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A DIRECT PROOF OF THE CAFFARELLI-KOHN-NIRENBERG THEOREM

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Abstract. In the present paper we give a new proof of the Caffarelli-Kohn-Nirenberg theorem based on a direct approach. Given a pair $(\mathbf{u}, p)$ of suitable weak solutions to the Navier-Stokes equations in $\mathbf{R}^3\times]0,\infty[$ the velocity field $\mathbf{u}$ satisfies the following property of partial regularity: The velocity $\mathbf{u}$ is Lipschitz continuous in a neighbourhood of a point $(x_0, t_0) \in \Omega\times]0,\infty[$ if

$$\limsup_{R\to 0^+} \frac{1}{R} \int_{Q_R(x_0,t_0)} \left|\operatorname{curl} \mathbf{u} \times \frac{\mathbf{u}}{|\mathbf{u}|}\right|^2 \mathrm{d}x\, \mathrm{d}t \le \varepsilon_\star$$

for a sufficiently small $\varepsilon_\star > 0$.

1. Introduction. The aim of the present paper is the study of the local regularity of weak solutions to the Navier-Stokes equations in $\mathbf{R}^3$,

$$\text{(N-S)} \qquad \begin{cases} \operatorname{div} \mathbf{u} = 0, \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} = -\nabla p \quad \text{in} \quad \mathbf{R}^3\times]0,\infty[, \\ \lim\limits_{|x|\to\infty} \mathbf{u} = 0, \\ \mathbf{u} = \mathbf{a} \quad \text{on} \quad \mathbf{R}^3\times\{0\}, \end{cases}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and p resp. denotes the unknown velocity and pressure resp., while $\mathbf{a} = (a_1, a_2, a_3)$ denotes the given initial velocity at the initial time $t = 0$.

The system (N-S) has been introduced first by Navier [10] and later rederived by Stokes [17]. The first mathematical treatment goes back to Leray [7], where he studied the existence of weak solutions to (N-S) for the case $\Omega = \mathbf{R}^3$ (for the notion of a weak solutions see below). Later Hopf [5] proved the existence of weak solutions in a general domain by using a Galerkin approximation. For more details and further approaches we refer to the monographs of Temam [20] and Sohr [15].

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However despite many efforts until now one is unable to construct a classical solution to the Navier-Stokes equations for an arbitrarily given smooth initial velocity, which has been considered as one of the seven Millennium problems introduced by the Clay institute. Concerning the issue of regularity the best result which is known is the so-called Caffarelli-Kohn-Nirenberg theorem (cf. [2]), which states that the velocity field $\mathbf{u}$ of a suitable weak solution to the Navier-Stokes equations is bounded on a neighbourhood of $(x_0, t_0) \in Q$, where $Q := \mathbf{R}^3 \times]0, \infty[$, if

$$\limsup_{R \to 0^+} \frac{1}{R} \int_{Q_R(x_0,t_0)} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon_0 \tag{1.1}$$

for a sufficiently small $\varepsilon_0 > 0$. This shows that the singular set Σ of all points $(x_0, t_0) \in Q$, where (1.1) fails is closed such that

$$\mathcal{P}_1(\Sigma) = 0. \tag{1.2}$$

In fact, the first result of partial regularity is due to Scheffer. In his pioneering paper [11] he has introduced the notion of a suitable weak solution to (N-S) that is a pair $(\mathbf{u}, p)$ with

$$\mathbf{u} \in L^\infty(0,\infty; L^2(\mathbf{R}^3)^3) \cap L^2(0,\infty; \overset{\circ}{W}{}^{1,2}_\sigma(\mathbf{R}^3))\,^1, \qquad p \in L^{3/2}(Q)$$

satisfying $(\text{N-S})_2$ in sense of distribution, such that the local energy inequality

$$\begin{aligned} &\int_{\mathbf{R}^3} |\mathbf{u}(t)|^2 \phi \,\mathrm{d}x + 2\int_0^t \int_{\mathbf{R}^3} |\nabla \mathbf{u}|^2 \phi \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \leq \int_0^t \int_{\mathbf{R}^3} |\mathbf{u}|^2 \left\{ \frac{\partial \phi}{\partial t} + \Delta \phi \right\} \mathrm{d}x\,\mathrm{d}t + \int_0^t \int_{\mathbf{R}^3} (|\mathbf{u}|^2 + 2p)\mathbf{u} \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{1.3}$$

holds for all $\phi \in C_0^\infty(Q)$, for a.e. $t \in]0, \infty[$. For such suitable weak solutions the author defined a set $\widetilde{\Sigma}$ of all possible singular points by a condition different than (1.1), proving that

$$\mathcal{H}_{5/3}(\widetilde{\Sigma}) = 0 \tag{1.4}$$

(for details see [11], [12]).

Later Lin [9] reproved the Caffarelli-Kohn-Nirenberg theorem by using pressure estimates obtained by Sohr and von Wahl [16]. Afterwards, Ladyzhenskaya and Seregin [6] carried out a more detailed proof of the partial regularity of a suitable weak solution to the Navier-Stokes equations including also the case when a force $\mathbf{f}$ is added to the right of $(\text{N-S})_2$. Recently, Vasseur [21] established a new proof of the Caffarelli-Kohn-Nirenberg theorem using a direct approach based on a Moser iteration.

In contrast to the results mentioned above the partial regularity of weak solutions to the Navier-Stokes equations in a general domain Ω is not known. This is due to the fact that there is no sufficient method to treat the pressure locally. However this problem one can overcome by the method introduced in [22]. In order to avoid technical complications we restrict ourselves to the Navier-Stokes equation in the full space. The general case will be treated in a forthcoming paper.

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[1] For the definition of $\overset{\circ}{W}{}^{1,2}_\sigma(\mathbf{R}^3)$ see below.

Weak solutions. Let $\Omega \subset \mathbf{R}^3$ be a Lipschitz domain. First, let us introduce the function spaces and some notations which will be used in what follows. By $W^{m,q}(\Omega)$, $W_0^{m,q}(\Omega)$ $(m = 1, 2, \dots; 1 \le q < \infty)$ we denote the usual Sobolev spaces. By $\mathcal{D}_\sigma(\Omega)$ we denote the vector space of all $\varphi \in C_0^\infty(\Omega)^3$ with $\operatorname{div} \varphi = 0$. Then define

$$L^2_\sigma(\Omega) := \text{closure of } \mathcal{D}_\sigma(\Omega) \text{ in } L^2(\Omega)^3,$$
$$\mathring{W}^{1,2}_\sigma(\Omega) := \text{closure of } \mathcal{D}_\sigma(\Omega) \text{ in } W^{1,2}(\Omega)^3.$$

In case $\Omega = \mathbf{R}^3$ we write $L^q, W^{m,q}$, etc. in place of $L^q(\mathbf{R}^3), W^{m,q}(\mathbf{R}^3)$ etc.

Given a normed vector space X with norm $\|\cdot\|$, we denote by $L^s(a, b; X)$ $(1 \le s \le \infty)$ $(-\infty \le a < b \le \infty)$ the vector space of all Bochner measurable functions $z :]a, b[\to X$ such that

$$\int_a^b \|z(t)\|^s \, \mathrm{d}t < \infty \quad \text{if } 1 \le s < \infty, \qquad \operatorname*{ess\,sup}_{t\in[0,T]} \|z(t)\| < \infty \quad \text{if } s = \infty$$

(see, e.g., [15; Chap. IV,1] for details).

DEFINITION 1.1. Let $\mathbf{a} \in L^2_\sigma$. A vector function $\mathbf{u} : Q \to \mathbf{R}^3$ is called a *weak solution* to (N-S) if

$$\mathbf{u} \in L^2(0, \infty; \mathring{W}^{1,2}_\sigma) \cap L^\infty(0, \infty; L^2_\sigma)$$

and the integral identity

$$\int_Q \{-\mathbf{u} \cdot \partial_t \varphi + (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \varphi + \nabla \mathbf{u} : \nabla \varphi\} \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbf{R}^3} \mathbf{a} \cdot \varphi(0) \, \mathrm{d}x \tag{1.5}$$

holds for all $\varphi \in C_0^\infty([0, \infty[; \mathcal{D}_\sigma)$.

REMARK 1.1. Let $\mathbf{u} : Q \to \mathbf{R}^3$ be a weak solution to (N-S). Then $\mathbf{u}$ can be redefined on a Lebesgue set with measure zero such that $\mathbf{u} \in C_w([0, \infty[; L^2_\sigma)$, i.e. for all $t > 0$ we have $\mathbf{u}(t) \in L^2_\sigma$ such that

$$\lim_{s\to t} \int_\Omega \mathbf{u}(s) \cdot \xi \, \mathrm{d}x = \int_\Omega \mathbf{u}(t) \cdot \xi \, \mathrm{d}x \quad \forall \, \xi \in L^2_\sigma.$$

By the following definition we introduce the notion of a suitable weak solution.

DEFINITION 1.2. A weak solution $\mathbf{u} : Q \to \mathbf{R}^3$ to (N-S) is called a *suitable weak solution* if there exist $p \in L^{4/3}(0, \infty; L^2)$ such that

$$\int_Q \left\{ -\mathbf{u} \cdot \frac{\partial \varphi}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \varphi + \nabla \mathbf{u} : \nabla \varphi \right\} \mathrm{d}x \, \mathrm{d}t = \int_Q p \operatorname{div} \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{1.6}$$

holds for all $\varphi \in C_0^\infty(Q)^3$ together with the local energy inequality (1.3).

