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SQUARE SUBGROUPS OF RANK TWO ABELIAN GROUPS

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Abstract. Let G be an abelian group and $\square G$ its square subgroup as defined in the introduction. We show that the square subgroup of a non-homogeneous and indecomposable torsion-free group G of rank two is a pure subgroup of G and that $G/\square G$ is a nil group.

1. Introduction. In this paper, all groups are abelian and written additively. A ring R is said to be a *ring on* G if the group G is isomorphic to the additive group of R. In this situation we write $R = (G, *)$ where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. In general, we call a group G a *nil group* if there is no ring on G other than the zero ring. A generalization of the notion of nil group was considered by Feigelstock [5]. Let H be a subgroup of G; then G is *nil modulo* H if $G * G \subseteq H$ for every ring $(G, *)$ on G. Clearly G is a nil group if and only if G is nil modulo $\{0\}$. Feigelstock [5] shows that if H is a divisible subgroup of G and G is nil modulo H, then G/H is a nil group. Also he asks if this is true in general. In other words, does G nil modulo H imply that G/H is a nil group? Stratton and Webb [8] show that the general answer must be no. However, the question has a positive answer if either G is a torsion group or H is a direct summand of G.

It is clear that if G is nil modulo H_1 and modulo H_2 then it is nil modulo $H_1 \cap H_2$. This suggests the following definition of the *square subgroup* $\square G$:

$$\square G = \bigcap \{H \subseteq G \mid G \text{ is nil modulo } H\}.$$

Clearly $\square G$ is the smallest subgroup with the property that G is nil modulo $\square G$. For the first time the square subgroup was studied by Stratton and Webb [8]. The basic question is whether $G/\square G$ is a nil group, and if this is not true in general then under what conditions it is true and why it fails. Aghdam [2] shows that if G is an arbitrary group, then

$$G/\square G \cong (D/T) \oplus (N/\square N), \qquad \square D \leq T \leq D,$$

where D and N are the maximal divisible subgroup and the reduced part

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of G respectively. Moreover, if G is a reduced torsion group then $G = \Box G$, and if G is non-torsion, then

$$G/\Box G \cong N/\Box N.$$

In this paper we show that the square subgroup of any non-homogeneous and indecomposable torsion-free group G of rank two is a pure subgroup of G and that $G/\Box G$ is a nil group. For this, we study $\Box G$ by classifying G according to the cardinality of the type set of G.

2. Notations and preliminaries. Let G be a torsion-free abelian group. The *type set* of G is the partially ordered set of types, i.e.,

$$T(G) = \{t(x) \mid 0 \neq x \in G\},$$

where $t(x)$ denotes the type of x. We also write $h_p^G(x)$ for the p-height of x and $\langle x\rangle^*$ for the pure subgroup of G generated by x. A type $t \in T(G)$ is said to be *maximal* if for all $\mu \in T(G)$, $\mu \geq t$ implies that $\mu = t$. A good reference for basic facts about type and other undefined concepts is [6, pp. 109ff].

PROPOSITION 2.1. *Let G be a torsion-free group of finite rank. Then the length of every chain in $T(G)$ is at most equal to the rank of G.*

Proof. See [4, Proposition 1]. ∎

THEOREM 2.2. *A torsion-free ring of rank one is either a zero ring or isomorphic to a subring of the rational number field. A torsion-free group of rank one is not a nil group if and only if its type is idempotent.*

Proof. See [6, Theorem 121.1]. ∎

LEMMA 2.3. *Let G be a subgroup of $\mathbb{Q}$. If $1/b, 1/d \in G$ with $(b, d) = 1$ then $1/bd \in G$.*

Proof. Obvious. ∎

We recall some definitions and results from [3]. Let x, y be independent elements of a torsion-free group G of rank two. Each element w of G has a unique representation $w = ux + vy$, where u, v are rational numbers. Let

$$U_0 = \{u_0 \in \mathbb{Q} : u_0x \in G\}, \quad U = \{u \in \mathbb{Q} : ux + vy \in G \text{ for some } v \in \mathbb{Q}\},$$
$$V_0 = \{v_0 \in \mathbb{Q} : v_0y \in G\}, \quad V = \{v \in \mathbb{Q} : ux + vy \in G \text{ for some } u \in \mathbb{Q}\}.$$

Then U_0 and V_0 are subgroups of U and V respectively. U, U_0, V, V_0 are called the *groups of rank one belonging to the independent set* $\{x, y\}$.

