n = H(u^n,u,\rho^n,\rho) - \nabla(\pi^n - \pi) - \tau^n\partial_t v^n. \tag{43}$$

It is easy to trace $H(\cdot)$ and prove that $H(u^n,u,\rho^n,\rho) \to 0$ as $n\to\infty$ in $L^2((0,T;H^{-1}(\Omega))$.

Now, multiplying (43) by V^n, after integration by parts, we get

$$\int_0^t |\nabla V^n|_2^2 d\tau + |\sqrt{\rho}V^n(t)|_2^2 \le c\int_0^t (\|H\|_{H^{-1}(\Omega)}^2 + |\tau^n|_3^2|\partial_\tau v^n|_2^2)d\tau.$$

Gronwall's lemma produces $V^n \to 0$ in $L^2(Q_{\bar{T}})$. Thus, the map G is continuous in $L^2(Q_{\bar{T}})$. The uniqueness can be proved with the same procedure as that used for the continuity setting $v_1 \equiv v^n$, $v_2 \equiv v$, $\rho_1 \equiv \rho^n$, $\rho_2 \equiv \rho$ in the continuity procedure of G where $(v_1, \rho_1), (v_2, \rho_2)$ are two solutions with the same data. We omit details. The existence of a local solution is completely proved.

6. Periodic problem. This section is devoted to the existence of periodic solution of a problem related to (2). We notice that for the diffusion equation the periodicity of solution yields $\rho = const$, in other words, we find the periodicity of the classical Navier-Stokes equations. For further development, we consider a semi-homogeneous diffusion equation that we write

$$\partial_t \rho + v \cdot \nabla \rho - \lambda \Delta \rho + \alpha \rho = g,\ \partial_n \rho(x,t) = 0 \text{ on } \Gamma,\ \rho(0) = \rho_0. \tag{44}$$

We assume that $m \le \rho_0 \le M$, $\alpha \ge 1$ and $\alpha m \le g \le \alpha M$. This model is used as a parabolic approximation of the transport equation in an iterative approach, for example assuming $\alpha = 1, g = \rho^{n-1}$ and $\lambda = 1/n$. Concerning the existence of a periodic solution of the problem $(2)_1$+ (44) with period T, we adopt the proof's scheme of section 4 assuming that the data are periodic functions.

First, let g be an L^2 T-periodic function, we consider the existence of a solution of the problem

$$\partial_t \rho + \bar{u} \cdot \nabla \rho - \lambda \Delta \rho + \alpha \rho = g,\ \partial_n \rho(x,t) = 0 \text{ on } \Gamma,\ \rho(0) = \rho_0. \tag{45}$$

Here $\bar{u}$ is built as in section 4.1 with $u \in L^2(0,T;V), u(0) = u(T)$. In this section we drop the superscript ϵ.

The existence of a solution of problem (45) is well known in literature. We need some estimates of the solution ρ.

First, we prove that $m \le \rho \le M$. In fact, multiplying (45) by $(\rho - m)^- = \min(0, \rho - m)$, after integration by parts, we have

$$d_t |(\rho - m)^-|_2^2 + \lambda |\nabla(\rho - m)^-|_2^2 + \alpha |(\rho - m)^-|_2^2 = ((g - \alpha m), (\rho - m)^-) \le 0.$$

Consequently, $\rho \ge m$. Analogously, multiplying (45) by $(\rho - M)^+ = \sup(0, \rho - M)^+$ and after integration by parts we get

$$d_t |(\rho - M)^+|_2^2 + \lambda |\nabla(\rho - M)^+|_2^2 + \alpha |(\rho - M)^+|_2^2 = ((g - \alpha M), (\rho - M)^+) \le 0,$$

thus $\rho \le M$. Using the same procedure of section 4.1 we have

$$\lambda |\nabla \rho(t)|_2^2 + \lambda^2 \int_0^t |\Delta \rho|_2^2 d\tau + \lambda \alpha \int_0^t |\nabla \rho|_2^2 d\tau \le \lambda |\nabla \rho_0|_2^2 + cM^2 \int_0^t (\|\bar{u}\|^2 + |g|_2^2) d\tau.$$

The H^3-estimate of ρ is obtained as in section 4.1, anyway we do not use it so we omit details.

The existence of a solution of (45) permits to define a map S

$$S\rho(0) = \rho(T). \tag{46}$$

S is a continuous map in L^2. In fact, let ρ^1, ρ^2 solutions of (45) corresponding to initial conditions ρ_0^1, ρ_0^2, respectively. From (45), we get

$$|S\rho_0^1 - S\rho_0^2|_2 = |\rho^1(T) - \rho^2(T)|_2^2 \le |\rho_0^1 - \rho_0^2|_2^2 e^{-\alpha T},$$

consequently S is a continuous map. We notice that we have not used Poincaré's inequality.

Moreover, from (45) we deduce

$$|\rho(T)|_2^2 \le e^{-\alpha T}\left(|\rho_0|_2^2 + \int_0^T e^{\alpha t}|g|_2^2 dt\right). \tag{47}$$

Now, let $B(R)$ be a ball in $L^2(\Omega)$ with center the origin and radius $R \ge (1 - e^{-\alpha T})^{-1}\int_0^T |g|_2^2 dt$. Thanks to (47) we have $SB(R) \subseteq B(R)$. The fixed point of S yields the periodic solution of (45).

We now pass to consider the existence of a periodic solution of the momentum equation.

In section 4.2 we proved the existence of the initial-boundary value problem of the system

$$\begin{cases} \rho(\partial_t v + \bar{u}\cdot\nabla v - f) - \mu\Delta v + gv - \lambda((\bar{u}\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)v) \\ -\nabla\cdot\left(\dfrac{\lambda^2}{\rho}\nabla\rho\otimes\nabla\rho\right) + \nabla\pi = 0, \\ \nabla\cdot v = 0, \end{cases} \tag{48}$$

and the uniform estimates obtained in section 4.2 hold for suitable λ. We have added the term gv in (48) to avoid additional assumptions on g. Now, multiplying (48) by v, integrating by parts and using Poincaré's inequality we get

$$|\rho(T)v(T)|_2^2 \le e^{-cT}\left(|\rho(0)v(0)|_2^2 + \int_0^T e^{ct}|\rho f(t)|_2^2 dt\right). \tag{49}$$

Now, we consider the map $S : \sqrt{\rho(0)}v(0) \to \sqrt{\rho(T)}v(T)$. Let $B(R)$ be a ball with radius R. If $R \ge (1 - e^{-cT})^{-1}\int_0^T |\rho f|_2^2 dt$ we get $SB(R) \subseteq B(R)$. To conclude the fixed point argument we need the continuity of S on $L^2(Q_T)$.

Let v_1, v_2 be solutions of problem (48) with initial conditions $v_1(0), v_2(0)$, respectively. Thus, $V = v_1 - v_2, \Pi = \pi_1 - \pi_2$ satisfies

$$\rho(\partial_t V + \bar{u}\cdot\nabla V) - \mu\Delta V - \lambda(\nabla\rho\cdot\nabla)V + gV + \nabla\Pi = 0. \tag{50}$$

Multiplying (50) by V, Gronwall's lemma implies

$$|\sqrt{\rho(T)}V(T)|_2 \le e^{ct}|\sqrt{\rho(0)}V(0)|_2^2.$$

Thus, a fixed point theorem implies $v(0) = v(T)$. From now on the proof of the existence of periodic weak solution of the modified model can be concluded with the procedure of section 4.

In the case of unbounded domains it is not possible to simply extend the methods used for the bounded domains, since these involve, in general, tools such as Poincaré's inequality, compact embedding, etc., that no longer holds for unbounded domains, in general. Consequently, it is necessary to resort to other arguments. In [18] the author solved the open problem of the existence of weak and strong periodic solutions for the Navier-Stokes equations in exterior domains using a different approach. However, the uniqueness remains on open problem for this type of solutions. Here we briefly present the method introduced in [18] giving a sketch of the existence proof of periodic solution which can be adapted to unbounded domains.

6.1. *Elliptic regularization.* We assume that ρ^ϵ is a T-periodic solution of equation (44). Now, we look for a T-periodic solution of the modified momentum equation (48). We consider the following integral relation

$$\begin{aligned}(51)\qquad &\int_0^T \Big(\epsilon(\partial_t v^\epsilon, \partial_t\phi) + (\rho^\epsilon\partial_t v^\epsilon, \phi) + \mu(\nabla v^\epsilon, \nabla\phi) + (\rho^\epsilon \bar{v}^\epsilon \cdot \nabla v^\epsilon, \phi) + (g v^\epsilon, \phi)\\ &- \lambda((\bar{v}^\epsilon \cdot \nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon \cdot \nabla)v^\epsilon, \phi) + \left(\frac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon \otimes \nabla\rho^\epsilon, \nabla\phi\right) - (\rho^\epsilon f, \phi)\Big)dt = 0.\end{aligned}$$

Here $\bar{v}^\epsilon$ is defined in section 4. We shall find a solution of (51) in the Hilbert space

$$\mathcal{H} = \{\phi|\phi \in L^2(0,T;V), \partial_t\phi \in L^2(Q_T), \phi(0) = \phi(T)\}.$$

We introduce

$$(52)\qquad \begin{cases} E(v^\epsilon, v^\epsilon;\phi) = \displaystyle\int_0^T (\epsilon(\partial_t v^\epsilon, \partial_t\phi) + (\rho^\epsilon\partial_t v^\epsilon, \phi) + \mu(\nabla v^\epsilon, \nabla\phi) + (g v^\epsilon, \phi) \\ \qquad\qquad + (\rho^\epsilon\bar{v}^\epsilon \cdot \nabla v^\epsilon, \phi) - \lambda((\bar{v}^\epsilon \cdot \nabla)\nabla\rho^\epsilon + (\nabla\rho^\epsilon \cdot \nabla)v^\epsilon, \phi))dt, \\ L(f;\phi) = -\displaystyle\int_0^T \left(\left(\frac{\lambda^2}{\rho^\epsilon}\nabla\rho^\epsilon \otimes \nabla\rho^\epsilon, \nabla\phi\right) - (\rho^\epsilon f, \phi)\right)dt, \end{cases}$$

for any smooth solenoidal ϕ such that $\phi(0) = \phi(T)$. We write (51)

$$(53)\qquad E(v^\epsilon, v^\epsilon;\phi) = L(f;\phi).$$

Bearing in mind that

$$\begin{aligned}(\rho^\epsilon\partial_t v^\epsilon, v^\epsilon) &= \frac{1}{2}(d_t|\sqrt{\rho^\epsilon}v^\epsilon|_2^2 - (\partial_t\rho^\epsilon v^\epsilon, v^\epsilon))\\ &= \frac{1}{2}(d_t|\sqrt{\rho^\epsilon}v^\epsilon|_2^2 - \lambda(\Delta\rho^\epsilon v^\epsilon, v^\epsilon) + (\bar{v}^\epsilon \cdot \nabla\rho^\epsilon v^\epsilon, v^\epsilon) + ((\alpha\rho - g)v^\epsilon, v^\epsilon)),\end{aligned}$$

there exists a constant c such that (for a suitable value of λ)

$$E(v^\epsilon, v^\epsilon; v^\epsilon) \geq c\|v^\epsilon\|_{\mathcal{H}}.$$

Moreover, for any $\phi \in \mathcal{H}$, the mapping $v^\epsilon \to E(v^\epsilon, v^\epsilon;\phi)$ is sequentially weakly continuous on $\mathcal{H}$. Consequently, using the classical fixed-point theorem due to Brouwer, we derive the existence of a solution of (53).