Notations. Let $x_0 \in \mathbf{R}^3$, $t_0 \in \mathbf{R}$ and $0 < R < \infty$. We define the ball

$$B_R(x_0) := \{x \in \mathbf{R}^3 \,|\, |x - x_0| < R\}$$

and the parabolic cylinder

$$Q_R(x_0, t_0) := B_R(x_0) \times]t_0 - R^2, t_0[.$$

Our main result concerns the partial regularity of suitable weak solutions to the Navier-Stokes equations.

THEOREM 1 (Main Theorem). *There exists an absolute constant $\varepsilon_\star > 0$, such that every suitable weak solution* $\mathbf{u}$ *to* (N-S) *is continuous in* $Q \setminus \operatorname{sing}(\mathbf{u})$, *where*

$$\operatorname{sing}(\mathbf{u}) := \left\{ (x_0, t_0) \in Q \,\middle|\, \limsup_{R\to 0^+} \frac{1}{R} \int_{Q_R(x_0,t_0)} \left| \operatorname{curl}\mathbf{u} \times \frac{\mathbf{u}}{|\mathbf{u}|} \right|^2 \mathrm{d}x\,\mathrm{d}t > \varepsilon_\star \right\}$$

is a closed subset of Q *with* $\mathcal{P}_1(\operatorname{sing}(\mathbf{u})) = 0$.

The proof of Theorem 1 relies essentially on the following two propositions.

PROPOSITION 1. *There exists an absolute constant $k_\star > 0$ such that if $(\mathbf{u}, p)$ is a suitable weak solution to* (N-S) *then*

$$\Xi(x_0, t_0; \tau R) \le k_\star [\tau^2 + \tau^{-3} \Theta(x_0, t_0; R)] \Xi(x_0, t_0; R) \tag{1.7}$$

for every $0 < \tau < \dfrac{1}{4}$, *for all* $(x_0, t_0) \in Q$ *and* $0 < R < \sqrt{t_0}$, *where*

$$\begin{aligned}
&\Xi(x_0, t_0; \rho) \\
&:= \frac{1}{\rho}(\|\nabla \mathbf{u}\|^2_{L^2(Q_\rho(x_0,t_0))} + \|\mathbf{u}\|^2_{L^{10/3}(Q_\rho(x_0,t_0))} + \|\mathbf{u}\|^2_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}) \\
&\qquad + \frac{1}{\rho^2} \|\hat{p} - \hat{p}_{B_\rho}\|^2_{L^{4/3}(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}, \\
&\Theta(x_0, t_0; \rho) := \frac{1}{\rho} \int_{Q_\rho(x_0,t_0)} \left| \operatorname{curl}\mathbf{u} \times \frac{\mathbf{u}}{|\mathbf{u}|} \right|^2 \mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

$0 < \rho \le \sqrt{t_0}$.

PROPOSITION 2. *There exists an absolute constant $\hat{k}_\star > 0$ such that if $(\mathbf{u}, p)$ is a suitable weak solution to* (N-S) *then*

$$\widehat{\Xi}(x_0, t_0; \tau R) \le \hat{k}_\star [\tau^3 + \tau^{-3} \mathcal{O}(x_0, t_0; R)] \widehat{\Xi}(x_0, t_0; R) \tag{1.8}$$

for every $0 < \tau < 1/4$, *for all* $(x_0, t_0) \in Q$ *and* $0 < R < \sqrt{t_0}$, *where*

$$\begin{aligned}
&\widehat{\Xi}(x_0, t_0; \rho) := \frac{1}{\rho} \|\nabla \mathbf{u}\|^2_{L^2(Q_\rho(x_0,t_0))} + \frac{1}{\rho^2} \|\hat{p} - \hat{p}_{B_\rho}\|^2_{L^{4/3}(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}, \\
&\mathcal{O}(x_0, t_0; \rho) := \frac{1}{\rho} \|\mathbf{u}\|^2_{L^4(t_0-\rho^2,t_0;L^3(B_\rho(x_0)))},
\end{aligned}$$

$0 < \rho \le \sqrt{t_0}$.

REMARK 1.2. 1. By Theorem 1 we improve the known results concerning the interior regularity of weak solutions to the Navier-Stokes equations in the following sense:

- The boundedness of $\mathbf{u}$ in a neighbourhood of a regular point (x_0, t_0) can be replaced by the Lipschitz continuity of $\mathbf{u}$, which seems to be the best possible regularity with respect to the time variable;
- Replacing the condition (1.1) by the more physical condition

$$\limsup_{R\to 0^+} \frac{1}{R} \int_{Q_R(x_0,t_0)} \left| \operatorname{curl}\mathbf{u} \times \frac{\mathbf{u}}{|\mathbf{u}|} \right|^2 \mathrm{d}x\,\mathrm{d}t < \varepsilon_\star \tag{1.9}$$

  improves the estimate of the singular set;

- The direct method based on Prop. 1 and Prop. 2 simplifies the proof and in addition enables us to specify the numerical value $\varepsilon_\star > 0$ in order to define the set of all singularities (see (1.9) above).

2. The statement of Theorem 1 continues to hold if one adds on the right hand side of (N-S) a body force $-\operatorname{div} \mathbf{f}$ with

$$\mathbf{f} \in \mathcal{L}^{2,5}(Q)^{9} \, ^2 .$$

The paper is organized as follows. In Section 2 we establish fundamental estimates for functions being almost caloric which will be used for both the proof of Prop. 1 and Prop. 2. In Section 3 we provide an appropriate model system together with an a-priori estimate for weak solutions to such systems in a given cylinder $Q_R(x_0, t_0)$. The subject of Section 4 is the proof of Prop. 1 and Prop. 2. These fundamental estimates will be achieved after having established a Caccioppoli-type inequality combined with a Campanato-type estimates for the pressure $\hat{p}$ along with the Campanato-type estimates for semi caloric functions and the estimates established for the model system. The proof of the Main Theorem will be completed in Section 5. Finally, in the appendix of the paper we list several lemmas which have been used throughout the paper.

2. Fundamental estimates for semi-caloric functions

DEFINITION 2.1. A function $\mathbf{V} \in L^2(Q_1)^3$ is said to be *semi-caloric* if there exists $P \in L^1(-1, 0; L^2(B_1))$ with $\Delta P = 0$ in Q_1 such that

$$\frac{\partial \mathbf{V}}{\partial t} - \Delta \mathbf{V} = -\nabla P \quad \text{in} \quad Q_1 \tag{2.1}$$

in sense of distributions. Then P is called the caloric pressure related to $\mathbf{V}$.

Let us start with the following Caccioppoli-type inequality.

LEMMA 2.1. *Let $\mathbf{V} \in L^2(Q_1)^3$ be a semi-caloric function with its caloric pressure $P \in L^1(-1, 0; L^2(B_1))$. Then $\nabla^m \mathbf{V} \in L^2_{\text{loc}}(B_1 \times]-1, 0])^3$ for all $m \in \mathbf{N}$ and there exists a constant c_1 depending on $m \in \mathbf{N}$ only such that*

$$\|\nabla^m \mathbf{V}\|^2_{L^\infty(-1/4,0;L^2(B_{1/2}))} \leq c_1 \{\|\mathbf{V}\|^2_{L^2(Q_1)} + \|P\|^2_{L^1(-1,0;L^2(B_1))}\}. \tag{2.2}$$

Proof. By using a standard mollification argument one easily proves

$$\mathbf{V} \in L^\infty(-\rho^2, 0; W^{m,2}(B_\rho)^3),$$
$$\frac{\partial \mathbf{V}}{\partial t} \in L^1(-\rho^2, 0; W^{m,2}(B_\rho)^3), \quad m = 1, 2, \ldots$$

for all $0 < \rho < 1$. Furthermore for each multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we have

$$\frac{\partial D^\alpha \mathbf{V}}{\partial t} - \Delta D^\alpha \mathbf{V} = -\nabla D^\alpha P \quad \text{in} \quad Q_1. \tag{2.3}$$

Fix $m \in \mathbf{N}$. Let $\phi \in C_0^\infty(B_1 \times]-1, 0])$ be a cut-off function with $0 \leq \phi \leq 1$ in Q_1 and $\phi \equiv 1$ on $Q_{1/2}$. Multiplying (2.3) by $\phi^{2m+2} D^\alpha \mathbf{V}$ summing up over $|\alpha| = m$,

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[2] Here $\mathcal{L}^{2,\lambda}(Q)$ stands for the usual parabolic Campanato space (cf. [3]).

integrating over $B_1\times]-1,t[\,(t\in]-1,0[)$ and applying integration by parts along with Cauchy-Schwarz's and Young's inequality yields

$$\begin{aligned}\int_{B_1}\phi^{2m+2}(x,t)|\nabla^m\mathbf{V}(x,t)|^2\,\mathrm{d}x+2\int_{-1}^t\int_{B_1}\phi^{2m+2}|\nabla^{m+1}\mathbf{V}|^2\,\mathrm{d}x\,\mathrm{d}s\\ \le\int_{-1}^t\int_{B_1}\Big(\frac{\partial}{\partial t}+\Delta\Big)(\phi^{2m+2})|\nabla^m\mathbf{V}|^2\,\mathrm{d}x\,\mathrm{d}s\\ +2\Big\{\int_{-1}^t\Big(\int_{B_1}\phi^{2m+2}|\nabla^{m+1}P|^2\,\mathrm{d}x\Big)^{1/2}\mathrm{d}s\Big\}^2\\ +\frac{1}{2}\operatorname*{ess\,sup}_{s\in]-1,0[}\int_{B_1}\phi^{2m+2}(x,s)|\nabla^m\mathbf{V}(x,s)|^2\,\mathrm{d}x.\end{aligned}$$

