SOLUTIONS TO THE EQUATION $\operatorname{div} u = f$ IN WEIGHTED SOBOLEV SPACES

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Abstract. We consider the problem $\operatorname{div} u = f$ in a bounded Lipschitz domain Ω, where f with $\int_\Omega f = 0$ is given. It is shown that the solution u, constructed as in Bogovski's approach in [1], fulfills estimates in the weighted Sobolev spaces $W_w^{k,q}(\Omega)$, where the weight function w is in the class of Muckenhoupt weights A_q.

1. Introduction and main results. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, $n \geq 2$. We consider a given function f with $\int_\Omega f = 0$ and we are looking for solutions u to the divergence equation

$$(1) \qquad \operatorname{div} u = f \ \text{ in } \ \Omega \ \text{ and } \ u|_{\partial\Omega} = 0.$$

It is an immediate consequence of Green's formula that the condition that f has mean value 0 is necessary for the existence of a solution u to (1).

This problem has been studied by many authors, in particular by Bogovski [1], von Wahl [12], Galdi [8] and Sohr [10]. They prove existence and estimates of a solution u to (1) in the framework of classical L^p- and Sobolev spaces.

We investigate this problem in weighted function spaces. More precisely, we consider weighted Lebesgue spaces $L_w^q(\Omega)$ and Sobolev spaces $W_w^{k,q}(\Omega)$ which means that we integrate with respect to the measure $w\,dx$ for an appropriate weight function w.

All weight functions that we use are contained in the Muckenhoupt class A_q. This is the class of nonnegative and locally integrable weight functions for which the expression

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}}\,dx \right)^{q-1}$$

is finite, where the supremum is taken over all cubes Q in $\mathbf{R}^n$.

As shown by Farwig and Sohr in [4] examples of Muckenhoupt weights are $w(x) = (1+|x|)^\alpha$, with $-n < \alpha < n(q-1)$ or $\text{dist}(x, M)^\alpha$, $-(n-k) < \alpha < (n-k)(q-1)$, where M is a compact k-dimensional Lipschitzian manifold. Thus such weight functions can be used for a better description of the solution close to the boundary, in a neighborhood of a point or for $|x| \to \infty$.

The class of Muckenhoupt weights is appropriate for analysis, since the maximal operator is continuous in weighted L^q-spaces if and only if the weight function is a Muckenhoupt weight. Thus the powerful tools of harmonic analysis may be applied, cf. García-Cuerva and Rubio de Francia [9] and Stein [11].

In this paper we follow Bogovski's approach in [1] and [8] and start with star shaped domains where we give an explicit construction of a solution to (1) in terms of a singular integral operator. Thereafter one uses a decomposition of Lipschitz domains into starshaped domains. The solution is represented by a non-translation-invariant singular integral operator. Before showing its continuity with the help of Proposition 3 below, it is necessary to modify the kernel of this operator in a way which does not affect the solution depending on the compactly supported right hand side f.

We denote by $W_{w,0}^{k,q}(\Omega)$ the closure of all smooth and compactly supported functions $C_0^\infty(\Omega)$ in the norm of the weighted Sobolev space $W_w^{k,q}(\Omega)$ that is introduced in Section 2, below. Then our main result reads as follows.

THEOREM 1. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded and locally lipschitzian domain. Assume $f \in W_{w,0}^{k,q}(\Omega)$ such that $\int f = 0$. Then there exists a function $u \in W_{w,0}^{k+1,q}(\Omega)$ such that*

$$\text{div } u = f \quad and \quad \|u\|_{k+1,q,w} \leq c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k) > 0$ depends A_q-consistently on w. Moreover, u can be chosen such that it depends linearly on f and such that $u \in C_0^\infty(\Omega)$ if $f \in C_0^\infty(\Omega)$.

For the definition and the relevance of the A_q-consistence of constants see Definition 1 below and the discussion below it.

The proof of Theorem 1 is given in Section 3. The result is first proven for strictly starshaped domains in Theorem 4 and then generalized to general Lipschitz domains with the help of Lemma 5.

2. Weighted function spaces. In this section we collect the basic definitions of weight functions and function spaces, which are needed in this text. Moreover, we quote the main theorem about the boundedness of maximal operator.

As in [9,11] we use the following notation.

DEFINITION 1. Let A_q, $1 < q < \infty$, the set of *Muckenhoupt weights*, consist of all $0 \leq w \in L_{loc}^1(\mathbf{R}^n)$ for which

$$(2) \qquad A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} \, dx \right)^{q-1} < \infty.$$

The supremum is taken over all cubes Q in $\mathbf{R}^n$. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

A constant $C = C(w)$ is called A_q-consistent if for every $c_0 > 0$ it can be chosen uniformly for all $w \in A_q$ with $A_q(w) < c_0$.

