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# ON THE ERGODIC DECOMPOSITION FOR A COCYCLE

BY

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**Abstract.** Let $(X, \mathfrak{X}, \mu, \tau)$ be an ergodic dynamical system and $\varphi$ be a measurable map from $X$ to a locally compact second countable group $G$ with left Haar measure $m_G$. We consider the map $\tau_\varphi$ defined on $X \times G$ by $\tau_\varphi : (x, g) \mapsto (\tau x, \varphi(x)g)$ and the cocycle $(\varphi_n)_{n\in\mathbb{Z}}$ generated by $\varphi$.

Using a characterization of the ergodic invariant measures for $\tau_\varphi$, we give the form of the ergodic decomposition of $\mu(dx) \otimes m_G(dg)$ or more generally of the $\tau_\varphi$-invariant measures $\mu_\chi(dx) \otimes \chi(g)m_G(dg)$, where $\mu_\chi(dx)$ is $\chi \circ \varphi$-conformal for an exponential $\chi$ on $G$.

**Contents**



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## 1. INTRODUCTION

We consider a dynamical system $(X, \mathfrak{X}, \mu, \tau)$, where $(X, \mathfrak{X})$ is a standard Borel space, $\mu$ a $\sigma$-finite measure on $\mathfrak{X}$, and $\tau$ an invertible measurable transformation on $X$ such that $\mu$ is *quasi-invariant* and *ergodic* for the action of $\tau$.

Let $G$ be a locally compact second countable (lcsc) group. We denote by $\mathfrak{B}_G$ the $\sigma$-algebra of its Borel sets, $m_G(dg)$ (or simply $dg$) a left Haar measure on $G$, and $e$ its identity element.

Let $\varphi$ be a measurable function on $X$ taking its values in $G$ and $\tau_\varphi$ the map on $X \times G$ (*skew product*) defined by

$$\tau_\varphi : (x, g) \mapsto (\tau x, \varphi(x)g). \tag{1}$$

The corresponding $G$-valued cocycle $(\varphi_n)_{n\in\mathbb{Z}}$ over $(X, \mu, \tau)$ (denoted also $(\varphi, \tau)$) is

$$\varphi_n(x) = \begin{cases} \varphi(\tau^{n-1}x) \cdots \varphi(x) & \text{for } n > 0, \\ e & \text{for } n = 0, \\ \varphi(\tau^n x)^{-1} \cdots \varphi(\tau^{-1}x)^{-1} & \text{for } n < 0. \end{cases}$$

If $\mu$ is $\tau$-invariant, the map $\tau_\varphi$ leaves invariant the product measure $\lambda_1 := \mu \otimes m_G$. The cocycle $(\varphi_n)$ can be seen as a *stationary walk* in $G$ over the dynamical system $(X, \mu, \tau)$.

More generally, let $\chi$ be an *exponential* on $G$, i.e. a continuous map from $G$ to $]0, +\infty[$ such that $\chi(g\,g') = \chi(g)\,\chi(g')$ for all $g, g' \in G$. If $\mu_\chi$ is a $\chi \circ \varphi$-*conformal* $\sigma$-finite measure on $X$, i.e. such that

$$(\tau\mu_\chi)(dx) = \chi(\varphi(\tau^{-1}x))\,\mu_\chi(dx), \tag{2}$$

then the measure $\lambda_\chi(dx, dg) := \mu_\chi(dx) \otimes \chi(g)\,m_G(dg)$ (sometimes called Maharam measure) is a $\sigma$-finite measure on $X \times G$ which is $\tau_\varphi$-invariant.

The study of cocycles was the subject of many papers since K. Schmidt ([Sc77]) and J. Feldman and C. C. Moore ([FeMo77]). There has recently been a new interest in the invariant measures for skew products (cf. [ANSS02], [Sa04], [LeSa07]).

Our main goal is to give the precise form of the ergodic decomposition (for the skew product $\tau_\varphi$) of the measures $\lambda_\chi$ on $X \times G$. In the first section we give the statement of the results on this ergodic decomposition, then some consequences in terms of regularity, boundedness and essential values of the cocycle $(\varphi_n)_{n\in\mathbb{Z}}$. The following sections are devoted to the proof of the main results. We also discuss a conjugacy equation for the closed subgroups of $G$ which arises in the ergodic decomposition. In the appendix, we recall and specify some results on ergodic decompositions and regular conditional probabilities.

## 2. STATEMENT OF THE MAIN RESULTS

**2.1. Ergodic decomposition.** Before we state the main results, we recall some facts about a topology on the set $\mathcal{F}(G)$ of closed subsets of $G$ and give some notations.

*A topology on* $\mathcal{F}(G)$. Let $G$ be a lcsc group. We equip the set $\mathcal{F}(G)$ of closed subsets of $G$ with the so-called *Chabauty's topology* [Ch50]. In this topology the open sets are defined by

$$U(\mathcal{O}, C) = \{S \in \mathcal{F}(G) : \forall U \in \mathcal{O},\ S \cap U \neq \emptyset \text{ and } S \cap C = \emptyset\},$$

where $\mathcal{O}$ is a finite family of open sets of $G$, and $C$ is a compact subset of $G$.

It can be shown that a sequence $(F_n)$ of closed subsets of $G$ converges to a closed subset $F$ in Chabauty's topology if and only if the following two properties are satisfied:

- Let $\xi : \mathbb{N} \to \mathbb{N}$ be an increasing sequence and let $(g_n)_{n\in\mathbb{N}}$ be a sequence such that $g_n \in F_{\xi(n)}$ for every $n \geq 0$. If $(g_n)_{n\in\mathbb{N}}$ converges to $g \in G$, then the limit $g$ is in $F$.
- Each $g \in F$ is the limit of a sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n \in F_n$ for every $n \geq 0$.

The Borel structure associated to this topology is generated by the sets $\{S \in \mathcal{F}(G) : S \subseteq F\}$ where $F \in \mathcal{F}(G)$. The lcsc group $G$ is metrizable. We denote by $d$ a metric on $G$ which defines the topology of $G$. For any dense sequence $(g_n)_{n\in\mathbb{N}}$ of elements of $G$, the family of continuous functions $\{d(g_n, \cdot), n \in \mathbb{N}\}$ separates the points of $\mathcal{F}(G)$ (see [AuMo66, Ch. II, Section 2]).

*Notations*

NOTATIONS 2.1.1. For a locally compact second countable group $H$, we denote by $m_H(d\gamma)$ (or simply $d\gamma$) a *left Haar measure* on the Borel sets of $H$, and by $\delta_u$ the Dirac measure at a point $u \in H$. The identity element is denoted by $e$.

If $\rho_1$ and $\rho_2$ are positive measures on the Borel subsets of $H$, we denote by $\rho_1 * \rho_2$ their convolution (i.e. the image of the product measure $\rho_1 \otimes \rho_2$ under the map $(g, g') \in H \times H \mapsto gg' \in H$).