A priori estimates. As in section 4, in (53) replacing ϕ by v^ϵ, assuming (23) we obtain

$$(54)\qquad \int_0^T (\epsilon|\partial_t v^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2)dt + c\int_0^T |v^\epsilon|_2^2 dt \leq c\int_0^T |f|_2^2 dt.$$

Now, if necessary, we consider the periodic extension of v^ϵ to the interval $(-\infty, +\infty)$. In (53) replacing ϕ by $\frac{1}{h}\int_t^{t+h} v^\epsilon(s)ds$, using the above estimates we get

$$(55)\qquad \int_0^T |v^\epsilon(t+h)-v^\epsilon(t)|_2^2 dt$$
$$\le h\Big(\frac{\epsilon}{h}\int_0^T |(\partial_t v^\epsilon, v^\epsilon(t+h)-v^\epsilon(t))|dt + Mh^{-1}\int_0^T \Big(|\nabla v^\epsilon|_2+|\nabla v^\epsilon|_2^2$$
$$+|f|_2+\lambda|\nabla v^\epsilon|_2|\nabla\rho^\epsilon|_3\int_t^{t+h}|\nabla v^\epsilon|_2 ds+\lambda^2|\Delta\rho^\epsilon|_2\Big)\Big|\int_t^{t+h}\nabla v^\epsilon ds\Big|_2 dt\Big)\le ch^{1/2}.$$

Estimates (54), (55) and Ascoli-Arzelà -Riesz-Kolmogorov theorem imply that v^ϵ belongs to a compact set in $L^2(Q_T)$ (in unbounded domains in $L^2(0,T;L^2_{loc}(\Omega))$. The periodicity of the diffusion equation can be proved analogously. Now, we can pass to the limit $\epsilon\to 0$ in (51) and in (45)(with $\bar u\equiv\bar v^\epsilon$) and the existence of a periodic weak solution of the modified model

$$(56)\qquad \begin{cases}\displaystyle\int_0^T\Big((\rho v,\partial_t\phi)+(\rho v\cdot\nabla\phi,v)-\mu(\nabla v,\nabla\phi)\\ -(\alpha\rho,\phi)-\lambda(((v\cdot\nabla)\phi,\nabla\rho)+((\nabla\rho\cdot\nabla)\phi,v))\\ -\lambda^2\Big(\dfrac{1}{\rho}\nabla\rho\otimes\nabla\rho,\nabla\phi\Big)+(\rho f,\phi)\Big)dt=0,\\ \partial_t\rho-\lambda\Delta\rho+v\cdot\nabla\rho+\alpha\rho=g,\end{cases}$$

is proved.

We remark that if $\alpha=0$ and $g\equiv 0$ in (45) then $\rho=$ constant and the periodic solution of the classical Navier-Stokes equations is obtained. To obtain the periodic solution of the reduced model we can use an iterative model type

$$\partial_t\rho^n-\frac{1}{n}\Delta\rho+v^{n-1}\cdot\nabla\rho+\rho^n=\rho^{n-1}$$

and the related momentum equation.

7. Proof of Theorem 3. This section is devoted to a development of the procedure of the existence of weak solution to prove that there exists a weak solution that has regularity properties under some conditions on the data or the existence time. Since we have not a weak-strong uniqueness theorem, we cannot extend the results to an arbitrary weak solution. We assume $f\equiv 0$ for simplicity of exposition. We prove the existence of a strong solution of (2) using the arguments of section 4.3 starting from the approximate problem,

$$(57)\qquad \begin{cases}\rho^\epsilon(\partial_t v^\epsilon+\bar v^\epsilon\cdot\nabla v^\epsilon-f)-\mu\Delta v^\epsilon-\lambda((\bar v^\epsilon\cdot\nabla)\nabla\rho^\epsilon+(\nabla\rho^\epsilon\cdot\nabla)v^\epsilon)\\ -\frac{\lambda^2}{\rho^\epsilon}((\nabla\rho^\epsilon\cdot\nabla)\nabla\rho^\epsilon-\frac{1}{\rho^\epsilon}(\nabla\rho^\epsilon\cdot\nabla\rho^\epsilon)\nabla\rho^\epsilon+\Delta\rho^\epsilon\nabla\rho^\epsilon)+\nabla\pi^\epsilon=0,\\ \nabla\cdot v^\epsilon=0,\\ \partial_t\rho^\epsilon-\lambda\Delta\rho^\epsilon+\bar v^\epsilon\cdot\nabla\rho^\epsilon=0,\end{cases}$$

assuming in addition that $v_0\in V,\rho_0\in\tilde H^2$ (recall that $\bar v^\epsilon=Pv^\epsilon_\epsilon$). Multiplying the first equation in (57) by v^ϵ and working as in section 4 we obtain the estimate

$$(58)\qquad d_t|\sqrt{\rho^\epsilon}v^\epsilon|_2^2+\mu|\nabla v^\epsilon|_2^2\le\lambda(\rho^\epsilon D_i\bar v^\epsilon_j,D_iv^\epsilon_j)+\frac{\lambda^2}{m}|\nabla\rho^\epsilon|_4^2|\nabla v^\epsilon|_2$$
$$\le\frac{\lambda(M-m)}{2}|\nabla v^\epsilon|_2^2+c\frac{\lambda^2M}{m}|\nabla v^\epsilon|_2|\Delta\rho^\epsilon|_2.$$

From (58) we can obtain two estimates: the global estimate, in the sense that the constants are independent of t, and a local estimate without any condition on the parameters but the constants are dependent on t. Integrating (58) with respect to t we get

$$(59)\quad \frac{1}{2}|\sqrt{\rho^\epsilon(t)}v^\epsilon(t)|_2^2 + \mu\int_0^t |\nabla v^\epsilon|_2^2 d\tau$$
$$\le c|\sqrt{\rho(0)}v(0)|_2^2 + \lambda\frac{M-m}{2}\int_0^t |\nabla v^\epsilon|_2^2 d\tau + \frac{\lambda^2 M}{m}\Big(\int_0^t |\nabla v^\epsilon|_2^2 d\tau\Big)^{1/2}\Big(\int_0^t |\Delta\rho^\epsilon|_2^2 d\tau\Big)^{1/2}$$
$$\le c|\sqrt{\rho(0)}v(0)|_2^2 + \frac{\lambda^2}{m}|\nabla\rho(0)|_2^2 + \Big(\lambda\frac{M-m}{2} + c\frac{\lambda M^2}{m}\Big)\int_0^t |\nabla v^\epsilon|_2^2 d\tau.$$

Assuming $\lambda(M-m)/2 + c\frac{\lambda M^2}{m} \le \mu/2$ we obtain the global estimate

$$(60)\qquad |\sqrt{\rho^\epsilon(t)}v^\epsilon(t)|_2^2 + \mu\int_0^t |\nabla v^\epsilon|_2^2 d\tau \le c\Big(|\sqrt{\rho(0)}v(0)|_2^2 + \frac{\lambda^2}{m}|\nabla\rho(0)|_2^2\Big),$$

otherwise we obtain the local estimate

$$(61)\quad |\sqrt{\rho^\epsilon(t)}v^\epsilon(t)|_2^2 + \mu\int_0^t |\nabla v^\epsilon|_2^2 d\tau$$
$$\le |\sqrt{\rho(0)}v(0)|_2^2 + \lambda(M-m)t\sup_t |\nabla v^\epsilon(t)|_2^2 + c\frac{\lambda^2 M^2}{m}t\sup_t |\nabla v^\epsilon(t)|_2|\Delta\rho^\epsilon(t)|_2.$$

In the sequel of this section we do not use (60) and (61). Now, multiplying (57) by $\partial_t v + \frac{\mu}{M}Av$ and integrating over Ω we obtain

$$(62)\quad |\sqrt{\rho^\epsilon}\partial_t v^\epsilon|_2^2 + \mu d_t|\nabla v^\epsilon|_2^2 + \frac{\mu^2}{M}|Av^\epsilon|_2^2 \le M|v^\epsilon|_6^2|\nabla v^\epsilon|_3^2$$
$$+ c\lambda^2(|v^\epsilon|_\infty^2|\Delta\rho^\epsilon|_2^2 + |\nabla v^\epsilon|_3^2|\nabla\rho^\epsilon|_6^2) + c\lambda^4(|\nabla\rho^\epsilon|_6^6 + |\nabla\rho^\epsilon|_6^2|\Delta\rho^\epsilon|_2|\Delta\nabla\rho^\epsilon|_2).$$

Now, recalling that $|\nabla v^\epsilon|_3^2 \le c|\nabla v^\epsilon|_2|Av^\epsilon|_2$, $|v^\epsilon|_\infty^2 \le c|\nabla v^\epsilon|_2|Av^\epsilon|_2$, setting $\chi^\epsilon = |\nabla v^\epsilon|_2^2 + \lambda|\nabla\rho^\epsilon|_2^2 + \lambda^2|\Delta\rho^\epsilon|_2^2$ and adding (62), (8), (10) we obtain

$$(63)\qquad d_t\chi^\epsilon + |\sqrt{\rho^\epsilon}\partial_t v^\epsilon|_2^2 + \frac{\mu^2}{M}|Av^\epsilon|_2^2 + |\Delta\rho^\epsilon|_2^2 + |\Delta\nabla\rho^\epsilon|_2^2 \le g(\chi^\epsilon).$$

In view of the Proposition 3, there exists $T(\chi^\epsilon(0))$ such that

$$\chi^\epsilon \le F(t;\chi^\epsilon(0));\qquad \int_0^t\Big(|\sqrt{\rho^\epsilon}\partial_t v^\epsilon|_2^2 + \frac{\mu^2}{M}|Av^\epsilon|_2^2 + \|\rho^\epsilon\|_3^2\Big)d\tau \le \bar F(t,\chi^\epsilon(0)).$$

It is a routine matter to pass to the limit $\epsilon \to 0$ in (57) so we obtain that the weak solution built in Theorem 1 is a local strong solution of the problem (2).

7.1. *Global existence.* Now, we consider the global strong solution for small data and the weak solution of section 4 is globally strong. Let the initial data u_0, ρ_0 be such that $\chi^\epsilon(0) \le \gamma$ with

$$c\gamma - g(\gamma) > 0,$$

where c is such that $c\chi^\epsilon \le \frac{\mu^2}{M}|Av^\epsilon|_2^2 + \|\rho^\epsilon\|_3^2$. Then $\chi^\epsilon(t) \le \gamma$ for all t for which χ^ϵ exists. Suppose that the contrary is true. Then there exists a $\bar t$ such that $\chi^\epsilon(\bar t) > \gamma$. Define

$$t^* = \inf\{t|\chi^\epsilon(t) > \gamma\}.$$

Clearly $\chi^\epsilon(t^*) = \gamma$. Moreover from (63) $d_t\chi^\epsilon(t^*) < 0$. But this implies that $\chi^\epsilon(t_1) < \gamma$ for some $t_1 > t^*$ which is a contradiction to the definition of t^*.

