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A NEW PROOF FOR THE MULTIPLICATIVE PROPERTY OF THE BOOLEAN CUMULANTS WITH APPLICATIONS TO THE OPERATOR-VALUED CASE

BY

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Abstract. The paper presents several combinatorial properties of the boolean cumulants. A consequence is a new proof of the multiplicative property of the boolean cumulant series that can be easily adapted to the case of boolean independence with amalgamation over an algebra.

1. Introduction. Boolean probability theory, according to [11], is one of the three symmetric universal probability theories (the other two being classical and free probability theories). It has been in the literature at least since early '70's ([15]) with various developments, from stochastic differential equations ([14]) to measure theory ([13], [2]) and Appell polynomials ([1]).

Two important notions in free probability theory are the R and S-transforms. For x a non-commutative random variable of finite moments, $R_x(z)$ and $S_x(z)$ are power series that encode the information from the moment generating function of x and have the following additive, respectively multiplicative, properties: if x, y are free, then $R_{x+y} = R_x + R_y$ and $S_{xy} = S_x S_y$.

The notion of boolean cumulants has appeared, in various contexts, at least since early '70's ([15], [14], [8]). Their generating function, which we will call the B-transform, has an additive property similar to the R-transform ([13]): if x and y are two boolean independent non-commutative random variables, then $B_{x+y} = B_x + B_y$.

In [6], U. Franz remarked an interesting multiplicative property of the B-transform, which gives a valuable tool for studying the multiplicative boolean convolution. If x, y are boolean independent, then

$$B_{(1+x)(1+y)} = B_{1+x} B_{1+y}. \tag{1}$$

In the literature there are two proofs of (1). The original proof, in [6], uses algebraic properties of the resolvent function. The proof in [2], in the spirit of D. Voiculescu's proof of the property for the R and S-transforms,

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uses a Fock-type model and properties of the Cauchy transform. In the attempt to develop a boolean probability theory with amalgamation over an algebra, when scalar-valued functionals are replaced by conditional expectations, the above approaches are not suitable to prove a similar multiplicative property, since objects such as the Cauchy transform and resolvent function do not have natural analogues. The present paper gives a combinatorial proof of (1), which can be easily adapted to amalgamation over an algebra (see Section 4).

In [4], [5], K. Dykema develops an analogue of the S-transform in the operator-valued free probability theory. Due to non-commutativity, it satisfies a "twisted" multiplicative property. Namely, if $S = T^{-1}$, then

$$T_{xy} = [T_x \circ (T_y I T_y^{-1})] T_y.$$

Corollary 4.6 from Section 4 shows that the operator-valued B-transform has the same multiplicative property as in the scalar case, without any "twists".

2. Preliminaries. Let $\mathcal{A}$ and $\mathcal{C}$ be unital algebras and $\varphi : \mathcal{A} \to \mathcal{C}$ be a linear map with $\varphi(1) = 1$. The typical situation is when $\mathcal{C}$ is $\mathbb{C}$ or a subalgebra of $\mathcal{A}$, but we do not require it, nor that $\mathcal{C}$ is commutative.

The subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ are said to be *boolean independent with respect* to φ (or just boolean independent) if

$$\varphi(x_1 y_1 x_2 \cdots) = \varphi(x_1)\varphi(y_1)\varphi(x_2)\cdots \tag{2}$$

for all $x_1, x_2, \ldots \in \mathcal{A}_1$ and $y_1, y_2, \ldots \in \mathcal{A}_2$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are subalgebras of $\mathcal{A}$, we will denote by $\mathcal{A}_1 \vee \mathcal{A}_2$ the algebra they generate in $\mathcal{A}$. With this notation, (2) can be interpreted as follows: if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two boolean independent subalgebras of $\mathcal{A}$, then

$$\varphi(a_1 x y a_2) = \varphi(a_1 x)\varphi(y a_2) \tag{3}$$

for all $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$ and $a_1, a_2 \in \mathcal{A}_1 \vee \mathcal{A}_2$.

We define the *boolean cumulant* of order n as the multilinear function $B^n : \mathcal{A}^n \to \mathcal{C}$ given by the recurrence

$$\varphi(a_1 \cdots a_n) = \sum_{k=1}^{n} B^k(a_1, \ldots, a_k)\varphi(a_{k+1} \cdots a_n). \tag{4}$$

Note that (4) is equivalent to

$$\begin{aligned}\varphi(a_1 \cdots a_n) &= \sum_{k=1}^{n} \varphi(a_1 \cdots a_k) B^{n-k}(a_{k+1} \cdots a_n) \\ &= \sum B^{k_1}(a_1, \ldots, a_{k_1}) B^{k_2-k_1}(a_{k_1+1}, \ldots, a_{k_2}) \cdots B^{n-k_m}(a_{k_m+1}, \ldots, a_n)\end{aligned}$$

where the last summation is over all $0 \le m \le n-1$ and $1 \le k_1 < \cdots < k_m$.