As in the introduction, we consider a measurable map $\varphi$ from $X$ to $G$ and the skew product $\tau_\varphi$ defined by (1). Let $\lambda$ be a $\tau_\varphi$-quasi-invariant positive measure on $X \times G$. We denote by $\mathfrak{J}$ or $\mathfrak{J}_\varphi$ the $\sigma$-algebra of $\tau_\varphi$-*invariant subsets*. We are interested in the $\mathfrak{X} \times \mathfrak{B}_G$-measurable functions on $X \times G$ which are invariant under the map $\tau_\varphi$.

The following remark is useful. If $f$ is $\tau_\varphi$-invariant $\lambda$-a.e., then there is a $\tau_\varphi$-invariant function $g$ such that $f = g$ $\lambda$-a.e. Therefore it is enough to consider functions which are *everywhere* $\tau_\varphi$-*invariant*.

Recall that two $G$-valued cocycles $(\varphi, \tau)$ and $(\psi, \tau)$ over the dynamical system $(X, \mu, \tau)$ are *$\mu$-cohomologous* if there is a measurable map $u : X \to G$ such that

$$\varphi(x) = u(\tau x)\psi(x)(u(x))^{-1} \quad \text{for } \mu\text{-a.e. } x. \tag{3}$$

The function $u$ in (3) is called the *transfer function*. We write $\varphi \overset{(u,\mu)}{\sim} \psi$ when (3) is satisfied. A cocycle $(\varphi, \tau)$ is a *$\mu$-coboundary* if it is $\mu$-cohomologous to the constant function $\psi \equiv e$.

Notations 2.1.2. In what follows, we consider a $\tau_\varphi$-invariant measure $\lambda_\chi$ of the form $\lambda_\chi = \mu_\chi \otimes (\chi m_G)$, where $\chi$ is an exponential on $G$, and $\mu_\chi$ is a $\sigma$-finite measure which is $\chi \circ \varphi$-conformal and $\tau$-ergodic on $X$. When $\chi \equiv 1$, the measure $\mu_\chi$ is $\tau$-invariant.

Once and for all we choose *a measurable positive function $h$* on $X \times G$ such that

$$\int_{X\times G} h(x, g)\, \mu_\chi(dx)\, \chi(g)\, m_G(dg) = 1.$$

The existence of $h$ results from the facts that $\mu_\chi$ is $\sigma$-finite on $X$ and that $G$ is a lcsc group.

Let $P^h$ be a regular conditional probability with respect to the probability measure $h\lambda_\chi$ and the $\sigma$-algebra $\mathfrak{J}$ of $\tau_\varphi$-invariant subsets (i.e. $P^h$ is a transition probability on $X \times G$ such that, for every nonnegative measurable function $f$ on $X \times G$, $P^h f$ is a version of the conditional expectation $\mathbb{E}_{h\lambda_\chi}[f \mid \mathfrak{J}]$).

We define a positive kernel $M^h$ on $X \times G$ by

$$\forall (x, g) \in X \times G, \qquad M^h f(x, g) = P^h(f/h)(x, g)$$

for any measurable nonnegative function $f$ on $X \times G$.

If we replace $h$ by another density $h'$, we have

$$M^{h'}((x, g), \cdot) = P^h(h/h')(x, g) M^h((x, g), \cdot).$$

For $\lambda_\chi$-a.e. $(x, g) \in X \times G$, the positive measure $M^h((x, g), \cdot)$ on $X \times G$ is $\tau_\varphi$-invariant ergodic (see the appendix).

*Statement of the main result.* The formula $\mathbb{E}_{h\lambda_\chi}[\,\cdot\,] = \mathbb{E}_{h\lambda_\chi}[\mathbb{E}_{h\lambda_\chi}[\,\cdot \mid \mathfrak{J}]]$ can be written

$$\lambda_\chi(dy, dt) = \int_{X\times G} M^h((x, g), (dy, dt))\, h(x, g)\, \lambda_\chi(dx, dg),$$

which represents a decomposition of $\lambda_\chi$ into $\tau_\varphi$-ergodic components. Our goal is to give a precise description of these ergodic components. This is the content of the following theorem:

THEOREM 2.1.3 (Ergodic decomposition of $\lambda_\chi$). (i) *There exist*:

- *a family* $(\mu_x)_{x\in X}$ *of* $\sigma$*-finite* $\tau$*-quasi-invariant measures on* $X$ *defining a* $\sigma$*-finite positive kernel from* $(X,\mathfrak{X})$ *to* $(X,\mathfrak{X})$ (*i.e. for every* $x\in X$, $\mu_x$ *is a* $\sigma$*-finite positive measure on* $\mathfrak{X}$ *and for every* $A\in\mathfrak{X}$ *the map* $x\mapsto\mu_x(A)\in[0,+\infty]$ *is* $\mathfrak{X}$*-measurable*),
- *a family* $(H_x)_{x\in X}$ *of closed amenable subgroups of* $G$ *such that the map* $x\mapsto H_x$ *from* $X$ *to* $\mathcal{F}(G)$ *is measurable*,
- *a measurable map* $\eta: X\times G\to\mathbb{R}_+^*$ *such that, for each* $x\in X$, $\chi_x(\cdot):=\eta(x,\cdot)$ *defines an exponential on* $H_x$,
- *a measurable map* $u: X\times X\to G$ (*for* $x\in X$, *we set* $u_x(\cdot)=u(x,\cdot)$)

*satisfying for* $\mu_\chi$*-a.e.* $x\in X$ *and every* $g\in G$ *the following conditions* (4) *to* (10):

$$H_{\tau x}=\varphi(x)H_x\varphi(x)^{-1}, \tag{4}$$

$$\psi(y):=(u_x(\tau y))^{-1}\varphi(y)u_x(y)\in H_x \quad \text{for } \mu_x\text{-a.e. } y, \tag{5}$$

$$\tau\mu_x(dy)=\chi_x(\psi(\tau^{-1}y))\,\mu_x(dy), \tag{6}$$

$$\chi_x(\gamma)=\chi_{\tau x}(\varphi(x)\gamma(\varphi(x))^{-1}), \quad \forall\gamma\in H_x, \tag{7}$$

$$\zeta_x(y):=(u_x(y))^{-1}\,u_{\tau x}(y)\varphi(x)\in H_x \quad \text{for } \mu_x\text{-a.e. } y, \tag{8}$$

$$\mu_{\tau x}(dy)=c(x)\chi_x(\zeta_x(y))\,\mu_x(dy) \quad \text{for a positive constant } c(x), \tag{9}$$

$$M^h f(x,g)=\frac{\int_X\int_{H_x} f(y,u_x(y)\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma)\,\mu_x(dy)}{\int_X\int_{H_x} h(y,u_x(y)\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma)\,\mu_x(dy)}. \tag{10}$$

*If we take for* $m_{H_x}$, $x\in X$, *the unique left Haar measure on* $H_x$ *such that*

$$\int_{H_x\cap\{d(e,\cdot)\le 1\}}\chi_x(\gamma)\,m_{H_x}(d\gamma)=1,$$

*then* $K(x,dt):=m_{H_x}(dt)$ *is a positive kernel from* $(X,\mathfrak{X})$ *to* $(G,\mathfrak{B}_G)$.

