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STATIONARY STATES AND MOVING PLANES

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Abstract. Most of the paper deals with the application of the moving plane method to different questions concerning stationary accumulations of isentropic gases. The first part compares the concepts of stationarity arising from the points of view of dynamics and the calculus of variations. Then certain stationary solutions are shown to be unstable. Finally, using the moving plane method, a short proof of the existence of energy-minimizing gas balls is given.

1. Introduction. In an ideal gas with constant specific heat and constant entropy density, a so-called isentropic gas, the pressure p of the gas is a function of the mass density ρ alone, which is given by the formula

$$p = K\rho^{\varkappa}$$

with a number $\varkappa > 1$ and a positive constant K depending on the entropy density. As pointed out in [2], the kineti theory of gases implies that $\varkappa$ is determined by the structure of the molecules, e.g. for monatomic gases $\varkappa = 5/3$, for diatomic gases $\varkappa = 7/5$, and $\varkappa$ converges to one as the size of the molecules goes to infinity.

There is a number of different systems of equations one can use to model the motion of a quantity of such a gas occupying all or part of the space $\mathbb{R}^3$ and subject to the gravity field it generates. One of them is that for inviscid flow considered in [2]. If one confines oneself to equilibrium states without motion, then the differences between most of these models disappear, and one is left with only the equations

$$\nabla p = -\rho\nabla\Phi \tag{1}$$

and

$$\Delta\Phi = 4\pi k\rho \tag{2}$$

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together with the condition $\Phi(x) \to 0$ as $|x| \to \infty$ for the gravity potential Φ and the gravity constant k. Both equations may need to be interpreted in the sense of distributions. This provides one possibility for the definition of a stationary state for such a system.

Another poses the problem in the framework of the calculus of variations, as a critical, or stationary, point of the energy, subject to all relevant conservation laws, as defined more precisely in Definition 1. The energy of the mass distribution ρ in in the absence of motion is given by

$$E(\rho) = \frac{K}{\varkappa - 1} \int_{\mathbb{R}^3} \rho^{\varkappa}(x)dx - \frac{k}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dxdy. \tag{3}$$

The first integral is the thermodynamic energy, the second one the gravitational energy. The only conservation law we have to keep in mind here is that of the total mass of the distribution ρ, given by

$$\mathcal{M}(\rho) = \int_{\mathbb{R}^3} \rho(x)dx. \tag{4}$$

From the point of view of physics one would expect these two concepts of stationarity to be equivalent, as indeed they are. This is, however, not immediately obvious, owing, as we shall see, to the fact that for such solutions the density ρ typically equals zero over much of space.

Such problems have been studied extensively. In most cases it is not only the motionless equilibrium that is being considered, but also the equilibria in which the mass distribution rotates like a rigid body around an axis, a significantly harder problem. Celebrated classical studies of this subject, in part also for incompressible fluids, include those of Poincaré [10] and Lichtenstein [7]. Among the more recent papers are [1], [6], [12], [8], [11], [2] and [9]. The reader is referred to these for additional references. In most of these papers the point of view is that of the calculus of variations.

Stationarity in the sense that equations (1) and (2) are fulfilled leads to the equations

$$K\varkappa\rho^{\varkappa-1}\nabla\rho = -\rho\nabla\Phi,$$

and where $\rho > 0$ this implies

$$-\nabla\Phi = K\varkappa\rho^{\varkappa-2}\nabla\rho = \nabla\left(\frac{K\varkappa}{\varkappa-1}\rho^{\varkappa-1}\right)$$

and

$$\nabla\left(\Phi + \frac{K\varkappa}{\varkappa-1}\rho^{\varkappa-1}\right) = 0.$$

Thus in every connected component of the set

$$\Omega = \{x \in \mathbb{R}^3 \mid \rho(x) > 0\} \tag{5}$$

we have

$$\Phi + \frac{K\varkappa}{\varkappa-1}\rho^{\varkappa-1} = C. \tag{6}$$

The constant C may assume different values in different subdomains of Ω. If, on the other hand, we look for stationary points of the energy functional, then the constant enters the calculation as a Lagrange parameter originating from the conservation of mass as a constraint, and therefore is the same throughout Ω, as stated in Theorem 7. If there is only one constant, we can solve this equation (6) for ρ inside Ω, substitute ρ in (2) to obtain a kind of Lane-Emden equation, and then it is not particularly hard to prove by applying the result of [3] to Φ that ρ is spherically symmetric and Ω is a sphere. In Theorem 4, which is probably the most interesting result of the paper, we prove that is the case, and that Ω is therefore connected, even if we permit these constants to be different in different components of Ω. For this purpose we have to modify the moving plane argument from [3],[4] somewhat. Among other considerations we also give a proof of the existence of a minimizing element for E with a method using the moving plane argument as well. Although we will be forced to enlarge that space somewhat for some of our analyses, we will largely consider density functions ρ belonging to $C_0^0(\mathbb{R}^3)$, the space of continuous functions with compact support. By equation (2) then Φ must immediately be a continuously differentiable function, whose Laplacian in the sense of distributions is continuous. Let $C^k(\mathbb{R}^3)$ be the space of k times continuously differentiable functions. We summarize the results of this paper in points 1 and 2 below.

1. All solutions $\rho \in C_0^0(\mathbb{R}^3)$ and $\Phi \in C^1(\mathbb{R}^3)$ of (1) and (2) with $\rho(x) \geq 0$ for $x \in \mathbb{R}^3$ are spherically symmetric with respect to some point $x_0 \in \mathbb{R}^3$.
2. For $\varkappa \in (1, 6/5)$ there are no solutions of positive finite mass, while for $\varkappa \in (6/5, \infty)$ such solutions exist, and for given mass they are unique up to translation. For $\varkappa < 4/3$ they are unstable as critical points of the energy functional defined in (3), for $\varkappa > 4/3$ they are global minima of this energy.

For more precise statements of the results see Theorems 4, 6, 7, 8 and 12.