We will write B^n_x for $B^n(x,\dots,x)$. For the moment cumulant and boolean cumulant generating power series of x we will use the notations $M_x(z)$, respectively $B_x(z)$, i.e.

$$M_x(z)=\sum_{n=1}^{\infty}\varphi(x^n)z^{n-1},\qquad B_x(z)=\sum_{n=1}^{\infty}B^n_x z^{n-1}.$$

The computations in the following two sections will often involve functions of many arguments. For brevity, if $\mathbf{a}=(a_1,\dots,a_n)$ is an element from $\mathcal{A}^n$, we will write $\mathrm{prod}(\mathbf{a})$ for the product $a_1\cdots a_n$ and $|\mathbf{a}|$ for the length of $\mathbf{a}$ (here n). If $\mathbf{a}_1$ and $\mathbf{a}_2$ are two elements from $\mathcal{A}^n$, respectively $\mathcal{A}^m$, then their concatenation will be denoted by $(\mathbf{a}_1,\mathbf{a}_2)\in\mathcal{A}^{n+m}$.

3. Properties of boolean cumulants. The following property is analogous to the vanishing of free cumulants with free independent entries. For the boolean case, only a weaker property is true.

PROPOSITION 3.1. *Let $\mathcal{A}_1,\mathcal{A}_2$ be boolean independent subalgebras of $\mathcal{A}$. If $n,m\ge 0$, and $\mathbf{a}_1\in(\mathcal{A}_1\vee\mathcal{A}_2)^n$, $\mathbf{a}_2\in(\mathcal{A}_1\vee\mathcal{A}_2)^m$, $x\in\mathcal{A}_1$, $y\in\mathcal{A}_2$, then*

$$B^{n+m+2}(\mathbf{a}_1,x,y,\mathbf{a}_2)=0.$$

Proof. The proof is by induction on $n+m$. For $n=m=0$, (2) implies

$$\varphi(xy)=\varphi(x)\varphi(y)=B^1(x)\varphi(y),$$

while (4) gives

$$\varphi(xy)=B^2(xy)+B^1(x)\varphi(y),$$

hence $B^2(xy)=0$.

Suppose now the assertion is true for $n+m<N$ and let us prove it for $n+m=N$. Write $\mathbf{a}=(\mathbf{a}_1,x,y,\mathbf{a}_2)$. From (4) we have

$$\begin{aligned}\varphi(\mathrm{prod}(\mathbf{a}_1,x,y,\mathbf{a}_2))&=\sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}}B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))\\&=\sum_{\substack{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}\\|\mathbf{d}_1|\le n+1}}B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))+\sum_{\substack{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}\\|\mathbf{d}_1|> n+1}}B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2)).\end{aligned}$$

On the other hand, from (3), we have

$$\begin{aligned}\varphi(\mathrm{prod}(\mathbf{a}_1,x,y,\mathbf{a}_2))&=\varphi(\mathrm{prod}(\mathbf{a}_1,x))\varphi(\mathrm{prod}(y,\mathbf{a}_2))\\&=\sum_{(\mathbf{a}_1,x)=(\mathbf{d}_1,\mathbf{d}_2)}B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))\varphi(\mathrm{prod}(y,\mathbf{a}_2)).\end{aligned}$$

Since the last component of $\mathbf{d}_2$ is $x\in\mathcal{A}_1$ and the first component of $(y,\mathbf{a}_2)$ is $y\in\mathcal{A}_2$, property (3) gives

$$\varphi(\mathrm{prod}(\mathbf{d}_2))\varphi(\mathrm{prod}(y,\mathbf{a}_2))=\varphi(\mathrm{prod}(\mathbf{d}_2,y,\mathbf{a}_2)).$$

Therefore

$$\varphi(\mathrm{prod}(\mathbf{a}_1, x, y, \mathbf{a}_2)) = \sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))$$
$$= \sum_{\substack{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}\\ |\mathbf{d}_1|\le n+1}} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2)).$$

Consequently,

$$\sum_{\substack{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}\\ |\mathbf{d}_1|> n+1}} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2)) = 0,$$

and the conclusion follows from the induction hypothesis. ∎

COROLLARY 3.2. *If* x, y *are boolean independent*, *then*

$$B_{x+y}(z) = B_x(z) + B_y(z).$$

Proof. The property is implied by $B^n_{x+y} = B^n_x + B^n_y$, which is an immediate consequence of 3.1 and the multilinearity of the mappings B^n. ∎

The next proposition investigates properties of boolean cumulants with scalars among their entries.

PROPOSITION 3.3. *If* $n, m \ge 1$ *and* $\mathbf{a}_1 \in \mathcal{A}^n$, $\mathbf{a}_2 \in \mathcal{A}^m$ *then*

(i) $B^{m+1}(1, \mathbf{a}_2) = 0$,
(ii) $B^{m+1}(\mathbf{a}_1, 1) = 0$,
(iii) $B^{n+m+1}(\mathbf{a}_1, 1, \mathbf{a}_2) = B^{n+m}(\mathbf{a}_1, \mathbf{a}_2)$.