*An ergodic decomposition of the measure* $\lambda_\chi=\mu_\chi\otimes(\chi m_G)$ *is given by*

$$\lambda_\chi(dy,dt)=\int_{X\times G} M^h((x,g),(dy,dt))\,h(x,g)\,\lambda_\chi(dx,dg). \tag{11}$$

*For every nonnegative measurable* $\lambda_\chi$*-a.e.* $\tau_\varphi$*-invariant function* $f$, *we have,* $\lambda_\chi$*-a.e.,* $f=P^hf$ (*the last function being* $\tau_\varphi$*-invariant according to the definition of a regular conditional probability*).

(ii) *When there exist a fixed closed subgroup* $H$ *of* $G$ *and a measurable map* $a: X\to G$ *such that* $H_x=a(x)H(a(x))^{-1}$ *for* $\mu_\chi$*-a.e.* $x\in X$ (*which is the case when* $G$ *is a nilpotent connected Lie group* (*Theorem* 5.1.1)), *the ergodic measures can be written, with* $\tilde\chi_x(\gamma):=\chi_x(a_x\gamma a_x^{-1})$,

$$M^h f(x,g)=\frac{\int_X(\int_H f(y,u_x(y)a(x)\gamma(a(x))^{-1}g)\tilde\chi_x(\gamma)\,d\gamma)\,\mu_x(dy)}{\int_X(\int_H h(y,u_x(y)a(x)\gamma(a(x))^{-1}g)\tilde\chi_x(\gamma)\,d\gamma)\,\mu_x(dy)}. \tag{12}$$

(iii) *When $G$ is abelian, the subgroups $H_x$ are equal to a fixed closed subgroup $H$ of $G$, the exponentials $\chi_x$ are equal to the exponential $\chi$, and the ergodic measures are given by*

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u_x(y)\gamma g)\chi(\gamma)\, d\gamma)\, \mu_x(dy)}{\int_X (\int_H h(y, u_x(y)\gamma g)\chi(\gamma)\, d\gamma)\, \mu_x(dy)}. \tag{13}$$

The proof of Theorem 2.1.3 will be given in Section 3.

## 2.2. Notion of regularity for a cocycle

*Regularity*

DEFINITION 2.2.1. We say that the cocycle defined by $\varphi$ is *$\mu_\chi$-regular* if there exist a closed subgroup $H$ of $G$ and a measurable map $u : X \to G$ such that the cocycle $\psi := (u \circ \tau)^{-1}\varphi u$ takes $\mu_\chi$-a.e. its values in $H$ and $\tau_\psi : (x,h) \mapsto (\tau x, \psi(x)h)$ is ergodic for the product measure $\mu_\chi \otimes (\chi m_H)$.

The measure $(\chi \circ u)\mu_\chi \otimes \chi m_H$ is $\tau_\psi$-invariant. In the regular case we have a "good" ergodic decomposition of $\mu_\chi \otimes (\chi\, dg)$, and the subgroups $H_x$ of Theorem 2.1.3 are conjugate to $H$: $H_x = u(x)H(u(x))^{-1}$.

THEOREM 2.2.2. (i) *For $x_0 \in X$, the set $\{x \in X : \mu_x \sim \mu_{x_0}\}$ is measurable and has zero or full $\mu_\chi$-measure.*

(ii) *Assume that the cocycle $(\varphi, \tau)$ is $\mu_\chi$-regular. Then every measurable $\tau_\varphi$-invariant function $f$ can be written $f(x,g) = F_f((u(x))^{-1} g)$, $\mu_\chi \otimes m_G$-a.e., where $F_f$ is a left $H$-invariant function on $G$. The ergodic components of $\lambda_\chi$ (see (10)) can be written*

$$M^h f(x,g) = \frac{\int_X (\int_H f(y, u(y)\gamma(u(x))^{-1}g)\chi(\gamma)\, d\gamma)\chi(u(y))\, \mu_\chi(dy)}{\int_X (\int_H h(y, u(y)\gamma(u(x))^{-1}g)\chi(\gamma)\, d\gamma)\chi(u(y))\, \mu_\chi(dy)}.$$

*In other words, $H_x = u(x)H(u(x))^{-1}$ and $\chi_x(\gamma) = \chi(u(x)\gamma(u(x))^{-1})$. We can take $u_x(y) = u(y)(u(x))^{-1}$ and $\mu_x(dy) = \chi(u(y))\, \mu_\chi(dy)$.*

(iii) *Assume that the cocycle $(\varphi, \tau)$ is not $\mu_\chi$-regular. Then for $\mu_\chi$-a.e. $x$, the measures $\mu_x$ of the ergodic decomposition of $\mu_\chi \otimes (\chi m_G)$ are singular with respect to $\mu_\chi$. There are uncountably many of them pairwise mutually singular. If $G$ is abelian and $\mu_\chi$ is finite, then, for $\mu_\chi$-a.e. $x \in X$, the measure $\mu_x$ is infinite.*

The proof of Theorem 2.2.2 will be given in Section 4.

Examples of nonregular cocycles over rotations were given by Lemańczyk in [Le95]. In Remark 5.2.2, we give an example of a nonregular cocycle over a rotation which is the difference $1_{[0,\beta]} - 1_{[0,\beta]}(\cdot + r)$ for some $\beta$ and $r$ on the circle.

*Boundedness.* In the proposition below, we discuss the boundedness of the map $u$ and of the cocycle $(\varphi_n)$. The notations are those of Theorem 2.1.3.

As the group $G$ is lcsc, we can write $G = \bigcup_n U_n$ for an increasing sequence of open sets such that $K_n = \overline{U}_n$ is compact. Consequently, $G = \bigcup_{n\in\mathbb{N}} K_n$ and for any compact subset $K$ of $G$ there exists $n \in \mathbb{N}$ such that $K \subset K_n$.

Lemma 2.2.3. (i) *Let $u$ be the measurable map from $X \times X$ to $G$ defined in Theorem* 2.1.3. *For any compact subset $K$ of $G$ we define the following subset of $X$*:

$$\begin{aligned} X_K &= \{x \in X : u_x(y)H_x \subset KH_x \text{ for } \mu_x\text{-a.e. } y \in X\} \\ &= \{x \in X : \operatorname{supp}(u_x(\mu_x)) \subset KH_x\}. \end{aligned}$$

*Then $X_K$ is measurable and $x \in X_K \Rightarrow \tau x \in X_{K(\varphi(x))^{-1}}$.*

*The set $\bigcup_{n\in\mathbb{N}} X_{K_n}$ is a $\tau$-invariant measurable subset of $X$ and* (*by ergodicity of $\mu_\chi$*) *has zero or full $\mu_\chi$-measure.*

(ii) *If there exists a compact subset $K$ of $G$ such that $\mu_\chi(X_K) > 0$, then $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full $\mu_\chi$-measure. In this case*, *we can replace the measurable map $u$ by another measurable map $u$ satisfying*, *for any $n \in \mathbb{N}$*,

(14) *for $\mu_\chi$-a.e. $x \in X_n = X_{K_n} \setminus X_{K_{n-1}}$, $u_x(y) \in K_n$ for $\mu_x$-a.e. $y \in X$.*

(iii) *In particular*, *the set $\{x : G/H_x$ is compact$\}$ is measurable and has zero or full measure. If this set has full measure*, *we are in the above situation.*

*Proof.* (i) If $K$ is a fixed compact set in $G$, the map $F \mapsto KF$ from the set $\mathcal{F}(G)$ of closed subsets of $G$ into itself is continuous. Since $x \mapsto H_x$ is measurable, the map $x \mapsto KH_x$ is measurable. In Section 3, we will see that the map $(x, y) \in X \times X \mapsto u(x, y)H_x \in \mathcal{F}(G)$ is measurable. We also know that, for any $g \in G$, the map $F \in \mathcal{F}(G) \mapsto d(g, F) \in \mathbb{R}_+$ is continuous. It follows that the set $\{(x, y) \in X \times X : d(g, KH_x) \le d(g, u(x, y)H_x)\}$ is measurable. Let $(g_n)_{n\in\mathbb{N}}$ be a dense sequence in $G$. Then we have

$$\begin{aligned} X_K = \{x \in X : \forall n \in \mathbb{N},& \\ &\nu_{(x,e)}(\{y \in X : d(g_n, KH_x) \le d(g_n, u(x, y)H_x)\}) = 1\}. \end{aligned}$$

This shows that $X_K$ is measurable.

