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RELATIVE THEORY IN SUBCATEGORIES

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Abstract. We generalize the relative (co)tilting theory of Auslander–Solberg in the category $\operatorname{mod}\Lambda$ of finitely generated left modules over an artin algebra Λ to certain subcategories of $\operatorname{mod}\Lambda$. We then use the theory (relative (co)tilting theory in subcategories) to generalize one of the main result of Marcos et al. [Comm. Algebra 33 (2005)].

Introduction. Let Λ be an artin algebra, and let $\operatorname{mod}\Lambda$ denote the category of finitely generated left Λ-modules. Auslander and Solberg [9, 10] developed a relative (co)tilting theory in $\operatorname{mod}\Lambda$ which is a generalization of standard (co)tilting theory [3], [12], [14], [23]. In this paper we develop a relative (co)tilting theory in extension-closed functorially finite subcategories of $\operatorname{mod}\Lambda$.

Let T be an ordinary tilting module over Λ. Then the module DT, where D is the usual duality between left and right modules, is a cotilting module over the endomorphism ring $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. If T is a relative tilting module, in the sense of [9, 10], then the Γ-module DT is a direct summand of the cotilting module $T^0 = \operatorname{Hom}_\Lambda(T, I)$ over Γ, where $\operatorname{add} I$ are the relative injective modules for the relative theory. Here we define relative (co)tilting modules relative to a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$. The module $\operatorname{Hom}_\Lambda(T, I)$, where I is as above, is not a cotilting module in general. However, we will show that when the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite (see below for the definition), then $\operatorname{Hom}_\Lambda(T, I)$ is a cotilting module. In addition, DT does not need to be a direct summand of T^0, but it has a finite resolution in $\operatorname{add} T^0$. Another main result is that for a relative tilting and cotilting module in $\mathcal{C}$, there exists an equivalence between the full subcategory $\widehat{\operatorname{add} T_{\mathcal{C}}}$ of $\mathcal{C}$ consisting of all modules having a finite resolution in $\operatorname{add} T$ and the full subcategory $\underbrace{\operatorname{add} T^0}$ consisting of all Γ-modules with finite coresolution in $\operatorname{add} T^0$. This is used to generalize Theorem 0.1 in [17].

Let T be an ordinary tilting Λ-module. Then the classical tilting functor $\operatorname{Hom}_\Lambda(T, \ )$ induces an equivalence between $T^\perp$, the category of all Λ-modules Y such that $\operatorname{Ext}^i_\Lambda(T, Y) = 0$ for all $i > 0$, and its image

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$\operatorname{Hom}_\Lambda(T, T^\perp)$ in $\operatorname{mod}\Gamma$, where the category $\operatorname{Hom}_\Lambda(T, T^\perp)$ is identified with ${}^\perp DT$, the category of all Γ-modules X such that $\operatorname{Ext}^i_\Gamma(X, DT) = 0$ for all $i > 0$. Similar results were established by Auslander–Solberg [10] for a relative tilting module T in $\operatorname{mod}\Lambda$. We want to establish a similar result for a relative tilting module in subcategories of $\operatorname{mod}\Lambda$. To do this we need to develop a relative theory in subcategories.

Let $\mathcal{C}'$ be an additive category which is closed under kernels and cokernels, and suppose $\mathcal{C}$ is a functorially finite subcategory of $\mathcal{C}'$. Iyama [15] introduced an invariant of $\mathcal{C}'$ given by $\mathcal{C}$, namely the right and left $\mathcal{C}$-resolution dimensions of $\mathcal{C}'$. When $\mathcal{C}'$ is $\operatorname{mod}\Lambda$, we refer to the right and left $\mathcal{C}$-resolution dimensions as the right and left $\mathcal{C}$-approximation dimensions. Let us call the maximum of the two invariants (the right and left $\mathcal{C}$-approximation dimensions) the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$.

Suppose $\mathcal{C}$ is closed under extensions, and assume that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is zero. Then it will be shown that $\mathcal{C}$ is naturally equivalent to a module category over an artin algebra. This means that a relative theory in $\mathcal{C}$ can be developed in the sense of [9, 10]. Let us refer to this theory as the relative theory in dimension "0". We develop a relative theory in dimension "n" for certain subfunctors F of the bifunctor $\operatorname{Ext}^1_\Lambda(\ ,\ )$, where n is the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions, and let $\mathcal{X}$ be a generator subcategory of $\mathcal{C}$ in the sense of [2] (i.e. $\mathcal{X}$ contains the Ext-projectives in $\mathcal{C}$). In Section 2 we investigate the subfunctors $F = F_\mathcal{X}$ in $\mathcal{C}$. Denote by C_X (resp. C^X) the right (resp. left) $\mathcal{C}$-approximation of X. Then we show that $\mathcal{P}_\mathcal{C}(F)$, the category of F-projectives in $\mathcal{C}$, and $\mathcal{I}_\mathcal{C}(F)$, the category of F-injectives in $\mathcal{C}$, are related by the formulas $\mathcal{P}_\mathcal{C}(F) = C^{\operatorname{TrD}\mathcal{I}_\mathcal{C}(F)} \cup \mathcal{P}(\mathcal{C})$ and $\mathcal{I}_\mathcal{C}(F) = C_{\operatorname{DTr}\mathcal{I}_\mathcal{C}(F)} \cup \mathcal{I}(\mathcal{C})$, where $\mathcal{P}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ denote the categories of Ext-projectives and Ext-injectives in $\mathcal{C}$ respectively. In Section 3 we state some results relating to approximation dimension. In particular, we show that the subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$ with $\mathcal{C}$-approximation dimension zero are equivalent to categories $\operatorname{mod}\Lambda/I$, where I is an ideal of Λ.

In Section 4 we investigate relative (co)tilting modules in extension-closed functorially finite subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$. Consider a subfunctor F in $\mathcal{C}$ with enough projectives and injectives in $\mathcal{C}$. Also suppose that T is an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$. In this setting we will generalize the classical tilting equivalence. Suppose that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is a nonnegative integer n. Then, if there is an F-tilting module in $\mathcal{C}$, we will show that $\mathcal{I}_\mathcal{C}(F)$ is of finite type. We assume from now on that $\mathcal{I}_\mathcal{C}(F)$ is of finite type. Denote the Γ-module associated to $\operatorname{Hom}_\Lambda(T, \mathcal{I}_\mathcal{C}(F))$ by $T^0_\mathcal{C}$. Then we will show that the image of the classical tilting functor restricted to $T^\perp_\mathcal{C}$, $\operatorname{Hom}_\Lambda(T, T^\perp_\mathcal{C})$, can be identified with ${}^\perp T^0_\mathcal{C}$, where $T^\perp_\mathcal{C}$ denotes the category $T^{\perp} \cap \mathcal{C}$. Moreover, the Γ-module $T_{\mathcal{C}}^{0}$ is cotilting. However, the Γ-module DT is not necessarily cotilting, and we give an example which shows that DT is not a direct summand of $T_{\mathcal{C}}^{0}$ either. Nevertheless, we show that DT has a finite $\operatorname{add} T_{\mathcal{C}}^{0}$-resolution. We also show that $\operatorname{gl.dim}_F \mathcal{C}$, the relative global dimension of $\mathcal{C}$, and $\operatorname{gl.dim} \Gamma$, the global dimension of Γ, are related by the formula $\operatorname{gl.dim}_F \mathcal{C} - \operatorname{pd}_F T \leq \operatorname{gl.dim} \Gamma \leq \operatorname{gl.dim}_F \mathcal{C} + \nu(n,r)$, where ν is a function of n and r.

If the $\mathcal{C}$-approximation dimension of $\operatorname{mod} \Lambda$ is infinite, then we have many examples where the Γ-module $T_{\mathcal{C}}^{0}$ is not cotilting. However, it is not known whether the $\mathcal{C}$-approximation dimension of $\operatorname{mod} \Lambda$ being finite is necessary for $T_{\mathcal{C}}^{0}$ to be cotilting.

Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Suppose T is an F-tilting F-cotilting module in $\mathcal{C}$. In Section 5 we generalize the aforementioned theorem from [17]. We show that the Γ-module $T_{\mathcal{C}}^{0}$ is tilting and that the tilting functor induces an equivalence between the subcategories $\widehat{\operatorname{add} T_{\mathcal{C}}}$ of $\mathcal{C}$ and $\widetilde{\operatorname{add} T_{\mathcal{C}}^{0}}$ of $\operatorname{mod} \Gamma$.

Unless otherwise stated, throughout this paper Λ is a basic artin algebra and $\operatorname{mod} \Lambda$ denotes the category of all finitely generated left Λ-modules. Given a subcategory $\mathcal{A}$ of $\operatorname{mod} \Lambda$, $\operatorname{add} \mathcal{A}$ is the full subcategory of $\operatorname{mod} \Lambda$ consisting of all Λ-modules which are direct summands of finite direct sums of modules in $\mathcal{A}$. Denote by D the duality between left and right modules as given in [6, II.3].

1. Properties of homological finite subcategories. In this section we recall some definitions from [7] and give some preliminary results. Among the results, we show that functorially finite subcategories $\mathcal{C}$ of $\operatorname{mod} \Lambda$ which are closed under extensions in $\operatorname{mod} \Lambda$ have enough Ext-projectives and Ext-injectives. Then we look at certain properties of covariantly and contravariantly finite subcategories of $\operatorname{mod} \Lambda$ which will be used, in the next section, to develop relative theory in subcategories.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod} \Lambda$. An *exact sequence* in $\mathcal{C}$ is an exact sequence in $\operatorname{mod} \Lambda$ with all terms in $\mathcal{C}$. A module Y in $\mathcal{C}$ is said to be Ext-*injective* if $\operatorname{Ext}_{\Lambda}^{1}(X,Y) = 0$ for all X in $\mathcal{C}$. We denote the subcategory of Ext-injective modules in $\mathcal{C}$ by $\mathcal{I}(\mathcal{C})$. A subcategory $\mathcal{C}$ is said to have *enough* Ext-*injectives* if for all C in $\mathcal{C}$ there is an exact sequence $0 \to C \xrightarrow{f} I \to C^{1} \to 0$ with I Ext-injective and C^{1} in $\mathcal{C}$. Note that if $\mathcal{C}$ has enough Ext-injectives and is closed under extensions in $\mathcal{C}$, then any map $g \colon C \to I'$ with I' in $\mathcal{I}(\mathcal{C})$ factors through f (i.e. there exists a map $h \colon I \to I'$ such that $g = hf$). The notions of Ext-*projective* module and *enough* Ext-*projectives* are defined dually. The subcategory of Ext-projective modules in $\mathcal{C}$ is denoted by $\mathcal{P}(\mathcal{C})$.

Let $\mathcal{D}$ be a subcategory of $\operatorname{mod}\Lambda$ containing a subcategory $\mathcal{C}$. Given a module M in $\mathcal{D}$, a sequence $0 \to Y \to C \xrightarrow{g} M$ with C in $\mathcal{C}$ is said to be a *right $\mathcal{C}$-approximation* of M if the sequence

$$0 \to (C', Y) \to (C', C) \xrightarrow{(C', g)} (C', M) \to 0$$

is exact in Ab for all C' in $\mathcal{C}$. A right $\mathcal{C}$-approximation is called a *minimal* right $\mathcal{C}$-approximation if g is *right minimal*, that is, if every endomorphism $s\colon C \to C$ satisfying $g = gs$ is an isomorphism. A minimal right $\mathcal{C}$-approximation is unique up to isomorphism. A module has a right $\mathcal{C}$-approximation if and only if it has a minimal right $\mathcal{C}$-approximation [5]. We denote the minimal right $\mathcal{C}$-approximation of M by $0 \to Y_M \to C_M \xrightarrow{g_M} M$. A subcategory of $\mathcal{C}$ of $\mathcal{D}$ is said to be *contravariantly finite* in $\mathcal{D}$ if every Λ-module in $\mathcal{D}$ has a right $\mathcal{C}$-approximation. Dually, one defines the notions of *left (minimal) $\mathcal{C}$-approximation* and *covariantly finite* subcategory of $\mathcal{D}$. A subcategory $\mathcal{C}$ of $\mathcal{D}$ is said to be *functorially finite* in $\mathcal{D}$ if it is both contravariantly and covariantly finite in $\mathcal{D}$.

Let $\mathcal{C}$ be a contravariantly finite subcategory of $\operatorname{mod}\Lambda$. Then by [7, Lemma 3.11], $\mathcal{C}$ has a *finite cocover*, that is, there is some Y in $\operatorname{add}\mathcal{C}$ such that $\mathcal{C}$ is contained in $\operatorname{Sub}Y$, the subcategory of $\operatorname{mod}\Lambda$ consisting of objects which are submodules of finite direct sums of copies of Y. Suppose $\mathcal{C}$ is closed under extensions in $\operatorname{mod}\Lambda$. Then we have the following analog of [7, Lemma 3.11].

Proposition 1.1. *Let $\mathcal{C}$ be a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then every X in $\mathcal{C}$ has an $\mathcal{I}(\mathcal{C})$-coresolution.*

To prove Proposition 1.1 we need to show that the full subcategory $\mathcal{E}$ of $\operatorname{mod}\Lambda$ consisting of all Y such that $\operatorname{Ext}^1_\Lambda(X, Y) = 0$ for all X in $\mathcal{C}$ is covariantly finite in $\operatorname{mod}\Lambda$. To do this, we use the following proposition which is the dual of [5, Proposition 1.8].

Proposition 1.2. *Suppose $\mathcal{J}$ is a subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions such that $\operatorname{Ext}^1_\Lambda(\ , A)|_{\mathcal{J}}$ is finitely generated for all A in* $\operatorname{mod}\Lambda$. *Then the subcategory $\mathcal{K} = \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\mathcal{J}, Y) = 0\}$ is covariantly finite in* $\operatorname{mod}\Lambda$.

It is not difficult to see that if $\mathcal{C}$ is contravariantly finite in $\operatorname{mod}\Lambda$, then $\operatorname{Ext}^1_\Lambda(\ , A)|_{\mathcal{C}}$ is finitely generated for all A in $\operatorname{mod}\Lambda$. Our subcategory $\mathcal{C}$ in Proposition 1.1 satisfies the conditions of Proposition 1.2. Hence the subcategory $\mathcal{E}$ is covariantly finite and contains the injective Λ-modules.

Proof of Proposition 1.1. Let X be in $\mathcal{C}$. Then we have a minimal left $\mathcal{E}$-approximation $0 \to X \to E^X \to Z^X \to 0$ of X, which is a monomorphism, since $D\Lambda$ is in $\mathcal{E}$. Then by [5, Corollary 1.7], Z^X is in $\mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, this implies that E^X is in $\mathcal{C} \cap \mathcal{E} = \mathcal{I}(\mathcal{C})$. Then the result follows by induction. ■

The following is a consequence of Propositions 1.1 and its dual.

COROLLARY 1.3. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then*:

(a) *$\mathcal{C}$ has enough* Ext*-projectives and* Ext*-injectives.*
(b) *The subcategory $\mathcal{P}(\mathcal{C})$ is contravariantly finite in $\mathcal{C}$.*
(c) *The subcategory $\mathcal{I}(\mathcal{C})$ is covariantly finite in $\mathcal{C}$.*

We now want to find Ext-projective and Ext-injective modules in functorially finite subcategories. The following lemma is part (b) of [16, Lemma 2.1]. It generalizes Wakamatsu's lemma [24].

LEMMA 1.4. *Let $\mathcal{C}$ be a contravariantly finite extension-closed subcategory of* $\operatorname{mod}\Lambda$ *and let Z be a Λ-module. Then the natural transformation* $\operatorname{Ext}^1_\Lambda(\ ,g_Z)\colon \operatorname{Ext}^1_\Lambda(\ ,C_Z)|_{\mathcal{C}} \to \operatorname{Ext}^1_\Lambda(\ ,Z)|_{\mathcal{C}}$ *restricted to $\mathcal{C}$ is a monomorphism of contravariant functors.*

The following consequence of [16, Theorem 3.4] gives us the Ext-injectives (the Ext-projectives are given dually).

