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NON-ABELIAN EXTENSIONS OF MINIMAL ROTATIONS

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Abstract. We consider continuous extensions of minimal rotations on a locally connected compact group X by cocycles taking values in locally compact Lie groups and prove regularity (i.e. the existence of orbit closures which project onto the whole basis X) in certain special situations beyond the nilpotent case. We further discuss an open question on cocycles acting on homogeneous spaces which seems to be the missing key for a general regularity theorem.

1. Introduction. Let T be a minimal homeomorphism of a compact metric space X, and G be a locally compact metrisable group. Any continuous function $f : X \to G$ defines an extension T_f of T via the equation

$$T_f^n(x, g) = (T^n x, f(n, x) \cdot g),$$

for every $x \in X$, $g \in G$ and $n \in \mathbb{Z}$, where $f(n, x)$ is the *cocycle* generated by f, i.e.

$$f(n,x) = \begin{cases} f(T^{n-1}x) \cdots f(Tx) \cdot f(x) & \text{if } n \geq 1, \\ e & \text{if } n = 0, \\ f(-n, T^n x)^{-1} & \text{if } n < 0, \end{cases}$$

with e being the identity in G. In this paper we investigate the problem of regularity of such an extension, i.e. whether there exist orbit closures which project onto the whole basis X (such orbit closures are called *surjective*). It is known that for arbitrary base transformations T such orbit closures may not exist (see [LM02]; this corresponds to the situation of type III_0 cocycles in the classical abelian case). However, if T is a minimal rotation on a locally connected group X then every topologically recurrent cocycle with values in a nilpotent locally compact group G does admit surjective orbit closures, and furthermore the entire product space $X \times G$ (or in geometric terminology the trivial G-bundle) decomposes into such orbit closures (which are closed subbundles of $X \times G$), see [GH05]. The essential idea involved goes back to G. Atkinson [At78] who proved regularity for the case $G = \mathbb{R}^d$, and

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was later generalised by M. Lemańczyk and M. Mentzen [LM02, Me03] to general locally compact abelian groups. Before the present paper no regularity results beyond the nilpotent case have been known, and our aim here is to develop methods which work in more general situations.

The difficulty in treating non-abelian (non-compact) extensions is that the local essential ranges $\{E_x\}_{x\in X}$ introduced in [GH05] alter along the orbits by conjugation:

$$E_{T^n x}(f) = f(n,x) \cdot E_x(f) \cdot f(n,x)^{-1},$$

for all $x \in X$ and $n \in \mathbb{Z}$. Furthermore, unlike the abelian case, these essential ranges might not be subgroups of G for points outside a dense G_δ-set in X. In this paper we show that understanding the behaviour of the identity component E_x^0 of E_x under conjugation of the cocycle is crucial to regularity: if x is any point in X and if the mapping

$$H_{T^n x} = f(n,x) \cdot E_x^0(f) \cdot f(n,x)^{-1},$$

which is only defined along the orbit of x, extends *continuously* to the entire space X, then the transformation T_f admits such a decomposition into surjective orbit closures. On the one hand, this improves the key tool used in [GH05], and secondly it directs our attention to the behaviour of these identity components under conjugation. This approach recalls the conjugacy problem of stabilizers for general Borel actions in S. G. Dani's paper [Da02], and in line with [Da02] we show that the identity components of E_x are conjugate on a dense G_δ-set in X. In some special situations we are able to prove that the identity components E_x^0 depend continuously on x, which implies regularity of the cocycle. However, in general this issue is still open and is closely related with the following open question:

Let T_f be a continuous G-extension of a minimal group rotation T, and H be any closed subgroup of G. Suppose $C \subseteq X \times G/H$ is a compact T_f-orbit closure which projects injectively onto a dense G_δ-subset of X (which means, in particular, that (C, T_f) is an almost one-to-one extension of the rotation (1)). Is it true that the projection $\pi : C \to X$ is then one-to-one on the whole set C?

This question was pointed out before in [GH05], but as its answer is positive for nilpotent groups G we did not realise its importance at that time.

The paper is organised as follows: first of all, we review basic facts on cocycles taken from [GH05]. In Section 3 we prove the generalised Atkinson Lemma for general locally compact groups G and draw some simple conclusions. In Section 4 we restrict our considerations to Lie groups, and adapt

(1) For the definition see [Gl].

the results from [Da02] to our setting in order to investigate the behaviour of the identity components E_x^0 under conjugation by the cocycle; we further discuss the importance of the above mentioned open question. Finally, in the last section we show the existence of surjective orbit closures in the situation of semidirect products $G = \mathbb{R}^d \rtimes \mathbb{R}$ where the action of $\mathbb{R}$ on $\mathbb{R}^d$ has no eigenvalue equal to one. The proof presented there is alternative to the approach in Section 4. However, it does not give a clearer picture of the general case; it is rather the simple group structure that allows us to reduce to situations that are easily understood.

It is worth noting that very likely all these results can be extended to a larger class of base transformations as is done in [Gr07] and [Gr08], but we will not focus on that issue in this paper.

2. Basic facts and notions. Let T be a minimal homeomorphism of a compact metric space X, and G a locally compact second countable (l.c.s.c.) group. A cocycle $f(n,x)$ is said to be (topologically) *recurrent* if for every open neighbourhood U of the identity in G and every open set $\mathcal{U} \subseteq X$ there is an integer $n \neq 0$ so that

$$T^{-n}\mathcal{U} \cap \mathcal{U} \cap \{x : f(n,x) \in U\} \neq \emptyset.$$

This property is equivalent to T_f being topologically conservative (or regionally recurrent in the terminology of [GoHe]), i.e. for every open set $\mathcal{O} \subseteq X \times G$ there is an integer $n \neq 0$ so that $T_f^n(\mathcal{O}) \cap \mathcal{O} \neq \emptyset$.