The A_q-consistence is of great importance since it is needed for the application of the Extrapolation Theorem [9, IV Lemma 5.18]. In particular this is used when showing the continuity of operator-valued Fourier multipliers and the maximal regularity of an operator; see e.g. Fröhlich [7] for details and applications.

We introduce some function spaces. For $1 < q < \infty$, $w \in A_q$ and an open set Ω we define the weighted Lebesgue and Sobolev spaces as follows.

- The weighted Lebesgue spaces $L_w^q(\Omega)$ are given by

$$L_w^q(\Omega) := \left\{ f \in L_{loc}^1(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w \, dx \right)^{\frac{1}{q}} < \infty \right\}.$$

- Assume in addition $k \in \mathbf{N}_0$, the set of nonnegative integers. The weighted Sobolev spaces are defined by

$$W_w^{k,q}(\Omega) = \left\{ u \in L_w^q(\Omega) \mid \|u\|_{k,q,w} := \sum_{|\alpha| \le k} \|D^\alpha u\|_{q,w} < \infty \right\}.$$

- Finally, we set

$$W_{w,0}^{k,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,q,w}}.$$

By [5], [7] and [3] the spaces $L_w^q(\Omega)$, $W_w^{k,q}(\Omega)$ and $W_{w,0}^{k,q}(\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$ or $C_0^\infty(\Omega)$, respectively, are dense.

For a locally integrable function f we define the maximal operator M by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(0)|} \int_{|y| \le r} |f(x-y)| dy.$$

One has the following close connection between the Muckenhoupt class A_q and the maximal operator.

PROPOSITION 2. *Let $1 < q < \infty$ and $w \in A_q$. Then the maximal operator M is continuous on $L_w^q(\mathbf{R}^n)$. More precisely, there exists an A_q-consistent constant c such that*

$$\|Mf\|_{q,w} \le c\|f\|_{q,w} \quad \text{for every} \ \ f \in L_w^q(\mathbf{R}^n).$$

Vice versa if μ is a nonnegative Borel measure and M is bounded on $L^q(\mathbf{R}^n, \mu)$, then μ is absolutely continuous and $d\mu = w \, dx$ for some $w \in A_q$.

Proof. See [9], Theorems 2.1 and 2.9. For the A_q-consistence of the constants one has to re-read the proof of [9], Theorem 2.9. The reverse inclusion can be found in [11, 2.2]. ∎

By [6] the following weighted analogue of the Poincaré inequality holds: there exists an A_q-consistent constant $c = c(q, w) > 0$ such that

(3) $$\|u\|_{q,w} \le c\|\nabla u\|_{q,w} \ \text{ for every } \ u \in W_{w,0}^{1,q}(\Omega).$$

3. Proof of Theorem 1. Throughout this section let $1 < q < \infty$ and $w \in A_q$.

The proof follows the same lines as the unweighted case [1], [8, Chapter III.3]. It uses non-translation-invariant singular integral operators. Thus we apply the following

theorem proved in [11, V.6.13] which ensures the continuity of a certain class of such operators.

PROPOSITION 3. *Let T be a bounded operator from $L^2(\mathbf{R}^n)$ into itself that is associated to a kernel K in the sense that*

$$(Tf)(x) = \int_{\mathbf{R}^n} K(x,y)f(y)dy$$

for all compactly supported $f \in L^2(\mathbf{R}^n)$ and all x outside the support of f. Suppose that for some $\gamma > 0$ and some $A > 0$ the kernel K satisfies the inequalities

(4) $$|K(x,y)| \le A|x-y|^{-n}$$

and

(5) $$|K(x,y) - K(x',y)| \le A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \quad if \ |x-x'| \le \frac{1}{2}|x-y|$$

as well as the symmetric version of the second inequality in which the roles of x and y are interchanged. Writing

$$(T_\varepsilon f)(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)dy \quad and \quad (T_* f)(x) = \sup_{\varepsilon>0}|(T_\varepsilon f)(x)|,$$

we have that

(6) $$\int [(T_* f)(x)]^q w(x)dx \le c\int [(Mf)(x)]^q w(x)dx,$$

where f is bounded and has compact support, $w \in A_q$, and $1 < q < \infty$. The constant c depends A_q-consistently on w.