Most of these results are not new. For $\varkappa > 4/3$ the existence of a function minimizing E with respect to all functions with a certain axisymmetry is a by-product of [1], it is proved without such restrictions, using spherically symmetric rearrangements, in [6], and by means of concentration-compactness arguments, in [8]. Here we give a different proof using the moving plane method. The existence and nonexistence of solutions claimed above is derived from the results in [2] for $\varkappa < 4/3$ and can, owing to the spherical symmetry of the solutions, also be obtained from numerous studies of the spherically symmetric Lane-Emden equation. (See references in [11].) In contrast to this, to the author's knowledge, the characterizations of stationary solutions in Theorem 4 and the instability result in Theorem 8 are new. In most results in this paper the moving plane method plays an important role, and typically some twist of the original argument in [3] and [4] is necessary. Theorem 4 may seem to contradict Theorem 1.3 in [2], but this is not the case as for the solutions obtained there the gravity potential is confined to the bounded domain being considered, and, as shown here, can in most cases not be extended to the entire space, as one might be inclined to require for a stationary solution in the physical sense.

2. Stationary solutions. We begin with solutions stationary in the sense that they fulfill equations (1) and (2). In the argument to prove the main result of this section, Theorem 4, the reflections T^λ across the planes $x_1 = \lambda$ given by the formula

$$T^\lambda(x_1, x_2, x_3) = (2\lambda - x_1, x_2, x_3) \tag{7}$$

play an important part. In preparation for the main argument we prove three lemmas relating to some of the asymptotic expansions of the Newton potential, which are collectively known in physics as multipole expansions. The results of these lemmas are by no means new. We initially consider arbitrary mass distributions with compact support with their center of gravity at zero. This means that for Lemmas 1, 2 and 3 we assume that $\rho \in L^1(\mathbb{R}^3)$, $\rho(x) \geq 0$ in $\mathbb{R}^3$ while $\rho(x) = 0$ for $|x| \geq r > 0$, ρ is not almost everywhere zero, and

$$\int_{\mathbb{R}^3} x_k \rho(x) dx = 0 \ (k = 1, 2, 3). \tag{8}$$

We also define

$$V(x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy. \tag{9}$$

Lemma 1. *We assume $\Gamma : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}, \Gamma \in C^2(\mathbb{R}^3 \backslash \{0\})$ is homogeneous of degree $k \in (-\infty, 2)$. Then there is a constant C such that for $|x| \geq 2r$ we have*

$$\left| \int_{\mathbb{R}^3} \Gamma(x-y)\rho(y)dy - \Gamma(x) \int_{\mathbb{R}^3} \rho(y)dy \right| \leq Cr^2 |x|^{k-2} \int_{\mathbb{R}^3} \rho(y)dy.$$

Proof. For $|x| \leq 2r$ and $|y| \leq r$ by Taylor's theorem

$$\Gamma(x-y) - \Gamma(x) = -\nabla\Gamma(x) \cdot y + \frac{1}{2} \sum_{i,j=1}^{3} \Gamma_{x_i x_j}(x - \sigma y) y_i y_j$$

for some $\sigma \in [0, 1]$. Multiplying by ρ, integrating the resulting equation over $\mathbb{R}^3$ and keeping assumption (8) in mind, we obtain

$$\left| \int_{\mathbb{R}^3} (\Gamma(x-y) - \Gamma(x))\rho(y)dy \right| \leq \frac{1}{2} \int_{\mathbb{R}^3} \left| \sum_{i,j=1}^{3} \Gamma_{x_i x_j}(x - \sigma y) y_i y_j \right| \rho(y)dy.$$

Now, as $\Gamma_{x_i x_j}$ is homogeneous of degree $k-2$, we have for $|y| \leq r < 2r \leq |x|$ that

$$\begin{aligned} \left| \sum_{i,j=1}^{3} \Gamma_{x_i x_j}(x-\sigma y) y_i y_j \right| &\leq \Big(\sum_{i,j=1}^{3} (\Gamma_{x_i x_j}(x - \sigma y))^2 \Big)^{1/2} \Big(\sum_{i,j=1}^{3} (y_i y_j)^2 \Big)^{1/2} \\ &= |x - \sigma y|^{k-2} \left(\sum_{i,j=1}^{3} \left(\Gamma_{x_i x_j} \left(\frac{x - \sigma y}{|x - \sigma y|} \right) \right)^2 \right)^{1/2} |y|^2 \\ &\leq (|x| - |y|)^{k-2} r^2 \max_{|\xi|=1} \sqrt{\sum_{i,j=1}^{3} (\Gamma_{x_i x_j}(\xi))^2} \leq C \left(|x| - \frac{|x|}{2} \right)^{k-2} r^2 \leq C|x|^{k-2} r^2. \end{aligned}$$

This proves our claim.

For the following two lemmas note that if $\lambda < 0, x_1 > \lambda$ and $|x| \geq 8|\lambda|$ then

$$|T^\lambda(x)| - |x| = \frac{|T^\lambda(x)|^2 - |x|^2}{|T^\lambda(x)| + |x|} = \frac{4\lambda(\lambda - x_1)}{|T^\lambda(x)| + |x|} > 0, \tag{10}$$

thus

$$|x| \leq |T^\lambda(x)| \leq |T^\lambda(x) - x| + |x| \leq 2|\lambda| + |x| \leq \frac{5}{4}|x|. \tag{11}$$

In Lemmas 2 and 3 and their proofs the constant C denotes the maximum of the two constants C from Lemma 1 for $\Gamma(x) = |x|^{-1}$ and $\Gamma(x) = x_1|x|^{-3}$.

LEMMA 2. *If* $\lambda < 0, r > 0$ *and* $x_1 \geq D_1 = \max(8|\lambda|, 3Cr^2|\lambda|^{-1}, 2r)$, *then* $V(T^\lambda(x)) < V(x)$.