The *local essential range* $E_x(f)$ of the cocycle f, as defined in [GH05], is the set of $g \in G$ such that for every open neighbourhood U of g and every open neighbourhood $\mathcal{U}$ of x there exists an integer $n \neq 0$ with

$$T^{-n}\mathcal{U} \cap \mathcal{U} \cap \{x : f(n,x) \in U\} \neq \emptyset.$$

Recall that $E_x(f)$ is a closed subset of G and it is symmetric, i.e. $E_x^{-1}(f) = E_x(f)$. For every x in X the set

$$P_x(f) = \{g \in G : (x,g) \in \overline{T_f^{\mathbb{Z}}(x,e)}\} \tag{1}$$

is a closed subsemigroup of G. We will simply write E_x and P_x whenever it is clear to which cocycle we refer. It is shown in [GH05, Proposition 1.7] that the set

$$\mathcal{D}(f) = \{x \in X : E_x = P_x\} \tag{2}$$

contains a dense G_δ-set, thus it is non-meagre in X. Thus for every x in $\mathcal{D}(f)$ the set E_x is a closed symmetric subsemigroup and hence a subgroup of G.

Recall that the essential ranges as well as the subsemigroups P_x satisfy the equation

$$E_{T^n x} = f(n,x) \cdot E_x \cdot f(n,x)^{-1} \tag{3}$$

for all $x \in X$ and $n \in \mathbb{Z}$, thus they are conjugate along orbits of T [GH05, Lemma 1.3]. The map $x \mapsto E_x$ is semicontinuous in the sense that if $x_n \to x$ and g_n are elements of E_{x_n} which converge to g, then $g \in E_x$.

If H is a closed subgroup of G, then the action of T_f (or its corresponding cocycle) on $X \times G/H$ is defined by setting

$$T_f^n(x, gH) = (T^n x, f(n,x) \cdot gH).$$

Any T_f-orbit closure in $X \times G/H$ is called *surjective* if it projects onto X. We shall make frequent use of the following lemma which is similar to [GH05, Lemma 2.3].

LEMMA 2.1. *Let $C \subseteq X \times G/H$ be a closed T_f-invariant set which projects onto a non-meagre set in X. Then there exists a compact set $K \subseteq G/H$ such that $(X \times K) \cap C$ projects onto the whole set X.*

Proof. Choose a sequence $\{K_n\}_{n \geq 1}$ of compact subsets of G/H such that $G/H = \bigcup_n K_n$. Then the sets $K'_n = \pi_X((X \times K_n) \cap C)$, where π_X is the projection onto X, are compact subsets of X and their union $\bigcup_{n \geq 1} K'_n$ is a non-meagre set. By Baire's category theorem there is an $m \geq 1$ such that K'_m contains a non-empty open set $\mathcal{U}$ of X. Since T is minimal and X is compact, $X = \bigcup_{n=1}^N T^{-n}(\mathcal{U})$ for some $N \geq 1$, and

$$\bigcup_{n=1}^{N} T_f^{-n}((X \times K_m) \cap C) = \bigcup_{n=1}^{N} T_f^{-n}(X \times K_m) \cap C$$

is a compact subset of C that projects onto X. ∎

A cocycle f is called *regular* if its skew product transformation T_f admits surjective orbit closures in $X \times G$. By [GH05, Theorem 2.1] any surjective orbit closure C is of the following form: If we set

$$H = \{g \in G : C \cdot g^{-1} = C\},$$

then C/H is compact regarded as a T_f-invariant subset of $X \times G/H$, and the restriction of T_f to C/H is minimal. Moreover, for every x in $\mathcal{D}(f)$ the vertical section of C consists of a single coset of H only: there exists $g_x \in G$ such that

$$C_x = \{g \in G : (x, g) \in C\} = g_x \cdot H.$$

Thus the system $(C/H, T_f)$ is an almost one-to-one extension of (X, T). It is further shown that the map

$$\gamma : \mathcal{D}(f) \to G/H, \quad x \mapsto C_x = g_x \cdot H.$$

is continuous and $E_x = g_x \cdot H \cdot g_x^{-1}$ for all $x \in \mathcal{D}(f)$ (see [GH05, Theorem 2.2]).

DEFINITION 2.2. We call an orbit closure C *strongly regular* if it is surjective and *every* vertical section C_x as above consists of a single left coset of H. A *strongly regular cocycle* is a cocycle f whose extension T_f admits strongly regular orbit closures.

REMARK 2.3. It is shown in [GH05, Theorem 3.1] that every regular cocycle with values in a nilpotent group is strongly regular, but for general groups this issue is still open even for a rotation as a base transformation T (cf. the open question mentioned in the introduction).

In other words, a strongly regular orbit closure is a subbundle of $X \times G$. Note that the entire product space (the trivial bundle) $X \times G$ then decomposes into such T_f-invariant subbundles, which are permuted via the right action of G on $X \times G$ defined by $R_h(x, g) = (x, g \cdot h^{-1})$. For such orbit closures the above statements on γ and E_x remain true with $\mathcal{D}(f)$ replaced by X: for every x in X the vertical section $C_x = \{g \in G : (x, g) \in C\}$ consists of a single left coset of $H = \{g : R_g(C) = C\}$, and the mapping

$$\gamma : X \to G/H, \quad x \mapsto C_x = g_x \cdot H,$$

is continuous on the whole set X. It is easy to see that then $E_x = g_x \cdot H \cdot g_x^{-1}$ for every x in X (cf. the proof of [GH05, Theorem 3.2]). Thus all essential ranges are subgroups conjugate to H, and if we identify H^G, the conjugacy class of H, with $G/N(H)$, where $N(H)$ is the normaliser of H, then

$$\varphi : X \to H^G, \quad x \mapsto E_x = g_x \cdot H \cdot g_x^{-1},$$

is continuous.

Finally, it should be noted that if f is continuously cohomologous to a topological transitive cocycle taking values in a closed subgroup H of G, then f is strongly regular, but not vice versa (if one does not allow discontinuities for the boundary function). More generally, if $b : X \to G$ is continuous and the cocycle

$$\widetilde{f}(n, x) = b(T^n x) \cdot f(n, x) \cdot b(x)^{-1}$$

is strongly regular, then f is also strongly regular.