Proof. This result is stated in [11, V.6.13]. The A_q-consistence of the constants is not explicitly mentioned there, however, it is established with the same arguments if one rereads the proof of Proposition 6 in [11, V.4.4]. ∎

Since the maximal operator $M : L_w^q(\mathbf{R}^n) \to L_w^q(\mathbf{R}^n)$ is bounded, the inequality (6) guarantees that the sublinear operator T_* can be extended to a continuous sublinear operator $T_* : L_w^q(\Omega) \to L_w^q(\Omega)$.

However, to make use of the above proposition we have to modify the singular integral operator which appears in the proof of Theorem 4 below outside the bounded set Ω such that it possesses the properties assumed in Proposition 3.

In the proof of the following Theorem the occurring integral operators have to be understood in the Cauchy principle value sense $\lim_{\varepsilon \to 0} T_\varepsilon f$.

THEOREM 4. *Let $\Omega \subset \mathbf{R}^n$, $n \ge 2$, be bounded and star-shaped with respect to every point of some closed ball $\overline{B}$ with $\overline{B} \subset \Omega$. Then for every $f \in W_{w,0}^{k,q}(\Omega)$ with $\int_\Omega f = 0$ there exists a $v \in W_{w,0}^{k+1,q}(\Omega)$ with*

$$\operatorname{div} v = f \quad and \quad \|v\|_{k+1,q,w} \le c\|f\|_{k,q,w},$$

where $c = c(\Omega, q, w, k) > 0$ depends A_q-consistently on w. The function v depends linearly on f and $f \in C_0^\infty(\Omega)$ implies $v \in C_0^\infty(\Omega)$.

Proof. Without loss of generality we may assume, using a coordinate transformation, that $B = B_1(0)$. First we assume that $f \in C_0^\infty(\Omega)$. We choose $a \in C_0^\infty(B_1(0))$ such that

$\int a = 1$ and define

(7)
$$v(x) := \int_\Omega f(y)(x-y) \left(\int_1^\infty a\left(y + \xi(x-y)\right) \xi^{n-1} d\xi \right) dy.$$

In the proof of [8, Lemma III.3.1] it is shown that $v \in C_0^\infty(\Omega)$ and div $v = f$.

It thus remains to prove the weighted estimates. To do this we use the following representation of $\partial_j v$ also shown in the proof of [Lemma III.3.1]:

(8)
$$\partial_j v_i(x) = \int_\Omega K_{i,j}(x, x-y)f(y)dy + f(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} a(y)dy$$
$$=: F_1(x) + F_2(x),$$

where

(9)
$$K_{i,j}(x, x-y) = \frac{\delta_{i,j}}{|x-y|^n} \int_0^\infty a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^{n-1} dr$$
$$+ \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty \partial_j a\left(x + r\frac{x-y}{|x-y|}\right) (|x-y|+r)^n dr,$$

for every $x, y \in \mathbf{R}^n$. To show the continuity of the integral operator $f \mapsto F_1$ its kernel must be modified. Set

$$E := \{z \in \Omega \mid z = \lambda z_1 + (1-\lambda)z_2, \ z_1 \in \text{supp } f, \ z_2 \in \overline{B_1(0)}, \ \lambda \in [0,1]\}.$$

Since Ω is star-shaped with respect to $\overline{B_1(0)}$, E is a compact subset of Ω. For $x \notin E$ and $y \in \text{supp } f$ we have

$$x + r\frac{x-y}{|x-y|} \notin \overline{B_1(0)} \quad \text{for all } r > 0,$$

which means $K_{i,j}(x, x-y) = 0$. Thus, if we choose a cut-off function $\psi \in C_0^\infty(\mathbf{R}^n)$ with $\psi(x) = 1$ on Ω and $\text{supp } \psi \subset B_R(0)$ for some $R > 0$, and set $\varphi(x, y) = \psi(x)\psi(y)$ we obtain

$$f(y)K_{i,j}(x, x-y) = f(y)\varphi(x, y)K_{i,j}(x, x-y) =: f(y)\tilde{K}_{i,j}(x, x-y),$$

for $x, y \in \mathbf{R}^n$, if f is assumed to be extended by 0 to $\mathbf{R}^n$. Moreover, for $x \in B_R(0)$ we have

$$r > R + 1 \implies \left| x + r\frac{x-y}{|x-y|} \right| \geq r - |x| > 1 \implies a\left(x + r\frac{x-y}{|x-y|}\right) = 0.$$

Thus for $x \in \Omega$ one has

$$\int_{\mathbf{R}^n} f(y)K_{i,j}(x, x-y)dy = \int_{\mathbf{R}^n} f(y)\tilde{K}_{i,j}(x, x-y)dy$$
$$= \int_{\mathbf{R}^n} f(y)\varphi(x, y)\left[\frac{\delta_{i,j}}{|x-y|^n} \int_0^{R+1} a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^{n-1}dr \right.$$
$$\left. + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^{R+1} \partial_j a\left(x + r\frac{x-y}{|x-y|}\right)(|x-y|+r)^n dr \right] dy.$$