COROLLARY 1.5. *Let $\mathcal{C}$ be a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let Y be in* $\operatorname{mod}\Lambda$, *and consider a succession of minimal right $\mathcal{C}$-approximations $Y_1 \hookrightarrow C_0 \to Y$, $Y_2 \hookrightarrow C_1 \to Y_1, \ldots$. Then for all $i > 0$, C_i is* Ext*-injective in $\mathcal{C}$.*

Note that if $Y = I$ is an injective Λ-module, then C_0 in Corollary 1.5 is Ext-injective in $\mathcal{C}$ [7, Lemma 3.5].

We recall the notions of a covariant and a contravariant defect of a short exact sequence [6]: Given a short exact sequence $\delta\colon 0 \to L \to M \to N \to 0$ in $\operatorname{mod}\Lambda$, the *covariant defect* δ_* and the *contravariant defect* δ^* of δ are the subfunctors of $\operatorname{Ext}^1_\Lambda(N,\ )$ and $\operatorname{Ext}^1_\Lambda(\ ,L)$ respectively, defined by the exact sequences

$$0 \to \operatorname{Hom}_\Lambda(N,\ ) \to \operatorname{Hom}_\Lambda(M,\ ) \to \operatorname{Hom}_\Lambda(L,\ ) \to \delta_* \to 0$$

and

$$0 \to \operatorname{Hom}_\Lambda(\ ,L) \to \operatorname{Hom}_\Lambda(\ ,M) \to \operatorname{Hom}_\Lambda(\ ,N) \to \delta^* \to 0.$$

The next result is given in [16], but we will give a different proof.

PROPOSITION 1.6 ([16, Proposition 2.5(b)]). *Let $\mathcal{C}$ be a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let $\delta\colon 0 \to L \xrightarrow{f} M \to N \to 0$ be an exact sequence in $\mathcal{C}$. For all Z in* $\operatorname{mod}\Lambda$, *the morphism* $\operatorname{Hom}_\Lambda(L,g_Z)\colon \operatorname{Hom}_\Lambda(L,Z_{\mathcal{C}}) \to \operatorname{Hom}_\Lambda(L,Z)$ *induces an isomorphism* $\delta_*(C_Z) \xrightarrow{\sim} \delta_*(Z)$.

The following consequence of Proposition 1.6 will be useful for finding the relative injectives in subcategories in the next section.

COROLLARY 1.7. *Let* $0 \to A \to B \to C \to 0$ *be exact in* $\mathcal{C}$, *and let* X *be in* $\operatorname{mod} \Lambda$. *Then the following are equivalent.*

(i) $\operatorname{Hom}_\Lambda(X, B) \to \operatorname{Hom}_\Lambda(X, C)$ *is an epimorphism.*
(ii) $\operatorname{Hom}_\Lambda(B, C_{\mathrm{DTr}\, X}) \to \operatorname{Hom}_\Lambda(A, C_{\mathrm{DTr}\, X})$ *is an epimorphism.*

We recall the following definition from [9]. A subcategory $\mathcal{X}$ of $\mathcal{C}$ is said to be a *generator* for $\mathcal{C}$ if it contains $\mathcal{P}(\mathcal{C})$. Dually one defines a *cogenerator* subcategory for $\mathcal{C}$.

LEMMA 1.8. *Let* $\mathcal{C}$ *be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. Let* $\mathcal{X}$ *be a contravariantly finite subcategory of* $\mathcal{C}$ *which is a generator for* $\mathcal{C}$. *Consider a right* $\mathcal{X}$*-approximation* $0 \to Y \to X \xrightarrow{g} C \to 0$ *of* C *in* $\mathcal{C}$. *Then* Y *is in* $\mathcal{C}$.

Proof. We know that $\mathcal{C}$ has enough Ext-projectives by Corollary 1.3. So, for any C in $\mathcal{C}$, there is an exact sequence $0 \to C_1 \to P \xrightarrow{p} C \to 0$ with P in $\mathcal{P}(\mathcal{C})$ and C_1 in $\mathcal{C}$. Therefore, we have the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C_1 & = & C_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & Y \oplus P & \to & P & \to & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle p} & & \\
0 & \to & Y & \to & X & \xrightarrow{g} & C & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

since g is a right $\mathcal{X}$-approximation of C. But since $\mathcal{C}$ is closed under extensions and summands, it follows that Y is in $\mathcal{C}$. ∎

2. Subfunctors in subcategories and their properties. Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod} \Lambda$ which is closed under extensions. In this section we study subfunctors in $\mathcal{C}$. We first recall some background on subfunctors in $\operatorname{mod} \Lambda$ from [9]. Then we study a special subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$, where $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{C}$.

2.1. *Background on subfunctors.* Let F be an additive sub-bifunctor of the additive bifunctor $\operatorname{Ext}^1_\Lambda(\ ,\ ) \colon (\operatorname{mod} \Lambda)^{\mathrm{op}} \times \operatorname{mod} \Lambda \to \mathrm{Ab}$, where $(\operatorname{mod} \Lambda)^{\mathrm{op}}$ denotes the opposite category of $\operatorname{mod} \Lambda$. Then F is said to be an additive subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ in $\operatorname{mod} \Lambda$. A short exact sequence $\eta \colon 0 \to$

$A \to B \to C \to 0$ is called an F-*exact* sequence if η is in $F(C, A)$. Any pullback, pushout and Baer sum of F-exact sequences are again F-exact [9]. In particular, a subfunctor F determines a collection of short exact sequences which is closed under pushouts, pullbacks and Baer sums. Conversely, any collection of short exact sequences which is closed under pushouts, pullbacks and Baer sums gives rise to a subfunctor of $\mathrm{Ext}^1_\Lambda(\ , \ )$ in the obvious way [9].

Let $\mathcal{P}(F)$ be the subcategory of $\operatorname{mod}\Lambda$ consisting of all Λ-modules P such that if $0 \to A \to B \to C \to 0$ is F-exact, then the sequence $0 \to (P, A) \to (P, B) \to (P, C) \to 0$ is exact in Ab. The objects in $\mathcal{P}(F)$ are called *projective modules* of the subfunctor F or F-*projectives*. If $\mathcal{P}(\Lambda)$ denotes the category of projective Λ-modules, then $\mathcal{P}(\Lambda)$ is contained in $\mathcal{P}(F)$. An additive subfunctor F is said to have *enough projectives* if for every A in $\operatorname{mod}\Lambda$ there exists an F-exact sequence $0 \to A' \to P \to A \to 0$ with P in $\mathcal{P}(F)$. The definitions of F-*injectives* and *enough injectives* are dual.

Let $\mathcal{Z}$ be a subcategory of $\operatorname{mod}\Lambda$. Define

$$F_{\mathcal{Z}}(C, A) = \{0 \to A \to B \to C \to 0 \mid (\mathcal{Z}, B) \to (\mathcal{Z}, C) \to 0 \text{ is exact}\}$$

for each pair of modules A and C in $\operatorname{mod}\Lambda$. Dually, one defines

$$F^{\mathcal{Z}}(C, A) = \{0 \to A \to B \to C \to 0 \mid (B, \mathcal{Z}) \to (A, \mathcal{Z}) \to 0 \text{ is exact}\}$$

for each pair of modules A and C in $\operatorname{mod}\Lambda$. It is shown in [9, Proposition 1.7] that these constructions give (additive) subfunctors of $\mathrm{Ext}^1_\Lambda(\ , \ )$.

2.2. *Subfunctors F in the subcategory $\mathcal{C}$.* Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions, and let F be a subfunctor in $\operatorname{mod}\Lambda$. When F-projectives and F-injectives are determined only by the F-exact sequences in $\mathcal{C}$, we say *F is a subfunctor in $\mathcal{C}$*. To study such subfunctors, we first find the subcategories of F-projectives and F-injectives in $\mathcal{C}$, denoted by $\mathcal{P}_{\mathcal{C}}(F)$ and $\mathcal{I}_{\mathcal{C}}(F)$ respectively.

Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{C}$. Then by Corollary 1.7 we know that for all $Z \in \operatorname{mod}\Lambda$, the sequence $(Z, B) \to (Z, C) \to 0$ is exact if and only if $(B, C_{\mathrm{DTr}\, Z}) \to (A, C_{\mathrm{DTr}\, Z}) \to 0$ is exact. This gives the following proposition.

Proposition 2.1. *Let $\mathcal{C}$ be a functorially finite subcategory which is closed under extensions. Then*:

(a) $\mathcal{I}_{\mathcal{C}}(F) = C_{\mathrm{DTr}\, \mathcal{P}_{\mathcal{C}}(F)} \cup \mathcal{I}(\mathcal{C})$.

(b) $\mathcal{P}_{\mathcal{C}}(F) = C^{\mathrm{TrD}\, \mathcal{I}_{\mathcal{C}}(F)} \cup \mathcal{P}(\mathcal{C})$.

Remark. Nothing can be said about the size of the subcategories $\mathcal{P}_{\mathcal{C}}(F)$ and $\mathcal{I}_{\mathcal{C}}(F)$ at the moment. But later we will see that if there exists an F-(co)tilting module in $\mathcal{C}$, then $\mathcal{P}_{\mathcal{C}}(F)$ and $\mathcal{I}_{\mathcal{C}}(F)$ are of finite type.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. We now study some properties of subfunctors in $\mathcal{C}$. A subfunctor

F in $\mathcal{C}$ is said to have *enough projectives* if for each C in $\mathcal{C}$ there exists an F-exact sequence $0 \to C_1 \to P \to C \to 0$ with P in $\mathcal{P}_{\mathcal{C}}(F)$ and C_1 in $\mathcal{C}$. The notion of *enough injectives* is defined dually.

Notation. Unless specified otherwise, F denotes a subfunctor $F_{\mathcal{X}}$, where $\mathcal{X}$ is a generator subcategory of $\mathcal{C}$.

Consider a subfunctor F with enough projectives. Then the following proposition shows that $\mathcal{C}$ is closed under kernels of F-*epimorphisms*.

PROPOSITION 2.2. *Let $\mathcal{C}$ be a functorially finite subcategory which is closed under extensions. Let F be a subfunctor in $\mathcal{C}$ with enough projectives in $\mathcal{C}$. Then $\mathcal{C}$ is closed under kernels of F-epimorphisms.*

Proof. Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an F-exact sequence with C_2, C_3 in $\mathcal{C}$. Then, since F has enough projectives in $\mathcal{C}$, we have an exact sequence $0 \to Y \to P \to C_3 \to 0$ with $P \in \mathcal{P}_{\mathcal{C}}(F)$ and $Y \in \mathcal{C}$. From the commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & Y & = & Y & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & C_1 & \to & E & \to & P & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

we see that E is in $\mathcal{C}$. The exact sequence $0 \to C_1 \to E \to P \to 0$ is F-exact, and it splits since $P \in \mathcal{P}_{\mathcal{C}}(F)$, so the claim follows. ∎

Now let $F = F_{\mathcal{X}}$, and consider the subfunctor $F^{\mathcal{I}_{\mathcal{C}}(F)}$ given by $\mathcal{I}_{\mathcal{C}}(F)$. Let M be a Λ-module with a surjective $\mathcal{C}$-approximation. Then we have the F-exact sequence $\eta\colon 0 \to Y_M \xrightarrow{g} C_M \to M \to 0$. If Y_M is in $\mathcal{C}$, then it is in $\mathcal{I}_{\mathcal{C}}(F)$ since $\mathcal{I}(\mathcal{C})$ is contained in $\mathcal{I}_{\mathcal{C}}(F)$. Assume Y_M is nonzero; then the identity map 1_{Y_M} does not factor through g. Therefore η is not $F^{\mathcal{I}_{\mathcal{C}}(F)}$-exact. Dually, given N in $\operatorname{mod} \Lambda$, the exact sequence $0 \to N \to C^N \to Z^N \to 0$ is not F-exact whenever Z^N is a nonzero Λ-module in $\mathcal{C}$. So outside $\mathcal{C}$ we may not have $F = F^{\mathcal{I}_{\mathcal{C}}(F)}$. But inside $\mathcal{C}$ we have the following result.

COROLLARY 2.3. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. Then $F|_{\mathcal{C}} = F^{\mathcal{I}_{\mathcal{C}}(F)}|_{\mathcal{C}}$.*

The following result shows that F has enough projectives and injectives under certain conditions.

PROPOSITION 2.4. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then*:

(a) *If $\mathcal{P}_{\mathcal{C}}(F)$ is contravariantly finite in $\mathcal{C}$, then F has enough projectives.*
(b) *If $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$, then F has enough injectives.*

Proof. (a) Follows from Lemma 1.8.

(b) Suppose $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$. Since $\mathcal{I}_{\mathcal{C}}(F)$ is a cogenerator for $\mathcal{C}$, for each C in $\mathcal{C}$ there is, by the dual of Lemma 1.8, an exact sequence $\eta\colon 0 \to C \to I \to C^1 \to 0$ with I in $\mathcal{I}_{\mathcal{C}}(F)$ and C^1 in $\mathcal{C}$, such that $0 \to (C^1, \mathcal{I}_{\mathcal{C}}(F)) \to (I, \mathcal{I}_{\mathcal{C}}(F)) \to (C, \mathcal{I}_{\mathcal{C}}(F)) \to 0$ is exact. Hence the sequence η is $F^{\mathcal{I}_{\mathcal{C}}(F)}$-exact. By Corollary 2.3 it follows that η is F-exact, since it is so in $\mathcal{C}$. Thus F has enough injectives. ∎

Suppose $\mathcal{I}_{\mathcal{C}}(F)$, where $F = F_{\mathcal{X}}$, is covariantly finite in $\mathcal{C}$. Then the following "dual" of Lemma 2.2 shows that $\mathcal{C}$ is closed under cokernels of $F^{\mathcal{I}_{\mathcal{C}}(F)}$-monomorphisms.

PROPOSITION 2.5. *Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an $F^{\mathcal{I}_{\mathcal{C}}(F)}$-exact sequence with C_1, C_2 in $\mathcal{C}$. Assume $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$. Then C_3 is in $\mathcal{C}$.*

3. Approximation dimension. Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. In this section we define $\mathcal{C}$-approximation dimension. Then we characterize the subcategories $\mathcal{C}$ with $\mathcal{C}$-approximation dimension equal to zero. Moreover, we prove that if the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite, then any long relative exact sequence in $\operatorname{mod}\Lambda$ with all middle terms in $\mathcal{C}$ is eventually in $\mathcal{C}$. This will be useful in the next section.

Let $\mathcal{C}$ be a contravariantly finite subcategory of $\operatorname{mod}\Lambda$. For any M in $\operatorname{mod}\Lambda$, consider a succession $0 \to Y_1 \to C_0 \xrightarrow{g_0} M$, $0 \to Y_2 \to C_1 \xrightarrow{g_1} Y_1, \ldots$ of minimal right $\mathcal{C}$-approximations. Then the complex

$$(*) \qquad \cdots \to C_t \xrightarrow{g_t} C_{t-1} \to \cdots \to C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} M$$

is called a right $\mathcal{C}$-*approximation resolution* of M. In [15] this was defined in general for a contravariantly finite subcategory $\mathcal{C}$ in an additive category $\mathcal{C}'$ with kernels and cokernels. There, a right $\mathcal{C}$-approximation resolution was called a right $\mathcal{C}$-resolution. Denote $\operatorname{Ker} g_i$ in $(*)$ by Y_{i+1}. We write $r\mathcal{C}$-app.dim$(M) = n$ if there exists a nonnegative integer n in a right $\mathcal{C}$-approximation resolution of M such that $Y_{n+1} = 0$ and $Y_i \neq 0$ for all $i \leq n$. If no such integer exists, we write $r\mathcal{C}$-app.dim$(M) = \infty$. We call $r\mathcal{C}$-app.dim(M) the *right $\mathcal{C}$-approximation dimension* of M. Then we define

$$r\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda) = \sup\{r\mathcal{C}\text{-app.dim}(M) \mid M \in \operatorname{mod}\Lambda\}.$$

Example 3.1. If $\mathcal{C}$ is closed under factor modules, then it is known that every right $\mathcal{C}$-approximation is a monomorphism [7, Proposition 4.8]. Hence $r\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda) = 0$.