From the formulas (4) and (8) of Theorem 2.1.3, we obtain $x \in X_K \Rightarrow \tau x \in X_{K(\varphi(x))^{-1}}$. Since for any compact subset $K$ of $G$, there exists $n \in \mathbb{N}$ such that $K \subset K_n$, we deduce that the measurable set $\bigcup_{n\in\mathbb{N}} X_{K_n}$ is $\tau$-invariant and (by ergodicity of $\mu_\chi$) has zero or full measure.

(ii) If $\mu_\chi(X_K) > 0$ for some compact subset $K$ of $G$, then the same argument shows that $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full $\mu_\chi$-measure. The last assertion follows from the construction of $u$ (cf. Lemma 7.1.1).

(iii) We have

$$\{x \in X : G/H_x \text{ is compact}\} = \bigcup_{n\in\mathbb{N}} \{x \in X : K_nH_x = G\},$$

which shows that the set is measurable. By the conjugacy relation (4) this set is $\tau$-invariant and (ergodicity of $\mu_\chi$) has zero or full $\mu_\chi$-measure. In the last case, for $\mu_\chi$-a.e. $x \in X$, we have $\bigcup_{n\in\mathbb{N}} K_n H_x = G$, which implies that $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full $\mu_\chi$-measure. ∎

PROPOSITION 2.2.4. (i) *Assume that the measure $\mu_\chi$ in the basis is a finite measure and that there exists a compact subset $K$ of $G$ such that $\mu_\chi(X_K) > 0$. If the map $u$ satisfies the boundedness condition* (14), *then the measures $\mu_x$ are finite for $\mu_\chi$-a.e. $x \in X$.*

(ii) *Assume that $G$ is abelian and there exists a compact subset $K$ of $G$ such that $\mu_\chi(X_K) > 0$. Then the cocycle is regular.*

(iii) *Assume that $G$ is abelian, $\mu_\chi$ finite and $\tau$ conservative for $\mu_\chi$. If $\mu_\chi(\{x \in X : \tilde{\mu}_x(X) < +\infty\}) > 0$, where*

$$\tilde{\mu}_x(dy) := (\chi(u_x(y)))^{-1}\,\mu_x(dy), \tag{15}$$

*then the cocycle is regular.*

(iv) *Assume that $\tau$ is conservative for $\mu_\chi$. If the cocycle $(\varphi_n)$ is $\mu_\chi$-bounded (i.e. there exists a compact subset $K$ of $G$ such that $\varphi_n(x) \in K$ for $\mu_\chi$-a.e. $x \in X$ and all $n \ge 0$), then $H_x$ is a compact subgroup of $G$ and the cocycle is cohomologous with a bounded transfer function to a cocycle taking its values in a compact subgroup of $G$.*

*Proof.* Let $r$ be a positive continuous function on $G$ with $\int_G r(t)\chi(t)\,dt = 1$. For any compact subset $K$ of $G$, we set $r_K(g) := \min_{u\in K} r(ug) > 0$. For all measurable nonnegative functions $f$ on $X$, we have (cf. (34))

$$\begin{aligned} M^h(f\otimes r)(x,g) &= c(x,g)\int_X f(y)\Big(\int_{H_x} r(u_x(y)\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma)\Big)\,\mu_x(dy) \\ &\ge c(x,g)\int_X f(y)\Big(\int_{H_x} 1_K(u_x(y))r(u_x(y)\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma)\Big)\,\mu_x(dy) \\ &= c(x,g)\Big(\int_{H_x} r_K(\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma)\Big)\int_X f(y)1_K(u_x(y))\,\mu_x(dy) \end{aligned}$$

and therefore

$$\mu_\chi(f) = \lambda_\chi(f\otimes r) \ge \int_X \Psi_K(x)\Big(\int_X f(y)1_K(u_x(y))\,\mu_x(dy)\Big)\,\mu_\chi(dx), \tag{16}$$

where $\Psi_K(x) := \int_G c(x,g)(\int_{H_x} r_K(\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma))h(x,g)\chi(g)\,dg > 0$.

(i) Under the assumptions of the first assertion, we have from (14) and (16), for each $n \in \mathbb{N}$,

$$\mu_\chi(f) \ge \int_{X_n} \Psi_{K_n}(x)\mu_x(f)\,\mu_\chi(dx),$$

and taking $f = 1_X$, we find that $\mu_x(X) < +\infty$ for $\mu_\chi$-a.e. $x \in X_n$, hence for $\mu_\chi$-a.e. $x \in X$ since $\bigcup_n X_n$ has full measure in $X$.

(ii) Recall that in items (ii) and (iii), we assume that $G$ is abelian. With the notations of Theorem 2.1.3, the exponentials in (13) do not depend on $x$ and the measures $\mu_x$ satisfy $(\tau\mu_x)(dy) = \chi(\psi(\tau^{-1}y))\,\mu_x(dy)$. One easily sees that the measures $\tilde{\mu}_x(dy)$ defined by (15) satisfy, as the measure $\mu_\chi$, the conformal property

$$\tau\tilde{\mu}_x(dy) = \chi(\varphi(\tau^{-1}y))\,\tilde{\mu}_x(dy). \tag{17}$$

By (16) we have, for any $n \in \mathbb{N}$,

$$\mu_\chi(f) \geq \int\limits_{X_n} \Phi_{K_n}(x)\tilde{\mu}_x(f)\,\mu_\chi(dx), \tag{18}$$

where $\Phi_{K_n}(x) = \Psi_{K_n}(x)\inf_{u\in K_n}\chi(u)$.