THEOREM 2.4. *Let G be a torsion-free abelian group of rank two. If U, U_0, V, V_0 are the groups of rank one belonging to $\{x, y\}$, then $U/U_0 \cong V/V_0$.*

Proof. See [3, p. 107]. ∎

PROPOSITION 2.5. *Let G be a torsion-free group of rank two and x, y independent elements of G. Assume U, U_0, V, V_0 are the rank one groups belonging to $\{x, y\}$. Then $G/\langle y\rangle^* \cong U$ and $G/\langle x\rangle^* \cong V$.*

Proof. Clearly $V_0 y = \langle y\rangle^*$ is the kernel of the epimorphism $\varphi : G \to U$ defined by $\varphi(ux + vy) = u$ for any $ux + vy \in G$, and thus the first assertion follows. The second is obtained similarly. ∎

PROPOSITION 2.6. *Let A, B be subgroups of $\mathbb{Q}$ such that $1 \in A \cap B$. Suppose there exists a non-zero integer n such that $nA \leq B$. If m is the least such positive integer, then the following statements hold:*

(a) *Let p be a prime number such that $\alpha = h_p^A(1) < \beta = h_p^B(1)$. Then $\frac{1}{p^{k-\alpha}}(mA) \leq B$ for all $k \leq \beta$. Furthermore, p does not divide m.*
(b) *If $B \leq A$ then $mA = B$ and $1/m \in A$.*
(c) *Let d be a positive integer such that d divides m and $1/d \in B$. If $B^2 = B$ then $d = 1$.*

Proof. (a) Let $k \leq \beta$. Then $k - \alpha \leq \beta$, so $1/p^{k-\alpha} \in B$, and if $1/p^e \in A$, then $e \leq \alpha$, hence $k + e - \alpha \leq k \leq \beta$, therefore $\frac{1}{p^{k-\alpha}} \cdot \frac{1}{p^e} = \frac{1}{p^{k+e-\alpha}} \in B$, which implies $\frac{1}{p^{k-\alpha}}\left(m\frac{1}{p^e}\right) \in B$. Also if $1/r \in A$ and $(r, p) = 1$, then $m/r \in B$, and since $1/p^{k-\alpha} \in B$, Lemma 2.3 yields $\frac{1}{p^{k-\alpha}}\left(\frac{m}{r}\right) \in B$. Consequently, $\frac{1}{p^{k-\alpha}}(mA) \leq B$ as required.

Let $k = \alpha + 1$ in $\frac{1}{p^{k-\alpha}}(mA) \leq B$. Then $\frac{1}{p}(mA) \leq B$. Now if p divided m then m/p would be an integer, so $\frac{m}{p}(A) \leq B$, which contradicts the hypothesis that m is the least positive integer with $mA \leq B$. Therefore p does not divide m.

(b) The hypothesis implies that A/mA is cyclic of order m, for if A/mA were of order d, d a proper divisor of m, then $dA \leq mA \leq B$, which contradicts the minimality of m. Furthermore, B/mA is cyclic of order s, where s is a divisor of m. Suppose that $m = sr$. Then

$$B/mA = r(A/mA) = rA/mA,$$

and consequently $B = rA$. Now the minimality of m implies that $m = r$, hence $B = mA$, and since $1 \in B$, we have $1/m \in A$.

(c) Suppose m' is an integer such that $m = dm'$. Then

$$m'A = (mA)\left(\frac{1}{d}\right) \leq B\left(\frac{1}{d}\right) \leq B^2 = B.$$

So if $d > 1$ then $m' < m$, which contradicts the minimality of m, hence $d = 1$. ∎

PROPOSITION 2.7. *Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$ with $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in G$ be such that $t(x) = t_1$ and*

$t(y) = t_2$. If t_1, t_2 are incomparable, then any ring on G satisfies $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ for some $a, b \in \mathbb{Q}$.