Proof. (i) The definition of B^n gives

$$\varphi(\mathrm{prod}(1, \mathbf{a}_2)) = B^1(1)\varphi(\mathrm{prod}(\mathbf{a}_2)) + \sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_2} B^{|\mathbf{d}_1|+1}(1, \mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2)).$$

But

$$\varphi(1 \cdot \mathrm{prod}(\mathbf{a}_2)) = \varphi(\mathrm{prod}(\mathbf{a}_2)) = B^1(1)\varphi(\mathrm{prod}(\mathbf{a}_2)),$$

hence

$$\sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_2} B^{|\mathbf{d}_1|+1}(1, \mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2)) = 0.$$

Since $\varphi(1x) = B^2(1, x) + B^1(1)\varphi(x)$, we have $B^2(1, x) = 0$, and (i) is proved by induction on m. The proof of (ii) is similar.

(iii) From (i), one has

$$\varphi(x \cdot 1 \cdot y) = B^3(x, 1, y) + B^2(x, 1)\varphi(y) + B^1(x)\varphi(1 \cdot y)$$
$$= B^3(x, 1, y) + B^1(x)\varphi(y).$$

Since $\varphi(xy) = B^2(x, y) + B^1(x)\varphi(y)$ we have $B^3(x, 1, y) = B^2(x, y)$.

The rest of the proof is again by induction on $n+m$:

$$\begin{aligned}\varphi(\mathrm{prod}(\mathbf{a}_1,1,\mathbf{a}_2)) &\\ = \sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_1} & B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2,1,\mathbf{a}_2)) + B^{|\mathbf{a}_1|+1}(\mathbf{a}_1,1)\varphi(\mathrm{prod}(\mathbf{a}_2))\\ + \sum_{(\mathbf{u}_1,\mathbf{u}_2)=\mathbf{a}_2} & B^{n+1+|\mathbf{u}_1|}(\mathbf{a}_1,1,\mathbf{u}_1)\varphi(\mathrm{prod}(\mathbf{u}_2)) + B^{n+m+1}(\mathbf{a}_1,1,\mathbf{a}_2).\end{aligned}$$

By applying (ii) and the induction hypothesis, this equation becomes

$$\begin{aligned}\varphi(\mathrm{prod}(\mathbf{a}_1,1,\mathbf{a}_2)) = &\sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_1} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2,1,\mathbf{a}_2))\\ &+ \sum_{(\mathbf{u}_1,\mathbf{u}_2)=\mathbf{a}_2} B^{n+|\mathbf{u}_1|}(\mathbf{a}_1,\mathbf{u}_1)\varphi(\mathrm{prod}(\mathbf{u}_2)) + B^{n+m+1}(\mathbf{a}_1,1,\mathbf{a}_2),\end{aligned}$$

and (2) gives the conclusion. ∎

COROLLARY 3.4. *For all $x \in \mathcal{A}$, one has $B^1_{1+x} = 1 + B^1_x$ and, for $n \geq 2$,*

$$B^n_{x+1} = \sum_{k=0}^{n-2} \binom{n-2}{k} B^{k+2}_x.$$

Proof. The multilinearity of B^n implies that

$$B^n_{1+x} = \sum_{\substack{a_j\in\{1,x\}\\ j=1,\dots,n}} B^n(a_1,\dots,a_n),$$

Terms having 1 as first or last entry also cancel from 3.3(i) and 3.3(ii). The remaining terms can have the entry 1 only on the remaining $n-2$ positions left, and 3.3(iii) gives the stated result. ∎

Finally, we will direct our attention toward boolean cumulants with products among their entries. The multiplicative property of the boolean cumulants will appear as a straightforward consequence of the following proposition.

PROPOSITION 3.5. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two boolean independent subalgebras of $\mathcal{A}$. For $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$ and $\mathbf{a}_1 \in (\mathcal{A}_1 \vee \mathcal{A}_2)^n$, $\mathbf{a}_2 \in (\mathcal{A}_1 \vee \mathcal{A}_2)^m$ we have*:

(i) $B^{n+m+1}(\mathbf{a}_1, xy, \mathbf{a}_2) = B^{n+1}(\mathbf{a}_1, x)B^{m+1}(y, \mathbf{a}_2)$,
(ii) $B^{n+m+2}(\mathbf{a}_1, xy, xy, \mathbf{a}_2) = 0$,
(iii) $B^{n+m+2}(\mathbf{a}_1, y, xy, \mathbf{a}_2) = B^{n+m+2}(\mathbf{a}_1, xy, x, \mathbf{a}_2) = 0$.

Proof. (i) For $n,m = 0$, $B^1(xy) = \varphi(xy) = \varphi(x\cdot y) = B^2(x,y) + B^1(x)B^1(y)$. But $B^2(x,y) = 0$ from 3.1, hence $B^1(xy) = B^1(x)B^1(y)$.