3. The generalised Atkinson Lemma. Let $\mathcal{S}(G)$ be the set of all closed subsets of G equipped with the Fell topology (= projective limit of the Hausdorff topology on every compactum). A basis for this topology is given by sets of the form $\{S \in \mathcal{S}(G) : S \cap K \neq \emptyset,\ S \cap O_i \neq \emptyset \text{ for } i = 1, \dots, k\}$, where K is any compact subset of G and every O_i is open. It is well known that $\mathcal{S}(G)$ is compact and metrisable, and the space $\mathcal{C}(G)$ of all closed subgroups of G is a closed subspace (see [Fe62]). A *consistent selection* of subgroups $\{H_x\}_{x \in X}$ is a continuous mapping from X into $\mathcal{C}(G)$ such that $H_x \subseteq E_x$ for every x in X, and which satisfies the consistency condition

$$H_{T^n x} = f(n, x) \cdot H_x \cdot f(n, x)^{-1} \tag{4}$$

for every x and $n \in \mathbb{Z}$. *In contrast to* [GH05] *we do not assume that all* H_x *belong to the same conjugacy class and assume continuity only with respect to the Fell topology.*

We will need the following auxiliary lemma on consistent selections:

LEMMA 3.1. *Let* $\{H_x\}$ *be a consistent selection as defined above, and let* U *be any relatively compact open set in* G. *Then the set* $\mathcal{M}_U = \{x \in X : E_x \cap \overline{U}H_x \setminus UH_x = \emptyset\}$ *is open.*

Proof. We show that the complement of $\mathcal{M}_U$ is closed. Indeed, suppose x_k is a sequence of points converging to x such that $E_{x_k} \cap \overline{U}H_{x_k} \setminus UH_{x_k} \neq \emptyset$. For any choice of relatively compact neighbourhoods V and W such that $\overline{V} \subseteq U$ and $\overline{U} \subseteq W$ one can find points z_k and $T^{n_k} z_k$ both converging to x such that

$$g_k = f(n_k, z_k) \in \overline{W}H_{z_k} \setminus VH_{z_k}.$$

Since H_x depends continuously (with respect to the Fell topology) on x we may assume without loss of generality that the points z_k and $T^{n_k} z_k$ are from our dense non-meagre set $\mathcal{D}(f)$, and therefore—after modifying the cocycle values along the essential ranges—the g_k stay in some fixed compactum. Thus the g_k converge along some subsequence to some element g which must be contained in the set $E_x \cap \overline{W}H_x \setminus VH_x$. As V and W were arbitrary, this implies that $E_x \cap \overline{U}H_x \setminus UH_x \neq \emptyset$. ∎

We omit the proof of the following lemma which is verbatim the one of Lemma 4.3 in [GH05]. That proof is in the same manner as the proof of the previous lemma.

LEMMA 3.2. *Let* $U \subseteq G$ *be an open subset and* $C \subseteq G$ *a compact subset. Then for any fixed integer* n *the sets* $\{y \in X : f(n, y) \cdot H_y \cap UH_y \neq \emptyset\}$ *and* $\{y \in X : f(n, y) \cdot H_y \cap CH_y = \emptyset\}$ *are both open.*

The following proposition which generalises a lemma of G. Atkinson [At78] will be the key to proving regularity of cocycles. It is an improvement of [GH05, Proposition 4.4], as we only need "cutting neighbourhoods" at a single point in X; moreover, we make no special assumptions on the group G.

PROPOSITION 3.3. *Suppose that* G *is a l.c.s.c. group and* $f : X \to G$ *is a recurrent cocycle over a minimal rotation* T *on a locally connected compact group* X, *and let* $\{H_x\}_{x \in X}$ *be a consistent selection of subgroups. If there exists a point* x_0 *for which the group* H_{x_0} *has a cutting neighbourhood in* E_{x_0}, *i.e. a relatively compact open neighbourhood* U *of the identity such that*

$$E_{x_0} \cap \overline{U}H_{x_0} \setminus UH_{x_0} = \emptyset,$$

then the T_f*-orbit closure of any point* (x, H_x) *is a compact subset of* $X \times G/H_x$.

Proof. First of all, note that it is sufficient to prove the existence of a single point x such that the T_f-orbit closure of (x, H_x) is compact, since this implies compactness of all other T_f-orbit closures. Indeed, if C is such a compact orbit closure then it projects onto the whole basis X. Thus for every y in X there exists a $g \in G$ such that $(y, gH_x) \in C$ and therefore we can find a sequence $\{n_k\}$ and elements $h_k \in H_x$ and $g \in G$ such that $T^{n_k}x \to y$ and $f(n_k, x) \cdot h_k \to g$. By continuity of the consistent selection we see that

$$H_y = \lim_{k\to\infty} f(n_k, x) \cdot H_x \cdot f(n_k, x)^{-1} = g \cdot H_x \cdot g^{-1}.$$

This shows that the compact T_f-orbit of (y, gH_x) in $X \times G/H_x$ translates under the right translation by g^{-1} to the T_f-orbit of (y, H_y) in $X \times G/H_y$. As a right translation is a homeomorphism, the orbit closure of (y, H_y) is also compact.

According to Lemma 3.1 the set $\mathcal{M}_U = \{x \in X : E_x \cap \overline{U}H_x \setminus UH_x = \emptyset\}$ is open for every relatively compact open neighbourhood U, and therefore the T-invariant non-empty set $\mathcal{M}_{\text{cut}} = \bigcup_U \mathcal{M}_U$, where the union is taken over all relatively compact open neighbourhoods of the identity, is also open. This yields $\mathcal{M}_{\text{cut}} = X$ and thus for every point y in X we can find a relatively compact cutting neighbourhood.