Proof. Let $z \in G$ with $t(z) = t_0$. Then $z \notin G(t_1)$. But since $G(t_1)$ is a pure subgroup of G, it is of rank one. Now since $t(x^2) \geq t(x) = t_1$, both x^2 and x belong to $G(t_1)$ so they are dependent, that is, $x^2 = ax$ for some $a \in \mathbb{Q}$. Similarly, $y^2 = by$ for some $b \in \mathbb{Q}$.

On the other hand, $t(yx) \geq t(x)$, so yx and x belong to $G(t_1)$, therefore $yx = ex$ for some $e \in \mathbb{Q}$ and similarly $yx = fy$ for some $f \in \mathbb{Q}$. Now if $yx \neq 0$ then $t(x) = t(xy) = t(y)$, contrary to our hypothesis, therefore $yx = 0$. By the same reasoning, $xy = 0$. ∎

LEMMA 2.8. *Let G be a torsion-free group of rank two. Let x, y be independent elements of G, and U, U_0, V, V_0 rank one groups belonging to $\{x, y\}$. Suppose that $U_0^2 = U_0$ and there exists an integer m such that $mU = U_0$. Then the multiplication*

$$x^2 = m^2 x, \qquad xy = yx = y^2 = 0$$

yields a ring on G such that $G^2 = U_0 x$.

Proof. Let $g_1 = u_1 x + v_1 y$ and $g_2 = u_2 x + v_2 y$ be arbitrary in G. Then $u_1, u_2 \in U$ and $g_1 g_2 = m^2 u_1 u_2 x$. Also, $m^2 u_1 u_2 = (mu_1)(mu_2) \in (mU)^2 = U_0^2 = U_0$, hence $m^2 u_1 u_2 x \in U_0 x \subseteq G$. Thus the product actually lies in G, which yields a ring structure on G such that $G^2 \leq U_0 x$. Now in view of $U_0^2 = U_0$ and $mU = U_0$ we have $(mU)^2 = U_0^2 = U_0$, hence any $u_0 \in U_0$ may be written in the form $u_0 = (mu_1)(mu_2)$ for some $u_1, u_2 \in U$. By definition of U there exist elements $u_1 x + v_1 y$ and $u_2 x + v_2 y$ in G such that $(u_1 x + v_1 y)(u_2 x + v_2 y) = u_0 x$, which yields $U_0 x \leq G^2$. Consequently, $G^2 = U_0 x$ as required. ∎

3. Type set has cardinality greater than or equal to three

LEMMA 3.1. *Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$ with $t_1^2 = t_1$, $t_2^2 \neq t_2$, $t_0 < t_1$, $t_0 < t_2$. Then $\square G$ is a pure subgroup of G and $G/\square G$ is a nil group.*

Proof. First, we observe that if G is a nil group then $\square G = 0$, so we are done. Now let $x, y \in G$ be such that $t(x) = t_1$, $t(y) = t_2$, and let U, U_0, V, V_0 be rank one groups belonging to $\{x, y\}$; we may assume that $U_0^2 = U_0$. Note that our hypotheses ensure that $t_1 \neq t_2$ and in view of Proposition 2.1, t_1 and t_2 are incomparable. Now let R be an arbitrary non-trivial ring on G. Then by Proposition 2.7,

$$x^2 = ax, \qquad xy = yx = 0, \qquad y^2 = by,$$

for some $a, b \in \mathbb{Q}$. If $b \neq 0$, then $t(y) = t(y^2) \geq t^2(y)$, which implies that $t(y)$ is idempotent, a contradiction to our hypothesis, so $y^2 = 0$. Furthermore, since R is non-trivial, a is non-zero. Now pick $z, z' \in G$. Then $z = ux + vy$ and $z' = u'x + v'y$ for some $u, v, u', v' \in \mathbb{Q}$, hence $zz' = auu'x$, which implies $G^2 \subseteq \langle x \rangle^*$. But since R is arbitrary, we have $\square G \subseteq \langle x \rangle^*$. Now suppose $u \in U$. Then there exists $v \in \mathbb{Q}$ such that $ux + vy \in G$, so $(ux + vy)x = aux$, hence $au \in U_0$ for all $u \in U$, thus $aU \leq U_0 \leq U$. It follows that there is a positive integer k such that $kU \leq U_0$; if m is the least such integer, Proposition 2.6(b) yields $mU = U_0$. We may now apply Lemma 2.8 to construct a ring on G satisfying $G^2 = U_0x$, thus $\langle x \rangle^* \subseteq \square G$ and consequently $\square G = \langle x \rangle^*$.