From the above results, there exists a solution of (63) in a interval $(0, T)$ where $T = T(v_0, \rho_0)$ is the maximal number for which exists a solution of (57). Suppose that $T < \infty$. Then there exists $T_n > 0$ such that $T_n \nearrow T$, and $\chi^\epsilon(T_n) \le \gamma$. Hence there exists an $r > 0$ such that $\|v^\epsilon(T_n)\| + \|\rho^\epsilon(T_n)\|_2 \le r$ for all n. According to Theorem 2 there exists an $\eta > 0$ such that for any n the solution exists on $[T_n, T_n + \eta)$. But this implies that $T_{n_0} + \eta > T$ for some n_0 which is a contradiction to the definition of T. Thus the global existence is proved.

7.2. *Global existence and decay.* In this section we derive further estimates of the approximate problem (57) and the solutions obtained from them in the context of L^2-theory. Notice, in the following lemma, that neither $|\nabla v_0|_2, |\Delta v_0|_2$ nor $|\Delta\rho_0|_2$ are assumed to be finite. Moreover, we assume $f \equiv 0$.

LEMMA. *Under the assumptions of Theorem 1 there exist numbers T^*, c_1, c_2, c_3 depending only on v_0, ρ_0 such that, for $t > T^*$,*

$$|\nabla v^\epsilon|_2 \le c_1 t^{-1},\ |Av^\epsilon|_2 < c_2 t^{-1/2},\ |\rho^\epsilon - C|_2 \le e^{-ct},\ |\nabla\rho^\epsilon|_2 \le c_3 e^{-ct}.$$

Proof. The energy inequality (60) implies

$$\mu\int_0^t |\nabla v^\epsilon|_2^2 d\tau \le c(|v_0|_2^2 + |\nabla\rho(0)|_2^2),$$

for $t > 0$, and therefore $\int_0^\infty |\nabla v^\epsilon|_2^2 dt \le c(|v_0|_2^2 + |\nabla\rho(0)|_2^2) = \bar{E}$.

We now apply Proposition 4 to the differential inequality

$$d_t(|\Delta\rho^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2) + \lambda|\Delta\nabla\rho^\epsilon|_2^2 + \frac{\mu^2}{M}|Av^\epsilon|_2^2 + |\sqrt{\rho}^\epsilon\partial_t v^\epsilon|_2^2 \le c(|\nabla v^\epsilon|_2^6 + |\Delta\rho^\epsilon|_2^6). \tag{64}$$

Setting $\phi = |\Delta\rho^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2$, $\psi = \lambda|\Delta\nabla\rho^\epsilon|_2^2 + \frac{\mu^2}{M}|Av^\epsilon|_2^2 + |\sqrt{\rho^\epsilon}\partial_t v^\epsilon|_2^2$ and $\int_0^\infty(|\Delta\rho^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2)dt = E$ we get

$$\partial_t\phi + \psi \le c\phi\phi^2. \tag{65}$$

From Proposition 4, taking $c_2 = \frac{1}{Ec}$ and $cc_2 = c_1$, we get

$$|\Delta\rho^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2 \le E\frac{e-1}{t} \equiv ct^{-1}, \tag{66}$$

and

$$\int_t^\infty \psi d\tau \equiv \int_t^\infty \left(\lambda|\Delta\nabla\rho^\epsilon|_2^2 + \frac{\mu}{M}|Av^\epsilon|_2^2 + |\sqrt{\rho^\epsilon}\partial_t v^\epsilon|_2^2\right)d\tau \le \frac{E(e^2-e)}{t} \equiv ct^{-1},$$

for $t > ceE^2 = T_0$. Thanks to the above estimates we deduce that

$$\int_t^\infty |\partial_t v^\epsilon|_2^2 d\tau \le ct^{-1}.$$

We can conclude that in $(T_0, +\infty)$ $|Av^\epsilon|_2, \|\rho^\epsilon\|_3, |\partial_t\rho^\epsilon|_2, |\partial_t v^\epsilon|_2$ are uniformly bounded in $L^2(T_0, +\infty)$.

Now, we continue to study the asymptotic behavior of the solution of problem (57).

Multiplying the diffusion equation by ρ^ϵ and integrating over Ω we obtain

$$d_t|\rho^\epsilon - C|_2^2 + |\nabla\rho^\epsilon|_2^2 = 0.$$

Consequently, from Poincaré's inequality,

$$|\rho^\epsilon - C|_2^2 \le e^{-ct}|\rho_0|_2^2. \tag{67}$$

Thanks to the Gagliardo-Nirenberg inequality and (67) we have

$$|\nabla\rho^\epsilon|_2^2 \le ce^{-ct}|\rho_0|_2|\Delta\rho^\epsilon|_2,\ |\Delta\rho^\epsilon|_2^2 \le ce^{-ct}|\rho_0|_2^{1/2}|\Delta\rho^\epsilon|_2^{1/2}\|\rho^\epsilon\|_3. \tag{68}$$

Now, multiplying $(57)_1$ by v^ϵ and integrating over Ω, one gets

$$d_t|\sqrt{\rho^\epsilon}v^\epsilon|_2^2 + \mu|\nabla v^\epsilon|_2^2 \le c|\nabla\rho^\epsilon|_2(|\nabla v^\epsilon|_2^{\frac{3}{2}}|Av^\epsilon|_2^{\frac{1}{2}} + |\nabla v^\epsilon|_6|\nabla\rho^\epsilon|_3). \tag{69}$$

Taking in account (68) we have, for $t > T_0$,

$$|v^\epsilon(t)|_2^2 \le ce^{-ct}.$$

Analogously, we multiply $(57)_1$ by Av^ϵ and $\partial_t v^\epsilon$ and after integration by parts and using the usual procedure we obtain

$$\begin{aligned}|\sqrt{\rho^\epsilon}\partial_t v^\epsilon|_2^2 + \mu d_t|\nabla v^\epsilon|_2^2 + \frac{\mu}{M}|Av^\epsilon|_2^2 &\le M|v^\epsilon|_2^{\frac{1}{2}}|\nabla v^\epsilon|_2^{\frac{1}{2}}|Av^\epsilon|_2^2\\ &+ \lambda(|\nabla\rho^\epsilon|_2^{\frac{1}{2}}|\nabla v^\epsilon|_2^{\frac{1}{2}}\|\rho^\epsilon\|_3^{\frac{1}{2}}|Av^\epsilon|_2^{\frac{3}{2}} + |\nabla\rho^\epsilon|_2^{\frac{1}{2}}\|\rho^\epsilon\|_2^{\frac{1}{2}}|Av^\epsilon|_2^2)\\ &+ c\lambda^2|\nabla\rho^\epsilon|_2^{1/2}(|\Delta\rho^\epsilon|_2^{1/2}\|\rho^\epsilon\|_3|Av^\epsilon|_2 + |\Delta\rho^\epsilon|_2^2\|\rho^\epsilon\|_3^{\frac{1}{2}}|Av^\epsilon|_2).\end{aligned} \tag{70}$$

Thanks to (67) we get

$$|\nabla v^\epsilon(t)|_2^2 \le ce^{-ct},$$

for $t > T_0$. In conclusion, there exists a $T^* > 0$ such that, uniformly with respect to ϵ,

$$\begin{cases} v^\epsilon \in L^2(T^*, +\infty; H^2(\Omega)) \cap L^\infty(T^*, +\infty; V);\ \partial_t v^\epsilon \in L^2(Q_{(T^*,\infty)}),\\ \rho^\epsilon \in L^2(T^*, +\infty; H^3(\Omega)) \cap L^\infty(T^*, +\infty; H^2(\Omega)),\\ \partial_t\rho^\epsilon \in L^2(T^*, +\infty; H^1(\Omega)),\ m \le \rho^\epsilon \le M, v^\epsilon \text{ and } \rho^\epsilon \text{ decay like}\\ (|v^\epsilon|_2, \|v^\epsilon\|, |\rho^\epsilon - C|_2, \|\rho^\epsilon\|_2) < ce^{-ct},\\ v\ (= \lim_{\epsilon\to 0} v^\epsilon), \rho\ (= \lim_{\epsilon\to 0}\rho^\epsilon)) \text{ satisfies (2) a.e. for } t > T^*.\end{cases} \tag{71}$$

7.4. $C^\infty((0,T)\times\bar{\Omega})$*-solution.* To conclude Theorem 3 we prove higher order derivatives for ρ^ϵ and v^ϵ solution of the approximate problem (57). Since we are mainly interested in a priori estimates we drop the superscript *epsilon* and bar of v, for simplicity of exposition.

Our main task is to prove, for any $\eta > 0$, the existence of continuous functions $F_{k,l}(t,\eta), \bar{F}_{k,l}(t,\eta)$ and $G_{k,l}(t,\eta), \bar{G}_{k,l}(t,\eta)$ of $t \in [\eta, T]$ such that

$$\begin{cases} |D_x^k D_t^l v|_2 \le F_{k,l}(t,\eta),\ \displaystyle\int_\eta^T |D_x^{k+1}D_t^l v|_2^2 dt \le \bar{F}_{k,l}(T,\eta),\\ |D_x^{k+1}D_t^l\rho(t)|_2 \le G_{k,l}(t,\eta),\ \displaystyle\int_\eta^T |D_x^{k+2}D_t^l\rho(t)|_2^2 dt \le \bar{G}_{k,l}(T,\eta),\end{cases} \tag{72}$$

for $t \in [\eta, T)$. This, in turn, implies $(v,\rho) \in C^\infty((0,T)\times\bar{\Omega})$.

In this section $R_l(v,\rho), S_l(v,\rho), \theta_l(v,\rho), \Theta_l(v,\rho), \Sigma_l(v,\rho), \Lambda(v,\rho)$ stand for continuous functions of t-derivatives up to the order l of ρ, v and their x-derivatives which appear in the context. In general $Z(\cdot)$ stands for a continuous functions of its argument. Now, we prove further estimates for the solution of the approximate problem (57) and the related diffusion equation. We assume $k = 1$ and prove (72) by induction on l. First, we will show that for every $l = 0, 1, 2, ...$ and every $\eta > 0$, there exist continuous functions $F_l(t;\eta), \bar{F}_l(t;\eta), G_l(t;\eta), \bar{G}_l(t;\eta)$ such that

$$(73)\qquad \begin{cases} |\nabla D_t^l v|_2^2 \le F_l(t;\eta), & \displaystyle\int_\eta^T |D_x^2 D_t^l v|_2^2 dt \le \bar{F}_l(t;\eta), \\ |\Delta D_t^l \rho|_2^2 \le G_l(t;\eta), & \displaystyle\int_\eta^T |D_x^3 D_t^l \rho|_2^2 dt \le \bar{G}_l(t;\eta), \end{cases}$$

for $t \in [\eta, T]$ and $F_l(t;\eta), \bar{F}_l(t;\eta), G_l(t;\eta), \bar{G}_l(t;\eta)$ will depend on l, η and the data.

We prove the estimates (73) by induction on l. For $l = 0$ Theorem 2 and section 7.1 imply (73).

Assume (73) is true for t-derivatives up to order l.