Dually, one can define a *left* $\mathcal{C}$*-approximation resolution* of M and *left* $\mathcal{C}$*-approximation dimension* of $\operatorname{mod}\Lambda$, denoted by $l\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda)$, for a covariantly finite subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$. We have the following proposition relating the two approximation dimensions when $\mathcal{C}$ is of finite type [15, Corollary 1.1.2].

Proposition 3.2. *Let* $\mathcal{C}$ *be a functorially finite subcategory of* $\operatorname{mod}\Lambda$. *Then* $r\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda)$ *is finite if and only if* $l\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda)$ *is finite. Moreover, in this case they differ by at most* 2.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$. The $\mathcal{C}$*-approximation dimension* of $\operatorname{mod}\Lambda$, $\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda)$, is defined to be

$$\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda) = \max\{l\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda), r\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda)\}.$$

The following is a nice corollary of Proposition 3.2.

Corollary 3.3. *Let* $\mathcal{C}$ *be a subcategory of* $\operatorname{mod}\Lambda$ *which is closed under factor modules. Then* $\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda) \leq 2$.

Note. Let $\mathcal{C}$ be equal to $\operatorname{mod}\Lambda$. Then $\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda) = 0$. However, $\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda)$ being zero does not necessarily mean that $\mathcal{C} = \operatorname{mod}\Lambda$, as shown below.

In general, $\mathcal{A}\text{-app.dim}(\mathcal{B})$ can be defined, where $\mathcal{A}$ is a functorially finite subcategory of a category $\mathcal{B}$ with kernels and cokernels [15].

3.1. *Approximation dimension zero.* In this section we want to characterize the functorially finite subcategories $\mathcal{C}$ with $\mathcal{C}$-approximation dimension zero.

The following result shows that functorially finite subcategories with approximation dimension zero are the same as those which are closed under factor modules and submodules.

Proposition 3.4. *Let* $\mathcal{C}$ *be an additive functorially finite subcategory of* $\operatorname{mod}\Lambda$. *Then* $\mathcal{C}\text{-app.dim}(\operatorname{mod}\Lambda) = 0$ *if and only if* $\mathcal{C}$ *is closed under factor modules and submodules.*

Now we want to characterize the subcategories of $\operatorname{mod}\Lambda$ closed under factor modules and submodules. But first we recall a well-known concept.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. Recall that the *annihilator* of $\mathcal{C}$, $\operatorname{ann}_\Lambda\mathcal{C}$, is equal to the intersection of the annihilators of the modules $C \in \mathcal{C}$, $\operatorname{ann}_\Lambda(C) = \{\lambda \in \Lambda \mid \lambda\cdot C = 0\}$. It is well known that $\operatorname{ann}_\Lambda\mathcal{C}$ is an ideal of Λ. The following result shows that the subcategories of $\operatorname{mod}\Lambda$ which are closed under submodules and factor modules are abelian.

Proposition 3.5. *Let $\mathcal{C}$ be an additive subcategory of* $\operatorname{mod}\Lambda$ *which is closed under factor modules and submodules. Then $\mathcal{C}$ is equivalent to* $\operatorname{mod}\Lambda/I$, *where $I = \operatorname{ann}_\Lambda \mathcal{C}$.*

Let $\mathcal{C}$ and I be as before and consider the algebra morphism $\varphi\colon \Lambda \to \Lambda/I$. Then φ induces an exact functor $G_\varphi\colon \operatorname{mod}(\Lambda/I) \to \operatorname{mod}\Lambda$, which is an embedding. We have $\operatorname{Im} G_\varphi = \mathcal{C}$. It is easy to see that G_φ and its inverse preserve exact sequences and exact diagrams. Hence they preserve pushouts, pullbacks and Baer sums. Since these last three operations determine subfunctors, it follows that G_φ and its inverse preserve subfunctors too. Hence $\mathcal{C}$ and $\operatorname{mod}(\Lambda/I)$ have the same relative theory.

Note that the factor category $\operatorname{mod}\Lambda/I$, in Proposition 3.5, is not necessarily closed under extensions in $\operatorname{mod}\Lambda$ [4]. However, if $\mathcal{C}$ is closed under extensions, then $\operatorname{mod}\Lambda/I$ is also closed under extensions in $\operatorname{mod}\Lambda$ (by using the functor G_φ above).

Now, we combine Propositions 3.4 and 3.5 to get the following crucial result for subcategories $\mathcal{C}$ with $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = 0$.

Corollary 3.6. *Let $\mathcal{C}$ be an additive functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Assume the* $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda)$ *is zero. Then $\mathcal{C}$ is canonically equivalent to* $\operatorname{mod}\Sigma$, *where Σ is a quotient algebra of Λ. Moreover*, $\operatorname{mod}\Sigma$ *inherits the relative theory in $\mathcal{C}$ and vice versa.*

3.2. *Approximation dimension $n > 0$.* Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Let $\mathcal{X}$ be a contravariantly finite generator subcategory of $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. In this subsection we study a relationship between $\mathcal{C}$ and $\operatorname{mod}\Lambda$ which will be useful later. We show that any long F-exact sequence in $\operatorname{mod}\Lambda$ with the middle terms in $\mathcal{C}$ is eventually in $\mathcal{C}$.

The following lemma is important.

Lemma 3.7. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Consider a minimal right $\mathcal{C}$-approximation resolution*

$$\cdots \to C_{i+s+1} \xrightarrow{g_{i+s+1}} C_{i+s} \to \cdots \to C_{i+1} \xrightarrow{g_{i+1}} C_i \xrightarrow{g_i} M_i$$

of M_i for some $i \geq 0$. Denote $\operatorname{Ker} g_{i+j}$ by Y_{i+j+1} for $j \geq 0$ and let $M_i = Y_i$. Let $0 \to M_{i+j+1} \to T_{i+j} \to M_{i+j} \to 0$ be an F-exact sequence with T_{i+j} in $\mathcal{C}$ for $j \geq 0$. Then there is a right $\mathcal{C}$-approximation $0 \to Y'_{i+j+1} \to C'_{i+j} \to M_{i+j}$ with $Y_{i+j+1} = Y'_{i+j+1}$ for $j \geq 0$.

Proof. We prove this by induction on j. For $j = 0$, we have $M_i = Y_i$, so $Y_{i+1} = Y'_{i+1}$.

For $j = 1$, consider the commutative F-exact diagram

$$\begin{array}{cccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_{i+1} & = & M_{i+1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & 0 \to & Y_{i+1} & \to & Y_{i+1} \oplus T_i & \to & T_i & \to & 0 \\
 & & \| & & \downarrow \alpha & & \downarrow & & \\
\theta_1\colon & 0 \to & Y_{i+1} & \to & C_i & \to & M_i & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and let $X \xrightarrow{p} C_i$ be an epimorphism with X in $\mathcal{X}$. Since $0 \to M_{i+1} \to Y_{i+1} \oplus T_i \xrightarrow{\alpha} C_i \to 0$ is F-exact, we deduce that p factors through α. Moreover, since

$$\eta\colon 0 \to Y_{i+2} \to C_{i+1} \oplus T_i \xrightarrow{(\, g_{i+1}\ 1_{T_i}\,)} Y_{i+1} \oplus T_i$$

is a right $\mathcal{C}$-approximation of $Y_{i+1} \oplus T_i$, we find that p factors through $f = \alpha \circ (g_i\ \ 1_{T_i})$. Hence f is onto, since p is onto. Then we use the F-exact sequence $0 \to M_{i+1} \to Y_{i+1} \oplus T_i \xrightarrow{\alpha} C_i \to 0$ to construct the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_{i+2} & = & Y_{i+2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & C'_{i+1} & \to & C_{i+1} \oplus T_i & \xrightarrow{f} & C_i & \to & 0 \\
 & & \downarrow g'_{i+1} & & \downarrow (\, g_i\ 1_{T_i}\,) & & \| & & \\
0 & \to & M_{i+1} & \xrightarrow{\delta} & Y_{i+1} \oplus T_i & \xrightarrow{\alpha} & C_i & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the earlier discussion, the exact sequence $0 \to C'_{i+1} \to C_{i+1} \oplus T_i \xrightarrow{f} C_i \to 0$ is F-exact. Then by Proposition 2.2, C'_{i+1} is in $\mathcal{C}$.

Our aim is to show that $\theta_2\colon 0 \to Y_{i+2} \to C'_{i+1} \xrightarrow{g'_{i+1}} M_{i+1}$ is a right $\mathcal{C}$-approximation of M_{i+1}. If C'_{i+1} were a pullback of δ and $(g_i\ \ 1_{T_i})$, then by the universal property of pullbacks, θ_2 would be a right $\mathcal{C}$-approximation, since η is a right $\mathcal{C}$-approximation of $Y_{i+1} \oplus T_i$. But it can be shown that C'_{i+1} is indeed a pullback of δ and $(g_i\ \ 1_{T_i})$. Hence the sequence θ_2 is a right $\mathcal{C}$-approximation, and we have $Y'_{i+2} = Y_{i+2}$.

For $j > 1$ we replace the sequence θ_1 in the first diagram by θ_j and continue as above. Then the result follows by induction. ∎

The following consequence of Lemma 3.7 shows that any long F-exact sequence in $\operatorname{mod} \Lambda$ with the middle terms in $\mathcal{C}$ is eventually in $\mathcal{C}$. This will be useful in the next section.

COROLLARY 3.8. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. Assume $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda) = n < \infty$. Fix an integer $t \geq 0$, and let $0 \to M_{i+1} \to T_i \to M_i \to 0$ be F-exact in* $\operatorname{mod} \Lambda$ *with T_i in $\mathcal{C}$ for all $i \geq t$. Then M_{t+n} is in $\mathcal{C}$. In general, M_i is in $\mathcal{C}$ for all $i \geq t+n$.*

Proof. By Lemma 3.7 we have the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C'_{t+n} & \longrightarrow & C_{i+n}\oplus T_{t+n-1} & \longrightarrow & C'_{t+n-1} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle g'_{t+n}} & & \downarrow & & \| & & \\
0 & \longrightarrow & M_{t+n} & \longrightarrow & Y_{t+n}\oplus T_{t+n-1} & \longrightarrow & C'_{t+n-1} & \longrightarrow & 0
\end{array}$$

where g'_{t+n} is a right $\mathcal{C}$-approximation of M_{t+n}. Since T_{t+n} maps onto M_{t+n}, it follows that g'_{t+n} is an epimorphism, and hence an isomorphism. Therefore M_{t+n} is in $\mathcal{C}$. Then by Lemma 2.2, M_i is in $\mathcal{C}$ for all $i \geq t+n$. ∎

4. Relative theory, approximation and global dimension. In this section, $\mathcal{C}$ is a functorially finite extension-closed subcategory of $\operatorname{mod} \Lambda$, and $\mathcal{X}$ is a contravariantly finite generator subcategory of $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. In this section we investigate a relative (co)tilting theory in $\mathcal{C}$. Suppose T is an F-tilting module in $\mathcal{C}$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. In 4.1 we show that the tilting functor $\operatorname{Hom}_\Lambda(T, \ )$ induces an equivalence between the subcategories $T_{\mathcal{C}}^{\perp}$ of $\mathcal{C}$ and $(T, T_{\mathcal{C}}^{\perp})$ of $\operatorname{mod} \Gamma$. Then we prove that $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module and use this to show that $\mathcal{P}_{\mathcal{C}}(F)$ is of finite type. In 4.2 we show that the image of the tilting functor restricted to $T_{\mathcal{C}}^{\perp}$, $(T, T_{\mathcal{C}}^{\perp})$, can be identified with the category ${}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$. Moreover, we prove that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is cotilting. In 4.3 we look at the relationship between the relative global dimension of $\mathcal{C}$ and the global dimension of Γ.

4.1. *Relative tilting in subcategories.* Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. We know that F has enough projectives in $\mathcal{C}$ (since $\mathcal{P}_{\mathcal{C}}(F) = \mathcal{X}$). Suppose $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$. Then by Proposition 2.4 we know that F has enough injectives in $\mathcal{C}$. So, from now on we assume that $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$.

First we define the concept of F-tilting in $\mathcal{C}$.

DEFINITION. A Λ-module T is called *F-tilting* in $\mathcal{C}$ if:

(i) T is in $\mathcal{C}$.
(ii) $\operatorname{Ext}^i_F(T,T) = 0$ for all $i > 0$.
(iii) $\operatorname{pd}_F T < \infty$.
(iv) For all P in $\mathcal{P}_{\mathcal{C}}(F)$ there is an F-exact sequence $0 \to P \to T_0 \to T_1 \to \cdots \to T_s \to 0$ with T_i in $\operatorname{add} T$.

An *F-cotilting* module in $\mathcal{C}$ is defined dually.

Let ω be a subcategory of $\operatorname{mod}\Lambda$. Then ω is said to be *F-selforthogonal* if $\operatorname{Ext}^i_F(\omega,\omega) = 0$ for all $i > 0$.

Let T be an F-selforthogonal Λ-module in $\mathcal{C}$. Define $T^\perp$ to be the full subcategory of $\operatorname{mod}\Lambda$ consisting of all modules Y with $\operatorname{Ext}^i_F(T,Y) = 0$ for all $i > 0$. It has been shown in [10] that $T^\perp$ is F-coresolving in $\operatorname{mod}\Lambda$. Denote $T^\perp \cap \mathcal{C}$ by $T^\perp_{\mathcal{C}}$, and let $\mathcal{Y}^{\mathcal{C}}_T$ be the full subcategory of all Λ-modules A in $T^\perp_{\mathcal{C}}$ such that there is an F-exact sequence

$$\cdots \to T_s \xrightarrow{f_s} T_{s-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \to A \to 0$$

with T_i in $\operatorname{add} T$ and $\operatorname{Im} f_i$ in $T^\perp_{\mathcal{C}}$.

A subcategory $\mathcal{J}$ of $\mathcal{C}$ is said to be *closed under F-extensions* in $\mathcal{C}$ if for each F-exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ with A and C in $\mathcal{J}$, also B is in $\mathcal{J}$. Then we have the following generalization of [5, dual of Proposition 5.1].

PROPOSITION 4.1. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. For an F-selforthogonal Λ-module T in $\mathcal{C}$ the subcategory $\mathcal{Y}^{\mathcal{C}}_T$ is closed under*

(a) *F-extensions*,
(b) *cokernels of F-monomorphisms*,
(c) *direct summands.*

A subcategory $\mathcal{Z}$ of $\mathcal{C}$ is said to be *F-resolving* in $\mathcal{C}$ if it satisfies the following conditions: (a) it is closed under F-extensions, (b) if $0 \to A \to B \to C \to 0$ is F-exact and B and C are in $\mathcal{Z}$, then A is in $\mathcal{Z}$, and (c) it contains $\mathcal{P}_{\mathcal{C}}(F)$. Dually, one defines *$F$-coresolving* subcategories in $\mathcal{C}$.