Suppose now 3.5(i) is true for $n \leq N$ and let us prove it for $n = N+1$. One has

$$\begin{aligned}\varphi(\mathrm{prod}(\mathbf{a}_1, x, y, \mathbf{a}_2)) &= \varphi(\mathrm{prod}(\mathbf{a}_1, x))\varphi(\mathrm{prod}(y, \mathbf{a}_2)) \\ &= \sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_1} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))\varphi(\mathrm{prod}(y, \mathbf{a}_2)) \\ &\quad + B^{n+1}(\mathbf{a}_1, x)\varphi(\mathrm{prod}(y, \mathbf{a}_2)) \\ &= \sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_1} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))\varphi(\mathrm{prod}(y, \mathbf{a}_2)) \\ &\quad + B^{n+1}(\mathbf{a}_1, x)B^1(y)\varphi(\mathrm{prod}(\mathbf{a}_2)) \\ &\quad + \sum_{\mathbf{u}_1,\mathbf{u}_2=\mathbf{a}_2} B^{n+1}(\mathbf{a}_1, x)B^{|\mathbf{u}_1|+1}(y, \mathbf{u}_1)\varphi(\mathrm{prod}(\mathbf{u}_2)) \\ &\quad + B^{n+1}(\mathbf{a}_1, x)B^{m+1}(y, \mathbf{a}_2).\end{aligned}$$

By applying the induction hypothesis, this equality becomes

$$\begin{aligned}\varphi(\mathrm{prod}(\mathbf{a}_1, x, y, \mathbf{a}_2)) &= \sum_{(\mathbf{d}_1,\mathbf{d}_2)=\mathbf{a}_1} B^{|\mathbf{d}_1|}(\mathbf{d}_1)\varphi(\mathrm{prod}(\mathbf{d}_2))\varphi(\mathrm{prod}(y, \mathbf{a}_2)) \\ + B^{n+1}(\mathbf{a}_1, xy)\varphi(\mathrm{prod}(\mathbf{a}_2)) &+ \sum_{(\mathbf{u}_1,\mathbf{u}_2=\mathbf{a}_2} B^{n+|\mathbf{u}_1|+1}(\mathbf{a}_1, xy, \mathbf{u}_1)\varphi(\mathrm{prod}(\mathbf{u}_2)) \\ + B^{n+1}(\mathbf{a}_1, x)B^{m+1}(y, \mathbf{a}_2).&\end{aligned}$$

Finally, comparing the above relation with the decomposition (3) for $\varphi(\mathrm{prod}(\mathbf{a}_1, x, y, \mathbf{a}_2))$, we obtain

$$B^{n+m+1}(\mathbf{a}_1, xy, \mathbf{a}_2) = B^{n+1}(\mathbf{a}_1, x)B^{m+1}(y, \mathbf{a}_2).$$

(ii) Part (i) gives

$$\begin{aligned}B^{n+m+2}(\mathbf{a}_1, x, y, \mathbf{a}_2) &= B^{n+1}(\mathbf{a}_1, n) \cdot B^{m+2}(y, xy, \mathbf{a}_2) \\ &= B^{n+1}(\mathbf{a}_1, x) \cdot B^2(y, x) \cdot B^{m+1}(y, \mathbf{a}_2) = 0.\end{aligned}$$

The rest of the proof is analogous. ■

COROLLARY 3.6. *For boolean independent algebras $\mathcal{A}_1, \mathcal{A}_2$, and $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$, one has*

$$B^n_{x+y+xy} = B^n_x + B^n_y + \sum_{k=1}^{n} B^k_x B^{n-k+1}_y.$$

Proof. The multilinearity of B^n implies

$$B^n_{x+y+xy} = \sum_{a_j \in \{x,y,xy\}} B^n(a_1, \ldots, a_n).$$

From 3.1, all the boolean cumulants above containing both x and y, but not xy vanish. From 3.5(ii)–(iii), among the boolean cumulants containing xy, only those of the form $B^n(x,\dots,x,xy,y,\dots,y)$ are nonzero. Therefore

$$B^n_{x+y+xy} = B^n_x + B^n_y + \sum_{k=0}^{n-1} B^n(\underbrace{x,\dots,x}_{k\text{ times}},xy,\underbrace{y,\dots,y}_{n-k}),$$

and the conclusion follows immediately from 3.5(i). ∎

COROLLARY 3.7. *If x, y are boolean independent, then*

$$B_{(1+x)(1+y)}(z) = B_{1+x}(z)\cdot B_{1+y}(z). \tag{5}$$

Proof. Let $w = x+y+xy$ and let α_n and β_n be the coefficients of z^{n+1} on the left hand side, respectively right hand side of (5). For $n=1$,

$$\begin{aligned}\alpha_1 = B^1_{(1+x)(1+y)} &= 1 + B^1_x + B^1_y + B^1_{xy}\\ &= 1 + B^1_x + B^1_y + B^1_x B^1_y\\ &= (1+B^1_x)(1+B^1_y) = B^1_{1+x}B^1_{1+y} = \beta_1.\end{aligned}$$

For $n=2$, we can write

$$\alpha_2 = \sum_{\substack{p,q\ge 0\\ p+q\ge 1}} \alpha_{p,q} B^p_x B^q_y \quad\text{and}\quad \beta_2 = \sum_{\substack{p,q\ge 0\\ p+q\ge 1}} \beta_{p,q} B^p_x B^q_y$$

and (5) reduces to the equality between $\alpha_{p,q}$ and $\beta_{p,q}$.