Proof. As $D_1 \geq 8|\lambda|$ we have $|x| \geq 8|\lambda|$ and (11) is valid and can be applied below. To be able to use Lemma 1 we consider

$$\begin{aligned}|T^\lambda(x)|^{-1} - |x|^{-1} &= \frac{|x| - |T^\lambda(x)|}{|T^\lambda(x)||x|} = \frac{4\lambda(x_1 - \lambda)}{|T^\lambda(x)||x|(|x| + |T^\lambda(x)|)} \\ &\leq -2\left(\frac{4}{5}\right)^2 \frac{|\lambda|(x_1 - \lambda)}{|x|^3} \leq -|\lambda| \frac{x_1}{|x|^3}.\end{aligned}$$

Thus, using $\Gamma(x) = |x|^{-1}$ in Lemma 1, we have

$$\begin{aligned}V(T^\lambda(x)) - V(x) &\leq \left[(|T^\lambda(x)|^{-1} - |x|^{-1}) + \frac{2Cr^2}{|x|^3}\right] \int_{\mathbb{R}^3} \rho(y)dy \\ &\leq \frac{2}{|x|^3}\left[-\frac{1}{2}|\lambda|x_1 + Cr^2\right] \int_{\mathbb{R}^3} \rho(y)dy,\end{aligned}$$

giving us our claim.

LEMMA 3. *Given* $\lambda < 0, r > 0$, *the constant* $D = \max(D_1, 2Cr^2|\lambda|^{-1} + 6D_1|\lambda|)$ *has the property that if* $|x| \geq D, x_1 > \lambda$, *then* $V(T^\lambda(x)) < V(x)$.

Proof. By Lemma 2 the conclusion is true if $x_1 \geq D_1$. As we chose $D \geq D_1$, we therefore may now assume $x_1 \in (\lambda, D_1)$ and $|x| \geq 2r$. Let $f(x) = V(T^\lambda(x)) - V(x)$. Then $f_{x_1}(x) = -V_{x_1}(T^\lambda(x)) - V_{x_1}(x)$, and

$$V_{x_1}(x) = -\int_{\mathbb{R}^3} \frac{x_1 - y_1}{|x - y|^3}\rho(y)dy$$

therefore, as $x_1/|x|^3$ is homogeneous of degree -2 we have, using Lemma 1,

$$f_{x_1}(x) \leq \left[\frac{(T^\lambda(x))_1}{|T^\lambda(x)|^3} + \frac{x_1}{|x|^3} + \frac{Cr^2}{|T^\lambda(x)|^4} + \frac{Cr^2}{|x|^4}\right] \int_{\mathbb{R}^3} \rho(y)dy.$$

Now, remembering (10) and (11), we have

$$\begin{aligned}&\frac{(T^\lambda(x))_1}{|T^\lambda(x)|^3} + \frac{x_1}{|x|^3} = \frac{x_1 + (T^\lambda(x))_1}{|T^\lambda(x)|^3} + \frac{x_1}{|x|^3} - \frac{x_1}{|T^\lambda(x)|^3} \\ &= \frac{-2|\lambda|}{|T^\lambda(x)|^3} + x_1\left[\frac{1}{|x|^3} - \frac{1}{|T^\lambda(x)|^3}\right] < -\frac{|\lambda|}{|x|^3} + 3|x_1|\frac{|T^\lambda(x)| - |x|}{|x|^4} \\ &\leq -\frac{|\lambda|}{|x|^3} + 6|x_1|\frac{|\lambda|}{|x|^4} \leq \frac{1}{|x|^3}\left(-|\lambda| + \frac{6D_1|\lambda|}{|x|}\right)\end{aligned}$$

and

$$f_{x_1}(x) < \frac{1}{|x|^3}\left[-|\lambda| + \frac{2Cr^2 + 6D_1|\lambda|}{|x|}\right] \int_{\mathbb{R}^3} \rho(y)dy.$$

Thus $f_{x_1}(x) < 0$ if $|x| \geq 2Cr^2|\lambda|^{-1} + 6D_1|\lambda|$ and $x_1 \in (\lambda, D_1)$. For $x_1 = \lambda$ we have $T^\lambda(x) = x$ and therefore $f(x) = V(T^\lambda(x)) - V(x) = 0$ for such x. This immediately implies the missing part of our claim.

Now we can prove the main theorem of this section.

THEOREM 4. *Assume* $\rho \in C_0^0(\mathbb{R}^3)$ *is a non-negative function,* $\Phi \in C^1(\mathbb{R}^3)$ *and that* $\Phi(x) \to 0$ *as* $x \to \infty$*. Assume they also solve equations (1) and (2) in the sense of distributions. Then* $\Phi \in C^2(\mathbb{R}^3)$ *and* $p = K\rho^\varkappa \in C^1(\mathbb{R}^3)$*, equations (1) and (2) are fulfilled in the classical sense,* $\Omega = B_R(x_0)$ *with some* $R > 0$*,* $x_0 \in \mathbb{R}^3$*, and* ρ, Φ *are spherically symmetric with respect to* x_0*.*

Proof. By equation (2) obviously Φ can be obtained from the convolution of $4\pi k\rho \in C_0^0(\mathbb{R}^3)$ with the Newton potential and therefore both the function and its gradient are uniformly bounded, and it is strictly negative on $\mathbb{R}^3$. Now let us define

$$G(x) = \Phi(x) + \frac{K\varkappa}{\varkappa - 1}\rho^{\varkappa-1}(x). \tag{12}$$