Let $\mathcal{Y}$ be F-covariantly finite and F-coresolving in $\mathcal{C}$. Then the *F-coresolution dimension* of a Λ-module C with respect to $\mathcal{Y}$ is defined to be the minimum of all n including infinity such that there exists an F-exact sequence $0 \to C \to Y^0 \to Y^1 \to \cdots \to Y^{n-1} \to Y^n \to 0$ with Y^i in $\mathcal{Y}$. We denote this dimension by $\mathcal{Y}\text{-coresdim}_F M$. If $\mathcal{W}$ is a subcategory of $\operatorname{mod}\Lambda$, then $\mathcal{Y}\text{-coresdim}_F(\mathcal{W})$ is defined to be $\sup\{\mathcal{Y}\text{-coresdim}_F Z \mid Z \in \mathcal{W}\}$.

When our F-selforthogonal module T is F-tilting in $\mathcal{C}$ we have the following generalization of [10, dual of Theorem 3.2]. Denote $\widehat{\operatorname{add} T} \cap \mathcal{C}$ by $\widehat{\operatorname{add} T_{\mathcal{C}}}$.

Proposition 4.2. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$. Then*:

(a) *The subcategory $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$ is F-coresolving and covariantly finite in $\mathcal{C}$ with $\mathcal{Y}_T^{\mathcal{C}}$-$\operatorname{coresdim}_F \mathcal{C}$ finite.*

(b) *The subcategory $\widetilde{\operatorname{add} T_{\mathcal{C}}} = {}^{\perp}(\mathcal{Y}_T^{\mathcal{C}}) \cap \mathcal{C}$ is F-resolving and contravariantly finite in $\mathcal{C}$ with $\operatorname{pd}_F \widetilde{\operatorname{add} T_{\mathcal{C}}}$ finite.*

Proof. The proof is similar to [10, dual of Theorem 3.2]. The only challenge is to ensure that some of the modules involved in the proof are in $\mathcal{C}$. We do that by using Proposition 2.2. ∎

We restate [20, Lemma 2.2] for the relative theory in subcategories. The proof is similar, so it will not be given. We denote $\widehat{\operatorname{add} T} \cap \mathcal{C}$ by $\widehat{\operatorname{add} T_{\mathcal{C}}}$.

Lemma 4.3. *Let T be an F-tilting module in $\mathcal{C}$. Then $T_{\mathcal{C}}^{\perp} \cap \mathcal{P}_{\mathcal{C}}^{<\infty}(F) = \widehat{\operatorname{add} T_{\mathcal{C}}}$.*

Next we show that the tilting functor is fully faithful on the category $\mathcal{Y}_T^{\mathcal{C}}$.

Let T be in $\mathcal{C}$ and $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. Consider the tilting functor

$$\operatorname{Hom}_{\Lambda}(T, \ ) \colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma.$$

Then we have the following analog of [10, dual of Lemma 3.3].

Lemma 4.4. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. If T is an F-tilting Λ-module in $\mathcal{C}$, then the functor* $\operatorname{Hom}_{\Lambda}(T, \ ) \colon \mathcal{Y}_T^{\mathcal{C}} \to \operatorname{mod}\Gamma$ *is an F-exact fully faithful covariant functor.*

The following is a consequence of Lemma 4.4.

Corollary 4.5. *Let T be an F-tilting module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. Then* $\operatorname{Hom}_{\Lambda}(T, \ ) \colon \operatorname{Ext}_F^i(Y, Y') \to \operatorname{Ext}_{\Gamma}^i((T, Y), (T, Y'))$ *is an isomorphism for all Y and Y' in $\mathcal{Y}_T^{\mathcal{C}}$, functorial in both variables.*

Let T be a tilting module in $\operatorname{mod}\Lambda$, $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ and DT the corresponding cotilting Γ-module. It is well known that the tilting functor $(T, \ ) \colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ induces an equivalence between the categories $T^{\perp}$ ($= \mathcal{Y}_T$ by the dual of [5, Theorem 5.4]) of $\operatorname{mod}\Lambda$ and $(T, T^{\perp})$ of $\operatorname{mod}\Gamma$, where the image $(T, T^{\perp})$ is identified with the subcategory ${}^{\perp}DT$. This was also established for relative tilting modules in $\operatorname{mod}\Lambda$ [10].

Let F be a subfunctor in $\operatorname{mod}\Lambda$. Let T be an F-tilting module in $\operatorname{mod}\Lambda$ and denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. Then it can be shown (by using duality in [10]) that the tilting functor induces the same equivalence as in the standard case. But this time the image $(T, T^{\perp})$ is identified with the category ${}^{\perp}(T, \mathcal{I}(F))$, where $(T, \mathcal{I}(F))$ is a cotilting Γ-module.

Our aim is to show that the same also holds for relative tilting modules T in subcategories. In the present subsection we prove the existence of an equivalence between the subcategory $\mathcal{Y}_T^{\mathcal{C}}$ of $\mathcal{C}$ and its image $(T, \mathcal{Y}_T^{\mathcal{C}})$ in $\operatorname{mod}\Gamma$.

Assume that $\mathcal{C}$-app.dim(mod Λ) is finite. In 4.2 we identify the subcategory which corresponds to the image $(T, \mathcal{Y}_T^{\mathcal{C}})$ of $(T,\ )$.

Let T be an F-tilting Λ-module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. We have seen that $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$. Since $\operatorname{Hom}_\Lambda(T,\ )\colon \mathcal{Y}_T^{\mathcal{C}} \to \operatorname{mod}\Gamma$ is a fully faithful functor by Lemma 4.4, we have

$$DY = \operatorname{Hom}_\Lambda(Y, D\Lambda) \simeq \operatorname{Hom}_\Gamma((T,Y),(T,D\Lambda)) \simeq \operatorname{Hom}_\Gamma((T,Y),DT)$$

for all Y in $\mathcal{Y}_T^{\mathcal{C}}$. Applying the duality D to the above isomorphism we get the isomorphism $Y \simeq D\operatorname{Hom}_\Gamma((T,Y),DT) \simeq T \otimes_\Gamma \operatorname{Hom}_\Lambda(T,Y)$. Hence $\mathcal{Y}_T^{\mathcal{C}} \simeq T \otimes_\Gamma (T, \mathcal{Y}_T^{\mathcal{C}})$. Therefore $\mathcal{Y}_T^{\mathcal{C}}$ is equivalent to $(T, \mathcal{Y}_T^{\mathcal{C}})$ in $\operatorname{mod}\Gamma$. The following result, which summarizes the above discussion, shows that there is an equivalence between the subcategories $\mathcal{Y}_T^{\mathcal{C}}$ of $\mathcal{C}$ and $(T, \mathcal{Y}_T^{\mathcal{C}})$ of $\operatorname{mod}\Gamma$. This is a generalization of the dual of [10, Corollary 3.6].

THEOREM 4.6. *Let $\mathcal{C}$ be a functorially finite subcategory of* mod Λ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$.*

(a) *The functor* $\operatorname{Hom}_\Lambda(T,\ )\colon \mathcal{C} \to \operatorname{mod}\Gamma$ *induces an equivalence between $\mathcal{Y}_T^{\mathcal{C}}$ and $(T, \mathcal{Y}_T^{\mathcal{C}})$.*
(b) *The functor* $\operatorname{Hom}_\Lambda(T,\ )\colon \mathcal{C} \to \operatorname{mod}\Gamma$ *induces an equivalence between $\mathcal{I}_{\mathcal{C}}(F)$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$.*

If T is a standard tilting Λ-module, then the Γ-modules $(T, D\Lambda_\Lambda)$ and $D(\Lambda, T)$ coincide. But for relative tilting modules this is not always the case.

We want to show that the Γ^{op}-module $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module. This will imply that the module $D(\mathcal{P}_{\mathcal{C}}(F), T)$ is a cotilting Γ-module by duality. But first we need the following results.

LEMMA 4.7. *For all W in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ and all C in* mod Λ *the homomorphism* $\operatorname{Hom}_\Lambda(\ ,T)\colon (C,W) \to {}_{\Gamma^{\mathrm{op}}}((W,T),(C,T))$ *is an isomorphism functorial in both variables.*

The following is a consequence of the above result; the proof is similar to that of [10, Proposition 3.7].

COROLLARY 4.8. *For W in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ and C in ${}^{\perp}T_{\mathcal{C}}$ the homomorphism*

$$\operatorname{Hom}_\Lambda(\ ,T)\colon \operatorname{Ext}_F^i(C,W) \to \operatorname{Ext}_{\Gamma^{\mathrm{op}}}^i((W,T),(C,T)) \quad \text{for all } i > 0$$

is an isomorphism functorial in both variables.

Now we show that $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module.

PROPOSITION 4.9. *Let $\mathcal{C}$ be a subcategory of* mod Λ *which is closed under extensions. Let T be an F-tilting Λ-module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$. Denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. Then $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module. Moreover, $(\mathcal{P}_{\mathcal{C}}(F), T)$ is of finite type.*

Proof. Since $\mathcal{P}_{\mathcal{C}}(F) \subseteq \widetilde{\operatorname{add} T_{\mathcal{C}}} \subseteq {}^{\perp}T_{\mathcal{C}}$, we have $0 = \operatorname{Ext}^i_F(\mathcal{P}_{\mathcal{C}}(F), \mathcal{P}_{\mathcal{C}}(F)) \simeq \operatorname{Ext}^i_{\Gamma^{\mathrm{op}}}((\mathcal{P}_{\mathcal{C}}(F), T), (\mathcal{P}_{\mathcal{C}}(F), T))$ for all $i > 0$. Hence $(\mathcal{P}_{\mathcal{C}}(F), T)$ is self-orthogonal. Since T is F-tilting we infer that $\operatorname{pd}_{\Gamma^{\mathrm{op}}}(\mathcal{P}_{\mathcal{C}}(F), T)$ is finite. Since $\operatorname{pd}_F T$ is finite it is not difficult to see that Γ^{op} is in $\operatorname{add}(\widetilde{\mathcal{P}_{\mathcal{C}}(F), T})$. Therefore $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module.

By the corollary to [19, Proposition 1.18], for all P in $\mathcal{P}_{\mathcal{C}}(F)$, the module (P, T) is a direct summand of

$$\operatorname{add} \bigoplus_{i=0}^{r} (P_i, T),$$

where the P_i are in $\mathcal{P}_{\mathcal{C}}(F)$. Hence $(\mathcal{P}_{\mathcal{C}}(F), T)$ is of finite type. ∎

Now we want to show that $\mathcal{P}_{\mathcal{C}}(F)$ is of finite type whenever there is an F-tilting module in $\mathcal{C}$. We need the following analog of [10, Proposition 5.4].

LEMMA 4.10. *Consider the functor* $\operatorname{Hom}_\Lambda(\ ,T)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$. *Then*:

(a) $\operatorname{Hom}_\Lambda(\ ,T)$ *induces a duality between* $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ *and* $(\widetilde{\operatorname{add} T_{\mathcal{C}}}, T)$.
(b) $\operatorname{Hom}_\Lambda(\ ,T)$ *induces a duality between* $\mathcal{P}_{\mathcal{C}}(F)$ *and* $(\mathcal{P}_{\mathcal{C}}(F), T)$.

The following result is a consequence of Proposition 4.9.

COROLLARY 4.11. *The subcategory* $\mathcal{P}_{\mathcal{C}}(F)$ *is of finite type.*

4.2. *Relative tilting and finite approximation dimension.* Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Suppose T is an F-tilting module in $\mathcal{C}$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. In this section we show that the image of the equivalence given in the previous section, namely $(T, \mathcal{Y}_T^{\mathcal{C}})$, can be identified with the subcategory ${}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$. Moreover, we show that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is cotilting.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and assume the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is zero. Then, by Corollary 3.6, $\mathcal{C}$ is canonically equivalent to $\operatorname{mod}\Sigma$, where Σ is a quotient algebra of Λ. Moreover, $\mathcal{C}$ and $\operatorname{mod}\Sigma$ have the same relative theory. Let T be an F-tilting module in $\mathcal{C}$ and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. Then by the duals of [10, Proposition 3.8] and [10, Theorem 3.13] we know that $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a cotilting Γ-module.

For $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = \infty$, we give examples which show that $(T, \mathcal{I}_{\mathcal{C}}(F))$ is not always a cotilting Γ-module.

Now assume that the $\mathcal{C}$-approximation of $\operatorname{mod}\Lambda$ is greater than zero, but finite. Let T be an F-tilting module in $\mathcal{C}$ and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. We want to show that $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a cotilting Γ-module.

But first we need several preliminary results. The following is an analog of [10, dual of Lemma 2.9].

LEMMA 4.12. *Let $\mathcal{C}$ be a functorially finite extension-closed subcategory of* $\operatorname{mod} \Lambda$. *Let T be an F-tilting module in $\mathcal{C}$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then the map* $\Psi\colon \operatorname{Hom}_\Lambda(W,T) \otimes_\Gamma \operatorname{Hom}_\Lambda(T,Y) \to \operatorname{Hom}_\Lambda(W,Y)$ *given by* $\psi(f \otimes g) = g \circ f$ *is an isomorphism for all W in* $\widehat{\operatorname{add} T_{\mathcal{C}}}$ *and Y in* $\mathcal{Y}_T^{\mathcal{C}}$ *and is functorial in both variables.*

The following result is an analog of [10, dual of Lemma 3.10].

LEMMA 4.13. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. If T is F-tilting in $\mathcal{C}$, then* $\operatorname{id}_\Gamma D(\widehat{\operatorname{add} T_{\mathcal{C}}}, T) \le \operatorname{pd}_F T$, *where* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *In particular*, $\operatorname{id}_\Gamma D(\mathcal{P}(\mathcal{C}), T) \le \operatorname{pd}_F T$.

We have the following nice corollary.

COROLLARY 4.14. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *and assume that* $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda) = n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with* $\operatorname{pd}_F T = r$ *and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then* $\operatorname{id}_\Gamma DT \le r + n$.

Proof. We prove this by induction on n. For $n = 0$, see Corollary 3.6 and the dual of [10, Lemma 3.10]. For $n = 1$, we have a left $\mathcal{C}$-approximation resolution (presentation) $\Lambda \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \to 0$ of Λ. The dual of Corollary 1.5 shows that C^0 and C^1 are in $\mathcal{P}(\mathcal{C})$. Applying $D(\ ,T)$ to the sequence we get the exact sequence $0 \to D(\Lambda,T) \to D(C^0,T) \to D(C^1,T) \to 0$. By Lemma 4.13 we have $\operatorname{id}_\Gamma D(C^i,T) \le r$ for $i = 0,1$. Hence, by [19, Lemma 2.1] (see also [22]) we conclude that $\operatorname{id}_\Gamma DT \le r + 1$.

Now suppose that $n > 1$. Then we have a left $\mathcal{C}$-approximation resolution $\Lambda \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \to \cdots \to C^n \to 0$ of Λ. Applying $D(\ ,T)$ to it we get the exact sequence $0 \to DT \to D(C^0,T) \to D(C^1,T) \to \cdots \to D(C^n,T) \to 0$. Denote $\operatorname{Ker} D(f^i,T)$ by L^i. Then by induction we find that $\operatorname{id}_\Gamma L^1 \le r+n-1$. Again by [19, Lemma 2.1] it follows that $\operatorname{id}_\Gamma DT \le r + n$. ∎

The following lemma will be useful.

LEMMA 4.15. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions and assume* $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda) = n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with* $\operatorname{pd}_F T = r$. *Let M be a Λ-module and consider a succession* $M_1 \hookrightarrow T_0 \to M$, $M_2 \hookrightarrow T_1 \to M_1, \ldots$ *of minimal right* $\operatorname{add} T$*-approximations. Then* $0 \to M_{i+1} \to T_i \to M_i \to 0$ *is F-exact for* $i \ge r + n + 1$.