First, differentiating the diffusion equation in (57) l-times with respect to t, we prove it for order $l+1$. Writing $\partial_t^l v \equiv v_t^l,\ \partial_t^l \rho \equiv \rho_t^l$, one obtains

$$(74)\qquad \rho_t^{l+1} - \lambda\Delta\rho_t^l = -v_t^l \cdot \nabla\rho - v \cdot \nabla\rho_t^l + ... + cv_t^{l-1} \cdot \nabla\rho_t^1 \equiv R_l.$$

By assumptions, we have

$$(75)\qquad \begin{cases} v^l \in L^2(\eta, T; H^2(\Omega)) \cap L^\infty(\eta, T; H^1(\Omega)), \\ \rho_t^l \in L^2(\eta, T; H^3(\Omega)) \cap L^\infty(\eta, T; H^2(\Omega)), \\ (\rho_t^i, v_t^i) \in C(\eta, T; H^3(\Omega)) \times C(\eta, T; H^2(\Omega)),\ 0 \le i \le l-1, \end{cases}$$

thus the right-hand side R_l belongs to $L^2(\eta, T; H^1(\Omega))$ then

$$\int_\eta^T |\rho_t^{l+1}|_2^2 dt \le Z(G_l(\eta, T), \bar{G}_l(\eta, T), \bar{F}_l(\eta, T), F_l(\eta, T)) \equiv Z(\eta, T).$$

Here Z is a continuous function of its arguments. So we can conclude that there exists a number τ, $\eta < \tau < 2\eta$ such that

$$|\rho_t^{l+1}(\tau)|_2^2 \le \eta^{-1} Z(\eta, 2\eta).$$

Now, we differentiate (74) with respect to t and obtain

$$(76)\qquad \partial_t\rho_t^{l+1} - \lambda\Delta\rho_t^{l+1} = -v_t^{l+1} \cdot \nabla\rho - v \cdot \nabla\rho_t^{l+1} + S_l(v, \rho).$$

Thanks to the assumptions, we have

$$S_l \in L^2(\eta, T; H^1(\Omega))$$

and its norm is bounded by a function type $Z(\eta, T)$. Now, setting $\rho_t^{l+1} \equiv \phi, v^{l+1} = \psi$, problem (74) can be written

$$(77)\qquad \partial_t\phi - \lambda\Delta\phi = -\psi \cdot \nabla\rho - v \cdot \nabla\phi + S_l(v, \rho).$$

with $\phi(\eta) \in L^2(\Omega)$ and $\partial_n\phi = 0$ on Γ.

We treat (77) in the same manner of the problem (5). We consider the first level of regularity.

Multiplying (77) by $\lambda\phi$ and integrating by parts we have

$$\lambda d_t|\phi|_2^2 + \lambda^2|\nabla\phi|_2^2 \le c|\psi|_2^2 + |S_l|_2^2,$$

then

$$|\phi(t)|_2^2 + \int_\eta^t |\nabla\phi|_2^2 d\tau \le c\Big(|\phi(\eta)|_2^2 + \int_\eta^t (|\psi|_2^2 + |S_l|_2^2)d\tau\Big). \tag{78}$$

As usual, there exists a number τ with $\eta \le \tau \le 2\eta$ such that

$$|\nabla\phi(\tau)|_2^2 \le \eta^{-1}\Big(|\phi(\tau)|_2^2 + \int_\eta^{2\eta} (|\psi|_2^2 + |S_l|_2^2)dt\Big). \tag{79}$$

Now, we deduce the second level of regularity for ϕ. We multiply (77) by $-\Delta\phi$ and after integration by parts we get

$$d_t|\nabla\phi|_2^2 + \lambda|\Delta\phi|_2^2 \le c(|\nabla\psi|_2^2|\nabla\rho|_3^2 + |v|_\infty^2|\nabla\phi|_2^2 + |S_l|_2^2). \tag{80}$$

Then applying Gronwall's lemma we obtain

$$|\nabla\phi(t)|_2^2 + \int_\eta^t |\Delta\phi|_2^2 d\tau \le Z(G_0(t,\eta), \bar{F}_0(t,\eta))\Big(|\nabla\phi(\eta)|_2^2 + \int_\eta^t (|\nabla\psi|_2^2 + |S_l|_2^2)d\tau\Big)$$

and

$$|\Delta\phi(\tau)|_2^2 \le \eta^{-1} Z(G_0(2\eta,\eta), \bar{F}_0(2\eta,\eta))\Big(|\nabla\phi(\tau)|_2^2 + \int_\eta^{2\eta} (|\nabla\psi|_2^2 + |S_l|_2^2)dt\Big) \tag{81}$$

for $\eta \le \tau \le 2\eta$.

Finally, we consider the third level of regularity of ϕ applying the ∇ operator to (77) and in the manner of (10) we get

$$d_t|\Delta\phi|_2^2 + |\nabla\Delta\phi|_2^2 \le c(|\nabla\psi\cdot\nabla\rho|_2^2 + |\psi\cdot\nabla\nabla\rho|_2^2 + |\nabla v\cdot\nabla\phi|_2^2 + |v\cdot\nabla\nabla\phi|_2^2 + |\nabla S_l|_2^2). \tag{82}$$

Immediately, we get

$$\begin{aligned} |\Delta\phi(t)|_2^2 &+ \int_\eta^t |\nabla\Delta\phi^2|_2^2 d\tau \\ &\le |\Delta\rho(\eta)|_2^2 + \int_\eta^t (|\nabla\psi|_2^2|\Delta\rho|_2\|\rho\|_3 + (|\nabla v|_3^2 + |v|_\infty^2)|\Delta\phi|_2^2 + |\nabla S_l|_2^2)d\tau. \end{aligned} \tag{83}$$

We now work on the momentum equation. Differentiating l-times with respect to t the first equation in (57) we get

$$\begin{aligned} \rho\partial_t v_t^l - \mu\Delta v_t^l &= -\rho_t^l v_t^1 - \rho_t^l v\cdot\nabla v - \rho v_t^l\cdot\nabla v - \rho v\cdot\nabla v_t^l + \lambda((v_t^l\cdot\nabla)\nabla\rho + (v\cdot\nabla)\nabla\rho_t^l \\ &+ (\nabla\rho\cdot\nabla)v_t^l + (\nabla\rho_t^l\cdot\nabla)v) + \theta_{l-1}(v,\rho) + \partial_t^l\Big(\nabla\cdot\Big(\frac{\lambda^2}{\rho}\nabla\rho\otimes\nabla\rho\Big)\Big) \equiv \Theta_l(v,\rho). \end{aligned} \tag{84}$$

Thanks to the assumptions, the right-hand side of (84) belongs to $L^2(\eta, T; L^2(\Omega))$, thus

$$\int_\eta^t |\partial_t^{l+1} v|_2^2 d\tau \le c\int_\eta^t (|\Delta v_t^l|_2^2 + |\Theta_l|_2^2)d\tau \le \bar{F}_l(t,\eta) + \int_\eta^t |\Theta_l|_2^2 d\tau.$$

So there exists τ with $\eta \le \tau \le 2\eta$ such that

$$|v_t^{l+1}(\tau)|_2 \le \eta^{-1}\Big(\bar{F}_l(2\eta,\eta) + \int_\eta^{2\eta} |\Theta_l|_2^2 dt\Big). \tag{85}$$

Following the procedure used for ρ, writing $v_t^{l+1} \equiv \psi, \rho_t^{l+1} \equiv \phi$, differentiating with respect to t (84) we get

$$
\begin{aligned}
\rho\partial_t\psi - \mu\Delta\psi = -\phi v_t^1 - \phi v\cdot\nabla v - \rho\psi\cdot\nabla v - \rho v\cdot\nabla\psi + \lambda((\psi\cdot\nabla)\nabla\rho + (\nabla\phi\cdot\nabla)v \\
+ (v\cdot\nabla)\nabla\phi + (\nabla\rho\cdot\nabla)\psi) - \nabla\cdot\left(\left(\frac{\lambda}{\rho}\right)^2\phi\nabla\rho\otimes\nabla\rho\right) \\
+ \nabla\cdot\left(\frac{\lambda^2}{\rho}\nabla\phi\otimes\nabla\rho\right) + \nabla\cdot\left(\frac{\lambda^2}{\rho}\nabla\rho\otimes\nabla\phi\right) + \Sigma_l(v_t,\rho).
\end{aligned}
\tag{86}
$$

Thanks to the assumptions

$$\|\Sigma_l\|_{L^2(\eta,T;H^1(\Omega))} \le Z(F_l(T,\eta), \bar F_l(T,\eta), G_l(T,\eta), \bar G_l(T,\eta) \equiv Z(T,\eta).$$

We notice that

$$
\begin{cases}
|\phi v\cdot\nabla v|_2 \le |v|_\infty|\nabla v|_2|\phi|_\infty \le |Av|_2^{1/2}\|v\|^{3/2}|\nabla\phi|_2^{1/2}|\Delta\phi|_2^{1/2}, \\
|\rho\psi\cdot\nabla v|_2 \le M|\psi|_\infty|\nabla v|_2 \le c|\nabla\psi|_2^{1/2}|A\psi|_2^{1/2}F_0, \\
|(\psi\cdot\nabla)\nabla\rho|_2 \le |\psi|_\infty|\Delta\rho|_2 \le c|\nabla\psi|_2^{1/2}|A\psi|_2^{1/2}G_0, \\
|(\nabla\rho\cdot\nabla)\psi|_2 \le |\nabla\psi|_2|\nabla\rho|_\infty \le |\nabla\psi|_2|\Delta\rho|_2^{1/2}\|\rho\|_3^{1/2}, \\
|(v\cdot\nabla)\nabla\phi|_2 \le \|v\|^{1/2}|Av|_2^{1/2}|\Delta\phi|_2, \\
|\nabla\phi\cdot\nabla)v|_2 \le |\nabla v|_3|\nabla\phi|_6 \le \|v\|^{1/2}|Av|_2^{1/2}|\Delta\phi|_2, \\
|\Delta\phi\nabla\rho|_2 \le c|\Delta\phi|_2|\nabla\rho|_\infty \le |\Delta\phi|_2|\Delta\rho|_2^{1/2}\|\rho\|_3^{1/2}, \\
|\phi\partial_t v|_2 \le |\phi|_\infty|\partial_t v|_2 \le |\nabla\phi|_2^{1/2}|\Delta\phi|_2^{1/2}|\partial_t v|_2.
\end{cases}
\tag{87}
$$

The estimates of the other terms can be found analogously. Now, working in the manner of (12), we obtain

$$
\begin{aligned}
|\sqrt{\rho}\partial_t\psi|_2^2 + \mu d_t|\nabla\psi|_2^2 + |A\psi|_2^2 \le c|\nabla\psi|_2^2(\|v\||Av|_2 + |\Delta\rho|_2\|\rho\|_3) \\
+ |\Delta\phi|_2^2((|\nabla v|_2 + |\nabla v|_2^3)|Av|_2 + |\partial_t v|_2^2 + (|\Delta\rho|_2^3 + |\Delta\rho|_2)\|\rho\|_3) + |\Sigma_l|_2^2.
\end{aligned}
\tag{88}
$$

Adding (82) and (88) yields

$$
\begin{aligned}
d_t(|\Delta\phi(t)|_2^2 + |\nabla\psi|_2^2) + |\sqrt{\rho}\partial_t\psi|_2^2 + |A\psi|_2^2 + |\nabla\Delta\phi|_2^2 \\
\le c|\nabla\psi|_2^2(\|v\||Av|_2 + |\Delta\rho|_2\|\rho\|_3) + |\Delta\phi|_2^2((|\nabla v|_2 + |\nabla v|_2^3)|Av|_2 \\
+ |\partial_t v|_2^2 + (|\Delta\rho|_2^3 + |\Delta\rho|_2)\|\rho\|_3) + |\nabla S_l|_2^2 + |\Sigma_l|_2^2.
\end{aligned}
\tag{89}
$$

Gronwall's lemma, (79), (81), (85) imply that ϕ and ψ satisfy (72) for $k=1$ and $l+1$, consequently (72) holds for every l with $k=1$.