The function G is uniformly continuous on $\mathbb{R}^3$, it is continuously differentiable in Ω and $\nabla G(x) = 0$ for $x \in \Omega$. Thus ρ is differentiable where it is positive. As ρ is continuous we have $\rho(x) = 0$ on $\partial\Omega$ and therefore $G(x) < 0$ for $x \in \partial\Omega$ and even for $x \in \overline{\Omega}$ as $\nabla G(x) = 0$ in Ω. Solving equation (12) for $\rho^{\varkappa-1}$ we obtain

$$\rho^{\varkappa-1}(x) = \frac{\varkappa-1}{K\varkappa}(G(x) - \Phi(x)) = \frac{\varkappa-1}{K\varkappa}(G(x) - \Phi(x))^+$$

for $x \in \mathbb{R}^3$. From this it is easy to see that $\rho^{\varkappa-1}$ is Lipschitz continuous, as Φ has this property and ρ equals zero where G is not constant. Therefore ρ is Hölder continuous and $\Phi \in C^2(\mathbb{R}^3)$ by Lemma 4.2 in [5]. As $\rho^{\varkappa-1}$ is Lipschitz continuous we obtain that $\rho^\varkappa = (\rho^{\varkappa-1})^{\frac{\varkappa}{\varkappa-1}}$ is differentiable, with gradient zero where $\rho(x) = 0$, and then it is easy to see that $p \in C^1(\mathbb{R}^3)$. Also we have

$$\rho(x) = \left(\frac{\varkappa-1}{K\varkappa}(G(x) - \Phi(x))^+\right)^{1/(\varkappa-1)}.$$

As Ω is an open set, it consists of a countable number of connected components $\Omega_m, m \in \mathbb{P} = \{q \in \mathbb{N}\,|q \le N\}$, where N could be finite or infinite. In each Ω_m the function $G(x)$ is constant. Let $C_m = G(x)$ for $x \in \Omega_m$. Now let $u = -\Phi$ and

$$g(s) = 4\pi k\left(\frac{\varkappa-1}{K\varkappa}s^+\right)^{1/(\varkappa-1)}.$$

Then g is an increasing and non-negative continuous function, $g(s) = 0$ for $s \le 0$, and we have

$$-\Delta u = g(G(x) + u(x)).$$

After a translation we may assume that the center of gravity of the entire mass distribution is located at $x = 0$, i.e.

$$\int_{\mathbb{R}^3} x_n\rho(x)dx = 0$$

for $n = 1, 2, 3$. We follow the same strategy as Gidas, Ni, Nirenberg in [3],[4]. Remembering the reflection T^λ defined in equation (7) we introduce $u^\lambda(x) = u(T^\lambda(x))$. Now let

$$S_+^\lambda = \{x \in \mathbb{R}^3 \mid x_1 > \lambda\}, \qquad S_-^\lambda = \{x \in \mathbb{R}^3 \mid x_1 < \lambda\}$$

and

$$A = \{\lambda \le 0 \mid \text{If } \mu \le \lambda, \text{ then } (u - u^{\mu}) \mid \overline{S}^{\mu}_{+} \ge 0 \text{ and } T^{\mu}(\overline{\Omega \cap S^{\mu}_{-}}) \subset \overline{\Omega}\},$$

and define $q_0 = \min_{x \in \overline{\Omega}}(x_1)$. Then for $\mu < q_0$ the set $\overline{\Omega \cap S^{\mu}_{-}}$ is empty. Therefore also $\Delta u^{\mu} = 0$ and $\Delta u \le 0$ in S^{μ}_{+}. As $u^{\mu}(x) = u(x)$ for $x \in \partial S^{\mu}_{+}$ and both functions go to zero as $|x| \to \infty$, the maximum principle implies $(u - u^{\mu}) \mid \overline{S}^{\mu}_{+} \ge 0$ for such μ. Therefore $(-\infty, q_0) \subset A$.

Letting $\lambda_1 = \sup(A)$, we have by the definition of A that $(-\infty, \lambda_1) \subset A \subset (-\infty, \lambda_1]$. We will show that $\lambda_1 \in A$.

If $\lambda_0 < \lambda_1$ and $x \in \overline{S}^{\lambda_0}_{+}$ then for all $\lambda \in [\lambda_0, \lambda_1), x \in \overline{S}^{\lambda}_{+}$ we have $u^{\lambda}(x) \le u(x)$, and therefore this is true also for $\lambda = \lambda_1$, as $u^{\lambda}(x)$ depends continuously on λ and $\overline{S}^{\lambda}_{+}$ is a decreasing set-valued function of λ.

If $x \in \overline{\Omega \cap S^{\lambda_0}_{-}}$ for some $\lambda_0 < \lambda_1$, then $x \in \overline{\Omega \cap S^{\lambda}_{-}}$ for $\lambda \ge \lambda_0$ and by assumption also $T^{\lambda}(x) \in \overline{\Omega}$ for $\lambda \in [\lambda_0, \lambda_1)$. As $\overline{\Omega}$ is closed, this is also true for $\lambda = \lambda_1$. If $x \in \overline{\Omega \cap S^{\lambda_1}_{-}}$ and $x \notin \overline{\Omega \cap S^{\lambda}_{-}}$ for all $\lambda < \lambda_1$, then $x_1 = \lambda_1$ and therefore $T^{\lambda_1}(x) = x \in \overline{\Omega}$. Thus we have shown that $\lambda_1 \in A$ and that $A = (-\infty, \lambda_1]$.

If $\lambda \in A$, then we have $T^{\lambda}(\Omega \cap S^{\lambda}_{-}) \subset \overline{\Omega}$, and as T^{λ} is a homeomorphism, it takes interior points to interior points, and therefore $T^{\lambda}(\Omega \cap S^{\lambda}_{-}) \subset \Omega$. Now we want to prove that for $\lambda \le \lambda_1$ we have $T^{\lambda}(\Omega_m \cap S^{\lambda}_{-}) \subset \Omega_m$ for all $m \in \mathbb{P}$. To see this let $x \in \Omega_m$ and $\lambda_0 = x_1$. Then $T^{\lambda_0}(x) = x$ and $x \in S^{\lambda}_{-}$ for $\lambda \in (\lambda_0, \infty)$, and for $\lambda \in [\lambda_0, \lambda_1]$ we have $T^{\lambda}(x) \in \Omega$ as $x \in \Omega$. As $T^{\lambda_0}(x) = x \in \Omega_m$ and $T^{\lambda}(x)$ for $\lambda \in [\lambda_0, \lambda_1]$ is a line segment wholly inside Ω, it must also lie in the same connected component of Ω as x, thus $T^{\lambda}(x) \in \Omega_m$.