Proof. Denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. From the complex $\cdots \to T_2 \to T_1 \to T_0 \to M$ we get a minimal projective resolution $\cdots \to (T,T_1) \to (T,T_0) \to (T,M) \to 0$ of (T,M) over Γ. We see that $\operatorname{Ext}^j_\Gamma((T,M_i), D(\widehat{\operatorname{add} T_{\mathcal{C}}},T)) = 0$ for all $j > 0$ and $i > r$, by Lemma 4.13. So if one applies the functor $\operatorname{Hom}_\Gamma(\ ,D(W,T))$, for $W \in \widehat{\operatorname{add} T_{\mathcal{C}}}$, to the sequence $\cdots \to (T,T_{r+1}) \to \cdots \to (T,T_r) \to (T,M_r) \to 0$ it remains exact. Let $W \in \widehat{\operatorname{add} T_{\mathcal{C}}}$. Then we have the following commutative diagram by the adjoint isomorphism and Lemma 4.12:

$$\begin{array}{ccccccc}
((T,M_r),D(W,T)) & \longrightarrow & ((T,T_r),D(W,T)) & \longrightarrow & ((T,T_{r+1}),D(W,T)) & \longrightarrow & \cdots \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
D((W,T)\otimes_\Gamma (T,M_r)) & \to & D((W,T)\otimes_\Gamma (T,T_r)) & \to & D((W,T)\otimes_\Gamma (T,T_{r+1})) & \to & \cdots \\
\uparrow & & \uparrow \wr & & \uparrow \wr & & \\
D((W,M_r)) & \longrightarrow & D((W,T_r)) & \longrightarrow & D((W,T_{r+1})) & \longrightarrow & \cdots
\end{array}$$

Since the middle row in the above diagram is exact, the sequence

$$0 \to (W, M_{i+1}) \to (W, T_i) \to (W, M_i) \to 0 \tag{1}$$

is exact for $i \geq r+1$. In particular, (1) is exact for $Q \in \mathcal{P}_{\mathcal{C}}(F)$, since $\mathcal{P}_{\mathcal{C}}(F) \subseteq \widetilde{\operatorname{add} T_{\mathcal{C}}}$.

Now, since $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda) = n$, for any $P \in \mathcal{P}(\Lambda)$ we have a minimal left $\mathcal{C}$-approximation resolution $P \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \to \cdots \to C^{l-1} \xrightarrow{f^l} C^l \to 0$ with $l \leq n$. Denote Coker f^{i-1} by Z^i for $0 < i < l$. Note that by the dual of Corollary 1.5 the C^i are in $\mathcal{P}_{\mathcal{C}}(F)$ for $0 \leq i \leq n$. We want to show that the sequence $0 \to (P, M_{i+1}) \to (P, T_i) \to (P, M_i) \to 0$ is exact for all $i \geq r+n+1$ by using induction on n. For $n = 0$, this follows from Corollary 3.6 and the dual of [10, Proposition 3.8].

For $n = 1$, we combine (1) and the resolution of P to get the exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (C^1, T_{r+2}) & \longrightarrow & (C^0, T_{r+2}) & \longrightarrow & (P, T_{r+2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (C^1, T_{r+1}) & \longrightarrow & (C^0, T_{r+1}) & \longrightarrow & (P, T_{r+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (C^1, M_{r+1}) & \longrightarrow & (C^0, M_{r+1}) & \longrightarrow & (P, M_{r+1}) & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By the long exact sequence (of complexes) [22], the sequence $0 \to (P, M_{i+1}) \to (P, T_i) \to (P, M_i) \to 0$ is exact for all $i \geq r+2$. Therefore the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is exact for $i \geq r+2$. Then by (1) it is F-exact.

Suppose $n > 1$. By induction and using (1) and the resolution of P, we find that the sequence $0 \to (Z^{n-k}, M_{i+1}) \to (Z^{n-k}, T_i) \to (Z^{n-k}, M_i) \to 0$ is exact for $i \geq r+1+k$ and $0 < k \leq n$. In particular, for $k = n$, the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is exact for $i \geq r+n+1$. Then by (1) it is F-exact. ∎

Remark. Let B be in $\operatorname{mod} \Gamma$ and consider a projective resolution of B. Then the Γ-module $\Omega^j_\Gamma(B)$ has a preimage in $\operatorname{mod} \Lambda$ for $j \geq 2$. However, $\Omega^1_\Gamma(B)$ does not necessarily have a preimage in $\operatorname{mod} \Lambda$.

Now we show that $(T, \mathcal{Y}^{\mathcal{C}}_T) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$ for a functorially finite subcategory $\mathcal{C}$ of $\operatorname{mod} \Lambda$ which is closed under extensions and has the property that $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda)$ is finite. This is a generalization of [10, dual of Proposition 3.8].

Proposition 4.16. *Let $\mathcal{C}$ be a functorially finite extension-closed subcategory of* $\operatorname{mod} \Lambda$ *and assume $\mathcal{C}$-*app.dim$(\operatorname{mod} \Lambda) = n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with* $\operatorname{pd}_F T = r$ *and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then* $\operatorname{Ext}^i_\Gamma(B, (T, \mathcal{I}_{\mathcal{C}}(F))) = 0$ *for all $i > 0$ if and only if* $B \in \operatorname{Hom}_\Lambda(T, \mathcal{Y}^{\mathcal{C}}_T)$.

Proof. We have $0 = \operatorname{Ext}^i_F(Y, \mathcal{I}_{\mathcal{C}}(F)) \simeq \operatorname{Ext}^i_\Gamma((T,Y),(T,\mathcal{I}_{\mathcal{C}}(F)))$ for $Y \in \mathcal{Y}^{\mathcal{C}}_T$, by Corollary 4.5. So $(T,Y) = B \in {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$.

Conversely, let B be a Γ-module such that $\operatorname{Ext}^i_\Gamma(B,(T,\mathcal{I}_{\mathcal{C}}(F))) = 0$ for $i > 0$. Let $\operatorname{Hom}_\Lambda(T,T_1) \xrightarrow{(T,f_1)} \operatorname{Hom}_\Lambda(T,T_0) \to B \to 0$ be a minimal projective presentation of B. By Lemma 4.4 this sequence is induced by $T_1 \xrightarrow{f_1} T_0$. Denote $\operatorname{Ker} f_1$ by M_2. Let $0 \to M_3 \to T_2 \to M_2$, $0 \to M_4 \to T_3 \to M_3, \ldots$ be a succession of minimal left $\operatorname{add} T$-approximations. Then we get a complex $\cdots \to T_4 \xrightarrow{f_4} T_3 \xrightarrow{f_3} T_2 \to M_2$, and the exact sequence

(2) $$\cdots \to (T,T_s) \to (T,T_{s-1}) \to \cdots \to (T,T_1) \to (T,T_0) \to B \to 0$$

is a minimal projective resolution of B over Γ. Denote $\Omega^1_\Gamma(B)$ by B_1. Applying $\operatorname{Hom}_\Gamma(\ ,(T,I))$, with $I \in \mathcal{I}_{\mathcal{C}}(F)$, to the resolution of B, we get the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_\Gamma(B,(T,I)) & \longrightarrow & {}_\Gamma((T,T_0),(T,I)) & \longrightarrow & {}_\Gamma((T,T_1),(T,I)) & \longrightarrow & \cdots \\
 & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \\
0 & \longrightarrow & \operatorname{Hom}_\Lambda(T \otimes_\Gamma B, I) & \longrightarrow & \operatorname{Hom}_\Lambda(T_0, I) & \longrightarrow & \operatorname{Hom}_\Lambda(T_1, I) & \longrightarrow & \cdots
\end{array}$$

by Lemma 4.4 and the adjoint isomorphism. The cohomology of the upper row is $\operatorname{Ext}^i_\Gamma(B,(T,\mathcal{I}_{\mathcal{C}}(F)) = 0$ for $i > 0$. So the sequence

(3) $$0 \to (T \otimes_\Gamma B, I) \to (T_0, I) \to \cdots \to (T_r, I) \to (T_{r+1}, I) \to \cdots$$

is exact.

On the other hand, since $\mathcal{C}$-app.dim$(\mathcal{I}(\Lambda)) = n$, we have, for all $I \in \mathcal{I}(\Lambda)$, a minimal right $\mathcal{C}$-approximation resolution $0 \to C_l \xrightarrow{g_l} \cdots \to C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} I$ with $l \leq n$. Denote $\operatorname{Ker} g_i$ by Y_{i+1} for $0 \leq i < n$. By Corollary 1.5 the modules C_i are in $\mathcal{I}(F)$ for $0 \leq i \leq n$. Then by the adjoint isomorphism, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & (T\otimes_\Gamma B, C_l) & \to \cdots \to & (T\otimes_\Gamma B, C_0) & \to & (T\otimes_\Gamma B, I) & & \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
0 \to & (B,(T,C_l)) & \to \cdots \to & (B,(T,C_0)) & \to & (B,(T,I)) & \to & \operatorname{Ext}^1_\Gamma(B,(T,Y_1))
\end{array}$$

with $l \leq n$. We then have $\operatorname{Ext}^1_\Gamma(B,(T,Y_1)) \simeq \operatorname{Ext}^n_\Gamma(B,(T,C_n)) = 0$ since $C_n \in \mathcal{I}_\mathcal{C}(F)$. So the top row in the above diagram is exact.

Now, combining (3) and the resolution of I we get the exact sequence of complexes

$$\begin{array}{ccccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & (T\otimes_\Gamma B, C_l) & \to \cdots \to & (T\otimes_\Gamma B, C_0) & \to & (T\otimes_\Gamma B, I) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & (T_0, C_l) & \to \cdots \to & (T_0, C_0) & \to & (T_0, I) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & (T_1, C_l) & \to \cdots \to & (T_1, C_0) & \to & (T_1, I) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

with $l \leq n$. By the long exact sequence (of complexes) [22], the sequence $0 \to (T\otimes_\Gamma B, I) \to (T_0, I) \to \cdots \to (T_r, I) \to \cdots$ is exact for all $I \in \mathcal{I}(\Lambda)$. Hence

$$0 \to M_{r+2n} \to T_{r+2n-1} \to \cdots \to T_0 \to T\otimes_\Gamma B \to 0 \tag{4}$$

is exact.

By Lemma 4.15 the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for all $i \geq r+n+1$. Hence Corollary 3.8 shows that $M_i \in \mathcal{C}$ for $i \geq r+2n+1$. But then by (3) the sequence (4) is $F^{\mathcal{I}_\mathcal{C}(F)}$-exact. Hence by Proposition 2.5, M_i for $2 \leq i \leq r+2n+1$, $T\otimes_\Gamma B_1$ and $T\otimes_\Gamma B$ are in $\mathcal{C}$. Since $F_\mathcal{X}|_\mathcal{C} = F^{\mathcal{I}_\mathcal{C}(F)}|_\mathcal{C}$ by Corollary 2.3, we infer that (4) is F-exact.

We deduce from (2) and (4) that $\operatorname{Ext}^1_F(T, M_i) = 0$ for $2 < i \leq r+2n+1$. The F-exact sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ gives

$$\operatorname{Ext}^{j+1}_F(T, M_{i+1}) \simeq \operatorname{Ext}^j_F(T, M_i) \quad \text{for } j > 0 \text{ and } 2 < i \leq r+2n+1.$$

By dimension shift, we have $\operatorname{Ext}^j_F(T, M_{r+2n+1}) = 0$ for $0 < j < r+1$. Since $\operatorname{pd}_F T = r$, it follows that $M_{r+2n+1} \in \mathcal{Y}^\mathcal{C}_T = T^\perp_\mathcal{C}$. By Proposition 4.2, the subcategory $\mathcal{Y}^\mathcal{C}_T$ is F-coresolving, hence, by using the fact that (4) is

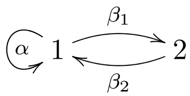
F-exact we find that $T \otimes_\Gamma B$, $T \otimes_\Gamma B_1$ and M_i, for $i = 2, \ldots, r + 2n + 1$, are in $\mathcal{Y}_T^{\mathcal{C}}$. Let $V = \operatorname{Ext}_F^1(T, T \otimes_\Gamma B_1)$. Then the commutative exact diagram

$$\begin{array}{ccccccccccc}
\cdots \to & (T, T_2) & \to & (T, T_1) & \to & (T, T_0) & \to & (T, T \otimes_\Gamma B) & \to & V & \to 0 \\
 & \| & & \| & & \| & & \uparrow & & \uparrow & \\
\cdots \to & (T, T_2) & \to & (T, T_1) & \to & (T, T_0) & \longrightarrow & B & \longrightarrow & 0 &
\end{array}$$

yields $(T, T \otimes_\Gamma B) \simeq B$, since $V = 0$. Therefore B is in $(T, \mathcal{Y}_T^{\mathcal{C}})$, and the result follows. ∎

REMARK. Note that $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda)$ being finite is sufficient but not necessary for the equality $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$ to hold, as illustrated below.

EXAMPLE 4.17. Let Λ be an algebra given by the quiver



with radical square-zero relations. Denote by P_i, I_i and S_i the indecomposable projective, injective and simple Λ-modules corresponding to the vertex i (the notations are fixed throughout the paper). Let $\mathcal{C} = \mathcal{F}(\Theta)$ where $\Theta = \{P_1/S_2, P_2\}$. Note that $\mathcal{C}$ is closed under summands, so it is closed under extensions by [21]. $\mathcal{C}$ is functorially finite since it is of finite type. A right $\mathcal{C}$-approximation resolution of S_1 is $\cdots \to P_1/S_2 \to P_1/S_2 \to S_1 \to 0$, so Proposition 3.2 yields $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda) = \infty$. We have $\mathcal{P}(\mathcal{C}) = \mathcal{I}(\mathcal{C}) = \mathcal{C}$. Let $F = F_{\mathcal{P}(\mathcal{C})}$. Then the only F-tilting module up to isomorphism is $T = P_1/S_2 \oplus P_2$. Let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and denote by Q_i and J_i the projective and injective Γ-modules corresponding to the vertex i (the notations are fixed throughout the paper). It can be shown that $(T, \mathcal{Y}_T^{\mathcal{C}}) = (T, \mathcal{C}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$.

Next we want to show that $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a standard cotilting Γ-module. The following result will help us to achieve our goal. The result also shows that $(T, \mathcal{Y}_T^{\mathcal{C}})$-coresdim$(\operatorname{mod} \Gamma)$ is finite when $\mathcal{C}$ is a functorially finite subcategory of $\operatorname{mod} \Lambda$ which is closed under extensions and has $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda)$ finite. This is a generalization of [10, Proposition 3.11].

PROPOSITION 4.18. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions and assume $\mathcal{C}$-app.dim$(\operatorname{mod} \Lambda) = n < \infty$. Let T be an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then $\widehat{(T, \mathcal{Y}_T^{\mathcal{C}})} = \operatorname{mod} \Gamma$ and*

$$(T, \mathcal{Y}_T^{\mathcal{C}})\text{-resdim}(\operatorname{mod} \Gamma) \leq \nu(n, r) = \begin{cases} 2 + n, & r = 0, \\ 3 + 2n, & r = 1, \\ r + 2n + 1, & r \geq 2. \end{cases}$$

Proof. Let $(T, T_{-1}) \to (T, T_{-2}) \to B \to 0$ be a minimal projective presentation of B in $\operatorname{mod}\Gamma$. By Lemma 4.4 the presentation is induced by $T_{-1} \xrightarrow{f} T_{-2}$. Denote $\operatorname{Ker} f$ by M_0. Then $\Omega^2_\Gamma(B) = (T, M_0)$.