Utilizing 3.4, we have

$$\begin{aligned}\alpha_n = B^n_{1+w} &= \sum_{k=2}^{n}\binom{n-2}{k-2}B^k_w\\ &= \sum_{k=2}^{n}\binom{n-2}{k-2}\Big(B^k_x + B^k_y + \sum_{l=1}^{k} B^l_x B^{k+l+1}_y\Big).\end{aligned}$$

Therefore

$$\alpha_{p,q} = \begin{cases}\dbinom{n-2}{p+q-2} & \text{if } p+q=n \text{ and } p=0 \text{ or } q=0,\\ \dbinom{n-2}{p+q-3} & \text{if } p,q\ge 1 \text{ and } p+q\le n+1,\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand,

$$\begin{aligned}\beta_n = \sum_{k=1}^{n} B^k_{1+x}B^{n-k+1}_{1+y} &= B^1_{1+x}B^n_{1+y} + B^n_{1+x}B^1_{1+y}\\ &\quad+ \sum_{k=2}^{n-1}\Big(\sum_{l=2}^{k}\binom{k-2}{l-2}B^l_x \sum_{s=2}^{n-k+1}\binom{n-k-1}{s-2}B^s_y\Big).\end{aligned}$$

Utilizing again 3.4 and $B^1_{1+x} = 1 + B^1_x$, we have

$$\beta_n = \sum_{k=2}^{n} \binom{n-2}{k-2}[B^k_x + B^k_y + B^1_x B^k_y + B^k_x B^1_y] + \sum_{k=2}^{n-1} \left(\sum_{l=2}^{k} \sum_{s=2}^{n-k+1} \binom{k-2}{l-2}\binom{n-k-1}{s-2} B^l_x B^s_y \right)$$

and therefore

$$\beta_{p,q} = \begin{cases} \binom{n-2}{p+q-2} & \text{if } p+q=n \text{ and } p=0 \text{ or } q=0, \\ \binom{n-2}{p+q-3} & \text{if } p=1 \text{ or } q=1 \text{ and } p+q \le n+1, \\ \sum_{k=p}^{n-q+1} \binom{k-2}{s-2}\binom{n-k-1}{q-2} & \text{if } p,q \ge 2, \text{ and } p+q \le n+1, \\ 0 & \text{otherwise.} \end{cases}$$

The property (5) reduces to the equality

$$\sum_{k=p}^{n-q+1} \binom{k-2}{s-2}\binom{n-k-1}{q-2} = \binom{n-2}{p+q-3},$$

which is just an avatar of the well-known identity (see, for example, [7, Chapter 5])

$$\sum_{k=a}^{n-b} \binom{k}{a}\binom{n-k}{b} = \binom{n+1}{a+b+1}. \ \blacksquare$$

4. Boolean independence with amalgamation over an algebra

4.1. *Preliminaries.* We will need to consider an extended notion of non-unital complex algebra. If $\mathfrak{A}, \mathfrak{B}$ are algebras, $\mathfrak{A}$ will be called a $\mathfrak{B}$-*algebra* if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ or there is an algebra $\widetilde{\mathfrak{A}}$ containing $\mathfrak{B}$ as a subalgebra such that $\widetilde{\mathfrak{A}} = \mathfrak{A} \sqcup \mathfrak{B}$. (The symbol $\sqcup$ stands for disjoint union.)

Let now $\mathfrak{A}$ and $\mathfrak{C}$ be algebras containing a unital subalgebra $\mathfrak{B}$, and let $\Phi : \mathfrak{A} \to \mathfrak{C}$ be a linear mapping such that $\Phi(\beta_1 a \beta_2) = \beta_1 \Phi(a) \beta_2$ for all $\beta_1, \beta_2 \in \mathfrak{B}$ and $a \in \mathfrak{A}$.

Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two $\mathfrak{B}$-subalgebras of $\mathfrak{A}$. Similarly to Section 2, we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *boolean independent with amalgamation over* $\mathfrak{B}$ (or just *boolean independent over* $\mathfrak{B}$) if

$$\Phi(x_1 y_1 x_2 \cdots) = \Phi(x_1)\Phi(y_1)\Phi(x_2)\cdots$$

for all $x_1, x_2, \ldots \in \mathfrak{A}_1$ and $y_1, y_2, \ldots \in \mathfrak{A}_2$.

Let $x \in \mathfrak{A}$ (if $\mathfrak{A}$ is a $*$-algebra, we also require x to be selfadjoint). If x is not commuting with $\mathfrak{B}$, the natural analogue (see [4], [10]) of the nth moment of x is the multilinear function $\widetilde{M^n}_x : \mathfrak{B}^{n-1} \to \mathfrak{B}$ given by

$$\widetilde{M^n}_x(\beta_1, \ldots, \beta_{n-1}) = \Phi(x\beta_1 x \cdots x\beta_{n-1}x).$$

The objects corresponding to power series are elements from Mul[[$\mathfrak{B}$]], the set of multilinear function series over $\mathfrak{B}$ (see [4]). A *multilinear function series* over $\mathfrak{B}$ is a sequence $F = (F^0, F^1, \ldots)$ such that $F^0 \in \mathfrak{B}$ and F^n is a multilinear function from $\mathfrak{B}^n$ to $\mathfrak{B}$, for $n \geq 1$. For $F, G \in$ Mul[[$\mathfrak{B}$]], the *sum* $F + G$ and the *formal product* FG are the elements from Mul[[$\mathfrak{B}$]] defined by

$$(F+G)^n(\beta_1, \ldots, \beta_n) = F^n(\beta_1, \ldots, \beta_n) + G^n(\beta_1, \ldots, \beta_n),$$
$$(FG)^n(\beta_1, \ldots, \beta_n) = \sum_{k=0}^{n} F^k(\beta_1, \ldots, \beta_k) G^{n-k}(\beta_{k+1}, \ldots, \beta_n).$$