Let $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denote the set of all integers > 0 and < 0, respectively. By recurrence both sets

$$\mathcal{R}_\pm = \{x \in X : (x, e) \in \overline{T_f^{\mathbb{Z}_\pm}(x, e)}\}$$

are comeagre subsets of X, and so is the intersection $\mathcal{R}_+ \cap \mathcal{R}_- \cap \mathcal{D}(f)$. Choose any point x from this non-empty intersection and set

$$C = \overline{T_f^{\mathbb{Z}}(x, H_x)}.$$

Let (y, gH_x) be any point belonging to the orbit closure C. By our choice of x there exists an increasing sequence of integers $n_k > 0$ such that $(y, g) = \lim_{k\to\infty} T_f^{n_k}(x, e)$. As above, we conclude that $H_y = g \cdot H_x \cdot g^{-1}$. Let U be a relatively compact cutting neighbourhood for H_y in E_y. Since $\mathcal{M}_U$ is open we can choose a connected open neighbourhood $\mathcal{U}$ of y such that

$$f(n, z) \in UH_z \cup (G \setminus \overline{U}H_z) \quad \text{whenever } z, T^n z \in \mathcal{U}.$$

By convergence of $T_f^{n_k}(x, e)$ to (y, g) we can find an integer k_0 such that $z = T^{k_0}x \in \mathcal{U}$ and

$$f(n_{k_0} - n_k, z) \in UH_z$$

for all $k \geq k_0$. As the neighbourhood $\mathcal{U}$ is connected it follows from Lemma 3.2 that the same is true with y replaced by z. Therefore all the cocycle values satisfy

$$\begin{aligned} f(-n_k, y) \cdot g &= f(-n_{k_0}, T^{-(n_k - n_{k_0})} y) \cdot f(n_{k_0} - n_k, y) \cdot g \\ &\in f(-n_{k_0}, T^{-(n_k - n_{k_0})} y) \cdot U \cdot \underbrace{H_y \cdot g}_{= g \cdot H_x}, \end{aligned}$$

and therefore stay within some compact subset of G/H_x as $k \to \infty$. Together with the fact that $T^{-n_k} y$ converges back to x this implies that the $T_f^{-n_k}(y, gH_x)$ converge along some subsequence to $(x, g'H_x)$ with g' in P_x. In other words, $(x, g'H_x)$ is in the negative orbit closure of (y, gH_x). In the same manner one sees that also (x, H_x) is in the negative orbit closure of $(x, g'H_x)$ and therefore it is contained in the negative orbit closure of (y, gH_x).

By the same argument one shows that (x, H_x) is also contained in the positive orbit closure of (y, gH_x). Together with recurrence, we conclude from [GoHe, Theorem 7.05] that T_f restricted to C is almost periodic and therefore C is compact. ∎

PROPOSITION 3.4. *Under the assumptions of Proposition* 3.3, *the* T_f*-orbit closure* C *of any point* (x, e), *with* x *from the set* $\mathcal{D}(f)$, *is strongly regular. Moreover*, H_x *is co-compact and normal in* E_x.

Proof. Let x be from the set $\mathcal{D}(f)$. Recall that $E_x = P_x$ is then a closed subgroup of G which contains H_x. According to Proposition 3.3 the T_f-orbit closure of (x, H_x) is compact. In particular, E_x/H_x is compact. The projection of the T_f-orbit is also compact and T-invariant, thus it equals X. As $H_x \subseteq P_x$, the same holds for the T_f-orbit closure C of (x, e).

Let (y, g_0) and (y, g_1) belong to C. As in the proof of Proposition 3.3 we deduce that

$$H_y = g_i \cdot H_x \cdot g_i^{-1} \subseteq g_i \cdot E_x \cdot g_i^{-1} \subseteq E_y$$

for both $i = 0, 1$. On the other hand, by compactness we can find a sequence $\{n_k\}_{k \geq 1}$ and g in G such that $T^{n_k} y \to x$ and $f(n_k, y) \cdot H_y \to g \cdot H_y$. Again, by the same reasoning as before (the $f(n_k, y)$ converge to g modulo $g_i \cdot E_x \cdot g_i^{-1}$),

$$g \cdot g_i \cdot E_x \cdot g_i^{-1} \cdot g^{-1} \subseteq E_x$$

for both $i = 0, 1$. Since E_x is a group, $g_1^{-1} \cdot g_0$ belongs to the normaliser $N(E_x)$ of E_x. The only thing left to prove is that every slice C_y consists of a single left coset of E_x, i.e. $g_1^{-1} \cdot g_0 \in E_x$. This is done by a simple "cohomology" argument. Since $C_x = E_x$ both the sequences $f(n_k, y) \cdot g_i \cdot E_x$ above converge to E_x. Let us define a "boundary function" on our countable set $\{y\} \cup \{T^{n_k} y\}_k$ by setting $b_0 = g_0$ and choosing

$$b_k \in f(n_k, y) \cdot g_0 \cdot N(E_x)$$

such that $b_k \to e$. Then

$$c_k = b_k^{-1} \cdot f(n_k, y) \cdot b_0 \in N(E_x)$$

and both the sequences $c_k \cdot (b_0^{-1} \cdot g_0) \cdot E_x$ and $c_k \cdot (g_0^{-1} \cdot g_1) \cdot E_x$ are contained in $N(E_x)$ and converge to E_x. As E_x is normal in $N(E_x)$ there exists a left-invariant metric for the topology in $N(E_x)/E_x$ and it follows that $(b_0^{-1} \cdot g_0) \cdot E_x = (b_0^{-1} \cdot g_1) \cdot E_x$. Thus $g_0 \cdot E_x = g_1 \cdot E_x$. ∎

COROLLARY 3.5. *Suppose that T is a minimal rotation on a locally connected compact group X, and f is a recurrent cocycle with values in a l.c.s.c. group G. If there exists a point $x_0 \in X$ for which the identity component of E_{x_0} is a normal subgroup of G, then f is strongly regular and E_x/E_x^0 is compact.*

Proof. We apply Proposition 3.4 to the consistent selection defined by setting $H_y = E_x^0$ for all y in X. ∎

COROLLARY 3.6. *Under the same assumption of Corollary* 3.5, *if there exists a point x_0 for which $E_{x_0} = \{e\}$ then f is a coboundary.*

Proof. By the previous corollary, the T_f-orbit closure C of any point (x, e) with $x \in \mathcal{D}(f)$ is regular and compact. Let us set $H = E_x$. By regularity every vertical section C_y equals $g_y \cdot H$ for some g_y in G, and moreover all essential ranges are conjugate to H (see Section 2). Since E_{x_0} is trivial so must be H, and therefore the set C projects injectively onto X. This implies that C is the graph of a continuous function $b : X \to G$ and $b(Ty) = f(y) \cdot b(y)$ for every y in X. Thus $f(y) = b(Ty) \cdot b(y)^{-1}$ is a coboundary. ∎

4. Regularity in general Lie groups. Throughout this section we will assume that G is a connected Lie group, and $\mathfrak{G}$ is its Lie algebra. As usual, the group $\mathrm{Aut}(G)$ of all bicontinuous automorphisms of G is considered as a (closed) subgroup of $\mathrm{GL}(\mathfrak{G})$. We denote by $\mathrm{Ad}(G)$ the image of G under the adjoint representation. Since G is connected, $\mathrm{Ad}(G)$ is contained in $\mathrm{Aut}(G)^0$, the identity component of the automorphism group, which is an almost algebraic subgroup of $\mathrm{GL}(\mathfrak{G})$ (i.e. of finite index in some algebraic subgroup of $\mathrm{GL}(\mathfrak{G})$; this is a theorem of D. Wigner, cf. [Da92]).