For $r = 0$, we have $T = \mathcal{P}_{\mathcal{C}}(F)$, so that $\mathcal{Y}^{\mathcal{C}}_T = \mathcal{C}$. From the right $\mathcal{C}$-approximation resolution of M_0, we have the sequence

$$\begin{array}{ccccccccccc} 0 \to C_l \to \cdots \to & C_1 & & \longrightarrow & & C_0 & & \longrightarrow & & T_{-1} \xrightarrow{f} T_{-2} \\ & & \overset{f_1}{\searrow} & & \nearrow & & \overset{f_0}{\searrow} & & \nearrow & \\ & & & Y_1 & & & & M_0 & & \end{array}$$

with $l \leq n$, since $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = n$. This yields the exact sequence

$$0 \to (T, C_l) \to \cdots \to (T, C_0) \to (T, T_{-1}) \to (T, T_{-2}) \to B \to 0.$$

But since $\mathcal{Y}^{\mathcal{C}}_T = \mathcal{C}$, it follows that $\widehat{(T, \mathcal{Y}^{\mathcal{C}}_T)} = \operatorname{mod}\Gamma$ and

$$(T, \mathcal{Y}^{\mathcal{C}}_T)\text{-resdim}(\operatorname{mod}\Gamma) \leq 2 + n.$$

For $r > 0$, let $0 \to M_1 \to T_0 \to M_0$, $0 \to M_2 \to T_1 \to M_1, \ldots$ be a succession of minimal right $\operatorname{add} T$-approximations. Then we get a complex $\cdots \to T_2 \to T_1 \to T_0 \to M_0$, and the exact sequence $\cdots \to (T, T_1) \to (T, T_0) \to (T, T_{-1}) \to (T, T_{-2}) \to B \to$ is a minimal projective resolution of B in $\operatorname{mod}\Gamma$.

Assume that $r \geq 2$. Since $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = n$, it follows by Lemma 4.15 that the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for all $i \geq r + n - 1$. Then Corollary 3.8 shows that $M_i \in \mathcal{C}$ for $i \geq r + 2n - 1$. Moreover, by (1) in the proof of Lemma 4.15, we have $\operatorname{Ext}^1_F(\widetilde{\operatorname{add} T_{\mathcal{C}}}, M_i) = 0$ for $i > r + 2n - 1$. Using the fact that $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for $i \geq r + 2n - 1$ and $\widetilde{\operatorname{add} T_{\mathcal{C}}} \subseteq {}^{\perp}T$, we obtain

$$\operatorname{Ext}^j_F(\widetilde{\operatorname{add} T_{\mathcal{C}}}, M_i) \simeq \operatorname{Ext}^{j+1}_F(\widetilde{\operatorname{add} T_{\mathcal{C}}}, M_{i+1})$$

for $j > 0$ and $i \geq r + 2n - 1$. By dimension shift, $\operatorname{Ext}^i_F(\widetilde{\operatorname{add} T_{\mathcal{C}}}, M_{2r+2n-1}) = 0$ for $0 < i < r + 1$. Since $\widetilde{\operatorname{add} T_{\mathcal{C}}} \subseteq \mathcal{P}^r(F)$ we have $M_{2r+2n-1} \in (\widetilde{\operatorname{add} T_{\mathcal{C}}})^{\perp} \simeq \mathcal{Y}^{\mathcal{C}}_T$. But since $\mathcal{Y}^{\mathcal{C}}_T$ is F-coresolving and $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for $i \geq r + 2n$, it follows that $M_i \in \mathcal{Y}^{\mathcal{C}}_T$ for $r + 2n - 1 \leq i \leq 2r + 2n - 1$. Hence $(T, M_{r+2n-1}) = \Omega^{r+2n+1}_\Gamma(B) \in (T, \mathcal{Y}^{\mathcal{C}}_T)$. Therefore $(T, \mathcal{Y}^{\mathcal{C}}_T)$-resdim$(\operatorname{mod}\Gamma) \leq r + 2n + 1$. If $r = 1$, the proof of the case $r \geq 2$ plus the remark after Lemma 4.15 can be used to show that $M_{2n+1} \in \mathcal{Y}^{\mathcal{C}}_T$. Hence $(T, M_{2n+1}) = \Omega^{3+2n}_\Gamma(B) \in (T, \mathcal{Y}^{\mathcal{C}}_T)$ and we conclude that $(T, \mathcal{Y}^{\mathcal{C}}_T)$-resdim$(\operatorname{mod}\Gamma) \leq 3 + 2n$. ∎

Remark. $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda)$ being finite is sufficient for the equality $\widehat{(T, \mathcal{Y}^{\mathcal{C}}_T)} = \operatorname{mod}\Gamma$ to hold, but it is not known if the assumption is necessary.

We are now in a position to show that $\mathrm{Hom}_\Lambda(T,\mathcal{I}_\mathcal{C}(F))$ is a cotilting module in $\mathrm{mod}\,\Gamma$ when $\mathcal{C}$ is a functorially finite subcategory of $\mathrm{mod}\,\Lambda$ which is closed under extensions and $\mathcal{C}$-app.dim$(\mathrm{mod}\,\Lambda)$ is finite. This is a generalization of [10, dual of Theorem 3.13].

THEOREM 4.19. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under extensions and assume* $\mathcal{C}$-app.dim$(\mathrm{mod}\,\Lambda) = n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with* $\mathrm{pd}_F T = r$ *and let* $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. *Then*:

(a) *The subcategory* $(T,\mathcal{Y}_T^\mathcal{C}) = {}^\perp(T,\mathcal{I}_\mathcal{C}(F))$ *is resolving and contravariantly finite in* $\mathrm{mod}\,\Gamma$ *with* $(T,\mathcal{Y}_T^\mathcal{C})$-resdim$(\mathrm{mod}\,\Gamma) \le \nu(n,r)$.
(b) *The subcategory* $(T,\mathcal{Y}_T^\mathcal{C})^\perp = \widehat{(T,\mathcal{I}_\mathcal{C}(F))}$ *is coresolving and covariantly finite in* $\mathrm{mod}\,\Gamma$ *with* $\mathrm{id}_\Gamma(T,\mathcal{I}_\mathcal{C}(F)) \le \nu(n,r)$.
(c) $(T,\mathcal{Y}_T^\mathcal{C}) \cap (T,\mathcal{Y}_T^\mathcal{C})^\perp = (T,\mathcal{I}_\mathcal{C}(F))$.
(d) *The subcategory* $(T,\mathcal{I}_\mathcal{C}(F))$ *equals* $\mathrm{add}\,T_\mathcal{C}^0$ *for a cotilting Γ-module $T_\mathcal{C}^0$ with* $\mathrm{id}_\Gamma T_\mathcal{C}^0 \le \nu(n,r)$. *In particular*, $(T,\mathcal{Y}_T^\mathcal{C}) = \mathcal{Y}_{T_\mathcal{C}^0} = {}^\perp T_\mathcal{C}^0$.

Proof. (a), (b) and (d) are similar to [10, dual of Theorem 3.13].

(c) We have $(T,\mathcal{Y}_T^\mathcal{C}) \cap (T,\mathcal{Y}_T^\mathcal{C})^\perp = (T,\mathcal{Y}_T^\mathcal{C}) \cap \widehat{(T,\mathcal{I}_\mathcal{C}(F))}$. So $(T,\mathcal{I}_\mathcal{C}(F)) \subseteq (T,\mathcal{Y}_T^\mathcal{C}) \cap (T,\mathcal{Y}_T^\mathcal{C})^\perp$. Let $(T,Y) \in (T,\mathcal{Y}_T^\mathcal{C}) \cap (T,\mathcal{Y}_T^\mathcal{C})^\perp$. Then there is an exact sequence

$$0 \to (T,I_s) \to \cdots \xrightarrow{(T,f_2)} (T,I_1) \xrightarrow{(T,f_1)} (T,I_0) \xrightarrow{(T,f_0)} (T,Y) \to 0 \tag{1}$$

with $I_j \in \mathcal{I}_\mathcal{C}(F)$ for all $j \le s$. Since $(T,\mathcal{Y}_T^\mathcal{C})$ is resolving, we deduce that $\mathrm{Coker}(T,f_i) = (T,Y_{i-1})$ with $Y_{i-1} \in \mathcal{Y}_T^\mathcal{C}$ for all $i > 0$. Since $(T,Y) \in {}^\perp(T,\mathcal{I}_\mathcal{C}(F))$, the functor $(\ ,(T,\mathcal{I}_\mathcal{C}(F)))$ is exact on (1). Applying $(\ ,(T,J))$, for $J \in \mathcal{I}_\mathcal{C}(F)$, to (1) we get the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & ((T,Y),(T,J)) & \to & ((T,I_0),(T,J)) & \to \cdots \to & ((T,I_s),(T,J)) & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & (Y,J) & \longrightarrow & (I_0,J) & \longrightarrow \cdots \longrightarrow & (I_s,J) &
\end{array}$$

By Lemma 4.4 the sequence

$$0 \to (Y,J) \to (I_0,J) \to \cdots \to (I_s,J) \to 0 \tag{2}$$

is exact.

Now, since $\mathcal{C}$-app.dim$(\mathrm{mod}\,\Lambda) = n < \infty$, we have a right $\mathcal{C}$-approximation resolution $0 \to C_l \to \cdots \to C_1 \to C_0 \to D\Lambda$ of $D\Lambda$ with $l \le n$. Combining (2) and the resolution of $D\Lambda$ we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Y, C_l) & \longrightarrow & \cdots & \longrightarrow & (Y, C_0) & \longrightarrow & (Y, D\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I_0, C_l) & \longrightarrow & \cdots & \longrightarrow & (I_0, C_0) & \longrightarrow & (I_0, D\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I_s, C_l) & \longrightarrow & \cdots & \longrightarrow & (I_s, C_0) & \longrightarrow & (I_s, D\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}$$

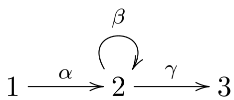
which is exact by the snake lemma. Hence the sequence

$$0 \to I_s \to \cdots \to I_1 \to I_0 \to Y \to 0 \tag{3}$$

is exact. Actually, it is F-exact by using (2) and Corollary 2.3. Since $I_s \in \mathcal{I}_{\mathcal{C}}(F)$, the sequence $0 \to I_s \to I_{s-1} \to Y_{s-1} \to 0$ splits and hence $Y_{s-1} \in \mathcal{I}_{\mathcal{C}}(F)$. By induction we have $Y \in \mathcal{I}_{\mathcal{C}}(F)$. Therefore $(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^{\perp} = (T, \mathcal{I}_{\mathcal{C}}(F))$. ∎

The following example illustrates the above theorem.

Example 4.20. Let Λ be an algebra given by the quiver



with relations $\gamma\alpha = 0 = \beta^2$ and $\gamma\beta\alpha = 0$. Let $\mathcal{C}$ be equal to the subcategory $\operatorname{add}\{S_2, P_2, I_2, L, M, N\}$, where L, M and N are given by the radical filtration $\begin{smallmatrix}2\\2\end{smallmatrix}$, $\begin{smallmatrix} & 2 & \\ 3 & & 2\end{smallmatrix}$, and $\begin{smallmatrix}2\\3\end{smallmatrix}$ respectively. Then $\mathcal{C}$ is closed under extensions. Moreover, $\mathcal{C}$ is functorially finite, since Λ is of finite type. It can be shown that $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) \leq 1$. Let $F = F_{\mathcal{X}}$, where $\mathcal{X} = \mathcal{P}(\mathcal{C}) \cup \operatorname{add} M$. Then we have $\mathcal{I}_{\mathcal{C}}(F) = \mathcal{I}(\mathcal{C}) \cup \operatorname{add} N$. The Λ-module $T = I_3 \oplus L \oplus M$ is an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = 1$. It can be shown that $\operatorname{id}_F T = \infty$, hence T is not F-cotilting in $\mathcal{C}$. Let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. It is easy to see that the Γ-module $V = P_1 \oplus P_2 \oplus S_3$, where $\operatorname{add} V = (T, \mathcal{I}_{\mathcal{C}}(F))$, is cotilting with $\operatorname{id}_\Gamma V = 2$.

The following immediate consequence of Theorem 4.19 is an analog of the dual of [10, Corollary 3.14].

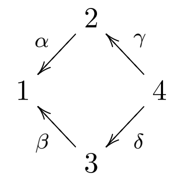
Corollary 4.21. *The subcategory $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type.*

Proof. Since $\mathcal{I}_{\mathcal{C}}(F)$ is equivalent to $(T, \mathcal{I}_{\mathcal{C}}(F))$ by Proposition 4.6(b) and $(T, \mathcal{I}_{\mathcal{C}}(F))$ is of finite type by Theorem 4.19(d), the subcategory $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type. ∎

By the above result, if $\mathcal{I}_{\mathcal{C}}(F)$ is of infinite type, then there is no F-tilting Λ-module in $\mathcal{C}$.

It can be shown that (by the dual of [10, Proposition 3.15]) if T is an F-tilting Λ-module in $\operatorname{mod}\Lambda$ and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$, then DT is a direct summand of a cotilting Γ-module T_0, where $\operatorname{add} T_0 = (T, \mathcal{I}(F))$. This is not necessarily the case for an F-tilting Λ-module T in a functorially finite subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ with $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = n$, where $0 \leq n < \infty$. We illustrate this by the following example.

EXAMPLE 4.22. Let Λ be given by the quiver



with relation $\alpha\gamma = 0$. Let $\mathcal{C} = \operatorname{add}\{P_1, P_2, S_2, P_4, C_1, C_2, I_1, I_2, I_4\}$, where the radical filtrations of C_1 and C_2 look like

$$\begin{smallmatrix} & & & 4 & \\ 2 & & 3 & & 2 \\ & 1 & & & \end{smallmatrix} \qquad\qquad \begin{smallmatrix} 4 \\ 3 \\ 1 \end{smallmatrix}$$

respectively. It can be (easily) shown that $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = 1$. Since $\operatorname{mod}\Lambda$ is of finite type, every subcategory of $\operatorname{mod}\Lambda$ is functorially finite ([5, Proposition 1.2]). Let $F = F_{\mathcal{X}}$ where $\mathcal{X} = \mathcal{P}(\mathcal{C}) \cup \operatorname{add} S_4$. Denote the direct sum of all indecomposable F-projective Λ-modules in $\mathcal{C}$ by P. Then P is the trivial F-tilting module in $\mathcal{C}$. Let $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$. By Theorem 4.19(d) the module $T^0_{\mathcal{C}} = J_1 \oplus Q_4 \oplus Q_2 \oplus Q_5 \oplus \begin{smallmatrix} 2 & & 3 & \\ & 1 & & 2 \end{smallmatrix}$ is a cotilting Γ-module. The module (T, I_3) is a direct summand of DT, but it is not a direct summand of $T^0_{\mathcal{C}}$. So DT is not a direct summand of $T^0_{\mathcal{C}}$.

Observe that in Example 4.22 the module DT is in $\widehat{\operatorname{add} T^0_{\mathcal{C}}}$. This is true in general, as shown by the following result.

PROPOSITION 4.23. *Let T be an F-tilting module in a functorially finite subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ with $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = n$, where $0 \leq n < \infty$. Then DT is in $\widehat{(T, \mathcal{I}_{\mathcal{C}}(F))}$.*

Proof. Consider the right $\mathcal{C}$-approximation resolution $0 \to C_l \to \cdots \to C_1 \to C_0 \to D\Lambda$ of $D\Lambda$, where $l \leq n$. Applying the functor $(T, \ )$ to it, we get the exact sequence

$$0 \to (T, C_l) \to \cdots \to (T, C_1) \to (T, C_0) \to (T, D\Lambda) \to 0.$$

Lemma 1.5 shows that C_i is in $\mathcal{I}_{\mathcal{C}}(F)$ for $0 \le i \le n$. Hence $(T, C_i) \in \operatorname{add} T^0_{\mathcal{C}}$ for $0 \le i \le n$. Therefore $DT \in \widehat{(T, \mathcal{I}_{\mathcal{C}}(F))}$. ∎

4.3. *Relative tilting and global dimension.* In this section we show some relationship between the F-global dimension of $\mathcal{C}$ and the global dimension of Γ, which generalizes [10]. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Throughout this section we assume that $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$. We fix an F-tilting module T in $\mathcal{C}$ with $\operatorname{pd}_F T = r$ and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ.