Now, we use induction on k. Of course, (72) holds true for $k=1$. Assuming that (72) holds for x-derivatives up to order k we now prove it for x-derivatives of order $k+1$.

We consider the mass diffusion equation

$$\lambda\Delta\rho_t^l = \rho_t^{l+1} + D^l(v\cdot\nabla\rho) \equiv g, \quad \partial_n\rho_t^l = 0 \text{ on } \Gamma.$$

First of all, in view of the equation (57) we note that the $i+1$-regularity level of ρ corresponds to the i-regularity level of v. In other words, for every $l \ge 1$, if $v_t^l \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ then $\rho_t^l \in L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$. Notice that the right-hand side g has, for every l, the x-derivatives up to order $k-1$ bounded in the manner of (72).

In fact the estimates of all terms in g can be obtained as follows.

$$
\left\{
\begin{array}{l}
|D_x^i v_t^j \cdot \nabla D_x^{k-(i+2)} \rho_t^{l-j}|_2 \le |D_x^i v_t^j|_6 |\nabla D_x^{k-(i+2)} \rho_t^{l-j}|_3, \ 0 \le i < k-3, \\
|v_t^j \cdot \nabla D_x^{k-1} \rho_t^{l-j}|_2 \le |v_t^j|_\infty |D^k \rho_t^{l-j}|_2, \\
|D_x^{k-1} v_t^j \cdot \nabla \rho_t^{l-j}|_2 \le |D_x^{k-1} v_t^j|_2 |\nabla \rho_t^{l-j}|_\infty .
\end{array}
\right. \tag{90}
$$

The theory of the Neumann problem for the Laplace operator implies that $(72)_2$ holds for k and every l. We now pass to the estimates for v. We prove the estimate (72) for v.

We consider the Stokes problem for v_t^l

$$
\left\{
\begin{array}{l}
\mu A v_t^l = -P\partial_t^l(\rho v_t^1 + \rho v \cdot \nabla v - \lambda((v \cdot \nabla)\nabla \rho \\
+(\nabla \rho \cdot \nabla) v) - \lambda^2 \nabla \cdot (\frac{1}{\rho} \nabla \rho \otimes \nabla \rho)) \equiv \Lambda(v, \rho), \\
v_t^l = 0 \text{ on } \Gamma.
\end{array}
\right. \tag{91}
$$

We note the the $(k-1)-x$ derivatives of v_t^l satisfy $(72)_1$ for every l. We estimate Λ in the manner of g. In fact, using the assumptions, we have for $i < k-1$

$$
\left\{
\begin{array}{l}
|\rho D^{k-1} v_t^l|_2 \le |\rho|_\infty |D^{k-1} v_t^l|_2, \\
|v_t D^{k-1} \rho_t^l|_2 \le |v_t|_\infty |D^{k-1} \rho_t^l|_2, \\
|D_x^i \rho_t^j D_x^{k-(i+2)} v_t^{l-j}| \le |D_x^i \rho_t^j|_3 |D_x^{k-(i+2)} v_t^{l-j}|_6 .
\end{array}
\right. \tag{92}
$$

Analogously, we can treat the remaining terms (we omit details).

Finally, the theory of Stokes problem gives $(72)_1$ for k and every l. The case $t > T^*$ can be treated in the same manner. Theorem 3 is completely proved.

8. Maximal L^q-regularity. This section is devoted to the maximal regularity of the problem (2) and to the proof of Theorem 4. Some notations used in this section. X will denote a real Banach space. If S is a linear operator on X, then $D(S)$ stands for its domain. As usual, $\sigma(S), \rho(S), R(\lambda, S)$ are the notation of the spectrum, resolvent set and resolvent of the operator S. For any $T > 0$ we denote by $W(0,T;X)$ the set of X-valued functions defined a.e. in $[0,T]$. If $W \equiv L^r$ they are summable in the sense of Bochner. Moreover, we continue to denote by S the operator $f \in W(0,T;X) \mapsto Sf(\cdot)$ with domain$(S) = \{f \in W(0,T;X) | f(t) \in D(S), \forall t \in (0,T)\}$. Our primary problem in this section is the solvability of the differential equation

$$
d_t v + Av = f, \ v(0) = x, \tag{93}
$$

where A is an (unbounded) linear operator (Stokes operator) acting on X with nonempty resolvent set. We study solvability of (93) by considering the sum of the two commuting operators $d/dt \equiv B, A$. The approach is of spectral type in the style of the Dore-Venni theory [6], and it is not considered the joint spectrum of n-commutating operators in sense of Taylor. In spectral approach it is crucial that

$$
\sigma(A+B) \subseteq \sigma(A) + \sigma(B), \ \sigma(AB) \subseteq \sigma(A)\sigma(B). \tag{94}
$$

If A, B are bounded operators the Gelfand transform proves this property. Unfortunately, for unbounded operators the property does not hold, in general.

We will denote by $\mathcal{R}$ the class of operators satisfying the first condition in (94) and by $\mathcal{R}_1$ satisfying the second condition in (94). Moreover, we assume that $A+B$ and AB are closable. Now we give some definition. We recall the notion of two commuting operators which will be used in the sequel.

DEFINITION 4. Let A and B be operators on a Banach space X with non-empty resolvent set. We say that A and B commute if one of the following equivalent conditions holds:

$$\begin{cases} (i) \quad R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A), \ \lambda \in \rho(A), \ \mu \in \rho(B); \\ (ii) \quad x \in D(A) \text{ implies } R(\mu, B)x \in D(A) \text{ and} \\ AR(\mu, B)x = R(\mu, B)Ax, \ \mu \in \rho(B). \end{cases} \tag{95}$$

For $\theta \in (0, \pi), r > 0$ we denote $\Sigma(\theta, r) = \{z \in \mathbf{C} : |z| \geq r, |argz| \leq \theta\}$.

DEFINITION 5. Let A and B be commutating operators. Then

(i) A is said to be of class $\Sigma(\theta + \pi/2, r)$ if there are positive constants θ, r such that $0 < \theta < \pi/2$, and

$$\Sigma(\theta + \pi/2, r) \subset \rho(A) \text{ and } \sup_{\lambda \in \Sigma(\theta+\pi/2,r)} \|\lambda R(\lambda, A)\| < \infty,$$

(ii) A and B are said to satisfy condition P if there are positive constants $\theta, \theta', r, \theta' < \theta$ such that A and B are of class $\Sigma(\theta + \pi/2, r)$ and $\Sigma(\pi/2 - \theta', r)$, respectively.

If A and B are commuting operators, $A + B$ is defined by $(A + B)x = Ax + Bx$ with domain $D(A + B) = D(A) \cap D(B)$ and AB with domain $D(AB) = \{x|x \in D(B)$ and $Bx \in D(A)\}$. In this paper we assume that $D(A), D(B)$ are dense in X.

The following assertions hold [1]:

i) if one of the operator A or B is bounded then $\sigma(A + B) \subseteq \sigma(A) + \sigma(B)$;

ii) If A, B satisfy the condition P, $A + B$ is closable and

$$\sigma(cl(A + B)) \subseteq \sigma(A) + \sigma(B).$$

This result implies that the class $\mathcal{R}$ is not empty.

Moreover, we remark that if

$$\Sigma(\theta + \pi/2, r) \subseteq \rho(A), \qquad \sup_{\lambda \in \Sigma(\theta+\pi/2,r)} |\lambda|^\alpha \| \ R(\lambda, A)\| < \infty,$$

and

$$\Sigma(\pi/2 - \theta', r) \subseteq \rho(B), \qquad \sup_{\lambda \in \Sigma(+\pi/2-\theta',r)} |\lambda|^\beta \| \ R(\lambda, B)\| < \infty,$$

the assertion (ii) holds if $\alpha + \beta > 1$.

The problem that we are going to study can be formulated in the following form.

Problem Q (Maximal regularity): Find for any $f \in X$ *a unique solution* $v \in D(A) \cap D(B)$ solving the problem

$$Av + Bv = f. \tag{96}$$

A spectral approach to the existence of a weak solution of problem (96) says that if $\rho(A) \cap \rho(B) \neq \emptyset$, $0 \in \rho(A)$ or $0 \in \rho(B)$ and A, B belong to the set $\mathcal{R}$ then there exists a weak solution of problem (96) (see later for the definition). Moreover, if $0 \in \rho(B)$, $\sigma(A), \sigma(B) \subseteq R^+$ and A, B^{-1} belong to the set $\mathcal{R}_1$ then $v \in D(A) \cap D(B)$ and $v = B^{-1}(AB^{-1} + I)^{-1}f$ is a solution of problem (Q).

Anyway, the study of the properties of the class of operators $\mathcal{R}$ and $\mathcal{R}_1$ is beyond the scope of this paper.

For the L^q-regularity of the solution of problem (2) we consider the theory of two commutating operators $-d/dt = B$ and $A \equiv A_q$, the Stokes operator. As described before, the main goals of that theory are the closeness and the invertibility of the operator $L = -d/dt + A$. We refer the reader to [7], [12] and the references therein for more information on the theory and applications of sums of commuting operators method to differential equations.

According to the properties of the operator L we recall the following definitions of the solution of the problem

$$v'(t) = Av(t) + f(t), \ v(0) = x. \tag{97}$$

Definition 6. We say that (97) has L^q-weak solution $1 < q < \infty$, on $[0, T), \ 0 < T \leq \infty$ if for every $f \in L^q(0, T; X)$, L is closable and $0 \in \rho(\bar{L})$.

Definition 7. Let A be the generator of a C_0-semigroup. We say that an X-valued continuous function $v(t)$ is a mild solution of (97) if for every $f \in BUC((0, T; X)$ (X-valued continuous uniformly bounded function), $v(t)$ satisfies

$$v(t) = e^{(t-s)A}v(s) + \int_s^t e^{(t-\tau)A} f d\tau, \quad \forall t \geq s.$$

Definition 8. We say that (97) has maximal L^q-regularity, $1 < q < \infty$, on $[0, T), \ 0 < T \leq \infty$ if for every $f \in L^q(0, T; X)$, $v(t) \in L^q(0, T; X)$, has value in $D(A)$ and there is a constant $C < \infty$ with

$$\|v'(t)\|_{L^q(0,T;X)} + \|Av(t)\|_{L^q(0,T;X)} \leq C\|f(t)\|_{L^q(0,T;X)}. \tag{98}$$

Concerning the existence or the existence and regularity as are conceived in the above definitions, to the best of our knowledge, so far the sum commuting method is mainly applied to the Cauchy problem (97) on a finite time interval. The extension of the method to the infinite time interval is conceived essentially in the direction of regularity of a solution as is explicated by the following definition.