On the other hand, recalling that $\Delta P=0$ and using integration by parts implies

$$\int_{B_1}\phi^{2m+2}(x,s)|\nabla^{m+1}P(x,s)|^2\,\mathrm{d}x=2^{-(m+1)}\int_{B_1}(\Delta^{m+1}\phi^{2m+2})(x,s)P^2(x,s)\,\mathrm{d}x$$

for a.e. $s\in]-1,0[$. Combining the last two statements shows that

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in]-1,0[}\int_{B_1}\phi^{2m+2}(x,t)|\nabla^m\mathbf{V}(x,t)|^2\,\mathrm{d}x+\int_{Q_1}\phi^{2m+2}|\nabla^{m+1}\mathbf{V}|^2\,\mathrm{d}x\,\mathrm{d}t\\ \le c\int_{Q_1}\phi^{2m}|\nabla^m\mathbf{V}|^2\,\mathrm{d}x\,\mathrm{d}t+c\Big\{\int_{-1}^0\Big(\int_{B_1}P^2\,\mathrm{d}x\Big)^{1/2}\mathrm{d}t\Big\}^2,\end{aligned}$$

where $c=\text{const}$ depending on m only. Iterating this inequality m-times gives

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in]-1,0[}\int_{B_1}\phi^{2m+2}(x,t)|\nabla^m\mathbf{V}(x,t)|^2\,\mathrm{d}x\\ \le c\int_{Q_1}|\mathbf{V}|^2\,\mathrm{d}x\,\mathrm{d}t+c\Big\{\int_{-1}^0\Big(\int_{B_1}P^2\,\mathrm{d}x\Big)^{1/2}\mathrm{d}t\Big\}^2,\end{aligned}$$

with $c=\text{const}$ depending only on m. This completes the proof of (2.2). ∎

We are now in a position to prove the following fundamental estimate

THEOREM 2.2. *Let $\mathbf{V}\in L^2(Q_1)^3$ be a semi-caloric function with its caloric pressure $P\in L^1(-1,0;L^2(B_1))$. Then*

$$\|\mathbf{V}\|^2_{L^2(Q_\tau)}\le c_2\tau^5\{\|\mathbf{V}\|^2_{L^2(Q_1)}+\|P\|^2_{L^1(-1,0;L^2(B_1))}\}\tag{2.4}$$

for every $0<\tau<1$ with an absolute constant $c_2>0$.

Additionally, if $P\in L^{4/3}(-1,0;L^2(B_1))$ then

$$\|\mathbf{V}-\mathbf{V}_{Q_\tau}\|^2_{L^2(Q_\tau)}\le c_3\tau^6\{\|\mathbf{V}-\mathbf{V}_{Q_1}\|^2_{L^2(Q_1)}+\|P\|^2_{L^{4/3}(-1,0;L^2(B_1))}\}\tag{2.5}$$

for every $0<\tau<1$ with an absolute constant $c_3>0$.

Proof. Clearly, both fundamental estimates are trivially fulfilled for $1/2\le\tau<1$. Therefore without loss of generality we may restrict ourselves to the case $0<\tau<1/2$. Using

Sobolev's embedding theorem together with (2.2) (with $m = 2$) yields

$$\begin{aligned}\|\mathbf{V}\|^2_{L^2(Q_\tau)} &\le |B_1|\,\tau^5\|\mathbf{V}\|^2_{L^\infty(Q_{1/2})} \le c\tau^5 \operatorname*{ess\,sup}_{t\in]-1/4,0[} \int_{B_{1/2}} |\nabla^2\mathbf{V}(x,t)|^2\,\mathrm{d}x\\ &\le c\tau^5 \int_{Q_1} |\mathbf{V}|^2\,\mathrm{d}x\,\mathrm{d}t + c\tau^5\bigg\{\int_{-1}^0 \bigg(\int_{B_1} P^2\,\mathrm{d}x\bigg)^{1/2}\mathrm{d}t\bigg\}^2.\end{aligned}$$[3]

Whence, (2.4).

In order to verify (2.5) we first make use of the Poincaré-type inequality (A.6) (cf. appendix; Lemma A.2 below) to get

$$\|\mathbf{V}-\mathbf{V}_{Q_\tau}\|^2_{L^2(Q_\tau)} \le c\tau^2\|\nabla\mathbf{V}_{Q_\tau}\|^2_{L^2(Q_\tau)} + \tau^2 c\bigg\{\int_{-\tau^2}^0 \bigg(\int_{B_\tau} |\nabla P|^2\,\mathrm{d}x\bigg)^{1/2}\mathrm{d}t\bigg\}^2. \tag{2.6}$$

As above by using Sobolev's embedding theorem one infers

$$\|\nabla\mathbf{V}_{Q_\tau}\|^2_{L^2(Q_\tau)} \le c\tau^5 \operatorname*{ess\,sup}_{t\in]-1/4,0[} \int_{B_{1/2}} |\nabla^3\mathbf{V}(x,t)|^2\,\mathrm{d}x.$$

Estimating the right hand side of this inequality with the aid of (2.2) with $m = 3$, replacing $\mathbf{V}$ by $\mathbf{V}-\mathbf{V}_{Q_1}$ therein, leads to

$$\|\nabla\mathbf{V}_{Q_\tau}\|^2_{L^2(Q_\tau)} \le c\tau^5 \int_{Q_1} |\mathbf{V}-\mathbf{V}_{Q_1}|^2\,\mathrm{d}x\,\mathrm{d}t + c\tau^5\bigg\{\int_{-1}^0 \bigg(\int_{B_1} P^2\,\mathrm{d}x\bigg)^{1/2}\mathrm{d}t\bigg\}^2. \tag{2.7}$$

Next, thanks to $\Delta P = 0$ using Caccioppoli's inequality and Hölder's inequality yields

$$\begin{aligned}\bigg\{\int_{-\tau^2}^0 \bigg(\int_{B_\tau} |\nabla P|^2\,\mathrm{d}x\bigg)^{1/2}\mathrm{d}t\bigg\}^2 &\le c\tau^4\bigg\{\int_{-1}^0 \|\nabla P(t)\|^{4/3}_{L^\infty(B_{1/2})}\,\mathrm{d}t\bigg\}^{3/2}\\ &\le c\tau^4\bigg\{\int_{-1}^0 \bigg(\int_{B_1} P^2\,\mathrm{d}x\bigg)^{2/3}\mathrm{d}t\bigg\}^{3/2}.\end{aligned} \tag{2.8}$$

Finally, estimating the right hand side of (2.6) from above by (2.7) and (2.8) completes the proof of (2.5). ■

3. Estimates for weak solutions to the model system. Let $(\mathbf{u}, p)$ be a suitable weak solution to the system (N-S). Using the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{u}\times\operatorname{curl}\mathbf{u} + \frac{1}{2}\nabla|\mathbf{u}|^2$$

setting $\hat{p} = p + \frac{1}{2}|\mathbf{u}|^2$ the identity (1.6) turns into

$$\int_Q \bigg\{ -\mathbf{u}\cdot\frac{\partial\varphi}{\partial t} + \mathbf{u}\times\operatorname{curl}\mathbf{u}\cdot\varphi + \nabla\mathbf{u} : \nabla\varphi\bigg\}\,\mathrm{d}x\,\mathrm{d}t = \int_Q \hat{p}\operatorname{div}\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{3.1}$$

for all $\varphi \in C_0^\infty(Q)^3$. By virtue of Sobolev's embedding theorem along with multiplicative inequalities one proves

$$L^2(0,\infty; \mathring{W}^{1,2}_\sigma(\Omega)) \cap L^\infty(0,\infty; L^2_\sigma(\Omega)) \subset L^{8/3}(0,\infty; L^4(\Omega)^3).$$

[3] In what follows c denotes an absolute positive constant, whose value may change from line to line and will be specified if necessary.

Consequently,

$$\hat{p} \in L^{4/3}(0,\infty; L^2(\Omega)).$$

Taking $\varphi = \nabla\phi$ in (3.1) and using integration by parts yields

$$\int_{\mathbf{R}^3} \mathbf{u}(t) \times \operatorname{curl}\mathbf{u}(t) \cdot \nabla\phi \,\mathrm{d}x = \int_{\mathbf{R}^3} \hat{p}(t)\Delta\phi \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(\mathbf{R}^3) \tag{3.2}$$

for a.e. $t > 0$.