For any cocycle f with values in G we define its *adjoint cocycle* by setting

$$\mathrm{Ad}(f)(n, x) = \mathrm{Ad}(f(n, x)),$$

which is a cocycle taking values in $\mathrm{Ad}(G) \subseteq \mathrm{GL}(\mathfrak{G})$. It is clear that if f is continuous and recurrent so is $\mathrm{Ad}(f)$.

The following proposition describes the behaviour of the identity component of an essential range under conjugation by the cocycle f. Its proof essentially uses the local closedness of the orbit of a connected subgroup H under the action of an almost algebraic group of automorphisms. From this point of view it does not contain much new compared to [Da02].

Proposition 4.1. *Suppose T is a minimal homeomorphism of a compact metric space X and f is a continuous cocycle with values in a connected Lie group G. Choose any almost algebraic and closed subgroup A in* $\mathrm{Aut}(G)^0$ *which contains* $\mathrm{Ad}(G)$. *If x is any point from $\mathcal{D}(f)$ and $I_A(H) = \{\alpha \in A : \alpha(H) = H\}$ is the stabiliser of the identity component $H = E_x^0$ in A, then the orbit closure*

$$C^* = \overline{T^{\mathbb{Z}}_{\mathrm{Ad}(f)}(x, I_A(H))}$$

taken in $X \times A/I_A(H)$ has the following properties:

(i) *it is compact and the action of $T_{\mathrm{Ad}(f)}$ restricted to C^* is minimal*,

(ii) *it projects onto X, and injectively onto the set $\mathcal{D}(f)$.*

In other words, the system $(C^*, T_{\mathrm{Ad}(f)})$ is an almost one-to-one extension of (X, T).

Proof. Let $\mathcal{H}(\mathfrak{G})$ be the Grassmannian manifold of all subalgebras of our Lie algebra $\mathfrak{G}$. Let x be as above, and $\mathfrak{H}_x$ be the subalgebra that corresponds to the identity component $H_x = E_x^0$. We choose open neighbourhoods $\mathfrak{U}$ of 0 in $\mathfrak{G}$ and U of e in G such that the exponential mapping is a diffeomorphism between $\mathfrak{U}$ and U. If $\{n_k\}_{k\geq 1}$ is any sequence of integers such that $T^{n_k}x \to y \in \mathcal{D}(f)$ then by compactness of $\mathcal{H}(\mathfrak{G})$ we have convergence (along some subsequence) of the conjugate subalgebras

$$\mathfrak{H}_k = \mathrm{Ad}(f(n_k, x))\mathfrak{H}_x \to \mathfrak{H}',$$

where $\mathfrak{H}'$ is some subalgebra of the same dimension as $\mathfrak{H}$. As $\exp(\mathfrak{H}_k \cap \mathfrak{U}) \subseteq E^0_{T^{n_k}x} \cap U$ we conclude from semicontinuity of the essential ranges that $\exp(\mathfrak{H}' \cap \mathfrak{U}) \subseteq E_y \cap U$, and as E_y is a closed group, E_y^0 contains the closed subgroup generated by $\exp(\mathfrak{H}')$. Thus if we denote by $\mathfrak{H}_y$ the algebra corresponding to $H_y = E_y^0$, then $\mathfrak{H}' \subseteq \mathfrak{H}_y$. By the same reasoning, if $\{m_k\}_{k\geq 1}$ is such that $T^{m_k}y \to x$ we may again assume convergence (along some subsequence) of

$$\mathrm{Ad}(f(m_k, y))\mathfrak{H}_y \to \mathfrak{H}'',$$

where $\mathfrak{H}''$ is a subalgebra of the same dimension as $\mathfrak{H}_y$, and that E_x^0 contains the closed subgroup generated by $\exp(\mathfrak{H}'')$. Therefore $\mathfrak{H}'' \subseteq \mathfrak{H}_x$ and since $\mathfrak{H}''$ has at least the dimension of $\mathfrak{H}_x$ we conclude that $\mathfrak{H}'' = \mathfrak{H}_x$ and also $\mathfrak{H}' = \mathfrak{H}_y$. In other words, $\mathfrak{H}_y$ is in the closure of the A-orbit of $\mathfrak{H}_x$ and vice versa. As A is almost algebraic its orbits on $\mathcal{H}(\mathfrak{G})$ are locally closed [Zi, Corollary 3.2.12], which is the same as saying that the factor map

$$A/I_A(\mathfrak{H}_x) \to \mathcal{H}(\mathfrak{G}), \qquad \alpha \cdot I_A(\mathfrak{H}_x) \mapsto \alpha(\mathfrak{H}_x),$$

with $I_A(\mathfrak{H}_x) = \{\alpha \in A : \alpha(\mathfrak{H}_x) = \mathfrak{H}_x\}$, is a homeomorphism between $A/I_A(\mathfrak{H}_x)$ and the orbit $\mathfrak{H}_x^A = \{\alpha(\mathfrak{H}_x) : \alpha \in A\}$ (cf. [Zi, Lemma 2.1.15]). We therefore deduce that $\mathfrak{H}_y$ must belong to $\mathfrak{H}_x^A$, as otherwise we obtain a contradiction to local closedness of the A-orbits. Hence $\mathfrak{H}_y = \alpha_y(\mathfrak{H}_x)$ for some α_y which is uniquely determined modulo $I_A(\mathfrak{H}_x)$, and

$$\operatorname{Ad}(f(n_k, x)) \cdot I_A(\mathfrak{H}_x) \to \alpha_y \cdot I_A(\mathfrak{H}_x)$$

along this subsequence of $\{n_k\}_{k\geq 1}$. This means that y is contained in the π_X-projection of the orbit closure $C^* = \overline{T^{\mathbb{Z}}_{\operatorname{Ad}(f)}(x, I_A(\mathfrak{H}_x))}$. Since y in $\mathcal{D}(f)$ was chosen arbitrarily, the orbit closure C^* projects onto $\mathcal{D}(f)$. By Lemma 2.1 we can find a compact subset K in G such that

$$(X \times K \cdot I_A(\mathfrak{H}_x)) \cap C^*$$

projects onto the whole set X. Since for every $y \in \mathcal{D}(f)$ the vertical section $C^*_y = \{\alpha_y \cdot I_A(\mathfrak{H}_x)\}$ is contained in the compact set $K \cdot I_A(\mathfrak{H}_x)$ we conclude that the whole closure C^* is contained in the compact set $X \times K \cdot I_A(\mathfrak{H}_x)$.