Now we have to prove that $\tilde{K}_{i,j}$ satisfies the assumptions of Proposition 3. By the Calderón-Zygmund Theorem [2] we find that

$$f \mapsto \int_{\mathbf{R}^n} \psi(x)K_{i,j}(x, x-y)f(y)dy : \quad L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

is continuous. Since the multiplication M_ψ with the C_0^∞-function ψ is a continuous operator on $L^2(\mathbf{R}^n)$ we obtain the continuity of

$$
f \mapsto \int_{\mathbf{R}^n} \tilde{K}_{i,j}(x, x-y) f(y) dy
$$
$$
= \int_{\mathbf{R}^n} \psi(x) K_{i,j}(x, x-y) M_\psi f(y) dy : \ L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n).
$$

It remains to prove the estimates (4) and (5). For (4) we may assume $|x|, |y| < R$. One has

$$
|x-y|^n |\tilde{K}_{i,j}(x, x-y)|
$$
$$
= \left| \varphi(x,y) \delta_{i,j} \int_0^{R+1} a\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^{n-1} dr \right.
$$
$$
\left. + \varphi(x,y) \frac{x_i - y_i}{|x-y|} \int_0^{R+1} \partial_j a\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^n dr \right|
$$
$$
\le c\left(\int_0^{R+1} (2R + r)^{n-1} dr + \int_0^{R+1} (2R + r)^n dr \right) = c'.
$$

To prove (5) we take $x, x', y \in \mathbf{R}^n$ with $|x - x'| \le \frac{1}{2}|x - y|$. If $(x,y), (x',y) \notin \operatorname{supp}\varphi$ nothing is to prove. Thus, without loss of generality we may assume that $|y| \le R$ and $|x| \le 3R$, since if $|y| \le R$ and $|x| \ge 3R$ then

$$
|x'| \ge |x| - |x - x'| \ge |x| - \frac{1}{2}(|x| + |y|) \ge \frac{3}{2}R - \frac{1}{2}R = R.
$$

Then using the triangle inequality together with the fact that a, φ and $(|x-y|+r)^n$ are Lipschitz continuous on compact sets we can estimate

$$
\left| \frac{x_i - y_i}{|x-y|^{n+1}} \varphi(x,y) \int_0^{R+1} \partial_j a\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^n dr \right.
$$
$$
\left. - \frac{x_i' - y_i}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \partial_j a\left(x' + r \frac{x'-y}{|x'-y|} \right) (|x'-y| + r)^n dr \right|
$$
$$
\le \left| \left(\frac{x_i - y_i}{|x-y|^{n+1}} - \frac{x_i' - y_i}{|x'-y|^{n+1}} \right) \varphi(x,y) \int_0^{R+1} \partial_j a\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^n dr \right|
$$
$$
+ \left| \frac{x_i' - y_i}{|x'-y|^{n+1}} (\varphi(x,y) - \varphi(x',y)) \int_0^{R+1} \partial_j a\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^n dr \right|
$$
$$
+ \frac{|x_i' - y_i|}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \left| \partial_j a\left(x + r \frac{x-y}{|x-y|} \right) - \partial_j a\left(x' + r \frac{x'-y}{|x'-y|} \right) \right|
$$
$$
(|x-y| + r)^n dr
$$
$$
+ \frac{|x_i' - y_i|}{|x'-y|^{n+1}} \varphi(x',y) \int_0^{R+1} \left| \partial_j a\left(x' + r \frac{x'-y}{|x'-y|} \right) \right| |(|x-y| + r)^n - (|x'-y| + r)^n| dr
$$
$$
= I_1 + I_2 + I_3 + I_4.
$$

Using the Lipschitz continuity of $\partial_i a$ and $|x - y| \le 4R$ we obtain

$$
I_3 \le \frac{c}{|x-y|^n} \int_0^{R+1} L\left(|x - x'| + \frac{|x - x'|}{|x - y|} \right) (|x-y| + r)^n dr \le c \frac{|x - x'|}{|x-y|^{n+1}}.
$$