Therefore $G/\square G = G/\langle x \rangle^*$ and by Proposition 2.5, $G/\square G \cong V$. On the other hand, Theorem 2.4 yields $U/U_0 \cong V/V_0$, and since $mU = U_0$, we have $mV \cong V_0$, hence $t(V) = t(V_0)$. Now if $G/\square G$ were not a nil group then by Theorem 2.2, $t(G/\square G) = t(V)$ would be idempotent, hence in view of $t(V) = t(V_0)$ we conclude that $t(V_0) = t_2$ is idempotent, contrary to assumption. Therefore $G/\square G$ is a nil group. ∎

Proposition 3.2. *Let A, B be subgroups of $\mathbb{Q}$ satisfying $1 \in B \leq A$ and $B^2 = B$. Suppose that $mA \leq B$ and m is the least positive integer with this property. Then $1/m + B$ generates A/B as a cyclic group.*

Proof. Our hypotheses together with Proposition 2.6(b) imply that $mA = B$, $1/m \in A$ and A/B is cyclic of order m. Thus it is sufficient to show that $1/m + B$ has order m in A/B. Suppose not, i.e., $1/m + B$ has order $d < m$. Then $d/m \in B$, hence $dA = \frac{d}{m}(mA) \leq B^2 = B$, contradicting the minimality of m. ∎

Lemma 3.3. *Let G be a torsion-free group of rank two and $T(G) = \{t_0, t_1, t_2\}$ with $t_0 < t_1$, $t_0 < t_2$, $t_1^2 = t_1$, $t_2^2 = t_2$ and t_1, t_2 incomparable. If G is not a nil group then $\square G = G$.*

Proof. Let $x, y \in G$ be such that $t(x) = t_1$, $t(y) = t_2$, and let U, U_0, V, V_0, be rank one groups belonging to $\{x, y\}$. Also, suppose that $U_0^2 = U_0$ and $V_0^2 = V_0$. Now if R is a non-trivial ring on G then by Proposition 2.7,

$$x^2 = ex, \quad xy = yx = 0, \quad y^2 = ry,$$

for some $e, r \in \mathbb{Q}$. We may assume, without loss of generality, that $e \neq 0$. For any $u \in U$ there exists $v \in \mathbb{Q}$ such that $ux + vy \in G$, hence $(ux + vy)x = eux$, which implies that $eu \in U_0$, and since u is arbitrary, we have $eU \leq U_0$. Consequently, there is an integer n such that $nU \leq U_0$; choosing m to be the least such integer we have $mU = U_0$. Now Lemma 2.8 allows us to construct a ring R on G satisfying $G^2 = U_0x$, so $U_0x \leq \square G$. Since $mU = U_0$ and $U/U_0 \cong V/V_0$, we have $mV = V_0$, and applying Lemma 2.8, we deduce that $V_0y \leq \square G$. Consequently,

$$U_0x \oplus V_0y \leq \square G. \tag{1}$$

Also,

$$1 \in U_0 \leq U,\, U_0^2 = U_0,\, mU \leq U_0, \quad 1 \in V_0 \leq V,\, V_0^2 = V_0,\, mV \leq V_0,$$

where in both cases m is the least positive integer with the given property. From Proposition 3.2 we deduce that $1/m + U_0$ generates U/U_0 and $1/m + V_0$ generates V/V_0. Furthermore, since $U/U_0 \cong V/V_0$ are cyclic groups of order m, the isomorphism $\phi : U/U_0 \to V/V_0$ must be defined by $\phi(\beta/m + U_0) = k\beta/m + V_0$, where k is a fixed integer coprime with m and β an integer such that $0 \leq \beta < m$. This leads to the following construction for G:

$$G = \{(\beta/m + u_0)x + (k\beta/m + v_0)y \mid 0 \leq \beta < m,\, u_0 \in U_0,\, v_0 \in V_0\},$$

and we shall denote an arbitrary element of G as

$$\beta\left(\frac{1}{m}x + \frac{k}{m}y\right) + u_0x + v_0y. \tag{2}$$

In particular, we set

$$g = \frac{1}{m}x + \frac{k}{m}y \in G. \tag{3}$$

Now define a multiplication $(G, *)$ over G as follows:

$$x * y = y * x = 0, \quad x * x = kmx, \quad y * y = my.$$

Let

$$y_1 = u_1x + v_1y, \quad y_2 = u_2x + v_2y, \quad y_3 = u_3x + v_3y,$$

be arbitrary elements of G. Then for $(G, *)$ to be a ring we must show:

(i) $y_1 * y_2 \in G$;
(ii) $y_1 * (y_2 + y_3) = y_1 * y_2 + y_1 * y_3$, $(y_1 + y_2) * y_3 = y_1 * y_3 + y_2 * y_3$.

To do this, in view of (2) suppose that

$$y_1 = \frac{\beta_1 + m\alpha_1}{m}x + \frac{\beta_1k + m\alpha_2}{m}y, \quad y_2 = \frac{\beta_2 + m\gamma_1}{m}x + \frac{\beta_2k + m\gamma_2}{m}y,$$

where $0 \leq \beta_1 < m$, $0 \leq \beta_2 < m$, $\alpha_1, \gamma_1 \in U_0$ and $\alpha_2, \gamma_2 \in V_0$. So we have

$$\begin{aligned} y_1 * y_2 &= \frac{k(\beta_1 + m\alpha_1)(\beta_2 + m\gamma_1)}{m}x + \frac{(\beta_2k + m\gamma_2)(\beta_1k + m\alpha_2)}{m}y \\ &= \frac{k\beta_1\beta_2 + mu_0}{m}x + \frac{k^2\beta_1\beta_2 + mv_0}{m}y \end{aligned}$$

for some $u_0 \in U_0$ and $v_0 \in V_0$. Hence,

$$\begin{aligned} y_1 * y_2 &= \frac{k\beta_1\beta_2}{m}x + \frac{k^2\beta_1\beta_2}{m}y + u_0x + v_0y \\ &= k\beta_1\beta_2\left(\frac{1}{m}x + \frac{k}{m}y\right) + u_0x + v_0y. \end{aligned}$$

By (3), $\frac{1}{m}x + \frac{k}{m}y \in G$, therefore $y_1 * y_2 \in G$. Also,

$$\begin{aligned} y_1 * (y_2 + y_3) &= (u_1x + v_1y) * ((u_2 + u_3)x + (v_2 + v_3)y) \\ &= kmu_1(u_2 + u_3)x + mv_1(v_2 + v_3)y \\ &= (kmu_1u_2x + mv_1v_2y) + (kmu_1u_3x + mv_1v_3y) \\ &= y_1 * y_2 + y_1 * y_3 \end{aligned}$$

and in a similar way $(y_1+y_2)*y_3 = y_1*y_3+y_2*y_3$, therefore $(G, *)$ is a ring over G. By (3), $g = \frac{1}{m}x + \frac{k}{m}y \in G$ with $0 < k < m$ and $(k, m) = 1$, so there exist integers a, b such that $ak + bm = 1$, and since $g^2 = \frac{k}{m}x + \frac{k^2}{m}y = kg$, we have $ag^2 = akg = (1 - bm)g = g - bmg$, therefore

(4) $$g = ag^2 + bmg = ag^2 + b(x + ky).$$

Now we take any $w \in G$. Then by (2) and (3) we have

$$w = (\beta/m + u_0)x + (\beta k/m + v_0)y = \beta g + u_0x + v_0y,$$

and by (4),

$$w = a\beta g^2 + b\beta(x + ky) + u_0x + v_0y = a\beta g^2 + (b\beta + u_0)x + (b\beta k + v_0)y.$$