Therefore $G(T^{\lambda}(x)) = G(x) = C_m$ for all $x \in \Omega_m \cap S^{\lambda}_{-}$ and as $\Omega \cap S^{\lambda}_{-}$ is the union of these sets we have $G(T^{\lambda}(x)) = G(x)$ for $x \in \Omega \cap S^{\lambda}_{-}$. By continuity we have

$$G(T^{\lambda}(x)) = G(x) \tag{13}$$

even for $x \in \overline{\Omega \cap S^{\lambda}_{-}}$ for all $\lambda \le \lambda_1$.

Now we prove, by contradiction, that $\lambda_1 = 0$, and therefore assume $\lambda_1 < 0$ from now on. Let us consider the function $u - u^{\lambda_1}$ restricted to the set $S^{\lambda_1}_{+}$. There we have

$$\begin{aligned} -\Delta(u(x) - u^{\lambda_1}(x)) &= -\Delta u(x) + \Delta(u \circ T^{\lambda_1})(x) \\ &= g(G(x) + u(x)) - g(G(T^{\lambda_1}(x)) + u(T^{\lambda_1}(x))) \ge 0. \end{aligned}$$

To prove the inequality note that as $x \in S^{\lambda_1}_{+}$, we have $T^{\lambda_1}(x) \in S^{\lambda_1}_{-}$. If $T^{\lambda_1}(x) \notin \Omega$, then $g(G(T^{\lambda_1}(x)) + u(T^{\lambda_1}(x))) = 0$, and then the inequality is clear. On the other hand, if $T^{\lambda_1}(x) \in \Omega$, then also $x = T^{\lambda_1}(T^{\lambda_1}(x)) \in \Omega$, and therefore $G(T^{\lambda_1}(x)) = G(T^{\lambda_1}(T^{\lambda_1}(x))) = G(x)$, and then

$$\begin{aligned} &g(G(x) + u(x)) - g(G(T^{\lambda_1}(x)) + u^{\lambda_1}(x)) \\ &= g(G(x) + u(x)) - g(G(x) + u^{\lambda_1}(x)) \ge 0, \end{aligned}$$

as $u^{\lambda_1}(x) \le u(x)$ and g is increasing. As also $u(x) = u^{\lambda_1}(x)$ for $x \in \partial S^{\lambda_1}_{+}$, by the strong maximum principle (Theorem 3.5 in [5]) now either $u(x) < u^{\lambda_1}(x)$ for $x \in S^{\lambda_1}_{+}$ or $u(x) = u^{\lambda_1}(x)$ for $x \in S^{\lambda_1}_{+}$. As the center of gravity is at zero, the second is an impossibility. The strong maximum principle (Lemma 3.4 in [5]) also implies $u_{x_1}((\lambda_1, x_2, x_3)) >$

$u^{\lambda_1}_{x_1}((\lambda_1, x_2, x_3))$ for all $x_2, x_3 \in \mathbb{R}$. Also obviously $u_{x_1}((\lambda_1, x_2, x_3)) = -u^{\lambda_1}_{x_1}((\lambda_1, x_2, x_3))$ and therefore $u_{x_1}((\lambda_1, x_2, x_3)) > 0$. It is easy to see that then for $R > 0$ there is a $\delta > 0$ such that for λ' with $|\lambda_1 - \lambda'| < \delta$ and $x \in B_R(0) \cap S^{\lambda'}_+$ we have $u^{\lambda'}(x) \le u(x)$ and even $u^{\lambda'}(x) < u(x)$ unless $x \in S^{\lambda'}_+$. As we assumed $\lambda_1 < 0$ the same inequality follows for large x by Lemma 3. Now assume that for every $\delta > 0$ there exists a λ' with $|\lambda_1 - \lambda'| < \delta$ such that $T^{\lambda'}(\overline{\Omega \cap S^{\lambda'}_-}) \not\subseteq \overline{\Omega}$. Then there would have to be a sequence $\mu_k > \lambda_1$ converging to λ_1 and a sequence of points $y_k \in \overline{\Omega \cap S^{\mu_k}_-}$ with $T^{\mu_k}(y_k) = x_k \notin \overline{\Omega}$ for all k. These sequences can be selected in such a way that $x_k \to \overline{x}$ and $y_k \to \overline{y}$, and even so that $y_k \in \Omega \cap S^{\mu_k}_-$. Then $\overline{x} = T^{\lambda_1}(\overline{y})$ and $\overline{x} \notin \Omega$, while $\overline{y} \in \overline{\Omega \cap S^{\lambda_1}_-}$. As $\lambda_1 \in A$ then also $\overline{x} \in \overline{\Omega}$ and therefore $\overline{x} \in \partial\Omega$ and $\rho(\overline{x}) = 0$ and, using (13),

$$\begin{aligned} u(\overline{x}) &= -G(\overline{x}) = -G(T^{\lambda_1}(\overline{y})) = -G(\overline{y}) \\ &= u(\overline{y}) - \frac{K\varkappa}{\varkappa - 1}\rho^{\varkappa-1}(\overline{y}) \le u(\overline{y}) = u^{\lambda_1}(\overline{x}) \le u(\overline{x}). \end{aligned}$$

Therefore $u(\overline{x}) = u^{\lambda_1}(\overline{x})$ and as a consequence $\overline{x} \in \partial S^{\lambda_1}$, which implies $u_{x_1}(\overline{x}) > 0$ as shown above. Also $T^{\lambda_1}(\overline{x}) = \overline{x}$, therefore $\overline{x} = \overline{y}$. Therefore also $u_{x_1}(y_k) > 0$ for sufficiently large k. Dropping all k for which this is not true and as $y_k \in \Omega$ there exists a $\tau > 0$ such that $y_k + \sigma e_1 \in \Omega$ for all $\sigma \in [0, \tau]$ with $e_1 = (1, 0, 0)$. Now

$$T^{\mu_k}(y_k) = 2(\mu_k - y_{k1})e_1 + y_k,$$

and as $y_{k1} \to \lambda_1$ as well as $\mu_k \to \lambda_1$ this implies $T^{\mu_k}(y_k) \in \Omega$ for large k contrary to our assumption. Thus $A = (-\infty, 0]$, and we have $u^0(x) \ge u(x)$ for $x_1 \ge 0$, which is only possible if they are equal in view of the assumption we made about the location of the center of gravity. As we can rotate our coordinate system in any way we wish, this means that u is symmetric with respect to any plane through the center of gravity, thus it is spherically symmetric. Then it is easy to see the remainder of our claim.