This implies that, for any $B \in \mathfrak{X}$ with $B \subset X_n$, there exists a nonnegative measurable function $\xi_B$ on $X$ such that

$$\int 1_B(x)\Phi_{K_n}(x)\,\tilde{\mu}_x(dy)\,\mu_\chi(dx) = \xi_B(y)\,\mu_\chi(dy).$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, $\mu_\chi$-a.e. As $\mu_\chi$ is $\tau$-ergodic, $\xi_B$ is $\mu_\chi$-a.e. equal to a constant $\nu(B)$. The map $B \mapsto \nu(B)$ defines a positive measure $\nu$ on $(X_n, X_n \cap \mathfrak{X})$ absolutely continuous with respect to the measure $\mu_\chi$. Therefore there exists a measurable nonnegative function $\xi$ on $X$ such that

$$\int 1_B(x)\Phi_{K_n}(x)\,\tilde{\mu}_x(dy)\,\mu_\chi(dx) = \nu(B)\mu_\chi(dy) = \Big(\int 1_B(x)\xi(x)\,\mu_\chi(dx)\Big)\,\mu_\chi(dy)$$

and, for $\mu_\chi$-a.e. $x \in X_n$,

$$\xi(x)\,\mu_\chi(dy) = \Phi_{K_n}(x)\;\tilde{\mu}_x(dy).$$

As $\bigcup_{n\in\mathbb{N}} X_n$ is of full measure, by gluing the $\Phi_{K_n}$, we obtain a function $\Phi$ such that, for $\mu_\chi$-a.e. $x \in X$,

$$\xi(x)\,\mu_\chi(dy) = \Phi(x)\;\tilde{\mu}_x(dy).$$

This shows the regularity of the cocycle.

(iii) We set $X_0 = \{x \in X : \tilde{\mu}_x(X) < +\infty\}$. For $x \in X_0$, we denote by $\hat{\mu}_x$ the probability measure $\tilde{\mu}_x/\tilde{\mu}_x(X)$. From (16), for any compact subset $K$ of $G$, we have

$$\mu_\chi(f) \geq \int\limits_{X_0} \Phi_K(x)\;\Big(\int\limits_X f(y)1_K(u_x(y))\,\hat{\mu}_x(dy)\Big)\,\mu_\chi(dx), \tag{19}$$

where $\Phi_K(x) := \Psi_K(x)\inf_{u\in K}\chi(u)\tilde{\mu}_x(X)$.

Let $h_1$ be a positive bounded measurable function on $X$. We know that $\tau$ is conservative, i.e. $\mu_\chi(\{\sum_{k\geq 0} h_1 \circ \tau^k < +\infty\}) = 0$. From (19), it follows that, for $\mu_\chi$-a.e. $x \in X_0$,

$$\forall n \in \mathbb{N}, \qquad \hat{\mu}_x\Big(\Big\{\sum_{k\geq 0} h_1 \circ \tau^k < +\infty\Big\} \cap \{u_x \in K_n\}\Big) = 0.$$

As $n \nearrow +\infty$, from the monotone convergence theorem, we obtain, for $\mu_\chi$-a.e. $x \in X$,

$$\hat{\mu}_x\Big(\Big\{\sum_{k\geq 0} h_1 \circ \tau^k < +\infty\Big\}\Big) = 0.$$

Since $h_1$ is bounded and thus $\hat{\mu}_x$-integrable for $x \in X_0$, we deduce that, for $\mu_\chi$-a.e. $x \in X_0$, $\tau$ is conservative for $\hat{\mu}_x$. Replacing $X_0$ by $X_0 \cap \{x \in X : \hat{\mu}_x(\{\sum_{k\geq 0} h_1 \circ \tau^k < +\infty\}) = 0\}$ we can assume that, for any $x \in X_0$, $\tau$ is conservative for $\hat{\mu}_x$.

From (19), there exists a measurable $[0,1]$-valued function $\xi_K$ such that

$$\xi_K(y)\,\mu_\chi(dy) = \int_{X_0} \Phi_K(x) 1_K(u_x(y))\,\hat{\mu}_x(dy)\,\mu_\chi(dx) \leq \int_{X_0} \Phi_K(x)\,\hat{\mu}_x(dy)\,\mu_\chi(dx).$$

Consequently, there exists a measurable $[0,1]$-valued function $\psi_K$ such that

$$\xi_K(y)\,\mu_\chi(dy) = \psi_K(y) \int_{X_0} \Phi_K(x)\,\hat{\mu}_x(dy)\,\mu_\chi(dx).$$

Since the measures $\mu_\chi$ and $\hat{\mu}_x$ have the same conformal property (cf. (17)), we have

$$\sum_{k=0}^{n-1} T^k \xi_K(y)\,\mu_\chi(dy) = \int_{X_0} \Phi_K(x) \sum_{k=0}^{n-1} T^k \psi_K(y)\,\hat{\mu}_x(dy)\,\mu_\chi(dx)$$

where $T$ is the operator defined by

$$Tf(y) = f \circ \tau^{-1}(y) \chi(\varphi(\tau^{-1} y)).$$

As $\tau$ is conservative for $\mu_\chi$ and for $\hat{\mu}_x$, $x \in X_0$, by Hurewicz's ergodic theorem, for any bounded measurable function $f$ on $X$, the sequence of functions

$$\Big(\sum_{k=0}^{n-1} T^k f \,/\, \sum_{k=0}^{n-1} T^k 1\Big)_{n\in\mathbb{N}}$$

converges $\mu_\chi$-a.e. to $\mu_\chi(f)$ and converges $\hat{\mu}_x$-a.e. to $\hat{\mu}_x(f)$, for $x \in X_0$. As the sequence of functions is bounded and the measures are finite, these convergences also hold in $\mathbb{L}^1$-norm.

Therefore, for any bounded measurable function $f$,

$$\int_X f(y) \frac{\sum_{k=0}^{n-1} T^k \xi_K(y)}{\sum_{k=0}^{n-1} T^k 1(y)}\,\mu_\chi(dy) \xrightarrow{n\to+\infty} \mu_\chi(\xi_K)\mu_\chi(f),$$

and for $\mu_\chi$-a.e. $x \in X$,

$$\alpha_n(x) = \int_X f(y) \frac{\sum_{k=0}^{n-1} T^k \psi_K(y)}{\sum_{k=0}^{n-1} T^k 1(y)}\,\hat{\mu}_x(dy) \xrightarrow{n\to+\infty} \hat{\mu}_x(\psi_K)\hat{\mu}_x(f).$$

The inequality (19) shows that $\Phi_K$ is $\mu_\chi$-integrable. Moreover, the sequence of functions $(\alpha_n)$ is bounded. By the dominated convergence theorem,

$$\int_{X_0} \Phi_K(x)\alpha_n(x)\,\mu_\chi(dx) \xrightarrow{n\to+\infty} \int_{X_0} \Phi_K(x)\hat{\mu}_x(\psi_K)\hat{\mu}_x(f)\,\mu_\chi(dx).$$

We deduce that

$$\mu_\chi(dy) = \int_{X_0} \hat{\Phi}_K(x)\,\hat{\mu}_x(dy)\,\mu_\chi(dx),$$

where $\hat{\Phi}_K(x) = \Phi_K(x)\hat{\mu}_x(\psi_K)/\mu_\chi(\xi_K)$.