But the fact that $g^2 \in \square G$ together with (1) imply that $w \in \square G$, therefore $\square G = G$. ∎

THEOREM 3.4. *Let G be a torsion-free group of rank two. If $T(G)$ has cardinality greater than or equal to three then $\square G$ is pure and $G/\square G$ is a nil group.*

Proof. If $T(G)$ has cardinality greater than three, then by [7, Theorem 3.3], G is a nil group, hence we are done. Suppose that $T(G)$ has cardinality three and G is a non-nil group. Then in view of [7, Theorem 3.3] we have two cases. First, suppose $T(G)$ contains one minimal type and two maximal ones, and precisely one of them is idempotent. Then by Lemma 3.1, $\square G$ is pure and $G/\square G$ is a nil group. In the other case, $T(G)$ contains one minimal type and two idempotent maximal types, so by Lemma 3.3, $\square G = G$. Consequently, $\square G$ is pure and $G/\square G$ is the trivial nil group. ∎

4. Type set has cardinality two

LEMMA 4.1. *Let G be an indecomposable torsion-free group of rank two and $T(G) = \{t_1, t_2\}$ with $t_1 < t_2$. If $\{x, y\}$ is an independent set such that $t(x) = t_1$, $t(y) = t_2$, then all non-trivial rings on G satisfy $x^2 = by$, $xy = yx = y^2 = 0$, for some rational number b.*

Proof. See [1, Lemma 3]. ∎

THEOREM 4.2. *Let G be an indecomposable torsion-free group of rank two. If $T(G) = \{t_1, t_2\}$ with $t_1 < t_2$, then the square subgroup of G is pure and $G/\square G$ is a nil group.*

Proof. If G is a nil group then we are done. Let R be a non-zero ring over G and $\{x, y\}$ a subset of G such that $t(x) = t_1$ and $t(y) = t_2$. Then by Lemma 4.1,

$$x^2 = by, \quad xy = yx = y^2 = 0, \quad b\ (\neq 0) \in \mathbb{Q}.$$

Let U, U_0, V, V_0 be rank one groups belonging to $\{x, y\}$ and $w = ux + vy$, $w' = u'x + v'y$ be arbitrary elements of G. Then $ww' = buu'y$, which means $ww' \in \langle y\rangle^*$, hence $G^2 \subseteq \langle y\rangle^*$. This happens for all rings, therefore

$$\Box G \subseteq \langle y\rangle^*. \tag{5}$$

Also from $ww' = buu'y$ and $b \neq 0$ we deduce $bU^2 \leq V_0$, hence

$$t(U^2) \leq t(V_0),$$

so there exists a least positive integer m such that

$$mU^2 \leq V_0, \quad mU^2 \leq U^2 \cap V_0 \leq U^2. \tag{6}$$

On the other hand, Proposition 2.6(b) implies

$$mU^2 = U^2 \cap V_0, \quad 1/m \in U^2. \tag{7}$$

Now let $\chi_{V_0}(1) = (n_1, n_2, \ldots)$ and $\chi_U(1) = (m_1, m_2, \ldots)$ be the height sequences of 1 in V_0 and U respectively. Then

$$\chi_{U^2}(1) = (2m_1, 2m_2, \ldots).$$

We prove $\left(\frac{1}{p_i^{\alpha_i}}\right)y \in \Box G$ for all α_i such that $0 \leq \alpha_i \leq n_i$ $(i = 1, 2, \ldots)$. To do this, we consider two cases for each fixed i: $n_i \leq 2m_i$ or $2m_i < n_i$. First, suppose that $n_i \leq 2m_i$. Then we define a multiplication over G by

$$x^2 = my, \quad xy = yx = y^2 = 0.$$

Let $w = ux + vy$ and $w' = u'x + v'y$ be arbitrary elements of G, so $ww' = muu'y$. By (6), $muu' \in V_0$, so the product actually lies in G, which yields a ring structure on G. Since $n_i \leq 2m_i$, we have $1/p_i^{\alpha_i} \in U^2 \cap V_0$ and in view of (7), $1/p_i^{\alpha_i} \in mU^2$. Consequently, $1/p_i^{\alpha_i} = mu_1u_2$ for some $u_1, u_2 \in U$. On the other hand, there exist $v_1, v_2 \in V$ such that $z = u_1x + v_1y$ and $z' = u_2x + v_2y$ belong to G, so $zz' = u_1u_2x^2 = mu_1u_2y = \frac{1}{p_i^{\alpha_i}}y$. That is, $\frac{1}{p_i^{\alpha_i}}y \in \Box G$.