3. Critical points of the energy functional. Now we complement the energy functional E defined in equation (3) by a set

$$A(M) = \{\rho \in C^0_0(\mathbb{R}^3) \mid \rho(x) \ge 0, \mathcal{M}(\rho) = M\}$$

of admissible density distributions of a given mass M with $\mathcal{M}$ as defined in (4). First we consider for which $\varkappa$ the functional E is bounded from below on $A(M)$. Then we show that all elements ρ of $A(M)$ with the gravity potentials Φ they generate are stationary solution of equations (1) and (2) exactly if they are stationary points of the functional in the sense defined in Definition 1, and we investigate their stability in the sense of the calculus of variations.

First we prove a lemma about rescaled density distributions.

LEMMA 5. *Let* $\varkappa \in (1, 4/3]$ *and for* $\rho \in A(M), t > 0$ *let*

$$\rho_1(x, t) = t^3\rho(tx)$$

and

$$e_\rho(t) = E(\rho_1(., t)).$$

Then $\rho_1(.,t) \in A(M)$ *as well and* $e_\rho \in C^\infty((0,\infty))$, $e_\rho''(t) \le 0$ *and if* $\varkappa < 4/3$ *then* $e_\rho''(t) < 0$ *and* $e_\rho(t) \to -\infty$ *as* $t \to \infty$.

Proof. It is easy to see that $\mathcal{M}(\rho_1(.,t)) = M$, therefore $\rho_1(.,t) \in A(M)$ and

$$e_\rho(t) = E(\rho_1)(t) = t^{3(\varkappa-1)}\frac{K}{\varkappa-1}\int_{\mathbb{R}^3}\rho^\varkappa(x)dx - t\frac{k}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}dxdy$$

for $t \in (0,\infty)$. From this equation it is easy to see the remainder of our claims.

THEOREM 6. *Let*

$$\mathcal{E}(M) = \inf_{\rho\in A(M)} E(\rho).$$

If $\varkappa \in (1, 4/3)$ *then* $\mathcal{E}(M) = -\infty$, *for* $\varkappa > 4/3$ *we have* $\mathcal{E}(M) > -\infty$, *for* $\varkappa = 4/3$ *there exists a constant* $M_0 > 0$ *such that for* $M > M_0$ *we have* $\mathcal{E}(M) = -\infty$, *while for* $M < M_0$ *this quantity is finite.*

Proof. By Proposition 6 in [1] we have

$$E(\rho) \ge \|\rho\|_{L^\varkappa}^\varkappa - C\|\rho\|_{L^{4/3}}^{4/3}M^{2/3}, \tag{14}$$

which is bounded from below for $\varkappa > 4/3$ and for $\varkappa = 4/3$ if M is small enough. For $\varkappa = 4/3$ it is easy to see that $\mathcal{E}(M) = -\infty$ implies the same for all larger masses. For $\varkappa < 4/3$ the fact $\mathcal{E}(M) = -\infty$ follows directly from Lemma 5.

Now we define what we mean by a stationary point of the functional E.

DEFINITION 1. We call $\rho_0 \in A(M)$ a *stationary point* of E if the following is true. Assume $v \in C_0^0(\mathbb{R}^n)$ and there is a number $\tau_0 > 0$ such that for $t \in [0, \tau_0]$ we also have $\rho_0 + tv \in A(M)$. Then there exists a continuous function $\gamma : [0,\tau_0] \to \mathbb{R}$ with $\gamma(0) = 0$ such that for $t \in [0,\tau_0]$ we have

$$E(\rho_0 + tv) \ge E(\rho_0) + t\gamma(t).$$

Then we have the following theorem.

THEOREM 7. *The function* $\rho_0 \in A(M)$ *is a stationary point of* E *exactly if*

$$\frac{K\varkappa}{\varkappa-1}\rho_0^{\varkappa-1}(x) = \left(k\int_{\mathbb{R}^3}\frac{\rho_0(y)}{|x-y|}dy + C\right)^+ \tag{15}$$

with a constant C. *Then*

$$\Phi_0(x) = -k\int_{\mathbb{R}^3}\frac{\rho_0(y)}{|x-y|}dy \tag{16}$$

and ρ_0 *solve equations (1) and (2). If functions* $\Phi_0 \in C^1(\mathbb{R}^3)$ *and* $\rho_0 \in C_0^0(\mathbb{R}^3)$ *fulfill the equations (1) and (2) and* $\Phi_0(x)$ *goes to zero at infinity, then* ρ_0 *fulfills equation (15).*

Proof. The considerations leading up to Theorem A in [1] are independent of the axial symmetry ordinarily assumed there. The fact that we are only considering continuous functions with compact support, and that our functional is less complicated, makes the arguments much easier. They lead to a proof that if ρ_0 is a stationary point of E, then it solves equation (15) and vice versa. Equation (2) then immediately follows from (16) and equation (6) from (15). Equation (15) can also be proved along the same lines as Lemma 10 in this paper. For the implication in the other direction note that if ρ_0, Φ_0

solve equations (1) and (2) then by Theorem 4 the set Ω is a ball, therefore (6) is true with a single constant, and from that it is easy to derive (15).