Now, as above, for any $B \in \mathfrak{X}$ with $B \subset X_0$, there exists a nonnegative measurable function $\xi_B$ such that

$$\xi_B(y)\,\mu_\chi(dy) = \int_B \hat{\Phi}_K(x)\,\hat{\mu}_x(dy)\,\mu_\chi(dx).$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, $\mu_\chi$-a.e. With the same argument as in (ii), since $\mu_\chi$ is $\tau$-ergodic, $\xi_B$ is $\mu_\chi$-a.e. equal to $\nu(B)$, where $\nu$ is a positive measure on $(X_n, X_n\cap\mathfrak{X})$ absolutely continuous with respect to $\mu_\chi$. Therefore there exists a measurable nonnegative function $\xi$ on $X$ such that

$$\int 1_B(x)\hat{\Phi}_K(x)\,\hat{\mu}_x(dy)\,\mu_\chi(dx) = \nu(B)\,\mu_\chi(dy) = \Big(\int_B \xi(x)\,\mu_\chi(dx)\Big)\,\mu_\chi(dy)$$

and, for $\mu_\chi$-a.e. $x \in X_0$,

$$\xi(x)\,\mu_\chi(dy) = \hat{\Phi}_K(x)\,\hat{\mu}_x(dy).$$

This shows the regularity of the cocycle.

(iv) Now assume that $\tau$ is conservative for $\mu_\chi$ and that there exists a compact subset $K$ such that, for $\mu_\chi$-a.e. $x \in X$, $\varphi_n(x) \in K$ for every $n \in \mathbb{N}$.

For any nonnegative measurable function $f$ on $X$ with $\mu_\chi(f) \in \,]0,+\infty[$, we have

$$\sum_{n\ge0} f(\tau^n x)1_K(\varphi_n(x)) = \sum_{n\ge0} f(\tau^n x) = +\infty \qquad \mu_\chi\text{-a.e.}$$

Hence $\tau_\varphi$ is conservative for $\lambda_\chi$. We deduce that, for $x \in X_0$ where $X_0$ is a set of full $\mu_\chi$-measure, and any $g \in G$, $\tau_\varphi$ is conservative for $M^h((x,g),\cdot)$.

We take $x \in X_0$. Let $s \in \operatorname{supp}(u_x(\mu_x))$ and $t \in H_x$. Then, for any neighborhoods $V$ and $W$ of $s$ and $t$, for $\mu_x$-a.e. $y \in X$, $\sum_{n\ge0} 1_V(\tau^n y)1_W(\varphi_n(y)) = +\infty$. From the inclusion

$$u_x(\tau^n y)\psi_n(y) = \varphi_n(y)u_x(y) \subset Ku_x(y) \qquad \text{for } \mu_x\text{-a.e. } y \in X,$$

it follows that $st \in Ku_x(y)$ for $\mu_\chi$-a.e. $x \in X$ and $\mu_x$-a.e. $y \in X$. Taking a fixed $s$ and a dense sequence $(t_n)$ in $H_x$, we infer that $t_n \in s^{-1}Ku_x(y)$ for

$\mu_\chi$-a.e. $x \in X$ and $\mu_x$-a.e. $y \in X$, for $n \geq 0$. Therefore $H_x \subset s^{-1}Ku_x(y)$ is a compact subgroup of $G$ and, with a similar argument, $\mathrm{supp}(u_x(\mu_x)) \subset Ku_x(y)H_x$ for $\mu_\chi$-a.e. $x \in X$ and $\mu_x$-a.e. $y \in X$. This implies that, for $\mu_\chi$-a.e. $x \in X$, there exists a compact subset $K_x$ of $G$ such that $\mathrm{supp}(u_x(\mu_x)) \subset K_x H_x$. Since any compact subset $K$ of $G$ satisfies $K \subset K_n$ for $n$ large enough, we deduce that $\bigcup_{n\in\mathbb{N}} X_{K_n}$ has full $\mu_\chi$-measure. So we can assume that $u$ satisfies the boundedness condition (14) (cf. Lemma 2.2.3).

By (16) we have, for any $n \in \mathbb{N}$,

$$\mu_\chi(f) \geq \int_{X_n} \Psi_{K_n}(x)\mu_x(f)\,\mu_\chi(dx). \tag{20}$$

This implies that there exists a $[0,1]$-valued measurable function $\xi$ such that

$$\int_{X_n} \Psi_{K_n}(x)\,\mu_x(dy)\,\mu_\chi(dx) = \xi(y)\,\mu_\chi(dy).$$

Observe that for any $x \in X$, the exponential $\chi_x$ on the compact group is trivial and consequently the measures $\mu_x$, $x \in X$, are $\tau$-invariant.

From the conformal property (17), it follows that $\xi \circ \tau^{-1} d\tau\mu_\chi/d\mu_\chi = \xi$, $\mu_\chi$-a.e. This shows that the measure $\xi\mu_\chi$ is $\tau$-invariant. Moreover, $\{\xi > 0\}$ is $\mu_\chi$-a.e. $\tau$-invariant and therefore has full $\mu_\chi$-measure.

For any $B \in \mathfrak{X}$ with $B \subset X_n$, there exists a $[0,1]$-valued measurable function $\xi_B$ such that

$$\int_B \Psi_{K_n}(x)\,\mu_x(dy)\,\mu_\chi(dx) = \xi_B(y)\xi(y)\,\mu_\chi(dy). \tag{21}$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, $\mu_\chi$-a.e. As in (ii) and (iii), $\xi_B$ is $\mu_\chi$-a.e. equal to $\nu(B)$, where $\nu$ is a positive measure on $(X_n, X_n \cap \mathfrak{X})$ absolutely continuous with respect to $\mu_\chi$. Therefore there exists a measurable nonnegative function $\psi$ on $X$ such that

$$\int_B \Psi_{K_n}(x)\,\mu_x(dy)\,\mu_\chi(dx) = \nu(B)\,\xi(y)\,\mu_\chi(dy) = \Big(\int_B \psi(x)\,\mu_\chi(dx)\Big)\xi(y)\,\mu_\chi(dy)$$

and, for $\mu_\chi$-a.e. $x \in X_n$,

$$\psi(x)\xi(y)\,\mu_\chi(dy) = \Psi_{K_n}(x)\,\mu_x(dy).$$

This shows the regularity of the cocycle hence the last assertion of (iv). ■

Remark. If $G$ is a compact group, then it is well known that every $G$-valued cocycle $\varphi$ is regular and therefore cohomologous to a cocycle $\psi$ taking its values in a compact subgroup $K$ of $G$ such that $\mu \otimes m_K$ is ergodic for $\tau_\psi$ (cf. [PaPo97], [Pa97] for the regularity of the cohomology when $G$ is compact and the cocycle $\varphi$ is Hölderian over a subshift of finite type).

See also [AaWe00] for results under the assumption of tightness for the cocycle $(\varphi_n)$.

**2.3. Essential values and periods of invariant functions.** The notion of essential values was introduced by K. Schmidt [Sc77] and J. Feldman and C. C. Moore [FeMo77]. See also [Sc75], [Sc79], [Sc81], [Aa97]. The results in this section, except Proposition 2.3.6, are not new, at least when $\mu$ is $\tau$-invariant. For the sake of completeness, we will give proofs. Note that we are here in the more general case of a quasi-invariant measure.