If T is F-tilting in $\operatorname{mod}\Lambda$, then it can be shown that (using duality [10, Section 4]) the relative (or F-) global dimension of Λ, $\operatorname{gl.dim}_F \Lambda$, and the global dimension of Γ, $\operatorname{gl.dim}\Gamma$, are related by the inequalities $\operatorname{gl.dim}_F \Lambda - \operatorname{pd}_F T \le \operatorname{gl.dim}\Gamma \le \nu(0, \operatorname{pd}_F T) + \operatorname{gl.dim}_F \Lambda$.

Denote by $\operatorname{gl.dim}_F \mathcal{C}$ the relative (or F-) global dimension of $\mathcal{C}$. We show that $\operatorname{gl.dim}_F \mathcal{C}$ and $\operatorname{gl.dim}\Gamma$ satisfy similar inequalities, namely $\operatorname{gl.dim}_F \mathcal{C} - \operatorname{pd}_F T \le \operatorname{gl.dim}\Gamma \le \nu(n, r) + \operatorname{gl.dim}_F \mathcal{C}$, where $\nu(n, r)$ is the upper bound of $\mathcal{Y}^{\mathcal{C}}_T$-resdim$(\operatorname{mod}\Gamma)$ (see Proposition 4.18).

The main result in this section, given below, is a generalization of [10, dual of Proposition 4.1].

PROPOSITION 4.24. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume $\mathcal{C}$-app.dim$(\operatorname{mod}\Lambda) = n < \infty$. Let T be an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then*

$$\operatorname{gl.dim}_F \mathcal{C} - \operatorname{pd}_F T \le \operatorname{gl.dim}\Gamma \le \nu(n, r) + \operatorname{gl.dim}_F \mathcal{C}.$$

Proof. First we prove that $\operatorname{gl.dim}\Gamma \le \nu(n, r) + \operatorname{gl.dim}_F \mathcal{C}$. If $\operatorname{gl.dim}_F \mathcal{C}$ is infinite, there is nothing to prove, so we assume that it is finite. For all $Y \in \mathcal{Y}^{\mathcal{C}}_T$ there is an F-exact sequence $0 \to Y \to I_0 \to I_1 \to \cdots \to I_s \to 0$ with $I_i \in \mathcal{I}_{\mathcal{C}}(F)$ and $s \le \operatorname{gl.dim}_F \mathcal{C}$. When we apply $\operatorname{Hom}_\Lambda(T,\ )$ to it we get the exact sequence $0 \to (T, Y) \to (T, I_0) \to \cdots \to (T, I_s) \to 0$. Theorem 4.19(b) shows that $\operatorname{id}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F)) \le \nu(n, r)$, hence $\operatorname{id}_\Gamma(T, \mathcal{Y}^{\mathcal{C}}_T) \le \nu(n, r) + \operatorname{gl.dim}_F \mathcal{C}$. By Proposition 4.18 we have $\Omega_F^{\nu(n,r)}(B) \in (T, \mathcal{Y}^{\mathcal{C}}_T)$ for all $B \in \operatorname{mod}\Gamma$. Hence $\operatorname{id}_\Gamma B \le \operatorname{id}_\Gamma(T, Y) \le \nu(n, r) + \operatorname{gl.dim}_F \mathcal{C}$ for all Y in $\mathcal{Y}^{\mathcal{C}}_T$, since Γ is in $(T, \mathcal{Y}^{\mathcal{C}}_T)$. Thus we have shown that $\operatorname{gl.dim}\Gamma \le \nu(n, r) + \operatorname{gl.dim}_F \mathcal{C}$.

Now we show that $\operatorname{gl.dim}_F \mathcal{C} \le \operatorname{pd}_F T + \operatorname{gl.dim}\Gamma$. If $\operatorname{gl.dim}\Gamma$ is infinite, there is nothing to prove, so we assume that it is finite. By the dual of [10, Proposition 3.7] we have $\operatorname{Ext}^i_F(C, A) \simeq \operatorname{Ext}^i_\Gamma((T, C), (T, A))$ for all A and $C \in \mathcal{Y}^{\mathcal{C}}_F$. So $\operatorname{Ext}^i_F(C, A) = 0$ for $i > \operatorname{gl.dim}\Gamma$.

We claim that if $\operatorname{Ext}^i_F(\mathcal{Y}^{\mathcal{C}}_F, B) = 0$ for all $i > j$ then $\operatorname{Ext}^i_F(\ , B) = 0$ for all $i > j$, equivalently $\Omega_F^{-j}(B) \in \mathcal{I}_{\mathcal{C}}(F)$. To prove the claim, let $N \in \mathcal{C}$. By Proposition 4.2, $\mathcal{Y}^{\mathcal{C}}_T$-coresdim$_F \mathcal{C} = r$ is finite, so we have an F-exact sequence $0 \to N \to Y_0 \to \cdots \to Y_r \to 0$ with $Y_i \in \mathcal{Y}^{\mathcal{C}}_T$. Applying $(\ , B)$ and using dimension shift, we get $\operatorname{Ext}^i_F(N, B) \simeq \operatorname{Ext}^{i+r}_F(Y_r, B) = 0$ for all $i > j$.

So $\mathrm{Ext}^i_F(N,B) = 0$ for all $i > j$ and $N \in \mathcal{C}$, which is equivalent to saying that $\Omega_F^{-j}(B) \in \mathcal{I}_{\mathcal{C}}(F)$. Hence the claim follows.

Now since $\mathrm{Ext}^i_F(C,A) = 0$ for $i > \mathrm{gl.dim}\,\Gamma$ for all C and $A \in \mathcal{Y}_T^{\mathcal{C}}$, the claim shows that $\Omega_F^{-\,\mathrm{gl.dim}\,\Gamma}(A) \in \mathcal{I}_{\mathcal{C}}(F)$. By Proposition 4.2 we have $\mathcal{Y}_T^{\mathcal{C}}\text{-coresdim}_F\,\mathcal{C} \le r$. Since $\mathcal{I}_{\mathcal{C}}(F) \subseteq \mathcal{Y}_T^{\mathcal{C}}$, we have an F-exact sequence $0 \to N \to I_0 \to \cdots \to I_{r-1} \to \Omega_F^{-r}(N) \to 0$ with $\Omega_F^{-r}(N) \in \mathcal{Y}_T^{\mathcal{C}}$ for all $N \in \mathcal{C}$. So $\mathrm{id}_F\, N \le r + \mathrm{gl.dim}\,\Gamma$ for all $N \in \mathcal{C}$. Therefore, $\mathrm{gl.dim}_F\,\mathcal{C} \le \mathrm{pd}_F\,T + \mathrm{gl.dim}\,\Lambda$, and the result follows. ∎

5. Relative theory and stratifying systems. Erdmann and Sáenz [13] introduced the concept of a stratifying system. The concept was studied further by Marcos et al. [17], who introduced the notion of an Ext-projective stratifying system. Suppose Θ is a stratifying system and let $\mathcal{F}(\Theta)$ denote the category of Λ-modules filtered by Θ. Let Q denote the direct sum of all nonisomorphic indecomposable Ext-projective modules in $\mathcal{F}(\Theta)$. One of the main results of [17] states that the algebra $B = \mathrm{End}_\Lambda(Q)^{\mathrm{op}}$ is standardly stratified and the functor $\mathrm{Hom}_\Lambda(Q,\ )$ induces an equivalence between the subcategories $\mathcal{F}_\Lambda(\Theta)$ and $\mathcal{F}_B(\Delta)$. Moreover, $\mathcal{F}_\Gamma(\Delta) = \widehat{\mathrm{add}\,{}_BT}$, where ${}_BT$ is the characteristic tilting B-module.

Throughout this section, $\mathcal{C}$ is a functorially finite subcategory of $\mathrm{mod}\,\Lambda$ which is closed under extensions, and $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{C}$ which is a generator for $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and denote $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ by Γ. In 5.1 we prove the main result of this section, which shows that the Γ-module $\mathrm{Hom}_\Lambda(T, \mathcal{I}_{\mathcal{C}}(F))$ is tilting. Moreover, there is an equivalence between the subcategories $\widehat{\mathrm{add}\,T_{\mathcal{C}}}$ of $\mathcal{C}$ and $\widehat{(T, \mathcal{I}_{\mathcal{C}}(F))}$ of $\mathrm{mod}\,\Gamma$. The main result of this section was inspired by the above-mentioned result from [17]. We look at the connection between relative theory and stratifying systems in 5.2. In 5.3 we first show that if the $\mathcal{C}$-approximation dimension of $\mathrm{mod}\,\Lambda$ is finite, then Γ is an artin Gorenstein algebra, which generalizes [11, Proposition 3.1]. We then construct quasihereditary algebras using relative theory in subcategories.

5.1. *Relative tilting cotilting modules in subcategories.* Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and denote $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. In the next result we show that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is tilting and the tilting functor induces an equivalence between $\widehat{\mathrm{add}\,T_{\mathcal{C}}}$ and $(T, \widehat{\mathrm{add}\,T_{\mathcal{C}}})$. This is the main result of this section.

THEOREM 5.1. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under extensions. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and let $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Then*:

(a) *The Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is tilting with projective dimension at most* $\mathrm{id}_F T$. *Moreover*, $(T, \mathcal{I}_{\mathcal{C}}(F))$ *is of finite type.*

(b) *The functor* $\mathrm{Hom}_\Lambda(T,\ ) \colon \mathrm{mod}\, \Lambda \to \mathrm{mod}\, \Gamma$ *induces an equivalence between* $\widehat{\mathrm{add}\, T_{\mathcal{C}}}$ *and* $(T, \widehat{\mathrm{add}\, T_{\mathcal{C}}})$.

Proof. (a) By Corollary 4.5, we have

$$\mathrm{Ext}^i_\Gamma((T, \mathcal{I}_{\mathcal{C}}(F)), (T, \mathcal{I}_{\mathcal{C}}(F))) \simeq \mathrm{Ext}^i_F(\mathcal{I}_{\mathcal{C}}(F), \mathcal{I}_{\mathcal{C}}(F)) = 0$$

since $\mathcal{I}_{\mathcal{C}}(F) \subseteq \mathcal{Y}^{\mathcal{C}}_T$. Since T is F-cotilting module in $\mathcal{C}$, we have an F-exact sequence. $0 \to T_m \to \cdots \to T_1 \to T_0 \to \mathcal{I}_{\mathcal{C}}(F) \to 0$ with $T_i \in \mathrm{add}\, T$ and $m \leq \mathrm{id}_F T$. Applying the functor $(T,\ )$ to it we deduce that $\mathrm{pd}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F))$ is finite. In particular, $\mathrm{pd}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F)) \leq \mathrm{id}_F T$. Applying $\mathrm{Hom}_\Lambda(T,\ )$ to the F-injective resolution of T we see that $\Gamma \in \widetilde{(T, \mathcal{I}_{\mathcal{C}}(F))}$. Therefore $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a standard tilting Γ-module.

By the corollary to [19, Proposition 1.8] we infer that (T, I), for all $I \in \mathcal{I}_{\mathcal{C}}(F)$, is a direct summand of

$$\mathrm{add} \bigoplus_{i=0}^{s} (T, I_i)$$

with all $I_i \in \mathcal{I}_{\mathcal{C}}(F)$. Hence $(T, \mathcal{I}_{\mathcal{C}}(F))$ is of finite type.

(b) This follows from Theorem 4.6, since $\widehat{\mathrm{add}\, T_{\mathcal{C}}} \subseteq T^\perp_{\mathcal{C}}$. ∎

The following result shows that in Theorem 5.1 it is sufficient to assume that $\mathrm{gl.dim}_F\, \mathcal{C} < \infty$ and T is F-tilting.

Corollary 5.2. *Let T be an F-tilting module in $\mathcal{C}$ and assume that* $\mathrm{gl.dim}_F\, \mathcal{C}$ *is finite. Then T is an F-cotilting module in $\mathcal{C}$.*

Proof. It follows that T is F-selforthogonal and has finite F-injective dimension, since T is F-tilting and $\mathrm{gl.dim}_F\, \mathcal{C}$ is finite. Since $\mathrm{gl.dim}_F\, \mathcal{C}$ is finite and T is an F-tilting module in $\mathcal{C}$, we have $T^\perp_{\mathcal{C}} = \widehat{\mathrm{add}\, T}$ by Lemma 4.3. Therefore $\mathcal{I}_{\mathcal{C}}(F)$ has a finite F-$\mathrm{add}\, T$-resolution. ∎

The following is also a consequence of Theorem 5.1.

Corollary 5.3. *Let T be an F-tilting F-cotilting module in $\mathcal{C}$. Then the subcategory $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type.*

Proof. Theorem 5.1(a) shows that $(T, \mathcal{I}_{\mathcal{C}}(F))$ is of finite type. By Theorem 5.1(b) there is an equivalence between $\mathcal{I}_{\mathcal{C}}(F)$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$. Hence the claim follows. ∎

Now we show that the subcategories $(T, \widehat{\mathrm{add}\, T_{\mathcal{C}}})$ and $\widetilde{(T, \mathcal{I}_{\mathcal{C}}(F))}$ coincide. We need the following results.

Lemma 5.4. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\, \Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$ and let* $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. *Assume* $\mathrm{pd}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F))$ *is finite. Then* $DT \in (T, \mathcal{I}_{\mathcal{C}}(F))^\perp$.

Proof. Since $\mathcal{C}$ is functorially finite in $\operatorname{mod} \Lambda$, we have a right $\mathcal{C}$-approximation resolution $\cdots \to C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} D\Lambda$ of $D\Lambda$. Denote $\operatorname{Ker} g_i$ by Y_{i+1} for $i \geq 0$. Applying $(T, \ )$ to the above sequence we get an exact sequence

$$\cdots \to (T, C_1) \to (T, C_0) \to (T, D\Lambda) \to 0. \tag{4}$$

since $T \in \mathcal{C}$. Consider the short exact sequence $0 \to (T, Y_{j+1}) \to (T, C_j) \to (T, Y_j) \to 0$. Applying $((T, \mathcal{I}_\mathcal{C}(F)), \ )$ we get the following commutative diagram by Lemma 4.4:

$$\begin{array}{ccccccc}
0 \to & ((T,I),(T,Y_{j+1})) & \to & ((T,I),(T,C_j)) & \to & ((T,I),(T,Y_j)) & \\
& \wr\uparrow & & \wr\uparrow & & \wr\uparrow & \\
0 \to & (I, Y_{j+1}) & \to & (I, C_j) & \to & (I, Y_j) & \to 0
\end{array} \tag{5}$$

Since $I \in \mathcal{C}$, the bottom row of (5) is exact. Hence the top row of (5) is exact. Thus the functor $((T, I), \ )$, for $I \in \mathcal{I}_\mathcal{C}(F)$, is exact on (4). Therefore $\operatorname{Ext}^1_\Gamma((T, I), (T, Y_j)) = 0$ for all $j > 0$. Let s be a nonnegative integer. Then by dimension shift, $\operatorname{Ext}^i_\Gamma((T, I), (T, Y_s)) = 0$ for all $i > 0$ and $s \geq \operatorname{pd}_\Gamma(T, I)$. But $\operatorname{pd}_\Gamma(T, \mathcal{I}_\mathcal{C}(F))$ is finite by the assumption. Hence $(T, Y_s) \in (T, \mathcal{I}_\mathcal{C}(F))^\perp$ for $s > \operatorname{pd}_\Gamma(T, I)$. Finally, by using the fact that $(T, \mathcal{I}_\mathcal{C}(F))^\perp$ is closed under cokernels of monomorphisms and (4), it follows by induction that $DT \in (T, \mathcal{I}_\mathcal{C}(F))^\perp$. ∎

As an immediate consequence of the above result we have the following.