To proceed we introduce the following notations. Let $G \subseteq \mathbf{R}^3$ be an open set. Define

$$A^2(G) := \{\Delta\phi \,|\, \phi \in W_0^{2,\,2}(G)\}, \quad B^2(G) := \{q_h \in L^2(G) \,|\, \Delta q_h = 0 \text{ in } G\}.$$

Clearly, both $A^2(G)$ and $B^2(G)$ are closed in $L^2(G)$. By virtue of Weyl's Lemma it is readily seen that

$$L^2(G) = A^2(G) \oplus B^2(G). \tag{3.3}$$

Let $(x_0, t_0) \in Q$ and $0 < R < \sqrt{t_0}$ be fixed. By the orthogonal decomposition (3.3) there exist

$$p_{0,R} \in L^{4/3}(0,\infty; A^2(B_R(x_0))), \qquad p_{h,R} \in L^{4/3}(0,\infty; B^2(B_R(x_0)))$$

such that

$$\hat{p}(t) - \hat{p}_{B_R(x_0)}(t) = p_{0,R}(t) + p_{h,R}(t) \quad \text{in} \quad B_R(x_0) \quad \text{for a.e. } t \in]0,\infty[.$$

Next, fix $t > 0$ such that

$$\mathbf{u}(t) \times \operatorname{curl}\mathbf{u}(t) \in L^{6/5}(B_R(x_0))^3, \quad p_{0,R}(t) \in A^2(B_R(x_0)) \tag{3.4}$$

(note that the set of $t > 0$ for which (3.4) fails has measure zero). According to the definition of $A(B_R(x_0))$ from the identity (3.2) one deduces

$$\int_{B_R(x_0)} \mathbf{u}(t) \times \operatorname{curl}\mathbf{u}(t) \cdot \nabla\phi \,\mathrm{d}x = \int_{B_R(x_0)} p_{0,R}(t)\Delta\phi \,\mathrm{d}x \tag{3.5}$$

for all $\phi \in C_0^\infty(B_R(x_0))$. Then, with the aid of Lemma A.1 one gets

$$p_{0,R}(t) \in W^{1,\,6/5}(B_R(x_0))$$

together with the estimate

$$\|\nabla p_{0,R}(t)\|^{4/3}_{L^{6/5}(B_R(x_0))} \le c\|\operatorname{curl}\mathbf{u}(t) \times \mathbf{u}(t)\|^{4/3}_{L^{6/5}(B_R(x_0))}. \tag{3.6}$$

In particular, because the function $t \mapsto \|\operatorname{curl}\mathbf{u}(t) \times \mathbf{u}(t)\|^{4/3}_{L^{6/5}(B_R(x_0))}$ belongs to $L^1(]0,\infty[)$ one obtains

$$\nabla p_{0,R} \in L^{4/3}(0,\infty; L^{6/5}(B_R(x_0))^3).$$

Moreover, the inequality (3.6) implies

$$\|\nabla p_{0,R}\|^2_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0)))} \le c_4\|\operatorname{curl}\mathbf{u} \times \mathbf{u}\|^2_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0)))} \tag{3.7}$$

with an absolute constant $c_4 > 0$.

In order to estimate the right hand side of (3.7) we will proceed in two different ways.

1) Firstly, using Hölder's inequality yields

$$\|\operatorname{curl}\mathbf{u}(t) \times \mathbf{u}(t)\|^{4/3}_{L^{6/5}(B_R(x_0))} \le \left\|\operatorname{curl}\mathbf{u}(t) \times \frac{\mathbf{u}(t)}{|\mathbf{u}(t)|}\right\|^{4/3}_{L^2(B_R(x_0))} \|\mathbf{u}(t)\|^{4/3}_{L^3(B_R(x_0))} \tag{3.8}$$

for a.e. $t \in]t_0 - R^2, t_0[$. Integrating both sides of (3.8) over the interval $]t_0 - R^2, t_0[$ once more using Hölder's inequality shows that

$$\|\operatorname{curl}\mathbf{u} \times \mathbf{u}\|^2_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0)))} \le R\Theta(x_0,t_0;R)\|\mathbf{u}\|^2_{L^4(t_0-R^2,t_0;L^3(B_R(x_0)))}. \tag{3.9}$$

Secondly, with the help of Hölder's inequality one estimates

$$\begin{aligned}\|\operatorname{curl}\mathbf{u} \times \mathbf{u}\|^2_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0)))} \\ \le 2\|\mathbf{u}\|^2_{L^4(t_0-R^2,t_0;L^3(B_R(x_0)))}\|\nabla\mathbf{u}\|^2_{L^2(Q_R(x_0,t_0))}.\end{aligned} \tag{3.10}$$

In order to fix the model problem we will use the following lemma which for the reader's convenience will be proved at the end of this section.

Lemma 3.1. *Given $f \in L^1(t_0 - R^2, t_0; L^\sigma(B_R(x_0)))$ $(1 < \sigma \le 2)$, there exists a unique function $w \in L^{\frac{5\sigma}{3+\sigma}}(t_0 - R^2, t_0; \mathring{W}^{1,\frac{5\sigma}{3+\sigma}}(B_R(x_0))) \cap L^{5\sigma/3}(Q_R(x_0,t_0))$ such that*

$$\int_{Q_R(x_0,t_0)} (-w\varphi_t + \nabla w \cdot \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_R(x_0,t_0)} f\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{3.11}$$

for all $\varphi \in C_0^\infty(B_R(x_0) \times [t_0 - R^2, t_0[)$. In addition,

$$\|w\|_{L^{5\sigma/3}(Q_R(x_0,t_0))} \le c\|f\|_{L^1(t_0-R^2,t_0;L^\sigma(B_R(x_0)))} \tag{3.12}$$

with a constant $c > 0$ depending only on σ.

The model problem. Applying Lemma 3.1 with $f = f_i = (\operatorname{curl}\mathbf{u} \times \mathbf{u})_i - \partial p_{0,R}/\partial x_i$ $(i = 1,2,3)$ and $\sigma = 6/5$ one gets a unique function $\mathbf{w} \in L^{10/7}(t_0 - R^2, t_0; \mathring{W}^{1,10/7}(B_R(x_0))^3) \cap L^2(Q_R(x_0,t_0))^3$ such that

$$\int_{Q_R(x_0,t_0)} (-\mathbf{w}\cdot\varphi_t + \nabla\mathbf{w} : \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_R(x_0,t_0)} (\operatorname{curl}\mathbf{u} \times \mathbf{u} - \nabla p_{0,R})\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{3.13}$$

for all $\varphi \in C_0^\infty(B_R(x_0) \times [t_0 - R^2, t_0[)$.

Clearly, using Jensen's inequality the estimate (3.12) yields

$$\|\mathbf{w}\|^2_{L^2(Q_R(x_0,t_0))} \le c\,R\|\operatorname{curl}\mathbf{u} \times \mathbf{u} - \nabla p_{0,R}\|^2_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0)))}.$$

Now, the right hand side of this inequality can be estimated by (3.7) together with (3.9). Thus,

$$\|\mathbf{w}\|^2_{L^2(Q_R(x_0,t_0))}\ c_5R^2\Theta(x_0,t_0;R)\|\mathbf{u}\|^2_{L^4(t_0-R^2,t_0;L^3(B_R(x_0)))}. \tag{3.14}$$

Alternatively, using (3.10) instead of (3.9) one gets

$$\|\mathbf{w} - \mathbf{w}_{Q_R}\|^2_{L^2(Q_R(x_0,t_0))} \le c_6R^2\mathcal{O}(x_0,t_0;R)\|\nabla\mathbf{u}\|^2_{L^2(Q_R(x_0,t_0))}. \tag{3.15}$$

Proof of Lemma 3.1. It will be sufficient to prove the assertion of the lemma for the case $(x_0,t_0) = (0,0)$ and $R = 1$. The general case can be deduced easily by introducing an appropriate transformation of coordinates.

Let $f \in L^1(-1,0;L^\sigma(B_1))$ be given. Clearly there exists a sequence $\{f_m\}$ in $L^2(Q_1)$ such that

$$f_m \to f \quad \text{in} \quad L^1(-1,0;L^\sigma(B_1)) \quad \text{as} \quad m \to \infty.$$

Let $m \in \mathbf{N}$. Consulting [8], there exists a unique function

$$w_m \in L^2(-1,0;\mathring{W}^{1,2}(B_1)) \cap C([-1,0];L^2(B_1)),$$

satisfying

$$\int_{Q_1}(-w_m\varphi_t+\nabla w_m\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_1}f_m\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{3.16}$$

for all $\varphi\in C_0^\infty(B_1\times[-1,0[)$. In particular, $\mathrm{d}w_m/\mathrm{d}t\in L^2(-1,0;W^{-1,2}(B_1))$, such that

$$\int_{-1}^t\Big\langle\frac{\mathrm{d}w_m}{\mathrm{d}t}(s),\varphi(s)\Big\rangle\,\mathrm{d}s+\int_{-1}^t\int_{B_1}\nabla w_m\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s=\int_{-1}^t\int_{B_1}f_m\varphi\,\mathrm{d}x\,\mathrm{d}s \tag{3.17}$$

for all $\varphi\in L^2(-1,0;\mathring{W}^{1,2}(B_1))$, for all $t\in]-1,0[$. Observing (3.16) and using integration by parts it follows that $w_m(-1)=0$. Into (3.17) inserting the admissible test function $\varphi=w_m|w_m|^{\sigma-2}$ and using integration by parts along with Hölder's inequality gives

$$\begin{aligned}\frac{1}{\sigma}\|w_m(t)\|^\sigma_{L^\sigma(B_1)}&+(\sigma-1)\int_{-1}^t\int_{B_1}|\nabla w_m|^2|w_m|^{\sigma-2}\,\mathrm{d}x\,\mathrm{d}s\\&\le\int_{-1}^t\int_{B_1}|f_m||w_m|^{\sigma-1}\,\mathrm{d}x\,\mathrm{d}s\le\|f_m\|_{L^1(-1,0;L^\sigma(B_1))}\|w_m\|^{\sigma-1}_{L^\infty(-1,0;L^\sigma(B_1))}.\end{aligned}$$

Next, define $\theta_m:=w_m|w_m|^{\frac{\sigma-2}{2}}$. From the inequality above one infers

$$\theta_m\in L^2(-1,0;\mathring{W}^{1,2}(B_1))\cap L^\infty(-1,0;L^2(B_1))$$

together with

$$\|\theta_m\|^2_{L^\infty(-1,0;L^2(B_1))}+\|\nabla\theta_m\|^2_{L^2(Q_1)}\le\frac{\sigma'}{2}\|f_m\|_{L^1(-1,0;L^\sigma(B_1))}\|\theta_m\|^{2/\sigma'}_{L^\infty(-1,0;L^2(B_1))}.$$

By means of Sobolev's embedding theorem together with Hölder's inequality applying Young's inequality implies

$$\|\theta_m\|_{L^{10/3}(Q_1)}+\|\nabla\theta_m\|_{L^2(Q_1)}\le c\|f_m\|^{\sigma/2}_{L^1(-1,0;L^\sigma(B_1))},$$

where $c=$ const >0 depending only on σ. Hence, recalling the definition of θ_m and using Hölder's inequality shows that

$$\|w_m\|_{L^{5\sigma/3}(Q_1)}+\|\nabla w_m\|_{L^{\frac{5\sigma}{3+\sigma}}(Q_1)}\le c\|f_m\|_{L^1(-1,0;L^\sigma(B_1))}.$$

Finally, passing to the limit $m\to\infty$ and using Banach-Steinhaus' theorem proves the assertion of Lemma 3.1. ■

4. Proof of Propositions 1/2. Let $(\mathbf{u},p)$ be a suitable weak solution to the system (N-S). As in the previous section we set $\hat{p}:=|\mathbf{u}|^2/2+p$.