Definition 9. We say that (97) has maximal L^q-regularity, $1 < q < \infty$, on $[0, T), \ 0 < T \leq \infty$ if for every $f \in L^q(0, T; X)$, $v(t)$ is almost everywhere differentiable, has value in $D(A)$ and there is a constant $C < \infty$ with

$$\|v'(t)\|_{L^q(0,T;X)} + \|Av(t)\|_{L^q(0,T;X)} \leq C\|f(t)\|_{L^q(0,T;X)}. \tag{99}$$

This definition is slightly weaker than Definition 8, which also requires $v \in L^q(0, T; X)$. But for $T = \infty$ this additional condition implies already $s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\} < 0$ and this includes the case $0 \in \rho(A)$; in other words, general unbounded domains are excluded in this approach.

An interpolation space and Stokes problem. Let X, Y be Banach spaces such that $Y \hookrightarrow X$. For $1 < q < \infty$ one may define the real interpolation space

$$(X, Y)_{\frac{1}{q'}, q} = \{u(0) | u(t) \in H^1_q(0, T; X) \cap L^q(0, T; Y)\},$$

where $\frac{1}{q'} + \frac{1}{q} = 1$. In particular, if A generates a holomorphic C_0- semigroup $T(\cdot)$ on X, then

$$(X, D(A))_{\frac{1}{q'},q} = \{x \in X | AT(\cdot)x \in L^q(0,T;X)\}.$$

Thus, the mild solution $v = T(\cdot)x$ of the Cauchy problem

$$\partial_t v + Av = 0, \; v(0) = x, \tag{100}$$

is in $H^1_q(0,T;X)$ if and only if $x \in (X, D(A))_{\frac{1}{q'},q}$. Thanks to this result, in the sequel, we will assume the initial data $v(0) \in (X, D(A))_{\frac{1}{q'},q}$.

In the sequel we need the estimates of the Stokes problem

$$\begin{cases} \partial_t v - \mu\Delta v + \nabla\pi = f, \\ \nabla \cdot v = g, \\ v|_{t=0} = v_0, \; v|_{\partial\Omega} = 0. \end{cases} \tag{101}$$

Since we will find solution in the space $L^q(0,T; H^2_q(\Omega) \cap V^1_q)$, we notice that if Y is the closure of the domain of the Stokes operator A_q in L^q under the norm $|Au|_q$ we can write

$$\mathcal{V}_q = (X,Y)_{\frac{1}{q'},q} = V^0_q \cap (L^q, H^2_q \cap H^1_{q,0})_{\frac{1}{q'},q}.$$

Moreover, the Stokes operator A_q and d_t belong to $\mathcal{R}$ and A_q, d_t^{-1} belong to $\mathcal{R}_1$ thus, there exists a solution (v, π) of the solenoidal Stokes problem

$$\begin{cases} \partial_t v - \mu\Delta v + \nabla\pi = f, \\ \nabla \cdot v = 0, \\ v|_{t=0} = v_0, \; v|_{\partial\Omega} = 0, \end{cases} \tag{102}$$

and the following estimate, with $v_0 \in H^{2-2/q}_q(\Omega)$ divergence-free and vanishing on the boundary, holds true

$$\|\partial_t v\|_{L^q(Q_T)} + \|\nabla^2 v\|_{L^q(Q_T)} + \sup_{0\le t\le T} \|v(t)\|_{\mathcal{V}_q} \le c(\|f\|_{L^q(Q_T)} + \|v_0\|_{\mathcal{V}_q(\Omega)}). \tag{103}$$

Here c depends on q, Ω.

For the general case $\nabla \cdot v = g$ the following result plays an important role.

Proposition 5 (Bogovskii [4]). *Let $\Omega \subset R^n$, $n \ge 2$ be a Lipschitz domain, and let $1 < q < \infty$, $m \in N$. Then, for each $f \in H^m_{q,0}$ with $\int_\Omega f dx = 0$ there exists at least one $z \in H^{m+1}_{q,0}(\Omega)$ satisfying*

$$\nabla \cdot z = f, \quad \|z\|_{H^{m+1}_q(\Omega)} \le c\|f\|_{H^m_q(\Omega)}.$$

Furthermore, for each $f \in H^{-1}_q(\Omega)$ there exists at least one $z \in L^q(\Omega)$ satisfying $\nabla \cdot z = f$ in the sense of distributions, and then

$$|z|_q \le c\|f\|_{H^{-1}_q(\Omega)}.$$

Here c depends on m, Ω.

The problem (101) can be reduced to the solenoidal Stokes problem (102) if we write $v = u + z$ where z is a solution of the problem in Proposition 5. With $v = u + z$ the problem (101) is transformed into

$$(104)\qquad \begin{cases} \partial_t u - \mu\Delta u + \nabla\pi = f - \partial_t z + \mu\Delta z, \\ \nabla\cdot u = 0, \\ u|_{t=0} = v_0,\ u|_{\partial\Omega} = 0, \end{cases}$$

and u satisfies the estimate

$$(105)\qquad \|\partial_t u\|_{L^q(Q_T)} + \|\nabla^2 u\|_{L^q(Q_T)} + \sup_{0\le t\le T}\|u(t)\|_{H_q^{2-2/q}(\Omega)} \le c(\|f\|_{L^q(Q_T)} + \|v_0\|_{H_q^{2-2/q}(\Omega)} + \|\partial_t g\|_{L^q(Q_T)} + \|\nabla g\|_{L^q(Q_T)}).$$

8.1. *Auxiliary problem.* This section is devoted to the following problem

$$(106)\qquad \begin{cases} \rho\partial_t v - \mu\Delta v + \nabla\pi = f, \\ \nabla\cdot v = 0, \\ v|_{t=0} = v_0,\ v|_{\partial\Omega} = 0. \end{cases}$$

We assume that π has mean zero. An existence and regularity theorem for (106) reads

THEOREM 5. *Let Ω be a $C^{2+\epsilon}$ bounded domain in R^3, $q > 3$, $v_0 \in \mathcal{V}_q$, $f \in L^q(Q_T)$ and $\rho \in L^\infty(Q_T)\cap L^q(0,T;H_q^2(\Omega))\cap C^\beta(0,T;L^\infty(\Omega))$ with $\beta\in(0,1)$, $\nabla\rho\in L^\infty(Q_T)$ and $m\le\rho\le M$. Then (106) has a unique solution (v,π) such that*

$$v\in L^q(0,T;H_q^2(\Omega)\cap V_{q,0}^1(\Omega)),\ \partial_t v\in L^q(Q_T),\ \pi\in L^q(0,T;H_q^1(\Omega)),$$

and

$$(107)\qquad \|\partial_t v\|_{L^q(Q_t)} + \|\nabla^2 v\|_{L^q(Q_t)} + \|\nabla\pi\|_{L^q(Q_t)} + \|v(t)\|_{\mathcal{V}_q(\Omega)} \le M_1 c^{th(M_1(t),M_2(t))}(\|f\|_{L^q(Q_t)} + \|v_0\|_{\mathcal{V}_q(\Omega)}).$$

Here c depends on q,Ω,m,M, and $M_1(t) = \|\nabla\rho(t)\|_{L^q(\Omega)}$,

$$M_2 = \sup_{x\in\Omega;\tau\ne\tau'\in[0,t]}\frac{|\rho(x,\tau)-\rho(x,\tau')|}{|\tau-\tau'|^\beta},$$

h is a bounded continuous function of M_1, M_2, and $0 < t \le T$.

Proof of Theorem 5. The proof of Theorem 5 imitates the proof of Theorem 2 in [10] and consists of five steps. The existence of problem (106) is given in section 4. We prove the estimate (107).

Step 1. Density is independent of time. To prove the estimate (107) we use the so-called freezing method (see [10]). Moreover, thanks to the remark in the above section, we assume $v_0 = 0$, for simplicity of exposition.

Let $(\Omega_k)_{1\le k\le n}$ be a covering of Ω by open sets with C^2 boundaries. Consider a partition of unity $(\phi)_{1\le k\le n}$ of class C^2 such that

1. $Supp\phi_k \subset \Omega$,
2. $\sum_1^n \phi_k = 1$,
3. $0\le\phi_k\le 1$,
4. $|D^i\phi|_\infty \le c_i|\delta|^{-i}, i = 1,2,\dots,$
5. $n = [(diam(\Omega)\delta^{-1})^3]$.

Let $f_k = \phi_k f$, $v_k = \phi_k v$, $\pi_k = \phi_k \pi$. Let x_k be a point (the point of minimum value of ρ) in $\bar\Omega_k \cap \Omega$, and $\rho_k = \rho(x_k)$, $\mu_k = \mu/\rho_k$. Then (v_k, ϕ_k) satisfies

$$\begin{cases} \partial_t v_k - \mu_k \Delta v_k + \nabla \frac{\pi_k}{\rho_k} \\ = \dfrac{f}{\rho_k} + \dfrac{\rho_k - \rho}{\rho_k}\partial_t v_k - \mu_k v \Delta\phi_k - 2\mu_k \nabla\phi_k \nabla v + \dfrac{\pi}{\rho_k}\nabla\phi_k \equiv F_k, \\ \nabla\cdot v_k = v\cdot\nabla\phi_k \equiv g_k, \\ v_k|_{t=0} = 0,\ v_k|_{\partial\Omega} = 0. \end{cases} \tag{108}$$

Let z_k be a solution of the problem in Proposition 5 with $g = g_k$ and let $v_k = u_k + z_k$. Then u_k satisfies

$$\begin{cases} \partial_t u_k - \mu_k \Delta v_k + \nabla \frac{\pi_k}{\rho_k} = F_k - \partial_t z_k + \mu_k \Delta z_k, \\ \nabla\cdot u_k = 0, \\ u_k|_{t=0} = 0,\ u_k|_{\partial\Omega} = 0. \end{cases} \tag{109}$$

First, to estimate $\|\partial_t z_k\|_{L^q(Q_T)}$ we write (with the use of the convention of repeated index)

$$\partial_t g_k = \frac{f}{\rho}\cdot\nabla\phi_k + \pi\partial_i\frac{\partial_i\phi_k}{\rho} - \partial_j v_i \partial_j\left(\frac{\mu}{\rho}\partial_i\phi_k\right) + \nabla\cdot\left(\frac{\mu}{\rho}\partial_i\phi_k\partial_j v_i - \frac{\pi}{\rho}\partial_i\phi_k\right) \equiv h + \nabla\cdot H.$$

Consequently, $\partial_t z_k$ satisfies

$$\nabla\cdot(\partial_t z_k - H) = h,$$

and thanks to Proposition 5 we get

$$\begin{aligned} \|\partial_t z_k\|_{L^q(Q_T^k)} \le{}& c(((m\delta^2)^{-1} + (m\delta)^{-1})(\|\pi\|_{L^q(Q_T^k)} + \|\nabla v\|_{L^q(Q_T^k)}) \\ &+ (m^2\delta)^{-1}(\|\pi\nabla\rho\|_{L^q(Q_T^k)} + \|\nabla v\cdot\nabla\rho\|_{L^q(Q_T^k)}) + (m\delta)^{-1}\|f\|_{L^q(Q_T^k)}). \end{aligned} \tag{110}$$