THEOREM 8. *Let $\varkappa < 4/3$. Then if ρ_0 is a stationary point for the energy functional E in $A(M)$, it is unstable in the sense that there are functions ρ in $A(M)$ arbitrarily close to ρ_0 in $C^0(\mathbb{R}^n)$ while $E(\rho) < E(\rho_0)$.*

Proof. From Theorem 7 we have

$$\rho_0(x) = \left(\left(\frac{k(\varkappa-1)}{K\varkappa}\int_{\mathbb{R}^3}\frac{\rho_0(y)}{|x-y|}dy + C\right)^+\right)^{1/(\varkappa-1)}.$$

As we assume $\varkappa < 4/3$, the function $f(x) = (x^+)^{1/(\varkappa-1)}$ belongs to $C^3(\mathbb{R})$, therefore $\rho_0 \in C^3(\mathbb{R}^3)$. As a consequence $\rho_1(.,t)$ is three time differentiable as a function of t. (For the definition of ρ_1 and e_ρ see Lemma 5.) Also

$$\frac{\partial\rho_1}{\partial t}(x,1) = 3\rho_0(x) + x\cdot\nabla\rho_0 = \operatorname{div}(x\rho_0(x)),$$

and therefore

$$\begin{aligned} e'_\rho(1) &= \frac{\varkappa K}{\varkappa-1}\int_{\mathbb{R}^3}\rho_0^{\varkappa-1}(x)\operatorname*{div}_x(x\rho_0(x))dx - k\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\operatorname{div}_x(x\rho_0(x))\rho_0(y)}{|x-y|}dxdy \\ &= \int_{\mathbb{R}^3}\left[\frac{\varkappa K}{\varkappa-1}\rho_0^{\varkappa-1}(x) - k\int_{\mathbb{R}^3}\frac{\rho_0(y)}{|x-y|}dy\right]\operatorname{div}(x\rho_0(x))dx = C\int_{\mathbb{R}^3}\operatorname{div}(x\rho_0(x))dx = 0. \end{aligned}$$

As $e''_\rho(t) < 0$ by Lemma 5, e_ρ has a global strict maximum at $t=1$, proving our claim.

4. The existence of an energy-minimizing density for $\varkappa > 4/3$. The space $C_0^0(\mathbb{R}^3)$ for mass-distributions is not very suitable for the purpose of proving the existence of energy-minimizing distributions. For this reason we consider the larger space $L^\varkappa(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$ and define as the set of admissible mass distributions

$$W(M) = \left\{\rho \in L^\varkappa(\mathbb{R}^3)\cap L^1(\mathbb{R}^3) \mid \rho(x)\ge 0 \text{ a.e. and } \int_{\mathbb{R}^3}\rho(x)dx = M\right\},$$

which is the closure of $A(M)$ in $L^\varkappa(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$. Here $L^\varkappa(\mathbb{R}^3)$ is the space of all measurable functions whose $\varkappa$-th power is integrable over $\mathbb{R}^3$. As the spaces on $\mathbb{R}^3$ lack desirable compactness properties we also define

$$W(M,R) = \{\rho \in W(M) \mid \rho(x) = 0 \text{ for } |x|\ge R\}$$

for $R\in(0,\infty]$. Again, from Proposition 6 in [1] we have that inequality (14) is true for $\rho\in W(M)$, therefore E is bounded from below for $\varkappa > 4/3$ even for this enlarged set of admissible mass distributions. Also any subset of $W(M)$ on which E is bounded is also bounded in $L^\varkappa$.

For $R\in(0,\infty)$ we define

$$\mathcal{E}(M,R) = \inf_{\rho\in W(M,R)} E(\rho).$$

Then obviously $\mathcal{E}(M,R)\ge\mathcal{E}(M)$ for all $R\in(0,\infty)$, and $\mathcal{E}(M,R)\to\mathcal{E}(M)$ as $R\to\infty$. Using the same arguments as in [1] we obtain the following two lemmas.

LEMMA 9. *For $R \in (0,\infty)$ there exists a function $\rho_R \in W(M,R)$ such that*

$$E(\rho_R) = \mathcal{E}(M,R).$$

LEMMA 10. *Let $R \le \infty$. If $\rho_0 \in W(M,R)$ is stationary in $W(M,R)$, then $\rho_0 \in L^\infty(\mathbb{R}^3)$ and there exists a constant C such that ρ_0 fulfills equation (15) for $x \in B_R$.*

Proof. We can again follow the arguments in [1] to prove this, but it is also easy to just give a shorter proof as our problem is less complicated. It is not difficult to see that if $2|\varphi| \le \rho_0$ and

$$\int_{\mathbb{R}^3} \varphi dx = 0$$

then

$$h(t) = E(\rho_0 + t\varphi)$$

is a continuously differentiable function in $[-1,1]$ and $h'(0) = 0$ as $\rho_0 + t\varphi \in W(M,R)$. Now for $|\phi| \le 1/2$ equaling zero outside B_R let

$$\varphi = \frac{1}{2}\Big(\phi - \frac{1}{M}\int_{\mathbb{R}^3} \phi\rho_0 dx\Big)\rho_0.$$

Then this φ fulfills the conditions above, and computing $h'(0)$ one obtains with

$$V(x) = \frac{K\varkappa}{\varkappa-1}\rho_0^{\varkappa-1}(x) - k\int_{\mathbb{R}^3} \frac{\rho_0(y)}{|x-y|}dy$$

that

$$h'(0) = \int_{\mathbb{R}^3} V(x)\Big(\phi(x) - \frac{1}{M}\int_{\mathbb{R}^3} \phi\rho_0 dy\Big)\rho_0(x)dx = 0$$

and therefore

$$\int_{\mathbb{R}^3} V(x)\phi(x)\rho_0(x)dx = \int_{\mathbb{R}^3} \phi\rho_0 dy \frac{1}{M}\int_{\mathbb{R}^3} V(x)\rho_0(x)dx,$$

or, with

$$C = \frac{1}{M}\int_{\mathbb{R}^3} V(x)\rho_0(x)dx$$

we have

$$\int_{\mathbb{R}^3} (V(x) - C)\phi(x)\rho_0(x)dx = 0,$$

implying our claim for all x for which $\rho_0(x) > 0$. A bootstrap argument using the Calderón-Zygmund estimates (see, e.g., [5], Theorem 9.9) and the Sobolev embedding theorem then implies ρ_0 is bounded. To obtain the remainder of the claim let $\phi \in L^\infty(\mathbb{R}^3)$ with $\phi(x) \in [0,1]$ and $\phi(x) = 0$ for $|x| \ge R$. Then for sufficiently small $t \ge 0$ the function φ given by

$$\varphi = \phi - M^{-1}\rho_0\int_{\mathbb{R}^3} \phi(x)dx \ge -M^{-1}\rho_0\int_{\mathbb{R}^3} \phi(x)dx$$

has the property that $\rho_0 + t\varphi \in W(M,R)$ and therefore $h'(0) \ge 0$. From this it is easy to obtain

$$\frac{K\varkappa}{\varkappa-1}\rho_0^{\varkappa-1}(x) \ge k\int_{\mathbb{R}^3} \frac{\rho_0(y)}{|x-y|}dy + C$$

for all $x \in B_R$. Where one of the two sides is bigger than zero they are equal. This proves the remainder of our claim.