DEFINITION 4.1. The object corresponding to the nth boolean cumulant B^n (as defined in Section 2) is the multilinear function $\widetilde{B}^n : \mathfrak{A}^n \times \mathfrak{B}^{n-1} \to \mathfrak{B}$ given by the recurrence

$$\Phi(x_1\beta_1 x_2 \cdots x_{n-1}\beta_{n-1}x_n) = \sum_{k=1}^{n} \widetilde{B}^k_{x_1,\ldots,x_k}(\beta_1, \ldots, \beta_{k-1})\beta_k \Phi(x_{k+1}\beta_{k+1} \cdots \beta_{n-1}x_n) \tag{6}$$

for all $x_1, \ldots, x_n \in \mathfrak{A}$ and all $\beta_1, \ldots, \beta_{n-1} \in \mathfrak{B}$. The n-uple $(x_1, \ldots, x_n)$ will be called the *lower argument* of $\widetilde{B}^n$, while $(\beta_1, \ldots, \beta_{n-1})$ will be called the *upper argument* of $\widetilde{B}^n$. If $x_1 = \cdots = x_n = x$, we will write $\widetilde{B}^n_x$ for $\widetilde{B}^n_{x_1,\ldots,x_n}$.

We will denote by $\widetilde{M_x}$, respectively $\widetilde{B}$ the multilinear function series for the multilinear moments and boolean cumulants of $x \in \mathfrak{A}$. With this notation, the recurrence in the definition of $\widetilde{B}^n$ can be rewritten as

$$\widetilde{M_x} = \widetilde{B}_x(1 + I\widetilde{M_x})$$

for $I = (0, \mathrm{Id}_{\mathfrak{B}}, 0, \ldots, 0)$.

REMARK 4.2. Boolean independence over the algebra $\mathfrak{B}$ implies boolean independence with respect to the functional Φ, in the sense of Section 2.

If we denote by B^n the nth boolean cumulant with respect to Φ in the sense of Section 2 and by $\boldsymbol{x}$ the n-uple $(x_1, \ldots, x_n) \in \mathfrak{A}^n$, comparing the recurrences (3) and (6), we obtain

$$B^n(\boldsymbol{x}) = \widetilde{B}^n_{\boldsymbol{x}}(1, \ldots, 1). \tag{7}$$

Also, since for all $\beta_0, \lambda_1, \ldots, \beta_{n-1}, \lambda_n \in \mathfrak{B}$,

$$\beta_0 \cdot \Phi(x_1\lambda_1\beta_1 x_2 \cdots \lambda_{n-1}\beta_{n-1}x_n) \cdot \lambda_n = \Phi((\beta_0 x_1\lambda_1) \cdot 1 \cdot (\beta_1 x_2 \lambda_2) \cdot 1 \cdots 1 \cdot (\beta_{n-1}x_n\lambda_n))$$

the recurrence (6) gives

$$\beta_0[\widetilde{B}^n_{\boldsymbol{x}}(\lambda_1\beta_1, \ldots, \lambda_{n-1}\beta_{n-1})]\lambda_n = B^n(\beta_0 x_1 \lambda_1, \ldots, \beta_{n-1}x_n\lambda_n). \tag{8}$$

4.2. *Main results*

PROPOSITION 4.3. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be independent over $\mathfrak{B}$ and $\mathfrak{A}_1 \vee_{\mathfrak{B}} \mathfrak{A}_2$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{B}$. Suppose that $x \in \mathfrak{A}_1$, $y \in \mathfrak{A}_2$ and $\mathbf{a}_1 \in (\mathfrak{A}_1 \vee_{\mathfrak{B}} \mathfrak{A}_2)^n$, $\mathbf{a}_2 \in (\mathfrak{A}_1 \vee_{\mathfrak{B}} \mathfrak{A}_2)^m$ with $(\mathbf{a}_1, \mathbf{a}_2) = (a_1, \ldots, a_{n+m})$. Then*

(i) $\widetilde{B}^{n+2}_{\mathbf{a}_1,x,y,\mathbf{a}_2} = 0$,

(ii) *for all $\beta, \beta_1, \ldots, \beta_{n+m} \in \mathfrak{B}$, one has $\widetilde{B}^{n+1}_{\beta,\mathbf{a}_1} = \widetilde{B}^{n+1}_{\mathbf{a}_1,\beta} = 0$, while*

$$\widetilde{B}^{n+m+1}_{\mathbf{a}_1,\beta,\mathbf{a}_2}(\beta_1, \ldots, \beta_n) = \widetilde{B}^n_{\mathbf{a}_1,\mathbf{a}_2}(\beta_1, \ldots, \beta_k\beta\beta_{k+1}, \ldots, \beta_n),$$

(iii) $\widetilde{B}^{n+m+1}_{\mathbf{a}_1,xy,\mathbf{a}_2}(\beta_1, \ldots, \beta_{n+m}) = \widetilde{B}^{n+1}_{\mathbf{a}_1,x}(\beta_1, \ldots, \beta_n)\widetilde{B}^{m+1}_{y,\mathbf{a}_2}(\beta_{n+1}, \ldots, \beta_m)$.