Minimality of C^* is clear since T is minimal and the vertical section C^*_x consists of a single point only. ∎

Remark 4.2. It follows immediately from the above proof that on the comeagre set $\mathcal{D}(f)$ all identity components E^0_y are A-conjugate, i.e. for every $x, y \in \mathcal{D}(f)$, E^0_y is the A-image of E^0_x.

The connection of Proposition 4.1 with a general regularity theorem as mentioned in the introduction is as follows: If we could prove that the almost one-to-one extension C^* in Proposition 4.1 projects injectively onto the whole set X, then the mapping

$$y \mapsto C^*_y = \alpha_y \cdot I_A(H)$$

is continuous and therefore

$$H_y = \alpha_y(H)$$

defines a consistent selection $\{H_y\}_{y\in X}$. Thus if T is a minimal rotation on a locally connected compact group X, we would be able to conclude with the help of the generalised Atkinson's Proposition 3.4 that every f admits strongly regular orbit closures. This makes the following open question so important for us:

Open question 4.3. Let T_f be a continuous G-extension of a minimal group rotation T (or more generally any minimal homeomorphism), and H be a closed subgroup of G. Suppose $C \subseteq X \times G/H$ is a T_f-invariant compact set such that for every x belonging to a dense G_δ-set in X the vertical section $C_x = \{gH \in G/H : (x, gH) \in C\}$ consists of a single coset $g_x H$. Is it true that then the same holds for every x in X?

This question can be answered positively for certain cases, as shown in [GH05]; for example, if for every $g \notin H$ we know that

$$e \notin \overline{HgH} = \overline{\{h_1 \cdot g \cdot h_2 : h_1, h_2 \in H\}},$$

which is always satisfied in any nilpotent (or virtually nilpotent) group G [GH05, Theorem 3.1], or if H is a normal subgroup of a (not necessarily nilpotent) group G. However, it is not clear to us whether the answer is affirmative for such a general formulation.

Now, let us provide a special version of Proposition 4.1, in which we replace the almost algebraic group A by $\mathrm{Ad}(G)$ itself. This version parallels the result on the identity components of the stabilisers of general Borel actions [Da02, Corollary 5.3].

PROPOSITION 4.4. *Suppose T is a minimal homeomorphism of a compact metric space X and f is a continuous cocycle with values in a connected Lie group G. Let H be the identity component of E_x at some point $x \in \mathcal{D}(f)$, and $N(H)$ its normaliser in G. Further, assume that one of the following properties from* [Da02, Theorem 3.2] *holds*;

(i) $\mathrm{Ad}(G)$ *is almost algebraic*;
(ii) *for all g from the radical of G, the eigenvalues of* $\mathrm{Ad}(g)$ *are real*;
(iii) *H is compact.*

Then the T_f-orbit closure $C^* = \overline{T_f^{\mathbb{Z}}(x, N(H))}$ in $X \times G/N(H)$ is minimal and compact and projects almost one-to-one onto X. Moreover, for all y in $\mathcal{D}(f)$, the identity component E_y^0 is conjugate to H.

REMARK 4.5. There are several criteria given in [Da02, Proposition 3.4] which guarantee that the group $\mathrm{Ad}(G)$ itself is almost algebraic, for example this is the case when G is an almost algebraic subgroup of $\mathrm{GL}(n, \mathbb{R})$ for $n \geq 2$ or G is semisimple.

Proof of Proposition 4.4. By [Da02, Theorem 3.2], if one of the three conditions is satisfied the conjugacy class $H^G = \{g \cdot H \cdot g^{-1} : g \in G\}$ is locally closed in the space $\mathcal{C}(G)$ of closed subgroups and therefore the map

$$G/N(H) \to H^G, \quad g \cdot N(H) \mapsto g \cdot H \cdot g^{-1},$$

is a homeomorphism. Using this fact—considering the adjoint action of f on $\mathcal{C}(G)$ rather than on the Grassmannian $\mathcal{H}(G)$—we conclude in the same manner ([2]) as in the proof of Proposition 4.1 that the orbit closure of $C^* = \overline{T_f^{\mathbb{Z}}(x, N(H))}$ in $X \times G/N(H)$ is minimal, compact and projects injectively onto the comeagre set $\mathcal{D}(f)$. Furthermore, these properties of C^* immediately imply the assertion on the identity components E_y^0 (cf. also Remark 4.2). ∎

Proposition 4.4 together with Proposition 3.4 also yields an alternative proof of the regularity result [GH05, Theorem 4.9].

([2]) The only detail which has to be considered additionally is the semicontinuity of dimension: if $g_k \cdot H \cdot g_k^{-1} \to H'$ with respect to the Fell topology, then $\dim H' \geq \dim H$.

COROLLARY 4.6 ([GH05, Theorem 4.9]). *Let T be a minimal rotation on a locally connected compact group X, and G a connected nilpotent Lie group. If f is a continuous and recurrent cocycle with values in G then f is strongly regular. Furthermore, all E_x are conjugate and E_x/E_x^0 is compact.*

Proof. Every $\mathrm{Ad}(g)$ has real eigenvalues only (actually, all eigenvalues are equal to one) and satisfies condition (ii) from [Da02, Theorem 3.2] listed in 4.4. Thus the orbit closure C^* of $(x, N(H))$ is compact and projects onto X, whereas it projects injectively onto the set $\mathcal{D}(f)$. Since Question 4.3 is answered positively for nilpotent groups, the set C^* must be a one-to-one extension of X and so $H_{T^n x} = f(n,x) \cdot H \cdot f(n,x)^{-1}$ extends to a consistent selection of conjugate subgroups. Now Proposition 3.4 yields the assertion of the corollary. ∎

Another consequence of Proposition 4.4 is the following partial result on regularity, which holds even for an arbitrary minimal compact system (X,T).