First, let us define the following quantities which are invariant under the natural scaling of the Navier-Stokes equations. For $(x_0,t_0)\in Q$ and $0<\rho<\sqrt{t_0}$ define

$$\begin{aligned}\mathcal{V}(x_0,t_0;\rho)&:=\frac{1}{\rho}[\|\mathbf{u}\|^2_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}+\|\mathbf{u}\|^2_{L^{10/3}(Q_\rho(x_0,t_0))}+\|\nabla\mathbf{u}\|^2_{L^2(Q_\rho(x_0,t_0))}],\\\mathcal{P}(x_0,t_0;\rho)&:=\frac{1}{\rho^2}\|\hat{p}-\hat{p}_{B_\rho(x_0)}\|^2_{L^{4/3}(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))},\\\mathcal{Z}(x_0,t_0;\rho|\varphi)&:=\frac{1}{\rho^3}\|\varphi\|^2_{L^2(Q_\rho(x_0,t_0))},\quad\varphi\in L^2(Q_\rho(x_0,t_0))^3.\end{aligned}$$

Fix $(x_0,t_0)\in Q$. For the sake of notational simplicity we will write $\mathcal{V}(\rho)$, $\mathcal{P}(\rho)$, $\mathcal{Z}(\rho|\varphi)$ etc. in place of $\mathcal{V}(x_0,t_0;\rho)$, $\mathcal{P}(x_0,t_0;\rho)$, $\mathcal{Z}(x_0,t_0;\rho|\varphi)$ etc.

We start by proving fundamental estimates which form the bases of the proof of the Prop. 1. First we establish a Caccioppoli-type inequality which immediately follows from the local energy inequality (1.3).

LEMMA 4.1. *There exist absolute positive constants c_7 and c_8 such that*

$$\mathcal{V}(\rho) \le c_7 \mathcal{Z}(2\rho|\mathbf{u}) + c_8 \mathcal{P}(2\rho) \quad \forall\, 0 < \rho < \tfrac{1}{2}\sqrt{t_0}. \tag{4.1}$$

Proof. Let $0 < \rho < \frac{1}{2}\sqrt{t_0}$. Let $\phi \in C^\infty(\mathbf{R}^4)$ be a cut-off function such that $0 \le \phi \le 1$ in $\mathbf{R}^4$, $\phi \equiv 0$ in $\mathbf{R}^4 \setminus B_{2\rho}(x_0) \times]-\infty, t_0 - 4\rho^2[$, $\phi \equiv 1$ in $B_\rho(x_0) \times]t_0 - \rho^2, \infty[$ and

$$|\nabla \phi|^2 + |\nabla^2 \phi| + |\partial_t \phi| \le \frac{c_0}{\rho^2} \quad \text{in} \quad \mathbf{R}^4.$$

From the local energy inequality (1.3) (replacing ϕ by ϕ^2 therein) using Hölder's and Young's inequality one easily estimates

$$\|\phi \mathbf{u}\|^2_{L^\infty(t_0-4\rho^2; L^2(B_{2\rho}))} + \|\phi \nabla \mathbf{u}\|^2_{L^2(Q_{2\rho})} \le c\|\mathbf{u}\|^2_{L^2(Q_{2\rho})} + c\rho^{-1} \int_{Q_{2\rho}} |\hat{p} - \hat{p}_{B_{2\rho}}|\, |\mathbf{u}| \phi \,\mathrm{d}x\,\mathrm{d}t.$$

With the help of Cauchy-Schwarz's inequality verifying

$$\int_{Q_{2\rho}} |\hat{p} - \hat{p}_{B_{2\rho}}|\, |\mathbf{u}| \phi \,\mathrm{d}x\,\mathrm{d}t \le \|p - p_{B_{2\rho}}\|_{L^1(t_0-2\rho^2, t_0; L^2(B_{2\rho}))} \|\phi \mathbf{u}\|_{L^\infty(t_0-4\rho^2, t_0; L^2(B_{2\rho}))}$$

applying Young's inequality one arrives at

$$\|\phi \mathbf{u}\|^2_{L^\infty(t_0-4\rho^2; L^2(B_{2\rho}))} + \|\phi \nabla \mathbf{u}\|^2_{L^2(Q_{2\rho})} \le c\rho \mathcal{Z}(2\rho|\mathbf{u}) + c\rho \mathcal{P}(2\rho). \tag{4.2}$$

Finally, with the aid of a multiplicative inequality, the Sobolev-Poincaré inequality and Young's inequality one obtains

$$\begin{aligned} \|\phi \mathbf{u}\|^2_{L^{10/3}(Q_{2\rho})} &\le \|\phi \mathbf{u}\|^{4/5}_{L^\infty(t_0-4\rho^2; L^2(B_{2\rho}))} \|\phi \mathbf{u}\|^{6/5}_{L^2(t_0-4\rho^2; L^6(B_{2\rho}))} \\ &\le c(\|\phi \mathbf{u}\|^2_{L^\infty(t_0-4\rho^2; L^2(B_{2\rho}))} + \|\phi \nabla \mathbf{u}\|^2_{L^2(Q_{2\rho})}) + c\rho \mathcal{Z}(2\rho|\mathbf{u}). \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) gives (4.1). ■

Next, we provide a fundamental estimate for the pressure.

LEMMA 4.2. *There exist absolute positive constants c_9 and c_{10} such that for every $0 < \tau < 1/2$ we have*

$$\mathcal{P}(2\tau R) \le c_9 \tau^3 \mathcal{P}(R) + c_{10} \tau^{-2} \Theta(R) \mathcal{V}(R) \quad \forall\, 0 < R < \sqrt{t_0}. \tag{4.4}$$

Proof. Let $0 < R < \sqrt{t_0}$ be arbitrarily chosen. By virtue of the orthogonal decomposition (3.3) one gets unique functions

$$p_{0,R} \in L^{4/3}(0, \infty; A^2(B_R(x_0))), \qquad p_{h,R} \in L^{4/3}(0, \infty; B^2(B_R(x_0)))$$

such that

$$\hat{p}(t) - \hat{p}(t)_{B_R(x_0)} = p_{0,R}(t) + p_{R,h}(t) \quad \text{in} \quad B_R(x_0)$$

for a.e. $t \in]0, \infty[$.

Next, fix $0<\tau<1/2$. With the help of the Sobolev-Poincaré inequality, Caccioppoli's inequality and properties of harmonic functions one easily deduces

$$
\begin{aligned}
(4.5)\qquad & \|\hat p(t)-\hat p(t)_{B_{2\tau R}}\|^{4/3}_{L^2(B_{2\tau R})}\\
&\le 2\|p_{h,R}(t)-p_{h,R}(t)_{B_{2\tau R}}\|^{4/3}_{L^2(B_{2\tau R})}+2\|p_{0,R}(t)-p_{0,R}(t)_{B_{2\tau R}}\|^{4/3}_{L^2(B_{2\tau R})}\\
&\le c\tau^{10/3}R^{10/3}\max_{x\in\overline{B}_{R/2}(x_0)}|\nabla p_{h,R}(x,t)|^{4/3}+c\|\nabla p_{0,R}(t)\|^{4/3}_{L^{6/5}(B_R)}\\
&\le c\tau^{10/3}\|p_{h,R}(t)\|^{4/3}_{L^2(B_R)}+c\|\nabla p_{0,R}(t)\|^{4/3}_{L^{6/5}(B_R)}
\end{aligned}
$$

for a.e. $t\in]t_0-R^2,t_0[$. Integrating both sides over the interval $]t_0-4\tau^2R^2,t_0[$ it follows that

$$
\begin{aligned}
&\|\hat p-\hat p_{B_{2\tau R}}\|^2_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0)))}\\
&\quad\le c\tau^5\|p_{h,R}\|^2_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0)))}+c\|\nabla p_{0,R}\|^2_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0)))}.
\end{aligned}
$$

Taking into account

$$
\|p_{h,R}\|_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0)))}\le\|\hat p-\hat p_{B_R(x_0)}\|_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0)))}
$$

using (3.7) along with (3.9), dividing the result by τ^2R^2 and applying Hölder's inequality one infers

$$
\begin{aligned}
\mathcal{P}(2\tau R)&\le c\tau^3\mathcal{P}(R)+c\tau^{-2}R^{-1}\Theta(R)\|\mathbf{u}\|^2_{L^4(t_0-R^2,t_0;L^3(B_R(x_0)))}\\
&\le c\tau^3\mathcal{P}(R)+c\tau^{-2}\Theta(R)\mathcal{V}(R).
\end{aligned}
$$

Whence, (4.4). ∎

Remark 4.3. Repeating the proof of Lemma 4.2 while estimating the pressure gradient $\nabla p_{0,R}$ by (3.10) instead of (3.9) one immediately gets the following alternative fundamental estimate

$$
(4.6)\qquad \mathcal{P}(2\tau R)\le\hat c_9\tau^3\mathcal{P}(R)+\hat c_{10}\tau^{-2}\mathcal{O}(R)\frac{1}{R}\int_{Q_R(x_0,t_0)}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t
$$

for all $0<R<\sqrt{t_0}$, with absolute constants $\hat c_9$ and $\hat c_{10}$.