Corollary 5.5. *The functor $T \otimes_\Gamma \ \simeq D(\ , DT)\colon \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda$ is exact on $(T, \widecheck{\mathcal{I}_\mathcal{C}(F)})$.*

Proof. Let $Y \in (T, \widecheck{\mathcal{I}_\mathcal{C}(F)})$. Applying $(\ , DT)$ to the $(T, \mathcal{I}_\mathcal{C}(F))$-coresolution of Y, and then using dimension shift and Lemma 5.4, we get $\operatorname{Ext}^i_\Gamma(Y, DT) \simeq \operatorname{Ext}^{i+q}_\Gamma((T, I_q), DT) = 0$ for all $i > 0$. Thus the claim follows. ∎

We now show that the subcategory $(T, \widehat{\operatorname{add} T_\mathcal{C}})$ can be identified with the subcategory $(T, \widecheck{\mathcal{I}_\mathcal{C}(F)})$.

Proposition 5.6. *Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod} \Lambda$ which is closed under extensions. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then $(T, \widehat{\operatorname{add} T_\mathcal{C}}) \simeq (T, \widecheck{\mathcal{I}_\mathcal{C}(F)})$.*

Proof. By Theorem 5.1(b), $Z \in \widehat{\operatorname{add} T_\mathcal{C}}$ if and only if $(T, Z) \in (T, \widehat{\operatorname{add} T_\mathcal{C}})$. Let $Z \in \widehat{\operatorname{add} T_\mathcal{C}}$. Then we have an F-exact sequence $0 \to Z \to T_0 \to T_1 \to \cdots \to T_m \to 0$ with $T_i \in \operatorname{add} T$. Since $\operatorname{id}_F T$ is finite, so is $\operatorname{id}_F Z$ by [19, Lemma 2.1(1)]. Let $0 \to Z \to I_0 \to \cdots \to I_s \to 0$ be an F-injective resolution of Z. Applying $(T, \ )$ to it we get an exact sequence $0 \to (T, Z) \to (T, I_0) \to (T, I_1) \to \cdots \to (T, I_s) \to 0$, and thus $(T, Z) \in (T, \widecheck{\mathcal{I}_\mathcal{C}(F)})$. Hence $(T, \widehat{\operatorname{add} T_\mathcal{C}}) \subseteq (T, \widecheck{\mathcal{I}_\mathcal{C}(F)})$.

Now let $Y \in \widetilde{(T, \mathcal{I}_{\mathcal{C}}(F))}$. Then we have an exact sequence $0 \to Y \to (T, I_0) \to (T, I_1) \to \cdots \to (T, I_s) \to 0$ with $I_i \in \mathcal{I}_{\mathcal{C}}(F)$. By Theorem 5.1(a), $\mathrm{pd}_\Gamma(T, I_j) < \infty$, hence $\mathrm{pd}_\Gamma Y < \infty$ (by [19, Lemma 2.1(4)]). Consider a projective resolution $0 \to P_t \to \cdots \to P_1 \to P_0 \to Y \to 0$ of Y over Γ. Denote $\Omega^i_\Gamma(Y)$ by Y_i. Note that all Y_i are in $\widetilde{(T, \mathcal{I}_{\mathcal{C}}(F))}$, since $(T, \mathcal{I}_{\mathcal{C}}(F))$ is tilting. Applying $T \otimes_\Gamma \cdot$ to the above sequence we get the exact sequence

$$0 \to T \otimes_\Gamma P_t \to \cdots \to T \otimes_\Gamma P_1 \to T \otimes_\Gamma P_0 \to T \otimes_\Gamma Y \to 0 \tag{6}$$

by Corollary 5.5. But since $T \otimes_\Gamma \Gamma \simeq T$ we see that (6) is isomorphic to

$$0 \to T_t \to \cdots \to T_1 \to T_0 \to T \otimes_\Gamma Y \to 0. \tag{7}$$

We need to show that (7) is F-exact. By using the adjoint isomorphism and the fact that the Y_j are in ${}^\perp(T, \mathcal{I}_{\mathcal{C}}(F))$, we infer that the functor $\mathrm{Hom}_\Lambda(\ , J)$, for J in $\mathcal{I}_{\mathcal{C}}(F)$, is exact on (6). Hence (7) is $F^{\mathcal{I}_{\mathcal{C}}(F)}$-exact. But then Proposition 2.5 implies that (7) is in $\mathcal{C}$. So (7) is F-exact by Corollary 2.3. Therefore $T \otimes_\Gamma Y$ is in $\widehat{\mathrm{add}\, T_{\mathcal{C}}}$. Then Theorem 5.1(b) shows that $(T, T \otimes_\Gamma Y) \in \widehat{(T, \mathrm{add}\, T_{\mathcal{C}})}$. But by [19, Lemma 1.9], we have $Y \simeq (T, T \otimes_\Gamma Y)$. Therefore $Y \in \widehat{(T, \mathrm{add}\, T_{\mathcal{C}})}$. This completes the proof. ■

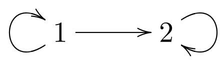
The following example illustrates the main result of this section. It also shows that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is not cotilting.

Example 5.7. Let Λ be an algebra given by the quiver in Example 4.17 with relations $\alpha^2 = 0$, $\beta_1\beta_2 = 0$ and $\beta_1\alpha = \alpha\beta_2 = 0$. Let $\theta_1 = P_1/P_2$ and $\theta_2 = P_2$. Then $\mathcal{C} = \mathcal{F}(\Theta) = \mathrm{add}\{\theta_1, P_1, P_2\}$ is closed under direct summands, hence also under extensions. A right $\mathcal{C}$-approximation resolution of S_2 is $\cdots \to P_1/P_2 \to P_1/P_2 \to P_2 \to S_2 \to 0$. Then by Proposition 3.2 we have $\mathcal{C}\text{-app.dim}(\mathrm{mod}\, \Lambda) = \infty$. Consider the subfunctor $F = F_{\mathcal{C}}$. There is only one F-tilting module in $\mathcal{C}$ up to isomorphism, namely the trivial F-tilting module $T = P_1 \oplus \theta_1 \oplus P_2$. Let $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. The module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is Γ itself, so it is a tilting Γ-module. It can be easily seen that $\mathrm{id}_\Gamma Q_3 = \infty$. Hence Γ is not a cotilting module over itself.

Question 1. Is $(T, \mathcal{I}_{\mathcal{C}}(F))$ a tilting Γ-module when T is an arbitrary F-tilting module in $\mathcal{C}$?

If T is an F-tilting F-cotilting module in $\mathcal{C}$, then the answer is given in Theorem 5.1. But if T is F-tilting but not F-cotilting, then we have the following example.

Example 5.8. Let Λ be an algebra given by the quiver



with radical square-zero relations. Let $\mathcal{C} = \mathrm{add}\{S_1, P_2, M, I_1, I_2\}$, where M is given by the radical filtration ${}_1{}^1{}_2{}^2$. The subcategory $\mathcal{C}$ is closed under extensions. The right $\mathcal{C}$-approximation resolution of S_2 is $\cdots \to I_2 \to I_2 \to$

$S_2 \to 0$. Then Proposition 3.2 yields $\mathcal{C}$-app.dim(mod Λ) $= \infty$. Since Λ is of finite type, all subcategories of mod Λ are functorially finite as in the previous example. Let $F = F_{\mathcal{P}(\mathcal{C})}$. Let T be the trivial F-tilting module in $\mathcal{C}$. It can be (easily) shown that $\mathrm{id}_F T = \infty$. Hence T is not an F-cotilting Γ-module. Let $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Denote by U the direct sum of all indecomposable modules in $\mathcal{I}_{\mathcal{C}}(F)$. It can be easily seen that $\mathrm{pd}_\Gamma J_1 = \infty$. Hence (T, U) is not a tilting Γ-module. It can also be seen that $\mathrm{id}_\Gamma Q_2/Q_1 = \infty$, hence (T, U) is not a cotilting module.

5.2. *Stratifying systems.* In this subsection we look at the relationship between relative theory and stratifying systems. We show how a relative theory can be defined in a subcategory associated with a stratifying system. Then we show that the main result of this section generalizes one of the main results of [17].

But first we recall the definition of a stratifying system.

Definition ([13, Definition 1.1]). Let R be a finite-dimensional algebra. A *stratifying system* $\Theta = (\Theta, \leq)$ of size t consists of a set $\Theta = \{\theta(i)\}_{i=1}^t$ of indecomposable R-modules and a total order $\leq$ on $\{1, \ldots, t\}$ satisfying the following conditions:

(i) $\mathrm{Hom}_R(\theta(j), \theta(i)) = 0$ for $j \succ i$,
(ii) $\mathrm{Ext}^1_R(\theta(j), \theta(i)) = 0$ for $j \succeq i$.

As before, $\mathcal{F}(\Theta)$ denotes the subcategory of mod R consisting of all modules having filtration with quotients isomorphic to the $\theta(i)$'s. The subcategory $\mathcal{F}(\Theta)$ is functorially finite in mod R [21]. If $\mathcal{F}(\Theta)$ is closed under direct summands, then it is closed under extensions [21].

Let Θ be a stratifying system and let $\mathcal{C} = \mathcal{F}(\Theta)$. Then $\mathcal{P}(\mathcal{C}) = \mathrm{add}\, Q$, where $Q = \bigoplus_{i=1}^t Q(i)$. The module $Q(i)$, for $i = 1, \ldots, t$, is given by the exact sequence $0 \to K(i) \to Q(i) \to \theta(i) \to 0$ such that $K(i) \in \mathcal{F}(\{\theta(j) : j \succ i\})$. Dually, $\mathcal{I}(\mathcal{C}) = \mathrm{add}\, Y$, where $Y = \bigoplus_{i=1}^t Y(i)$. The module $Y(i)$, for $i = 1, \ldots, t$, is given by the exact sequence $0 \to \theta(i) \to Y(i) \to L(i) \to 0$ such that $L(i)$ is in $\mathcal{F}(\{\theta(j) : j \prec i\})$ [17], [18].

Now, since $\mathcal{C}$ is functorially finite in mod Λ and closed under extensions, it has enough Ext-projectives and Ext-injectives by Corollary 1.3. Then gl.dim $\mathcal{C}$ is finite by [17, Corollary 2.11] and [13, Lemma 1.5]. It is easy to see that $\mathcal{P}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ are contravariantly and covariantly finite subcategories of $\mathcal{C}$, respectively.

Consider the subfunctor $F = F_{\mathcal{X}}$, where $\mathcal{X} = \mathcal{P}(\mathcal{C})$. Then F is the trivial subfunctor in $\mathcal{C}$ with $\mathrm{gl.dim}_F\, \mathcal{C}$ finite. We have $\mathcal{P}_{\mathcal{C}}(F) = \mathrm{add}\, Q$ and $\mathcal{I}_{\mathcal{C}}(F) = \mathrm{add}\, Y$. Let T be the trivial F-tilting module Q in $\mathcal{C}$ and let $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Then the following result is a consequence of Theorem 5.1 and Proposition 5.6.

THEOREM 5.9 ([17, Theorems 3.1, 3.2]). *Let Θ be a stratifying system and consider the category $\mathcal{F}(\Theta)$. Then*:

(a) $\mathrm{Hom}_{\Lambda}(T, Y)$ *is a tilting Γ-module.*
(b) *The functor* $\mathrm{Hom}_{\Lambda}(T, \ )\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ *induces an equivalence between* $\mathcal{F}(\Theta)$ *and* $\mathrm{Hom}_{\Lambda}(T, \mathcal{F}(\Theta))$.
(c) $(T, \mathcal{F}(\Theta)) = \widetilde{(T, Y)}$.

Proof. (a) and (b) follow from Theorem 5.1, while (c) follows from Proposition 5.6. ■

5.3. *Construction of Gorenstein and quasihereditary algebras.* In this section we construct Gorenstein algebras as endomorphism algebras of relative tilting and relative cotilting modules. We then construct quasihereditary algebras from stratifying systems.

Recall that an algebra Λ is said to be *Gorenstein* if $\mathrm{id}_{\Lambda}\Lambda$ and $\mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda^{\mathrm{op}}$ are both finite. If Λ is also artin (or an algebra which admits duality), then $\mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda^{\mathrm{op}}$ is finite if and only if $\mathrm{pd}_{\Lambda} D(\Lambda^{\mathrm{op}})$ is finite [11]. The following result is a generalization of [11, Proposition 3.1].

PROPOSITION 5.10. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under extensions and assume* $\mathcal{C}\text{-app.dim}(\mathrm{mod}\,\Lambda) = n < \infty$. *Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and* $\Gamma = \mathrm{End}_{\Lambda}(T)^{\mathrm{op}}$. *Then Γ is an artin Gorenstein algebra with both* $\mathrm{id}_{\Gamma}\Gamma$ *and* $\mathrm{pd}_{\Gamma} D(\Gamma^{\mathrm{op}})$ *at most* $\mathrm{id}_F T + \nu(n, r)$.

Proof. By Theorem 5.1, $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a tilting Γ-module such that $\mathrm{pd}_{\Gamma}(T, \mathcal{I}_{\mathcal{C}}(F)) \le \mathrm{id}_F T$. So we have an exact sequence

$$0 \to \Gamma \to (T, I_0) \to (T, I_1) \to \cdots \to (T, I_s) \to 0$$

with the (T, I_j) in $(T, \mathcal{I}_{\mathcal{C}}(F))$ and $s \le \mathrm{id}_F T$. Then Theorem 4.19 shows that $\mathrm{id}_{\Gamma}\Gamma \le \mathrm{id}_F T + \nu(n, r)$.

On the other hand, we have, by Theorem 4.19, an exact sequence

$$0 \to (T, I_t) \to \cdots \to (T, I_1) \to (T, I_0) \to D(\Gamma^{\mathrm{op}}) \to 0$$

with the (T, I_j) in $(T, \mathcal{I}_{\mathcal{C}}(F))$ and $t \le \nu(n, r)$, since $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a cotilting Γ-module. Hence $\mathrm{pd}_{\Gamma} D(\Gamma^{\mathrm{op}}) \le \mathrm{id}_F T + \nu(n, r)$. Therefore Γ is artin Gorenstein. ■

The following result gives us an important subclass of Gorenstein algebras, namely a class of algebras of finite global dimension.

PROPOSITION 5.11. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$. Assume* $\mathcal{C}\text{-app.dim}(\mathrm{mod}\,\Lambda)$ *and* $\mathrm{gl.dim}_F\,\mathcal{C}$ *are finite. Then* $\Gamma = \mathrm{End}_{\Lambda}(T)^{\mathrm{op}}$ *has finite global dimension.*

Proof. Follows easily from Proposition 4.24. ∎

The following consequence of Proposition 5.11 gives a sufficient condition for obtaining a quasihereditary algebra for a given stratifying system Θ. Let Q denote the direct sum of non-isomorphism indecomposable Ext-projective modules in $\mathcal{F}(\Theta)$.

Corollary 5.12. *Let Θ be a stratifying system and Q be as above. Assume $\mathcal{F}(\Theta)$-app.dim(mod Λ) is finite. Then $\mathrm{End}_\Lambda(Q)^{\mathrm{op}}$ is quasihereditary.*

Proof. Define a subfunctor $F = F_{\mathcal{X}}$, where $\mathcal{X} = \operatorname{add} Q$. Then the dimension $\mathrm{gl.dim}_F\, \mathcal{F}(\Theta)$ is finite. By [17, Theorem 0.1], $\mathrm{End}_\Lambda(Q)^{\mathrm{op}}$ is a standardly stratified algebra. But then Proposition 5.11 shows that $\mathrm{End}_\Lambda(Q)^{\mathrm{op}}$ has finite global dimension. Hence $\mathrm{End}_\Lambda(Q)^{\mathrm{op}}$ is quasihereditary by using [1, Theorem 2.4]. ∎

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