Moreover

$$\begin{aligned} \|F_k\|_{L^q(Q_T^k)} \le{}& c(\|f\|_{L^q(Q_T^k)} + (m\delta)^{-1}\|\nabla v\|_{L^q(Q_T^k)} \\ &+ (m\delta^2)^{-1}\|v\|_{L^q(Q_T^k)} + (m\delta)^{-1}\|\pi\|_{L^q(Q_T^k)} + \delta^\beta\|\partial_t v\|_{L^q(Q_T^k)} M_2). \end{aligned} \tag{111}$$

Bearing in mind the multiplicative inequalities in Proposition 1, in particular $\|\phi\|_{1,q} \le c(\epsilon^{-1}|\phi|_q + \epsilon|D^2\phi|_q)$, for u_k we have the estimate

$$\begin{aligned} &\|\partial_t u_k\|_{L^q(Q_T)} + \mu_k\|\nabla^2 u_k\|_{L^q(Q_T)} + \|\nabla\pi_k\|_{L^q(Q_T)} \\ &\quad\le c(\|F_k\|_{L^q(Q_T)} + \|\partial_t z_k\|_{L^q(Q_T)} + \mu_k\|\Delta z_k\|_{L^q(Q_T)}) \\ &\quad\le c(\delta^{-1}\|\nabla v\|_{L^q(Q_T^k)} + \delta^{-2}\|v\|_{L^q(Q_T^k)} + \delta^\beta\|\partial_t u_k\|_{L^q(Q_T^k)} M_2 \\ &\qquad + (\delta^{-2} + \delta^{-1})\|\pi\|_{L^q(Q_T^k)} + \delta^{-1}\|\nabla v\|_{L^q(Q_T^k)} \\ &\qquad + \delta^{-1}(\|\pi\nabla\rho\|_{L^q(Q_T^k)} + \|\nabla v\cdot\nabla\rho\|_{L^q(Q_T^k)}) + \delta^{-1}\|f\|_{L^q(Q_T^k)}). \end{aligned} \tag{112}$$

Assuming $\delta^\beta M_2 = \sigma$ with σ small enough, raising to the q-th power and summing on k of (112), taking into account that the covering has finite multiplicity s and

$\sum_1^K \|V_k\|_{L^p(Q_T^k)} \leq s\|V\|_{L^p(Q_T)}$ we obtain

$$\begin{aligned} (113) \quad & \|m\partial_t v\|_{L^q(Q_T)} + \mu\|\nabla^2 v(t)\|_{L^q(Q_T)} + \|\nabla\pi\|_{L^q(Q_T)} \\ & \leq c(\delta^{-2}\|v\|_{L^q(Q_T)} + \delta^{-1}\|\nabla v\|_{L^q(Q_T)} + (\delta^{-2}+\delta^{-1})\|\pi\|_{L^q(Q_T)} \\ & \quad + \delta^{-1}(\|\nabla v\cdot\nabla\rho\|_{L^q(Q_T)} + \|\pi\nabla\rho\|_{L^q(Q_T)}) + \delta^{-1}\|f\|_{L^q(Q_T)}). \end{aligned}$$

Combining the Hölder, Gagliardo-Nirenberg and Young inequalities we get

$$(114) \qquad \begin{cases} |\pi\nabla\rho|_q \leq |\nabla\rho|_\infty|\pi|_q, \\ |\nabla v\cdot\nabla\rho|_q \leq |\nabla\rho|_\infty(\epsilon^{-1}|v|_q + \epsilon|D^2v|_q). \end{cases}$$

Using the above estimates, with a suitable choice of ϵ in (114) we get

$$\begin{aligned} (115) \quad & \|m\partial_t v\|_{L^q(Q_T)} + \mu\|\Delta v\|_{L^q(Q_T)} + \|\nabla\pi\|_{L^q(Q_T)} \\ & \leq c(\|f\|_{L^q(Q_T)} + M_1^2\|v\|_{L^q(Q_T)} + M_1\|\pi\|_{L^q(Q_T)}). \end{aligned}$$

Step 2. Estimate of π. Let χ be the solution of the Neumann problem in divergence form

$$(116) \qquad \begin{cases} \nabla\cdot(\rho^{-1}\nabla\chi) = \theta, & \int_\Omega \chi dx = 0,\ \int_\Omega \theta dx = 0, \\ \partial_n\chi = 0, & \text{on } \Gamma. \end{cases}$$

We recall that ρ is bounded from below and its regularity insures the existence and uniqueness in $H^{2,q}(\Omega)$ with

$$(117) \qquad \|\chi\|_{2,q} \leq c|\theta|_q,$$

c depends on ρ and Ω, q.

In fact, since the problem (116) is in divergence form, the existence of a unique solution for example in $H^1(\Omega)$ is well known, and Poincaré's inequality produces

$$|\chi|_2 \leq c|\theta|_2.$$

Concerning the regularity, we write (116)

$$\Delta\chi - \nabla\log(\rho)\nabla\chi = \theta.$$

Assuming $\nabla\log\rho \in L^p$ ($p > n$ if $1 < q \leq n$, or $p = q$ if $q > n$) produces the estimate

$$\|\chi\|_{2,q} \leq c(|\theta|_q + |\chi|_q),$$

where c depends on $q, \Gamma, |\nabla\rho|_p$. We remark that if $p = \infty$

$$|D^2\chi|_q \leq (c + |\nabla\log\rho|_\infty^2)|\theta|_q,\ |\nabla\chi|_q \leq (c + |\nabla\log\rho|_\infty)|\theta|_q.$$

If $q = 2$ the above estimates give (117). For $q \neq 2$, a suitable mean of χ yields (117) (see [10]).

We now estimate π making use of the functional approach. We consider

$$\begin{aligned} (118) \quad & \int_\Omega \pi\theta dx = -\int_\Omega \rho^{-1}\nabla\pi\nabla\chi dx = \int_\Omega \nabla\chi(\partial_t v - \mu\rho^{-1}\Delta v - \rho^{-1}f)dx \\ & \leq \mu m^{-1}|\nabla v|_q|D^2\chi|_{q'} + \mu m^{-2}|\nabla\rho|_\infty|\nabla v|_q|\nabla\chi|_{q'} \\ & \quad + \mu m^{-1}\|\nabla v\|_{L^q(\Gamma)}\|\nabla\chi\|_{L^{q'}(\Gamma)} + m^{-1}|f|_q|\nabla\chi|_{q'}. \end{aligned}$$

The Hölder inequality combined with the Gagliardo-Nirenberg inequality yields

$$\begin{aligned} &\mu m^{-1}|\nabla v|_q|D^2\chi|_{q'} \le cC|\nabla v|_q|\theta|_{q'}, \\ &\|\nabla\chi\|_{L^{q'}(\Gamma)} \le c(|\nabla\chi|_{q'} + |D^2\chi|_{q'}) \le cC|\theta|_{q'}. \end{aligned} \tag{119}$$

We recall that the constant C in (119) depends on M_1. Thanks to (117) we get

$$\begin{aligned} |\pi|_q &\le c(M_1|f|_q + M_1^2|\nabla v|_q + M_1\|\nabla v\|_{L^q(\Gamma)}) \\ &\le c(M_1|f|_q + M_1^2(\epsilon^{-1}|v|_q + \epsilon|D^2 v|_q) + M_1(\eta^{-1}|v|_q + \eta|D^2 v|_q)). \end{aligned} \tag{120}$$

Thanks to the above estimates with ϵ, η small enough inequality (115) gives

$$\begin{aligned} &\|m\partial_t v\|_{L^q(Q_T)} + \mu\|\nabla^2 u\|_{L^q(Q_T)} + \|\nabla\pi\|_{L^q(Q_T)} \\ &\qquad \le c((1+M_1^2)\|f\|_{L^q(Q_T)} + (M_1^6 + M_1^3 + M_1^2)\|v\|_{L^q(Q_T)}). \end{aligned} \tag{121}$$

Step 3. Time dependent density. We remark that, if the density is time-dependent, the estimate (121) continues to hold true in a small interval. Indeed, (v, π) satisfies

$$\begin{cases} \rho(0)\partial_t v - \mu\Delta v + \nabla\pi = f + (\rho(t) - \rho(0))\partial_t v, \\ \nabla\cdot v = 0, \\ v|_{t=0} = 0,\ v|_{\partial\Omega} = 0. \end{cases} \tag{122}$$

Applying the results of steps 1, 2 with density $\rho(0)$, denoting $M_3 = M_1^6 + M_1^3 + M_1^2$, we get

$$\begin{aligned} &\|m\partial_t v\|_{L^q(Q_\tau)} + \mu\|\nabla^2 u\|_{L^q(Q_\tau)} + \|\nabla\pi\|_{L^q(Q_\tau)} \\ &\qquad \le c(M_3\|v\|_{L^q(Q_\tau)} + M_1^2\|f\|_{L^q(Q_\tau)} + M_1^2|\rho(t) - \rho(0)|_\infty\|\partial_t v\|_{L^q(Q_\tau)}). \end{aligned} \tag{123}$$

Using the Hölder continuity of ρ with respect to t and denoting $\tau = \min(T, (cM_1^2M_2)^{-\frac{1}{\beta}})$ we get (121) when $t \in [0, \tau]$,

$$\|m\partial_t v\|_{L^q(Q_\tau)} + \mu\|\nabla^2 u\|_{L^q(Q_\tau)} + \|\nabla\pi\|_{L^q(Q_\tau)} \le c(M_1^2\|f\|_{L^q(Q_\tau)} + M_3\|v\|_{L^q(Q_\tau)}). \tag{124}$$

Estimates on the whole interval $(0, T)$ with $T > \tau$ can be performed making use of a partition of unity (ψ_k), $(k \in N, 1 \le k \le n)$, with respect to t such that:

1) Supp $\phi_0 \subset [0, \tau)$ and $\psi_0 \equiv 1$ in a neighborhood of 0;
2) for $k \ge 1$ Supp $\psi_k \subset [\frac{k}{2}\tau, (\frac{k}{2}+1)\tau]$ and $|\partial_t\psi_k|_\infty \le \frac{c}{\tau}$.