Now, we are in a position to complete the proof of Prop. 1.

Proof of Proposition 1. Let $0<R<\sqrt{t_0}$ be fixed. According to Lemma 3.1 there exists a unique function

$$
\mathbf{w}\in L^{10/7}(t_0-R^2,t_0;\mathring{W}^{1,10/7}(B_R(x_0))^3\cap L^2(Q_R(x_0,t_0))^3
$$

satisfying the identity (3.13). Furthermore, from (3.14) one easily deduces

$$
(4.7)\qquad \mathcal{Z}(R|\mathbf{w})\le c_{11}\Theta(R)\mathcal{V}(R).
$$

Combining (3.1) and (3.13) yields

$$
(4.8)\qquad \int_{Q_R(x_0,t_0)}\{-\mathbf{v}\cdot\varphi_t+\nabla\mathbf{v}:\nabla\varphi\}\,\mathrm{d}x\,\mathrm{d}t=\int_{Q_R(x_0,t_0)}p_{h,R}\operatorname{div}\varphi\,\mathrm{d}x\,\mathrm{d}t
$$

where $\mathbf{v}=\mathbf{u}-\mathbf{w}$. Thus, setting

$$
\begin{aligned}
&\mathbf{V}(y,s):=\mathbf{v}(x_0+Ry,t_0+R^2s),\\
&P(y,s):=Rp_{h,R}(x_0+Ry,t_0+R^2s),\quad (y,s)\in Q_1
\end{aligned}
$$

shows that the function $\mathbf{V}$ is semi-caloric in Q_1 with corresponding pressure P. Applying Theorem 2.2 one finds

$$\|\mathbf{V}\|^2_{L^2(Q_{2\tau})} \le c\tau^5\{\|\mathbf{V}\|^2_{L^2(Q_1)} + \|P\|^2_{L^{4/3}(-1,0;L^2(B_1))}\}.$$

Then using the transformation formula of the Lebesgue integral, noticing that

$$\|p_{h,R}(t)\|_{L^2(B_R(x_0))} \le \|\hat{p}(t) - \hat{p}_{B_R(x_0)}(t)\|_{L^2(B_R(x_0))} \quad \text{for a.e.} \quad t \in]t_0 - R^2, t_0[,$$

one arrives at

$$\mathcal{Z}(2\tau R|\mathbf{v}) \le c_{12}\tau^2[\mathcal{Z}(R|\mathbf{v}) + \mathcal{P}(R)]. \tag{4.9}$$

Combining (4.7) and (4.9) yields

$$\begin{aligned}\mathcal{Z}(2\tau R|\mathbf{u}) &\le 2\mathcal{Z}(2\tau R|\mathbf{v}) + 2\mathcal{Z}(2\tau R|\mathbf{w}) \le c\tau^2[\mathcal{Z}(R|\mathbf{v}) + \mathcal{P}(R)] + \tau^{-3}\mathcal{Z}(R|\mathbf{w}) \\ &\le c[\tau^2 + \tau^{-3}\Theta(R)](\mathcal{V}(R) + \mathcal{P}(R)).\end{aligned} \tag{4.10}$$

Now from (4.1) with $\rho = \tau R$ using (4.10) together with (4.4) leads to

$$\mathcal{V}(\tau R) + \mathcal{P}(\tau R) \le c\mathcal{Z}(2\tau R|\mathbf{u}) + c\mathcal{P}(2\tau R) \le c[\tau^2 + \tau^{-3}\Theta(R)](\mathcal{V}(R) + \mathcal{P}(R)).$$

Whence, (1.7). ■

Next, given $\mathbf{\Lambda} \in \mathbf{R}^3$ using (3.1) from the local energy inequality (1.3) one easily gets

$$\begin{aligned}&\int_\Omega |\mathbf{u}(t) - \mathbf{\Lambda}|^2\phi(t)\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\phi\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le \int_0^t\int_\Omega |\mathbf{u} - \mathbf{\Lambda}|^2\Big\{\frac{\partial\phi}{\partial t} + \Delta\phi\Big\}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega 2\hat{p}(\mathbf{u} - \mathbf{\Lambda})\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + 2\int_0^t\int_\Omega (\nabla\tilde{p}_h - \mathbf{\Lambda}) \times \operatorname{curl}\mathbf{u}\cdot(\mathbf{u} - \mathbf{\Lambda})\phi\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{4.11}$$

for all nonnegative functions $\phi \in C_0^\infty(Q)$, for a.e. $0 < t < \infty$. Here, the set of all $t > 0$, where (4.11) fails, does not depend on $\mathbf{\Lambda}$.

Let $0 < R < \sqrt{t_0}$ be fixed. Arguing as in the proof of Lemma 3.1 using (4.11) with $\mathbf{\Lambda} := \mathbf{u}_{Q_{2\rho}(x_0,t_0)}$ one gets the following alternative Caccioppoli inequality.

Lemma 4.4. *There exists an absolute positive constant c_{13} such that*

$$\widehat{\mathcal{V}}(\rho) \le c_{13}[\widehat{\mathcal{Z}}(2\rho|\mathbf{u}) + \mathcal{O}(2\rho)\widehat{\mathcal{V}}(2\rho) + \mathcal{P}(2\rho)] \tag{4.12}$$

for all $0 < \rho < \frac{1}{2}\sqrt{t_0}$, where

$$\widehat{\mathcal{V}}(\rho) = \widehat{\mathcal{V}}(x_0, t_0; \rho) := \frac{1}{\rho}\|\nabla\mathbf{u}\|^2_{L^2(Q_\rho(x_0,t_0))},$$

$$\widehat{\mathcal{Z}}(\rho|\varphi) = \widehat{\mathcal{Z}}(x_0, t_0; \rho|\varphi) := \frac{1}{\rho^3}\|\varphi - \varphi_{Q_\rho(x_0,t_0)}\|^2_{L^2(Q_\rho(x_0,t_0))},$$

$\varphi \in L^2(Q_\rho(x_0, t_0))^3$.

Proof of Proposition 2. Let $0 < R < \sqrt{t_0}$ be fixed. Arguing as in the proof of Prop. 1, estimating $\mathbf{w}$ by (3.15) instead of (3.14) yields

$$\widehat{\mathcal{Z}}(R|\mathbf{w}) \le c_{14}\mathcal{O}(R)\widehat{\mathcal{V}}(R), \tag{4.13}$$

where $c_{14} > 0$ denotes an absolute constant.

Recalling that the function $\mathbf{V}$ which has been defined in proof of Prop. 1 is semi-caloric, applying the fundamental estimate (2.5) (see Theorem 2.2 above) and using the transformation formula of the Lebesgue integral it follows that

$$\widehat{\mathcal{Z}}(2\tau R|\mathbf{v}) \leq c_{15}\tau^3[\widehat{\mathcal{Z}}(R|\mathbf{v}) + \mathcal{P}(R)]. \tag{4.14}$$

Then combining (4.13) and (4.14) yields

$$\begin{aligned}\widehat{\mathcal{Z}}(2\tau R|\mathbf{u}) &\leq 2\widehat{\mathcal{Z}}(2\tau R|\mathbf{v}) + 2\widehat{\mathcal{Z}}(2\tau R|\mathbf{w}) \\ &\leq c\tau^3[\widehat{\mathcal{Z}}(R|\mathbf{v}) + \mathcal{P}(R)] + \tau^{-3}\mathcal{Z}(R|\mathbf{w}) \\ &\leq c[\tau^3 + \tau^{-3}\mathcal{O}(R)](\widehat{\mathcal{V}}(R) + \mathcal{P}(R)).\end{aligned} \tag{4.15}$$

Finally, taking $\rho = \tau R$ in (4.12), estimating the first term on the right by (4.15) and the pressure term by (4.6) gives

$$\widehat{\mathcal{V}}(\tau R) + \mathcal{P}(\tau R) \leq c[\tau^3 + \tau^{-3}\mathcal{O}(R)](\widehat{\mathcal{V}}(R) + \mathcal{P}(R)),$$

which proves the desired inequality (1.8). ■

5. Proof of the Main Theorem. To begin with, define

$$\varepsilon_\star := 0.18566\, k_*^{-2.5}, \qquad f(\tau) := k_\star(\tau^2 + \tau^{-3}\varepsilon_\star), \quad \tau > 0.$$

Let $\tau_\star > 0$ such that $\theta_\star := f(\tau_\star) = \min_{\tau>0} f(\tau)$. In fact, by an elementary calculus one finds

$$\tau_\star = (1.5\varepsilon_\star)^{1/5}, \qquad \theta_\star = \frac{5}{3}(1.5 \cdot 0.18566)^{2/5} = 0.999\ldots < 1.$$