DEFINITIONS 2.3.1. Let $\mu$ be a *$\tau$-quasi-invariant conservative* measure on $X$. An element $a \in G \cup \{\infty\}$ is an *essential value* of the cocycle $(\varphi, \tau)$ (with respect to $\mu$) if, for every neighborhood $V$ of $a$, and for every subset $B$ such that $\mu(B) > 0$, there is $n \in \mathbb{Z}$ such that

$$\mu(B \cap \tau^{-n} B \cap \{x : \varphi_n(x) \in V\}) > 0.$$

We denote by $\bar{\mathcal{E}}(\varphi)$ the set of essential values of the cocycle $(\varphi, \tau)$ and by $\mathcal{E}(\varphi) = \bar{\mathcal{E}}(\varphi) \cap G$ the *set of finite essential values*.

Let $B$ be a measurable set of positive $\mu$-measure. Let $\tau_B$ be the induced transformation on $B$ and $\varphi^B(x) := \varphi_{n(x)}(x)$, where $n(x) = n_B(x) := \inf\{j \geq 1 : \tau^j x \in B\}$ for $x \in B$. The "induced" cocycle is given, for $n \geq 1$, by $\varphi_n^B(x) := \varphi^B(x)\varphi^B(\tau_B x) \cdots \varphi^B(\tau_B^{n-1} x)$.

Equivalently to Definition 2.3.1, an element $a \in G \cup \{\infty\}$ is an essential value of the cocycle $(\varphi, \tau)$ if and only if, for every subset $B$ such that $\mu(B) > 0$, and for any neighborhood $V$ of $a$, $\mu(\{x : \varphi_n^B(x) \in V\}) > 0$ for some $n \in \mathbb{Z}$.

PROPOSITION 2.3.2. *Assume that $\tau$ is conservative for $\mu_\chi$. If $\infty \notin \bar{\mathcal{E}}(\varphi)$, then $\varphi$ is cohomologous to a cocycle taking its values in a compact subgroup of $G$. When $G$ is abelian we have $\bar{\mathcal{E}}(\varphi) = \{e\}$ if and only if $\varphi$ is a coboundary.*

*Proof.* If $\infty \notin \bar{\mathcal{E}}(\varphi)$, then there is $B$ with $\mu_\chi(B) > 0$ such that $(\varphi_n^B)_{n\in\mathbb{Z}}$ is a bounded sequence. This implies that $\varphi^B$ is $\tau_B$-cohomologous to a cocycle taking values in a compact subgroup of $G$ (cf. Proposition 2.2.4), i.e. there are measurable maps $\zeta^B$ from $B$ to $G$ and $\psi^B$ from $B$ to a compact subgroup of $G$ such that

$$\varphi^B = (\zeta^B \circ \tau_B)\psi^B(\zeta^B)^{-1}. \tag{22}$$

By ergodicity and conservativity of $(X, \mu_\chi, \tau)$, for $\mu_\chi$-a.e. $y \in X$ there are a unique $x \in B$ and an integer $k$ with $0 \leq k < n_B(x)$ such that $y = \tau^k x$. We define $\zeta$ on $X$ by taking, for $y = \tau^k x$ and $0 \leq k < n_B(x)$,

$$\zeta(y) = \varphi_k(x)\zeta^B(x)(\psi(y))^{-1}$$

with $\psi(y) = e$ if $k < n_B(x) - 1$, and $\psi(y) = \psi^B(x)$ for $k = n_B(x) - 1$.

For $0 \leq k < n_B(x) - 1$, the cocycle relation is clearly satisfied by construction. For $k = n_B(x) - 1$, it results from the cocycle relation (22) for the induced cocycle.

Now we consider the abelian case. Let us show that if $\bar{\mathcal{E}}(\varphi) = \{e\}$ then $\varphi$ is a coboundary. From the first assertion we know that the cocycle is cohomologous to a cocycle $\psi$ taking values in a compact subgroup $K$ of $G$. The set of essential values is the same for $\phi$ and $\psi$ (see below). As $\tau_\psi$ is ergodic conservative and $\bar{\mathcal{E}}(\psi) = \{e\}$, one has $K = \{e\}$. ∎

We now consider, as in Theorem 2.1.3, a measure $\lambda_\chi$.

NOTATION 2.3.3. Let $\mathcal{P}(\varphi)$ be the closed subgroup of $G$ of *left periods* of the $\tau_\varphi$-invariant measurable functions, i.e. the subgroup of elements $\gamma \in G$ such that, for every $\tau_\varphi$-invariant function $f$, $f(x, \gamma g) = f(x, g)$ for $\lambda_\chi$-a.e. $(x, g) \in X \times G$.

Note that we should write $\mathcal{P}(\varphi, \mu_\chi)$, since $\mathcal{P}(\varphi)$ and $\bar{\mathcal{E}}(\varphi)$ depend on the measure $\mu_\chi$. We will show that $\mathcal{P}(\varphi) = \mathcal{E}(\varphi)$ by using the following lemma from [ArNgOs].

Let $(Y, \rho)$ be a complete separable metric space with a continuous action $(g, y) \mapsto g.y$ of a group $G$ on it. Let $f$ be a measurable map from $X$ to $Y$. Given a $G$-valued cocycle $\varphi$, we say that $f$ is $(\varphi, \tau)$-*invariant* if $f(\tau x) = \varphi(x).f(x)$, $\mu$-a.e.

LEMMA 2.3.4 ([ArNgOs]). *If $f$ is $(\varphi, \tau)$-invariant, then $a.f(x) = f(x)$, $\mu$-a.e. for all $a \in \mathcal{E}(\varphi)$.*

*Proof.* $(Y, \rho)$ being a separable metric space, the set

$$X_f := \{x \in X : \mu(\{x' \in X : \rho(f(x'), f(x)) < \varepsilon\}) > 0 \text{ for every } \varepsilon > 0\}$$

has full $\mu$-measure since it contains $f^{-1}(\operatorname{supp} f(\mu))$. Let $x \in X_f$ and $a \in \mathcal{E}(\varphi)$. Let $\varepsilon > 0$ be arbitrary. Then the subset $E_x = \{x' : \rho(f(x'), f(x)) < \varepsilon\}$ has positive $\mu$-measure. Since $a \in \mathcal{E}(\varphi)$, for every $\varepsilon_1 > 0$ there exist $x_1 \in E_x$ and $n \in \mathbb{Z}$ such that $\tau^n x_1 \in E_x$ and $d(a, \varphi_n(x_1)) < \varepsilon_1$, where $d$ is a distance on $G$. By the invariance of $f$ we have

$$\begin{aligned}\rho(a.f(x), f(x)) \le \rho(a.f(x), a.f(x_1)) &+ \rho(a.f(x_1), \varphi_n(x_1).f(x_1)) \\ &+ \rho(f(\tau^n x_1), f(x)).\end{aligned}$$

Since $\varepsilon$ and $\varepsilon_1$ are arbitrary and the action of $G$ is continuous, we get $\rho(a.f(x), f(x)) = 0$. ∎

PROPOSITION 2.3.5. *$\mathcal{E}(\varphi) = \mathcal{P}(\varphi)$.*

*Proof.* If $a \notin \mathcal{E}(\varphi)$, there are a subset $A$ with $\mu(A) > 0$ and a neighborhood $V$ of $e$ such that

$$A \cap \tau^{-n} A \cap \{\varphi_n \in aVV^{-1}\} = \emptyset, \qquad \forall n \in \mathbb{Z}.$$

This implies that $a$ is not a period of the $\tau_\varphi$-invariant set $B = \bigcup_{n\in\mathbb{Z}} \tau_\varphi^n(A \times V)$.