Proof. The proposition is an immediate consequence of Remark 4.2 and Propositions 3.3 and 3.5.

For (i) we need to prove that, for all $\beta_1, \ldots, \beta_{n+m+1} \in \mathfrak{B}$,

$$\widetilde{B}^{n+2}_{\mathbf{a}_1 x,y,x_{k+1},\mathbf{a}_2}(\beta_1, \ldots, \beta_{n+m+1}) = 0.$$

From (8) we have

$$\widetilde{B}^{n+m+2}_{\mathbf{a}_1,x,y,\mathbf{a}_2}(\beta_1, \ldots, \beta_{n+m+1}) = B^{n+m+2}(a_1\beta_1, \ldots, x\beta_{n+1}, y\beta_{n+2}, \ldots, a_{n+m}).$$

Since $x\beta_{n+1} \in \mathfrak{A}_1$ and $y\beta_{n+2} \in \mathfrak{A}_2$, while $a_1\beta_1, \ldots, a_{n+m-1}\beta_{n+m+1}, a_{n+m} \in \mathfrak{A}_1 \vee_{\mathfrak{B}} \mathfrak{A}_2$, Proposition 3.1 implies the conclusion.

Parts (ii) and (iii) are analogous consequences of 4.2 and 3.3(i), 3.3(ii), respectively 3.5(i). ∎

To state the next results, we need a brief discussion of interval partitions.

DEFINITION 4.4. An *interval partition* γ on the set $\{1, \ldots, n\}$ is a collection of disjoint subsets, $D_1, \ldots, D_q$, called *blocks*, such that

(i) $\bigcup_{k=1}^q D_k = \{1, \ldots, n\}$,

(ii) if $i_1 < i_2$ and $k \in D_{i_1}$, $l \in D_{i_2}$, then $k < l$.

The set of all interval partitions on $\{1, \ldots, n\}$ will be denoted by $\mathcal{I}(n)$. The number of blocks of the interval partition γ will be denoted by $|\gamma|$. For $\gamma_1 \in \mathcal{I}(n)$ and $\gamma_2 \in \mathcal{I}(m)$, we will write $\gamma_1 \oplus \gamma_2$ for the interval partition from $\mathcal{I}(m+n)$ obtained by juxtaposing γ_1 and γ_2.

If $\gamma \in \mathcal{I}(n)$, $\gamma = (1, \ldots, p_1)(p_1+1, \ldots, p_2), \ldots, (p_{k-1}+1, \ldots, n)$, and $(\beta_1, \ldots, \beta_n)$ is a n-uple from $\mathfrak{B}$, then $\pi_\gamma(\beta_1, \ldots, \beta_n)$ will denote the following q-uple from $\mathfrak{B}^q$:

$$((\beta_1 \cdots \beta_{p_1}), (\beta_{p_1+1} \cdots \beta_{p_2}), \ldots, (\beta_{p_{q-1}+1} \cdots \beta_n)).$$

Proposition 4.5. *For all $x \in \mathfrak{A}$ and $n \geq 2$, one has*

$$\widetilde{B}^n_{1+x}(\beta_1, \ldots, \beta_{n-1}) = \sum_{\gamma \in \mathcal{I}(n-1)} \widetilde{B}^{|\gamma|+1}_x(\pi_\gamma(\beta_1, \ldots, \beta_{n-1})). \tag{9}$$

Proof. From the multilinearity of $\widetilde{B}^n$,

$$\widetilde{B}^n_{1+x} = \sum_{\substack{x_j \in \{1,x\} \\ j=1,\ldots,n}} \widetilde{B}^n_{x_1,\ldots,x_n} = \sum_{\substack{x_j \in \{1,x\} \\ j=2,\ldots,n-1}} \widetilde{B}^n_{x,x_2,\ldots,x_{n-1},x} \tag{10}$$

since, from 4.3(ii), the terms with 1 on the first and last position vanish.

Let $\gamma \in \mathcal{I}(n-1)$, $\gamma = (1, \ldots, p_1)(p_1+1, \ldots, p_2), \ldots, (p_{k-1}+1, \ldots, n-1)$. Denote by $\widetilde{B}^n_{\gamma,x}$ the boolean cumulant having the lower argument

$$x, \underbrace{1, \ldots, 1}_{p_1-1}, x, \underbrace{1, \ldots, 1}_{p_2-p_1-1}, \ldots, x, \underbrace{1, \ldots, 1}_{n-p_{k-1}-1}, x.$$

With this notation, (10) becomes

$$\widetilde{B}^n_{1+x}(\beta_1, \ldots, \beta_{n-1}) = \sum_{\gamma \in \mathcal{I}(n-1)} \widetilde{B}^n_{\gamma,x}(\beta_1, \ldots, \beta_{n-1}).$$