Next, let $(x_0, t_0) \in Q$ such that

$$\limsup_{R\to 0^+} \Theta(x_0, t_0; R) < \varepsilon_\star.$$

We select $0 < R_0 < \sqrt{t_0}$ such that

$$\Theta(x_0, t_0; R) < \varepsilon_\star \quad \forall\, 0 < R \leq R_0.$$

Then (1.7) reads

$$\Xi(x_0, t_0; \tau_\star R) \leq \theta_\star \Xi(x_0, t_0; R) \quad \forall\, 0 < R \leq R_0.$$

Consequently,

$$\lim_{R\to 0^+} \Xi(x_0, t_0; R) = 0. \tag{5.1}$$

In particular, having $|\operatorname{curl}\mathbf{u}| \leq 2|\nabla\mathbf{u}|$, there exists $0 < R_1 < R_0$ such that

$$\Theta(x_0, t_0; R_1) < \frac{\varepsilon_\star}{2}, \quad \Xi(x_0, t_0; R_1) < \frac{\varepsilon_\star}{4}.$$

By the absolute continuity of the Lebesgue integral we may choose $0 < \rho < \sqrt{t_0} - R_1$ such that for all $(y, s) \in Q_\rho(x_0, t_0)$:

$$\Xi(y, s; R_1) < \frac{\varepsilon_\star}{4}, \qquad \Theta(y, s; R_1) < \frac{\varepsilon_\star}{2}.$$

Now, fix $(y, s) \in Q_\rho(x_0, t_0)$. Once more applying (1.7) iteratively, replacing (x_0, t_0) by (y, s) and setting $R = R_1$ therein yields

$$\Xi(y, s; \tau_\star^k R_1) \leq \theta_\star^k \Xi(y, s; R_1).$$

Let $\alpha \in]0,1[$ be determined by $\tau_\star^\alpha = \theta_\star$. From the estimate above one gets

$$\Xi(y,s;R) \le \hat{C}_1 R^\alpha \quad \forall\, 0 < R \le R_1, \tag{5.2}$$

where $\hat{C}_1 = \text{const} > 0$ depending neither on (y,s) nor on R.

Next, fix $0 < \beta < 1$. Choose $0 < \tau < 1/2$ such that

$$\tau^\beta \hat{k}_\star \le \frac{1}{2}\,[4].$$

Furthermore, by virtue of (5.2) one can select $0 < R_2 \le R_1$ such that for all $0 < R \le R_2$ we have

$$\tau^{-3}\mathcal{O}(y,s;R) \le \frac{\tau^{3-\beta}}{2} \quad \forall\,(y,s) \in Q_\rho(x_0,t_0).$$

Now, fix $(y,s) \in Q_\rho(x_0,t_0)$. Applying (1.8) with (y,s) instead of (x_0,t_0) gives

$$\widehat{\Xi}(y,s;\tau R) \le \tau^{3-\beta}\widehat{\Xi}(y,s;R) \quad \forall\, 0 < R \le R_2.$$

This implies

$$\int_{Q_R(y,s)} |\nabla \mathbf{u}|^2 \,\mathrm{d}x \le \hat{C}_2 R^{4-\beta} \quad \forall\, 0 < R \le R_2,$$

where the constant $\hat{C}_2 > 0$ depends neither on $(y,s) \in Q_\rho(x_0,t_0)$ nor on R. Applying the Poincaré-type inequality (A.6) (cf. appendix; Lemma A.2) gives

$$\mathbf{u} \in \mathcal{L}^{6-\beta}(Q_\rho(x_0,t_0)),$$

which, by a well-known theorem of Da Prato [3], shows that $\mathbf{u}$ belongs to the Hölder space $C^{(1-\beta)/2,(1-\beta)/4}(\overline{Q_\rho(x_0,t_0)})^3$.

Now we proceed by a standard bootstrap argument using the method of differences to get the existence of the second weak derivatives of $\mathbf{u}$ with resp. to the spatial variable locally in $Q_\rho(x_0,t_0)$ belonging to $L^\infty(t_0,t_0-\rho^2;L^2(B_\rho(x_0)))$. Repeating this argument iteratively one gets

$$\nabla^k \mathbf{u}|_{Q_\rho(x_0,t_0)} \in L^\infty(t_0,t_0-\rho^2;L^2(B_\rho(x_0))) \quad \forall\, k \in \mathbf{N}.$$

In particular, by Sobolev's embedding theorem $\nabla \mathbf{u}$ is bounded in $Q_\rho(x_0,t_0)$. Therefore it only remains to prove that $\partial \mathbf{u}/\partial t$ is bounded in $Q_{\rho/2}(x_0,t_0)$.

Observing the decomposition (3.3) there exist functions

$$p_{0,\rho} \in L^\infty(t_0,t_0-\rho^2;A^2(B_\rho(x_0))), \qquad p_{h,\rho} \in L^{4/3}(t_0,t_0-\rho^2;B^2(B_\rho(x_0))),$$

such that

$$p = p_{0,\rho} + p_{h,\rho} \quad \text{a.e. in} \quad Q_\rho(x_0,t_0).$$

Since $p_{0,\rho}$ satisfies the equation $-\Delta p_{0,\rho} = \operatorname{div}\operatorname{div} \mathbf{u} \otimes \mathbf{u}$ one obtains

$$\nabla p_{0,\rho} \in L^\infty(Q_{\rho/2}(x_0,t_0))^3. \tag{5.3}$$

Set $\widetilde{\mathbf{u}} := \mathbf{u} + \nabla p_{h,\rho}$ in $Q_\rho(x_0,t_0)$. From (N-S)$_2$ it follows

$$\frac{\partial \widetilde{\mathbf{u}}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} - \nabla p_{0,\rho} \quad \text{in} \quad Q_\rho(x_0,t_0).$$

[4] Here $\hat{k}_\star$ denotes the constant which appears in (1.8).

Consequently, $\partial\widetilde{\mathbf{u}}/\partial t \in L^\infty(Q_{\rho/2}(x_0,t_0))^3$. Thus, to verify the assertion it will be enough to show that $\nabla p_{h,\rho} \in L^\infty(Q_{\rho/2}(x_0,t_0))^3$.

To begin with, we state the following pressure identity

$$q(x,t) := \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{p(y,t)}{|x-y|}\,\mathrm{d}y = -\frac{1}{8\pi}\int_{\mathbf{R}^3}\frac{|\mathbf{u}(y,t)\times(x-y)|^2}{|x-y|^3}\,\mathrm{d}y, \tag{5.4}$$

$(x,t)\in Q$, which one can obtain by using an elementary calculus from the equation

$$-\Delta p = \operatorname{div}\operatorname{div}\mathbf{u}\otimes\mathbf{u} \quad \text{in} \quad Q$$

using integration by parts.

Let $(x,t)\in B_{5\rho/8}(x_0)\times]t_0, t_0-\rho^2[$. According to the mean value formula for harmonic functions one gets

$$\begin{aligned} p_{h,\rho}(x,t) &= \frac{1}{|B_{\rho/4}|}\int_{B_{\rho/4}(x)} p_{h,\rho}(y,t)\,\mathrm{d}y \\ &= \frac{1}{|B_{\rho/4}|}\int_{B_{\rho/4}(x)} p(y,t)\,\mathrm{d}y - \frac{1}{|B_{\rho/4}|}\int_{B_{\rho/4}(x)} p_{0,\rho}(y,t)\,\mathrm{d}y. \end{aligned} \tag{5.5}$$

Observing (5.4) we have $p=\Delta q$. Applying integration by parts yields

$$\int_{B_{\rho/4}(x)} p(y,t)\,\mathrm{d}y = \frac{4}{\rho}\int_{\partial B_{\rho/4}(x)}\frac{\partial q}{\partial y_i}(y,t)y_i\,\mathrm{d}S.$$

On the other hand, one calculates

$$\begin{aligned} \frac{\partial q}{\partial y_i}(y,t) = {}& \frac{3}{8\pi}\int_{\mathbf{R}^3}\frac{(\mathbf{u}(z,t)\cdot y-z)^2(y_i-z_i)}{|y-z|^5}\,\mathrm{d}z \\ &- \frac{1}{8\pi}\int_{\mathbf{R}^3}\frac{|\mathbf{u}(z,t)|^2(y_i-z_i)+2\mathbf{u}(z,t)\cdot(y-z)u_i(z,t)}{|y-z|^3}\,\mathrm{d}z. \end{aligned}$$

Taking into account $\mathbf{u}\in L^\infty(0,\infty;L^2_\sigma)$ and $\mathbf{u}\in L^\infty(Q_\rho(x_0,t_0))^3$ gives

$$\left|\frac{\partial q}{\partial y_i}(y,t)\right| \le c \quad \text{for a.e. } (y,t)\in B_{7\rho/8}(x_0)\times]t_0-\rho^2,t_0[.$$

Hence,

$$\left|\int_{B_{\rho/4}(x)} p(y,t)\,\mathrm{d}y\right| \le c \quad \text{for a.e. } (x,t)\in B_{5\rho/8}(x_0)\times]t_0-\rho^2,t_0[,$$

where $c=$ const depending only on ρ.

In addition, verifying $-\Delta p_{0,\rho} = \operatorname{div}\operatorname{div}\mathbf{u}\otimes\mathbf{u}$ in $Q_\rho(x_0,t_0)$ and recalling the definition of the space $A^2(B_\rho(x_0))$ one estimates

$$\left|\int_{B_{\rho/4}(x)} p_{0,\rho}(y,t)\,\mathrm{d}y\right| \le c\|p_{0,\rho}(t)\|_{L^2(B_\rho(x_0))} \le c\,\|\mathbf{u}\|_{L^\infty(Q_\rho(x_0,t_0))}$$

for a.e. $(x,t)\in B_{5\rho/8}(x_0)\times]t_0-\rho^2,t_0[$. Here the constant c depends on ρ only.