Denoting $v_k \equiv v\psi_k$, $\pi_k \equiv \pi\psi_k$ and $f_k \equiv f\psi_k$, we have

$$\begin{cases} \rho\partial_t v_k - \mu_k\Delta v_k + \nabla\pi_k = f_k - v\rho\partial_t\psi_k, \\ \nabla\cdot v_k = 0, \\ v_k|_{t=0} = 0,\ v_k|_{\partial\Omega} = 0. \end{cases} \tag{125}$$

For $t \ge k\tau/2$, with $\tau^\beta(M_1^2(T)M_2(T)) = \frac{m}{2}$ we have

$$\|\rho v\partial_t\psi_k\|_{L^q(Q_{\frac{k\tau}{2}})} \le cM_1^2(M_1^2M_2)^{\frac{1}{\beta}}\|v\|_{L^q(Q_{\frac{k\tau}{2}})}.$$

Thanks to (124) we get

$$\begin{aligned} &\|m\partial_t v_k\|_{L^q(Q_{\frac{k\tau}{2}})} + \mu\|\nabla^2 u\|_{L^q(Q_{\frac{k\tau}{2}})} + \|\nabla\pi_k\|_{L^q(Q_{\frac{k\tau}{2}})} \\ &\qquad \le c(M_1^2\|f\|_{L^q(Q_{\frac{k\tau}{2}})} + M_3\|v_k\|_{L^q(Q_{\frac{k\tau}{2}})} + cM_1^2\tau^{-1}\|v_k\|_{L^q(Q_{\frac{k\tau}{2}})}). \end{aligned} \tag{126}$$

For $t > \frac{k}{2}\tau$, according to the above estimates, we get

$$
\begin{aligned}
(127)\quad & \|m\partial_t v_k\|_{L^q(Q_t)} + \mu\|\nabla^2 u\|_{L^p(Q_t)} + \|\nabla\pi\|_{L^q(Q_t)} \\
& \le c(M_1^2\|f\|_{L^q(Q_t)} + M_3\|v_k\|_{L^q(Q_t)} + M_1^2|\rho(t)-\rho(t_k)|_\infty\|\partial_t v_k\|_{L^q(Q_t)} \\
& \quad + M_1^2(M_1^2M_2)^{\frac{1}{\beta}}\|v_k\|_{L^q(Q_t)}) \\
& \le c(M_1^2\|f\|_{L^q(Q_t)} + M_3\|v_k\|_{L^q(Q_t)} + M_1^2(M_1^2M_2)^{\frac{1}{\beta}}\|v_k\|_{L^q(Q_t)} + M_1^2M_2\|\partial_t v_k\|_{L^q(Q_t)}).
\end{aligned}
$$

whenever $t \in [\frac{k}{2}\tau, (\frac{k}{2}+1)\tau]$.

Now performing the summation on $k \in \{0, \cdots, K\}$ (with $(K\tau \le T < (K+1)\tau)$ we obtain for (v, π)

$$
\begin{aligned}
(128)\quad & \|m\partial_t v\|_{L^q(Q_T)} + \mu\|\nabla^2 v\|_{L^q(Q_T)} + \|\nabla\pi\|_{L^q(Q_T)} \\
& \qquad \le c(M_1^2\|f\|_{L^q(Q_T)} + M_3\|v\|_{L^q(Q_T)} + c\bar{M}\|v\|_{L^q(Q_T)}).
\end{aligned}
$$

where $\bar{M}(T) = M_1^2(M_2M_1^2)^{\frac{1}{\beta}}$.

To conclude the proof we need to estimate v in $L^p(Q_t)$.

Step 4. Estimate of $|v(t)|_q$. We recall the inequality

$$\partial_t|v|_q \le |\partial_t v|_q.$$

For a solution of (106) we get

$$
\begin{aligned}
(129)\quad |v(t)|_q^q &= q\int_0^t |v(\tau)|_q^{q-1} d_\tau|v(\tau)|_q d\tau \\
&\le \epsilon\int_0^t |v(\tau)|_q^q d\tau + c\epsilon^{-1}\int_0^t |\partial_\tau v|_q^q d\tau \\
&\le (\epsilon + c\epsilon^{-1}(\bar{M}+M_3)^q)\int_0^t |v(\tau)|_q^q d\tau + c\epsilon^{-1}M_1^{2q}\int_0^t |f|_q^q d\tau,
\end{aligned}
$$

and applying Gronwall's lemma we get

$$\|v(t)\|_{L^q(Q_t)} \le cM_1^2(t)e^{h(M_1,M_2)t}\|f\|_{L^q(Q_t)},$$

($h(\cdot)$ is easily traced) and finally we conclude that

$$
(130)\quad \|m\partial_t v\|_{L^q(Q_t)} + \mu\|\nabla^2 v\|_{L^q(Q_t)} + \|\nabla\pi\|_{L^q(Q_t)} \le cM_1^2 e^{h(t)t}\|f\|_{L^q(Q_t)}.
$$

Step 5. General initial data. First, we remark that v satisfying (126) belongs to the space $C(0,T;\mathcal{V}_q)$. We observed at the beginning of this section that the solution v of the momentum equation can be written as a sum of two functions, i.e. $v = u + w$, that satisfy semi-homogeneous problems, precisely

$$
(131)\quad \begin{cases} m\partial_t u - \mu\Delta u + \nabla Q = 0, \quad u(0) = v_0, \\ \rho\partial_t w - \mu\Delta w + \nabla Q_1 = f + (m-\rho)\partial_t u, \\ \nabla\cdot u = \nabla\cdot w = 0, \quad w(0) = 0, \end{cases}
$$

with the homogeneous Dirichlet boundary condition, and $v_0 \in H_q^{2-\frac{2}{q}}(\Omega)$ (divergence free). Collecting the above results we get

$$(132)\qquad \|m\partial_t v\|_{L^q(Q_t)} + \mu\|\nabla^2 v\|_{L^q(Q_t)} + \sup_{0\le\tau\le t}\|v(\tau)\|_{\mathcal{V}_p(\Omega)} \le M_1^2(t)e^{h(t)t}(\|v_0\|_{\mathcal{V}_q} + \|f\|_{L^q(Q_t)}).$$

6. Existence in a small time interval. In this section we prove the L^q-regularity of problem (2) using the procedure of Theorem 2.

We consider the linear problem

$$(133)\qquad \begin{cases} \partial_t\rho - \lambda\Delta\rho = -u\cdot\nabla\rho, \\ \rho\partial_t v - \mu\Delta v + \nabla\pi = -\rho u\cdot\nabla u + \lambda((u\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)u) \\ +\frac{\lambda^2}{\rho}((\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho) + \rho f, \\ \nabla\cdot v = 0,\ v(0) = v_0,\ v = 0 \text{ on } \Gamma,\ \rho(0) = \rho_0,\ \partial_n\rho = 0 \text{ on } \Gamma. \end{cases}$$

Here u belongs to the set

$$\mathcal{B} = \{\phi \in H^{2,1}_{q,0} \,|\, \max(\|\partial_t\phi\|_{L^q(Q_T)}, \|D^2\phi\|_{L^q(Q_T)}) \le R\},$$

where R is an arbitrary positive number and $q > 3$.

Making use of density results in section 4 the existence for (133) for regular data is easily proved. Now the existence and uniqueness of the solution of system (133) enables us to define the map $v = Gu$ given by the composition of $g : u \to \rho$ and $h : (u,\rho) \to v$. The fixed point of G is the solution in L^q spaces of system (2). It is clear that $\mathcal{B}$ is a compact set in $L^q(Q_T)$. As we are going to use a fixed point theorem, we have to show that $G\mathcal{B} \subseteq \mathcal{B}$ and G is continuous in $\mathcal{B}$ with respect to the norm in $L^q(Q_T)$.

Next we prove $G\mathcal{B} \subseteq \mathcal{B}$ for suitable $\bar{T}$.

Assuming $\rho_0 \in (L^r(\Omega), H^2_r(\Omega))_{(\frac{1}{r'},r)}$ with $r > q$ and recalling

$$|u\cdot\nabla\rho|_r \le |u|_\infty|\nabla\rho|_r \le |u|_\infty|\rho|_\infty^{1/2}|D^2\rho|_{r/2}^{1/2} \le c(\Omega, M)|u|_\infty|D^2\rho|_r^{1/2}$$

the maximal L^q theory yields

$$(134)\qquad \lambda\|\partial_t\rho\|_{L^r(Q_T)} + \lambda^2\|\Delta\rho\|_{L^r(Q_T)} \le c\|\rho_0\|_{H_r^{2-\frac{2}{r}}(\Omega)} + T^{\frac{1}{r}}R^2,$$

and from interpolation results we have that

$$(135)\qquad \begin{cases} M_2(\rho) \le \|\rho\|_{H^{2,1}_r} \le c(\|\rho_0\|_{H_r^{2-\frac{2}{r}}(\Omega)} + T^{\frac{1}{r}}R^2), \\ M_1(\rho) \le \|\rho\|_{H^{2,1}_r} \le c(\|\rho_0\|_{H_r^{2-\frac{2}{r}}} + T^{\frac{1}{r}}R^2), \end{cases}$$

with $\beta = 1 - \frac{3+r}{2r}$, $r \ge 5$.

Now applying the results in Step 4 to system (133) we obtain

$$(136)\qquad \|m\partial_t v\|_{L^q(Q_t)} + \mu\|\nabla^2 v\|_{L^q(Q_t)} + \sup_{0\le\tau\le t}\|v(\tau)\|_{\mathcal{V}_p(\Omega)} + \|\nabla\pi\|_{L^q(Q_t)}$$
$$\le cM_1^2(t)e^{h(t)t}\Big(\|v_0\|_{\mathcal{V}_q} + \|f\|_{L^p(Q_t)}$$
$$+ \|u\cdot\nabla u\|_{L^q(Q_t)} + \lambda\|(\nabla\rho\cdot\nabla)u + (u\cdot\nabla)\nabla\rho\|_{L^q(Q_t)}$$
$$+ \lambda^2\Big\|\frac{1}{\rho}\Big((\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho\Big)\Big\|_{L^q(Q_t)}\Big)$$

$$\leq cM_1^2(t)e^{h(t)t}(\|v_0\|_{\mathcal{V}_q} + \|f\|_{L^q(Q_t)} + \|u\|_{L^\infty(Q_t)}\|\nabla u\|_{L^q(Q_t)}$$
$$+ \lambda(\|u\|_{L^\infty(Q_t)}\|D^2\rho\|_{L^q(Q_t)} + \|\nabla u\|_{L^\infty(0,t;L^q(\Omega))}\|\nabla\rho\|_{L^q(0,t;L^\infty(\Omega))})$$
$$+ \lambda^2\|D^2\rho\|_{L^q(Q_t)}\|\nabla\rho\|_{L^\infty(Q_t)})$$
$$\leq c(\|\rho_0\|_{H_r^{2-\frac{2}{r}}(\Omega)} + t^{\frac{1}{r}}R^2)^2e^{tR}(\|v_0\|_{\mathcal{V}_q} + \|\rho_0\|_{H_r^{2-\frac{2}{r}}} + \|f\|_{L^q(Q_t)} + R^2t^{\frac{1}{q}}$$
$$+ Rt^{\frac{r-q}{qr}}(\|\rho_0\|_{H_r^{2-\frac{1}{r}}(\Omega)} + t^{\frac{1}{r}}R^2) + t^{\frac{r-q}{qr}}(\|\rho_0\|_{H_r^{2-\frac{1}{r}}(\Omega)} + t^{\frac{1}{r}}R^2)^2).$$

Assuming $R \geq e^C(\|\rho_0\|_{H_r^{2-\frac{2}{r}}})^2(\|v_0\|_{\mathcal{V}_q} + \|\rho_0\|_{H_r^{2-\frac{2}{r}}}) + \|f\|_{L^q(Q_t)})$ and choosing $t = \bar{T}$ small enough (136) yields

$$G\mathcal{B} \subseteq \mathcal{B}.$$

Since $u \in L^\infty(Q_T)$ and $q > n$, the L^2-continuity of G implies the L^q-continuity of G, then Theorem 2 yields the continuity of G as well as the uniqueness of the solution.

The existence of an L^q-solution of system (2) is proved. Notice that (136) suggests that if the data are small enough the solution exists for any $T > 0$. We omit details.

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