Finally, applying 4.3(iii), we get

$$\begin{aligned}\widetilde{B}^n_{\gamma,x}(\beta_1, \ldots, \beta_{n-1}) &= \widetilde{B}^{|\gamma|+1}_x((\beta_1 \cdots \beta_{p_1}), (\beta_{p_1+1} \cdots \beta_{p_2}), \ldots, (\beta_{p_{k-1}+1} \cdots \beta_{n-1})) \\ &= \widetilde{B}^{|\gamma|+1}_x(\pi_\gamma(\beta_1, \ldots, \beta_n)). \quad \blacksquare\end{aligned}$$

Corollary 4.6. *Suppose $\mathfrak{A}_1, \mathfrak{A}_2$ are boolean independent over $\mathfrak{B}$ and $x \in \mathfrak{A}_1$ while $y \in \mathfrak{A}_2$. Then*

$$\widetilde{B}_{x+y} = \widetilde{B}_x + \widetilde{B}_y, \qquad \widetilde{B}_{(1+x)(1+y)} = \widetilde{B}_{1+x}\widetilde{B}_{1+y}.$$

Proof. The first relation amounts to showing that $\widetilde{B}^n_{x+y} = \widetilde{B}^n_x + \widetilde{B}^n_y$, which is an obvious consequence of Proposition 4.3(i) and of the multilinearity of $\widetilde{B}^n$.

For the second relation, first note that the nth component of $B_{(1+x)(1+y)}$ is

$$\begin{aligned}\widetilde{B}^n_{(1+x)(1+y)}(\beta_1, \ldots, \beta_{n-1}) &= \widetilde{B}^n_{1+(x+y+xy)}(\beta_1, \ldots, \beta_{n-1}) \\ &= \sum_{\gamma \in \mathcal{I}(n-1)} \widetilde{B}^{|\gamma|+1}_{x+y+xy}(\beta_1, \ldots, \beta_{n-1}).\end{aligned}$$

The results 4.3(i) and (iii) imply the analogue of Corollary 3.6:

$$\widetilde{B}^m_{x+y+xy}(\beta_1,\dots,\beta_{m-1}) = (\widetilde{B}^m_x(1+\widetilde{B}^1_y)+(1+\widetilde{B}^1_x)\widetilde{B}^m_y)(\beta_1,\dots,\beta_{m-1}) + \sum_{k=2}^{m-2}\widetilde{B}^k_x(\beta_1,\dots,\beta_{k-1})\widetilde{B}^{m-k}_y(\beta_k,\dots,\beta_{m-k-1}).$$

Fix $\gamma\in\mathcal{I}(n-1)$. The above equation gives

$$\begin{aligned}\widetilde{B}^{|\gamma|+1}_{x+y+xy}&(\pi_\gamma(\beta_1,\dots,\beta_{n-1}))\\ &= (\widetilde{B}^{|\gamma|+1}_x(1+\widetilde{B}^1_y)+(1+\widetilde{B}^1_x)\widetilde{B}^{|\gamma|+1}_y)(\pi_\gamma(\beta_1,\dots,\beta_{n-1}))\\ &\quad+\sum_{\substack{\gamma_1\oplus\gamma_2=\gamma\\ \gamma_1\in\mathcal{I}(p)\\ \gamma_2\in\mathcal{I}(n-p-1)}}\widetilde{B}^{|\gamma_1|+1}_x(\pi_{\gamma_1}(\beta_1,\dots,\beta_p))\widetilde{B}^{|\gamma_2|+1}_y(\pi_{\gamma_2}(\beta_{p+1},\dots,\beta_{n-1}))\end{aligned}$$

and therefore

$$\begin{aligned}\widetilde{B}^n_{(1+x)(1+y)}&(\beta_1,\dots,\beta_{n-1})\\ &= \sum_{\gamma\in\mathcal{I}(n-1)}\widetilde{B}^{|\gamma|+1}_{\gamma,x+y+xy}(\pi_\gamma(\beta_1,\dots,\beta_{n-1}))\\ &= \sum_{\gamma\in\mathcal{I}(n-1)}\widetilde{B}^{|\gamma|+1}_{x+y+xy}(\pi_\gamma(\beta_1,\dots,\beta_{n-1}))\\ &= \sum_{\gamma\in\mathcal{I}(n-1)}(\widetilde{B}^{|\gamma|+1}_x(1+\widetilde{B}^1_y)+(1+\widetilde{B}^1_x)\widetilde{B}^{|\gamma|+1}_y)(\pi_\gamma(\beta_1,\dots,\beta_{n-1}))\\ &\quad+\sum_{k=2}^{n-1}\Big(\sum_{\gamma\in\mathcal{I}(k-1)}\widetilde{B}^{|\gamma|+1}_x(\pi_\gamma(\beta_1,\dots,\beta_{k-1}))\Big)\\ &\quad\times\Big(\sum_{\sigma\in\mathcal{I}(n-k)}\widetilde{B}^{|\sigma|+1}_y(\pi_\sigma(\beta_k,\dots,\beta_{n-k}))\Big).\end{aligned}$$

The relation (9) implies that the above equation amounts to

$$\widetilde{B}^n_{(1+x)(1+y)}(\beta_1,\dots,\beta_{n-1}) = \sum_{k=1}^{n}\widetilde{B}^k_{1+x}(\beta_1,\dots,\beta_{k-1})\widetilde{B}^{n-k+1}_{1+y}(\beta_k,\dots,\beta_{n-1}),$$

hence the conclusion. ■

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