I_2 and I_4 can be estimated analogously. For I_1 we estimate

$$
\begin{aligned}
\left| \frac{x_i - y_i}{|x-y|^{n+1}} - \frac{x_i' - y_i}{|x'-y|^{n+1}} \right| &\leq \frac{|x_i - x_i'|}{|x-y|^{n+1}} + \left| \frac{1}{|x-y|^{n+1}} - \frac{1}{|x'-y|^{n+1}} \right| |x_i' - y_i| \\
&\leq \frac{|x-x'|}{|x-y|^{n+1}} + \left| \frac{|x'-y|^{n+1} - |x-y|^{n+1}}{|x-y|^{n+1}|x'-y|^{n+1}} \right| |x_i' - y_i| \\
&\leq \frac{|x-x'|}{|x-y|^{n+1}} + c\frac{\left| |x'-y| - |x-y| \right| \cdot |x-y|^n}{|x-y|^{2n+2}} |x_i' - y_i| \\
&\leq c\frac{|x-x'|}{|x-y|^{n+1}},
\end{aligned}
$$

where we used that $|x' - y| \geq \frac{1}{2}|x-y|$. The estimate

$$
\left| |x'-y|^{n+1} - |x-y|^{n+1} \right| \leq c \left| |x'-y| - |x-y| \right| \cdot |x-y|^n
$$

follows from an elementary induction with respect to n.

The first summand in (9) can treated in the same way. Moreover, interchanging the roles of x and y the same kind of estimates can be done.

Combining the above and using Proposition 3 we obtain

$$
\|F_1\|_{q,w} \leq \|T^* f\|_{q,w} \leq c\|Mf\|_{q,w} \leq c\|f\|_{q,w},
$$

where T^* is the operator given by Proposition 3 and associated to the kernel $\tilde{K}_{i,j}$. The function F_2 that appears in (8) is easily estimated since

$$
\int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} a(y) dy
$$

is bounded. Thus using the Poincaré inequality (3) we obtain $\|v\|_{1,q,w} \leq c\|f\|_{q,w}$.

Now the general case with $f \in L_w^q(\Omega)$ follows easily, since we can approximate f by C_0^∞-functions (f_n) with $\int f_n = 0$.

It remains to prove the estimate in the spaces $W_w^{k,q}(\Omega)$. Using Leibniz' formula one can show (see [8, Remark III.3.2])

$$
\partial^\alpha v(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_\Omega N_\beta(x,y) \partial^{\alpha-\beta} f(y) dy,
$$

where

$$
N_\beta(x,y) = (x-y) \int_1^\infty \partial^\beta a(y + r(x-y)) r^{n-1} dr.
$$

Clearly $\partial^\beta a \in C_0^\infty(B_1(0))$. Hence the same proof as above yields

$$
\|\partial^\alpha v\|_{1,q,w} \leq c\|f\|_{k,q,w}
$$

for $f \in C_0^\infty(\Omega)$ and every α with $|\alpha| \leq k$. As before one can approximate an arbitrary $f \in W_{w,0}^{k,q}(\Omega)$ with $\int f = 0$ by C_0^∞-functions (f_n) with $\int f_n = 0$. This argument finishes the proof. ∎

The following Lemma is the weighted analogue to [8, Lemma III.3.4.]. Its proof works in exactly the same way as in the case of unweighted function spaces.

LEMMA 5. *Let Ω be a bounded and locally lipschitzian domain.*

1. *There exist open sets $\Omega_1, \ldots, \Omega_m$ with $\Omega = \bigcup_{i=1}^{m} \Omega_i$ such that each Ω_i is star-shaped with respect to an open ball B_i with $\overline{B}_i \subset \Omega_i$.*

2. *For every $f \in C_0^\infty(\Omega)$ with $\int_\Omega f = 0$ there exist $f_i \in C_0^\infty(\Omega_i)$, $i = 1, \ldots, m$, with $f = \sum_{i=1}^{m} f_i$, $\int f_i = 0$ and $\|f_i\|_{k,q,w} \leq c\|f\|_{k,q,w}$ for every $k \in \mathbf{N}_0$ and $q \geq 1$ and an A_q-consistent constant $c = c(k, q, w, \Omega)$.*

Now we finish the proof of Theorem 1.

Proof. Let $f \in C_0^\infty(\Omega)$ with $\int f = 0$ and take Ω_i, f_i, $i = 1, \ldots, m$, as in Lemma 5. We denote by v_i the solution to div $v_i = f_i$ given by Theorem 4. Then we have

$$\|v_i\|_{k+1,q,w} \leq c\|f_i\|_{k,q,w} \leq c\|f\|_{k,q,w}.$$

Then $v = \sum_{i=1}^{m} v_i$ solves div $v = f$ with $\|v\|_{k+1,q,w} \leq c\|f\|_{k,q,w}$. For arbitrary $f \in W_{w,0}^{k,q}(\Omega)$ with $\int f = 0$ use again approximations with C_0^∞-functions. ∎

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