COROLLARY 4.7. *Suppose that G is a connected Lie group with one of the properties listed in Proposition 4.4. If for some point $x \in \mathcal{D}(f)$, the identity component $H = E_x^0$ equals its own normaliser in G, then the T_f-orbit closure of (x,e) is surjective and hence f is regular.*

Proof. The assertion of the corollary is evident from Proposition 4.4, since for every $x \in \mathcal{D}(f)$ we have $N(H) = H \subseteq P_x$. ∎

5. Regularity results for $\mathbb{R}^d \rtimes \mathbb{R}$. Let $\mathbb{R}$ act continuously by linear automorphisms A_u ($u \in \mathbb{R}$) on $\mathbb{R}^d$, and G be the semidirect product $G = \mathbb{R}^d \rtimes \mathbb{R}$ defined by the group operation

$$(v_1, u_1) \cdot (v_2, u_2) = (v_1 + A_{u_1}(v_2), u_1 + u_2).$$

With this definition the sets

$$U = \{e\} \times \mathbb{R} \quad \text{and} \quad N = \mathbb{R}^d \times \{e\}$$

are subgroups of G, with N normal in G, and conjugation by u in U equals the automorphism A_u on N. Let

$$\pi : G = \mathbb{R}^d \rtimes \mathbb{R} \to \mathbb{R}$$

denote the projection of G onto its second coordinate, and denote by

$$\pi(f)(n,x) = \pi(f(n,x))$$

the factor cocycle with values in $\mathbb{R}$.

Although Question 4.3 remains open even for this special group, we are able to prove the existence of surjective orbit closures, as the following theorem shows. Its proof involves a direct proof of compactness of the cocycle modulo the normaliser $N(H)$ of the identity component $H = E_x^0$, and uses the simple group structure to reduce to the case where H equals its own stabiliser.

THEOREM 5.1. *Let f be a continuous and recurrent cocycle over a minimal rotation on a locally connected compact group X with values in the semidirect product $G = \mathbb{R}^d \rtimes \mathbb{R}$. If the action of $\mathbb{R}$ on $\mathbb{R}^d$ has no eigenvalue equal to one* (3), *then f is regular.*

REMARK 5.2. Assuming the action of $\mathbb{R}$ has no eigenvalue equal to one implies (but is not equivalent to) the following local property: Let $\mathfrak{G}$ be the Lie algebra of G and $\mathfrak{N}$ be the subalgebra which corresponds to the normal abelian kernel N. Then for every vector $\mathfrak{h}$ which is not contained in the ideal $\mathfrak{N}$,

$$[\mathfrak{h}, \mathfrak{n}] = \mathrm{ad}(\mathfrak{h}, \mathfrak{n}) \neq 0$$

for all $\mathfrak{n}$ in $\mathfrak{N}$.

Proof. Step 1. Let x be any point from our non-meagre set $\mathcal{D} = \mathcal{D}(f) \cap \mathcal{D}(\pi(f))$, and let S be the essential range of the projected cocycle $\pi(f)$ at the point x. Then the inverse image

$$A = \pi^{-1}(\overline{T^{\mathbb{Z}}_{\pi(f)}(x, e)})$$

of the regular orbit closure of (x, e) with respect to the projected cocycle is regular in the sense that every slice $A_y = \{g \in G : (x, g) \in A\}$ of A consists of a single coset $g_y \cdot \pi^{-1}(S)$, and further the map

$$X \to G/\pi^{-1}(S), \quad y \mapsto A_y = g_y \cdot \pi^{-1}(S),$$

is continuous. For every g in $\pi^{-1}(S)$ we can find a sequence $\{n_k\}_{k\geq 1}$ and $v_k \in N$ such that $T^{n_k}x \to x$ and $f(n_k, x) \cdot v_k \to g$. Thus

$$\begin{aligned} E_{T^{n_k}x} \cap N &= f(n_k, x) \cdot (E_x \cap N) \cdot f(n_k, x)^{-1} \\ &= f(n_k, x) \cdot v_k \cdot (E_x \cap N) \cdot v_k^{-1} \cdot f(n_k, x)^{-1} \end{aligned}$$

since N is abelian; by letting $k \to \infty$ it follows that

$$E_x \cap N \supseteq g \cdot (E_x \cap N) \cdot g^{-1}.$$

Thus $\pi^{-1}(S)$ is contained in the normaliser $N(E_x \cap N)$ and the map $y \mapsto g_y \cdot N(E_x \cap N)$ is continuous. Use this map to define a consistent selection $\{N_y\}_{y \in X}$ of subgroups conjugate to $N_x = E_x \cap N$ by setting

$$N_y = g_y \cdot (E_x \cap N) \cdot g_y^{-1}.$$

It is important to note that by symmetry (4) $N_y = E_y \cap N$ for all y from our comeagre set $\mathcal{D}$.

(3) By which we mean that for every $u \in \mathbb{R}$ the transformation A_u has no eigenvalue equal to one.

(4) We could have started with any other y in $\mathcal{D}$.