LEMMA 11. *If $R > 0$ and $\rho_0 \in W(M,R)$ is a bounded function with the property that $E(\rho_0) = \mathcal{E}(M,R)$, then it is spherically symmetric, and the center of the spheres of symmetry is located in B_R.*

Proof. By Lemma 10 the function ρ_0 fulfills equation (15) and is bounded. We can rotate and translate the ball B_R in such a way that the center of gravity of the mass distribution is located at 0 and the center of the ball at $y = (y_1, 0, 0)$ with $y_1 \geq 0$. Again for $\lambda \in \mathbb{R}$ let T^λ be defined as in (7), with

$$u(x) = k \int_{\mathbb{R}^3} \frac{\rho_0(y)}{|x-y|} dy$$

let $u^\lambda(x) = u(T^\lambda(x))$ and

$$A = \{\lambda \leq 0 \mid \text{For all } \mu \leq \lambda \text{ and } x \in S_+^\mu \text{ we have } u^\mu(x) \leq u(x)\},$$

remembering the definitions of S_+^λ and S_-^λ from the proof of Theorem 4. Owing to the fact that u is a bounded function fulfilling the integral equation

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\chi_{B_R(y)}(z) g(u(z))}{|x-z|} dz,$$

with

$$g(u) = 4\pi k \left(\frac{\varkappa - 1}{K\varkappa} (C+u)^+ \right)^{1/(\varkappa - 1)},$$

we see that the distributional Laplacian of u belongs to L^∞. As before, A is easily seen to be closed and non-empty. If $\lambda \in A$ and $\lambda < 0$ we have for $x \in S_+^\lambda$ that, as $T^\lambda(B_R(y) \cap S_-^\lambda) \subset B_R(y)$ for $\lambda \leq y_1$,

$$-\Delta(u(x) - u^\lambda(x)) = \chi_{B_R(y)} g(u) - \chi_{T^\lambda(B_R(y))} g(u^\lambda) \geq [\chi_{B_R(y)} - \chi_{T^\lambda(B_R(y))}] g(u)(x) \geq 0,$$

thus

$$\Delta(u(x) - u^\lambda(x)) \leq 0.$$

By the strong maximum principle now either $u(x) < u^\lambda(x)$ for $x \in S_+^\lambda$ or $u(x) = u^\lambda(x)$ in the same set. The strong maximum principle can be applied as the Laplacian is in the weak sense and u, u^λ both are continuously differentiable, which easily allows to verify the mean value inequality for $u - u^\lambda$. As the center of gravity is at zero, the assumption $\lambda < 0$ excludes the possibility $u(x) = u^\lambda(x)$ for $x \in S_+^\lambda$. The strong maximum principle also implies $u_{x_1}((\lambda, x_2, x_3)) > u_{x_1}^\lambda((\lambda, x_2, x_3))$ for all $x_2, x_3 \in \mathbb{R}$. It is then easy to see that for $R_1 > 0$ there is a $\delta > 0$ such that for λ' with $|\lambda - \lambda'| < \delta$ and $x \in B_{R_1}(0) \cap S_+^\lambda$ we have $u^\lambda(x) \leq u(x)$. As $\lambda < 0$ the same inequality follows for large x by Lemma 3. Thus we have $u^0(x) \geq u(x)$ for $x_1 \geq 0$, which is only possible if they are equal in view of the assumption we made about the location of the center of gravity. If $y_1 > 0$ we can translate the function so that the plane of symmetry is located at y_1. Then ρ_0 is still a minimizing function and is zero near the boundary of $B_R(y)$ and the arguments presented so far apply to it as well. Therefore C must be non-positive, as otherwise ρ_0 could not be zero there, and as $u(x) \leq -C$ on $\partial B_R(y)$ this inequality is also true outside this ball by the maximum principle, and thus u solves $-\Delta u = g(u)$ in the entire space and is therefore symmetric by [3]. Otherwise $y_1 = 0$, which means the center of gravity coincides with the center of the circle, in which case we can carry out the same argument for planes with arbitrary normals and obtain spherical symmetry.

THEOREM 12. *Let $M > 0$, $\varkappa > 4/3$. Then there exists a function $\rho_0 \in A(M)$ which is spherically symmetric with respect to zero such that $E(\rho_0) = \mathcal{E}(M)$. It is Hölder continuous and, together with the function Φ_0 defined by the conditions $\Phi_0(x) \to 0$ as $|x| \to \infty$ and $\Delta\Phi_0 = 4\pi k\rho_0$, fulfills equations (1) and (2). Up to translation it is the only stationary point of the energy functional and ρ_0 and Φ_0 are the only stationary solution of these equations amongst functions for which ρ_0 has compact support.*

Proof. For $R > 0$ let ρ_R be the function minimizing E on $W(M,R)$, which exists by Lemma 9. By Lemma 11 it is spherically symmetric and can be chosen so that the center of the symmetry is at 0. By Lemma 14 in [11] it then follows that for R sufficiently large ρ_R is independent of R and therefore is a minimizer in $A(M)$. Our construction immediately assures it has all the properties claimed. Its uniqueness follows from Theorem 4 combined with Lemma 14 in [11].

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