Estimating the right hand side of (5.5) by the two inequalities we have just obtained it follows that $p_{h,\rho}\in L^\infty(Q_{5\rho/8}(x_0,t_0))$. Finally, once more using Caccioppoli's inequality one gets $\nabla p_{h,\rho}\in L^\infty(Q_{\rho/2}(x_0,t_0))^3$ which completes the proof of the theorem. ∎

6. Appendix

LEMMA A.1. *Let* $\mathbf{g} \in L^r(B_R)^3$ $(6/5 \le r \le 2)$ *and* $q_0 \in A^2(B_R)$ *with*

$$\int_{B_R} \mathbf{g} \cdot \nabla\phi \,\mathrm{d}x = \int_{B_R} q_0 \Delta\phi \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(B_R). \tag{A.1}$$

Then $q_0 \in W^{1,r}(B_R)$ *and there exists a constant* C_r *such that*

$$\|\nabla q_0\|_{L^r(B_R)} \le C_r \|\mathbf{g}\|_{L^r(B_R)}. \tag{A.2}$$

Proof. Without loss of generality we may assume $R = 1$, the general case easily follows by using an appropriate transformation of coordinates.

Consulting [4; Chap. III, Th. 3.4], there exists $\mathbf{h} \in W_0^{1,r}(B_1)^9$ such that $\operatorname{div} \mathbf{h} = \mathbf{g} - \mathbf{g}_{B_1}$ with

$$\|\nabla \mathbf{h}\|_{L^r(B_1)} \le c\|\mathbf{g}\|_{L^r(B_1)}. \tag{A.3}$$

Next, applying integration by parts, identity (A.1) turns into

$$-\int_{B_1} (\mathbf{h} - \mathbf{h}_{B_1}) : \nabla^2\phi \,\mathrm{d}x = \int_{B_1} q_0 \Delta\phi \,\mathrm{d}x \quad \forall\, \phi \in W_0^{2,2}(B_1).$$

By the definition of $A^2(B_1)$ there exists $u \in W_0^{2,2}(B_1)$ such that $q_0 = \Delta u$. Thus, from the identity above with $\phi = u$ using Sobolev-Poincaré's inequality along with (A.3) yields

$$\|q_0\|_{L^2(B_1)} \le c\|\mathbf{h} - \mathbf{h}_{B_1}\|_{L^2(B_1)} \le c\|\mathbf{g}\|_{L^r(B_1)}. \tag{A.4}$$

Next, applying [13; Th. 9.11, p.156] with $B[u,\phi] = \int_{B_1} \Delta u \Delta\phi \,\mathrm{d}x$ and $F(\phi) = \int_{B_1} \mathbf{g} \cdot \nabla\phi \,\mathrm{d}x$ it follows that $q_0 \in W^{1,r}(B_1)$ together with the estimate

$$\|\nabla q_0\|_{L^r(B_1)} \le c(\|\mathbf{g}\|_{L^r(B_1)} + \|u\|_{L^2(B_1)}).$$

Finally, taking into account

$$\|u\|_{L^2(B_1)} \le c\|\Delta u\|_{L^2(B_1)} = c\|q_0\|_{L^2(B_1)}$$

and making use of (A.4) shows (A.2).

For the general case $R > 0$ the assertion easily follows from the former case $R = 1$ by using an appropriate transformation of coordinates and applying the transformation formula of the Lebesgue integral. ∎

For the reader's convenience we will present a short proof of the Poincaré-type inequality we have used above.

LEMMA A.2. *Let* $u \in L^2(Q_R)$ *with* $\nabla u \in L^2(Q_R)$ *and* $\mathbf{h} \in L^1(Q_R)^3$ *such that*

$$-\int_{Q_R} u\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_R} \mathbf{h} \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, \varphi \in C_0^\infty(Q_R). \tag{A.5}$$

Then

$$⨍_{Q_R} |u - u_{Q_R}|^2 \,\mathrm{d}x\,\mathrm{d}t \le c_0 R^2 \bigg\{ ⨍_{Q_R} |u - u_{Q_R}|^2 \,\mathrm{d}x\,\mathrm{d}t + \bigg(⨍_{Q_R} |\mathbf{h}| \,\mathrm{d}x\,\mathrm{d}t \bigg)^2 \bigg\}, \tag{A.6}$$

where $c_0 = \text{const} > 0$ *depends on* n *only.*

Proof. 1) First, let us prove the assertion for the case $R = 1$.

For a given function $\zeta \in C_0^\infty(B_1)$ with $0 \le \zeta \le 1$ in B_1 and $\zeta \neq 0$ we define the functional $F \in L^1(B_1)^*$ by means of

$$\langle F, v\rangle := \frac{1}{\int_{B_1} \zeta \,\mathrm{d}x} \int_{B_1} \zeta v \,\mathrm{d}x, \quad v \in L^1(B_1).$$

Clearly, $\langle F, \mathbf{1}\rangle = 1$ and

$$\langle F, v - \langle F, v\rangle\rangle = 0 \quad \forall\, v \in L^1(B_1(0)).$$

Since $|||v||| := \|\nabla v\|_{L^2(B_1)} + |\langle F, v\rangle|$ defines an equivalent norm on $W^{1,q}(B_1)$ we have

$$\|v - \langle F, v\rangle\|_{L^{q^*}(B_1(0))} \le c\|\nabla v\|_{L^q(B_1)} \quad \forall\, v \in W^{1,q}(B_1). \tag{A.7}$$

Now let $u \in L^2(Q_1)$ with $\nabla u \in L^2(Q_1)^3$ and $\mathbf{h} \in L^1(Q_1)^3$ fulfilling (A.5). One easily calculates

$$\begin{aligned} u(x,t) - u_{Q_1} &= (u(x,t) - \langle F, u(t)\rangle) + (\langle F, u(t)\rangle - u_{Q_1}) \\ &= u(x,t) - \langle F, u(t)\rangle + \fint_{Q_1} \langle F, u(t)\rangle - u(y,s)\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

for a.e. $(x,t) \in Q_1$. Moreover, verifying

$$\begin{aligned} &\fint_{Q_1} \langle F, u(t)\rangle - u(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad = \frac{1}{\int_{B_1} \zeta\,\mathrm{d}y} \fint_{Q_1} \int_{B_1} \zeta(y')(u(y',t) - u(y,s))\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \\ &\quad = \int_{-1}^0 \langle F, u(t) - u(s)\rangle\,\mathrm{d}s + \frac{1}{2\displaystyle\int_{B_1} \zeta\,\mathrm{d}y} \int_{-1}^0 \fint_{B_1} \int_{B_1} \zeta(y')(u(y',s) - u(y,s))\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

for a.e. $-1 < t - 0$ gives

$$\begin{aligned} u(x,t) - u_{Q_1} = u(x,t) - \langle F, u(t)\rangle &+ \int_{-1}^0 \langle F, u(t) - u(s)\rangle\,\mathrm{d}s \\ &+ \frac{1}{2\int_{B_1} \zeta\,\mathrm{d}y} \int_{-1}^0 \fint_{B_1} \int_{B_1} \zeta(y')(u(y',s) - u(y,s))\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Hence, from the identity above one obtains

$$\begin{aligned} \|u(t) - u_{Q_1}\|_{L^2(B_1)} \le \|u(t) - \langle F, u(t)\rangle\|_{L^2(B_1)} &+ \left\| \int_{-1}^0 |\langle F, u(t) - u(s)\rangle|\,\mathrm{d}s \right\|_{L^2(B_1)} \\ &+ \frac{1}{\int_{B_1} \zeta\,\mathrm{d}y} \left\| \int_{-1}^0 \int_{B_1} |u(y,s) - u(s)_{B_1}|\,\mathrm{d}y\,\mathrm{d}s \right\|_{L^2(B_1)} \end{aligned}$$

for a.e. $t \in]-1, 0[$. Then with the aid of (A.7) and Poincaré's inequality one finds

$$\|u(t) - u_{Q_1}\|^2_{L^2(B_1)} \le c\|\nabla u(t)\|^2_{L^2(B_1)} + c\int_{-1}^0 |\langle F, u(t) - u(s)\rangle|^2\,\mathrm{d}s + c\|\nabla u\|^2_{L^1(Q_1)}$$

for a.e. $t \in]-1, 0[$. Next, integrating both sides of this inequality over the interval $]-1, 0[$

yields

$$\int_{-1}^{0} \|u(t)-u_{Q_1}\|^2_{L^2(B_1)}\,\mathrm{d}t \le c\int_{-1}^{0} \|\nabla u(t)\|^2_{L^2(B_1)}\,\mathrm{d}t + c\int_{-1}^{0}\int_{-1}^{0} |\langle F, u(t)-u(s)\rangle|^2\,\mathrm{d}s\,\mathrm{d}t.$$

Finally, inserting $\varphi(x,t)=\zeta(x)$ into (A.5) and using integration by parts implies

$$|\langle F, u(t)-u(s)\rangle|^2 = \left|\frac{1}{\int_{B_1}\zeta\,\mathrm{d}y}\int_s^t\int_{B_1}\mathbf{h}\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}t\right|^2 \le c\left(\int_{Q_1}|\mathbf{h}|\,\mathrm{d}x\,\mathrm{d}t\right)^2$$

for a.e. $s,t\in]-1,0[$. Whence, (A.6).

2) The general case easily follows from the above by means of a standard homothety argument using an appropriate transformation of coordinates. ∎

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