Step 2. We let $H = E_x^0$ be the identity component of E_x and $\hat{H} = N(H)^0$ the identity component of the normaliser $N(H)$ and claim that

$$C = \overline{T_f^{\mathbb{Z}}(x, \hat{H} \cap N)}$$

projects onto the whole space X. Let y be any point in $\mathcal{D}$ and choose $T^{n_k}x$ converging to y so that $f(n_k, x) = g_k \cdot v_k$, with $g_k \to g$ and v_k in the kernel $N = \ker(\pi)$. We denote by $\mathfrak{H}$ and $\hat{\mathfrak{H}}$ the subalgebras that correspond to H and $\hat{H}$. The conjugate subgroups

$$E_{T^{n_k}x} = f(n_k, x) \cdot H \cdot f(n_k, x)^{-1} = g_k \cdot v_k \cdot H \cdot v_k^{-1} \cdot g_k^{-1}$$

correspond to the subalgebras

$$\mathrm{Ad}(f(n_k, x))\mathfrak{H} = \mathrm{Ad}(g_k)\,\mathrm{Ad}(v_k)\mathfrak{H}.$$

For any $\mathfrak{v}$ from the subalgebra $\mathfrak{N}$ corresponding to N we know that $\mathrm{ad}(\mathfrak{v})(\cdot) \in \mathfrak{N}$, since N is normal. Moreover, $\mathrm{ad}(\mathfrak{v})(\cdot) = 0$ on $\mathfrak{N}$, as N is abelian. Thus $\mathrm{ad}(\mathfrak{v})^j = 0$ for all $j \geq 2$ and one can calculate

$$\mathrm{Ad}(v) = \exp(\mathrm{ad}(\mathfrak{v}))(\mathfrak{h}) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathrm{ad}(\mathfrak{v})^k = 1 + \mathrm{ad}(\mathfrak{v}),$$

where $v = \exp(\mathfrak{v})$. This implies that

$$\mathrm{Ad}(f(n_k, x))\mathfrak{H} = \mathrm{Ad}(g_k)(1 + \underbrace{[\mathfrak{v}_k, \,\cdot\,]}_{\in \mathfrak{N}})\mathfrak{H},$$

with any choice of $\mathfrak{v}_k \in \mathfrak{N}$ such that $v_k = \exp(\mathfrak{v}_k)$. Note that since $\mathfrak{H} \cap \mathfrak{N} = \mathfrak{N}_x$, where $\mathfrak{N}_x$ is the subalgebra that corresponds to N_x, we have

$$\hat{\mathfrak{H}} \cap \mathfrak{N} = \{\mathfrak{v} \in \mathfrak{N} : [\mathfrak{v}, \mathfrak{H}] \subseteq \mathfrak{N}_x\}.$$

Assume for the moment that the $\mathfrak{v}_k + (\hat{\mathfrak{H}} \cap \mathfrak{N})$ are unbounded in $\mathfrak{N}/(\hat{\mathfrak{H}} \cap \mathfrak{N})$. Then we can find ([5]) a vector $\mathfrak{h}$ in $\mathfrak{H}$ such that along some subsequence

$$[\mathfrak{v}_k, \mathfrak{h}] + \mathfrak{N}_x \to \infty$$

in the quotient space $\mathfrak{N}/\mathfrak{N}_x$. This implies that the one-dimensional spaces

$$(1 + [\mathfrak{v}_k, \cdot\,])(\langle \mathfrak{h} \rangle)$$

converge to some one-dimensional space $\langle \mathfrak{h}' \rangle$ contained in $\mathfrak{N}$ but not in $\mathfrak{N}_x$. As the $\{N_y\}_{y \in X}$ form a consistent selection,

$$\mathrm{Ad}(g)(\mathfrak{N}_x) = \lim_k \mathrm{Ad}(g_k)(\mathfrak{N}_x) = \lim_k \mathrm{Ad}(f(n_k, x))(\mathfrak{N}_x) = \mathfrak{N}_y,$$

and the subspaces

$$\mathrm{Ad}(g_k)(1 + [\mathfrak{v}_k, \cdot\,])(\langle \mathfrak{h} \rangle)$$

([5]) Choose any linear functional $\Lambda : \mathfrak{N} \to \mathbb{R}$ such that $\ker \Lambda = \mathfrak{N}_x$. Then every $\mathfrak{v}$ in $\mathfrak{N}$ defines a linear functional $\Lambda_{\mathfrak{v}}$ on $\mathfrak{H}$ by putting $\Lambda_{\mathfrak{v}}(\mathfrak{h}) = \Lambda([\mathfrak{v}, \mathfrak{h}])$. Then $\hat{\mathfrak{H}} \cap \mathfrak{N}$ is the kernel of the linear map $\mathfrak{v} \mapsto \Lambda_{\mathfrak{v}}$. As $\mathfrak{H}$ is finite-dimensional, boundedness of the $\mathfrak{v}_k$ modulo $\hat{\mathfrak{H}} \cap \mathfrak{N}$ is equivalent to boundedness of the $\Lambda_{\mathfrak{v}_k}(\mathfrak{h})$ for every $\mathfrak{h}$ in $\mathfrak{H}$.

converge to the one-dimensional subspace $\mathrm{Ad}(g)(\langle\mathfrak{h}'\rangle)$ which is contained in $\mathfrak{N}$ but not in $\mathfrak{N}_y$. By semicontinuity of the essential ranges, the immersed subgroup corresponding to this one-dimensional subspace is contained in E_y but not in N_y (as in the proof of Proposition 4.1, this follows easily from the fact that the exponential mapping is a local diffeomorphism). This contradicts the fact that $N_y = E_y \cap N$. Thus the $\mathfrak{v}_k + (\hat{\mathfrak{H}} \cap \mathfrak{N})$ stay in some compactum and the same is true for the $v_k \cdot (\hat{H} \cap N)$. This proves that the T_f-orbit closure modulo $\hat{H} \cap N$ projects onto $\mathcal{D}$ and Lemma 2.1 shows that it projects onto the whole space X.

Step 3. Now, we distinguish two cases: If H is contained in the normal subgroup N, then $N_x = H = E_x^0$ and there exists a cutting neighbourhood for N_x in E_x. Proposition 3.3 yields the existence of surjective closures.

If H is not contained in N, then there exists $\mathfrak{h} \in \mathfrak{H}$ outside $\mathfrak{N}$. By Remark 5.2 the linear transformation $[\mathfrak{h}, \cdot]$ maps $\mathfrak{N}$ bijectively onto itself; and the same is true for the invariant subspace $\mathfrak{N}_x$. Thus for any $\mathfrak{v} \in \mathfrak{N}$ outside $\mathfrak{N}_x$ we must also have $[\mathfrak{h}, \mathfrak{v}] \notin \mathfrak{N}_x$ and so $\mathfrak{v} \notin \hat{\mathfrak{H}}$. Therefore $\hat{H} \cap N = H \cap N$ and Step 2 together with the fact that $H \subseteq P_x$ shows that the T_f-orbit closure of (x, e) is surjective. ∎

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