In the other case i.e., $2m_i < n_i$, by Proposition 2.6(a), p_i does not divide m. By (7), $1/m \in U^2$, hence $1/m = 1/m'm''$ where $1/m', 1/m'' \in U$. If $m_i = \infty$ then $n_i = \infty$ and so $2m_i = n_i$, contrary to $2m_i < n_i$; thus $m_i < \infty$. Now since $1/p_i^{m_i} \in U$ and p_i does not divide m, we have $(p_i, m') = (p_i, m'') = 1$, hence by Lemma 2.3,

$$\frac{1}{p_i^{m_i}m'}, \frac{1}{p_i^{m_i}m''} \in U. \tag{8}$$

Define another multiplication over G by

$$x^2 = \frac{m}{p_i^{\alpha_i - 2m_i}}\, y, \qquad xy = yx = y^2 = 0.$$

Since $2m_i < n_i$, (6) and Proposition 2.6(a) imply

$$(mU^2)\frac{1}{p_i^{\alpha_i - 2m_i}} \leq V_0,$$

thus the product lies in G, which yields a ring structure on G. By (8) there exist $v_1, v_2 \in V$ such that

$$z = \frac{1}{p_i^{m_i} m'}\, x + v_1 y \in G, \qquad z' = \frac{1}{p_i^{m_i} m''}\, x + v_2 y \in G,$$

and since $m'm'' = m$, we have

$$zz' = \frac{m}{p_i^{\alpha_i} m'm''}\, y = \frac{1}{p_i^{\alpha_i}}\, y.$$

Consequently, in this case, $\frac{1}{p_i^{\alpha_i}} y \in \Box G$. Therefore $\langle y \rangle^* \subseteq \Box G$, and by (5), $\langle y \rangle^* = \Box G$, which means $\Box G$ is a pure subgroup of G.

Now we are going to prove that $G/\Box G$ is a nil group. Let $w \in G$. Then $w = ux + vy$ for some $u \in U$ and $v \in V$, hence $wx = ux^2 = uby$, which implies $bu \in V_0$ for all $u \in U$ and so $bU \leq V_0$. It follows that $t(U) \leq t(V_0)$.

Now if $G/\Box G$ were non-nil, then $t(U)$ would be idempotent, so $h_p^U(1) = 0$ or ∞ for almost all prime numbers p. We prove $t(U) = t(U_0)$. For this we note that if $h_p^U(1) = 0$ then since $U_0 \leq U$ we have $h_p^{U_0}(1) = 0$; hence we suppose $h_p^U(1) = \infty$. Then in view of $t(U) \leq t(V_0)$ we have $h_p^{V_0}(1) = \infty$. Now if $1/p^n \in U$ for some integer n, then there is $a/b \in V$ such that

$$w = \frac{1}{p^n}\, x + \frac{a}{b}\, y \in G.$$

Let $b = b'p^m$ where $(b', p) = 1$. Then $1/p^m \in V_0$ and $b'w = \frac{b'}{p^n} x + \frac{a}{p^m} y$, which yields $\frac{b'}{p^n} x = b'w - a\big(\frac{1}{p^m} y\big) \in G$, hence $1/p^n \in U_0$. Therefore if $h_p^U(1) = \infty$, then $h_p^{U_0}(1) = \infty$. We conclude that $t(U) = t(U_0)$ and consequently, in view of [1, Proposition 3], $\langle y \rangle^*$ is a direct summand of G, contrary to the hypothesis that G is indecomposable. Therefore $G/\Box G$ is a nil group. ■

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