Conversely, let $h$ be a strictly positive function on $G$ such that $\int h(g)\, m_G(dg) = 1$. We apply Lemma 2.3.4 to the $G$-space $Y$ of real measurable functions on $G$, with the metric defined by $\rho(f_1, f_2) = \int_X \inf(|f_1 - f_2|, 1)h\, dm_G$. A function on $X \times G$ can be viewed as a function on $X$ taking its values in $Y$. By Lemma 2.3.4, if a function $f$ on $X \times G$ is $\tau_\varphi$-invariant, then every element of $\mathcal{E}(\varphi)$ is a period for $f$. ■

The proposition shows that $\mathcal{E}(\varphi) = G$ if and only if $\lambda_\chi$ is ergodic for $\tau_\varphi$. With the notations of Theorem 2.1.3, we have:

PROPOSITION 2.3.6. *An element $\gamma$ in $G$ belongs to $\mathcal{P}(\varphi)$ if and only if $\gamma$ belongs to $H_x$ for $\mu_\chi$-a.e. $x \in X$. In the abelian case, $\mathcal{P}(\varphi)$ (and therefore $\mathcal{E}(\varphi)$) coincides with the subgroup $H$.*

*Proof.* For $(x, g) \in X \times G$, we set (cf. (34))

$$c(x,g) = \Big( \int\limits_X \Big( \int\limits_{H_x} h(y, u_x(y)\gamma g)\chi_x(\gamma)\, d\gamma \Big)\, \mu_x(dy) \Big)^{-1}.$$

According to Theorem 2.1.3, we have

$$\gamma \in \mathcal{P}(\varphi) \;\Leftrightarrow\; M^h((x,\gamma g), \cdot) = M^h((x,g),\cdot) \text{ for } \lambda_\chi\text{-a.e. } (x,g) \in X \times G.$$

For $\lambda_\chi$-a.e. $(x, g) \in X \times G$, the right member is equivalent to

$$c(x,g)\, \mu_x(dy)\, \delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_g = c(x,\gamma g)\, \mu_x(dy)\, \delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_{\gamma g},$$

that is, for $\mu_x$-a.e. $y \in X$,

$$c(x,g)\, \delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_\gamma = c(x,\gamma g)\, \delta_{u_x(y)} * (\chi_x m_{H_x}).$$

The equality of the supports of these measures implies $H_x\gamma = H_x$ for $\mu_\chi$-a.e. $x \in X$. Hence the result. ■

*Abelian groups.* If $\varphi$ and $\psi$ are two cohomologous cocycles, $\varphi \overset{(u,\mu)}{\sim} \psi$, then $f$ is $\tau_\varphi$-invariant if and only if $\tilde f$ is $\tau_\psi$-invariant, where $\tilde f(x,g) = f(x, u(x)g)$.

If $G$ is abelian, this implies that $\mathcal{P}(\varphi) = \mathcal{P}(\psi)$, so that two cohomologous cocycles have the same set of essential values. This is false in the nonabelian case (cf. [ArNgOs]).

When $G$ is abelian, the cocycle $\tilde\varphi := \varphi \bmod \mathcal{E}(\varphi)$ satisfies $\mathcal{E}(\tilde\varphi) = \{0\}$. If $\overline{\mathcal{E}}(\tilde\varphi) = \{0\}$, then by 2.3.2, $\varphi$ is $\mu_\chi$-cohomologous to a cocycle taking its values in $\mathcal{E}(\varphi)$. Therefore the regularity of the cocycle is equivalent to $\overline{\mathcal{E}}(\tilde\varphi) = \{0\}$. This last property, for an invariant measure, corresponds to the definition of regularity given by K. Schmidt for a cocycle (defined for a group action) taking its values in an abelian group.

If $G/\mathcal{E}(\varphi)$ is compact, then $\overline{\mathcal{E}}(\tilde\varphi) = \{0\}$ and $\varphi$ is regular. In particular, this is the case when $G = \mathbb{R}$ and $\mathcal{E}(\varphi) \neq \{0\}$.

Note that if $\varphi$ is cohomologous to $\varphi_1$ and to $\varphi_2$, two functions with values respectively in closed subgroups whose intersection reduces to the identity element $e$ of $G$, then $\mathcal{E}(\varphi) = \{e\}$.

For instance, if $\varphi$ is a $\mathbb{Z}$-valued cocycle such that there is $s \notin \mathbb{Q}$ for which the multiplicative equation $e^{2\pi i s\varphi} = \psi/\psi \circ \tau$ has a measurable solution $\psi$, then either $\varphi$ is a coboundary or the cocycle $\varphi$ is not regular. We will use this remark to give an example of a nonregular cocycle in Section 5.

## 3. PROOF OF THEOREM 2.3.1

**3.1. Characterization of the $\tau_\varphi$-invariant ergodic measures.** The key tool in the proof of Theorem 2.1.3 is the following result:

THEOREM 3.1.1 ([Ra07]). *Let $\lambda$ be a $\tau_\varphi$-invariant ergodic measure of the form $\lambda(dy, dg) = \mu(dy)\, N(y, dg)$, where $\mu$ is a probability measure on $X$ and $N$ a positive Radon kernel (i.e. such that, for every $y \in X$, $N(y, dg)$ is a positive Radon measure on the Borel subsets of $G$ and, for every Borel set $B$ in $G$, the map $y \mapsto N(y, B)$ is measurable). Then there exist a closed subgroup $H$ of $G$ and a measurable map $u$ from $X$ to $G$ such that:*

- *$\varphi_u(y) := (u(\tau y))^{-1}\, \varphi(y)\, u(y) \in H$ for $\mu$-a.e. $y \in X$;*
- *the measure $\tilde{\lambda}$ that is the image of $\lambda$ under the map $(y, g) \mapsto (y, (u(y))^{-1} g)$ is a $\tau_{\varphi_u}$-invariant ergodic measure with support $X \times H$ and has the form*

$$\tilde{\lambda}(dy, dh) = \tilde{\mu}(dy)\, \chi(h)\, dh, \tag{23}$$

*where $\chi$ is an exponential on $H$ and $\tilde{\mu}$ a positive $\sigma$-finite measure, equivalent to $\mu$ such that*

$$\tau\tilde{\mu}(dy) = \chi(\varphi_u(\tau^{-1} y))\, \tilde{\mu}(dy). \tag{24}$$

*If $H = G$, then $u(y) \equiv e$,*

$$\lambda(dy, dg) = \tilde{\mu}(dy)\, \chi(g)\, dg, \qquad \tau\tilde{\mu}(dy) = \chi(\varphi(\tau^{-1} y))\, \tilde{\mu}(dy).$$

**3.2. Ergodic decomposition of $\lambda_\chi$**

*Abstract ergodic decomposition.* Let $h$ be a positive measurable function on $X \times G$ such that $\lambda_\chi(h) = 1$ (cf. 2.1.2). We apply the results of the appendix to the Borel standard space $(X \times G, \mathfrak{X} \times \mathfrak{B}_G)$ and to the probability measure $h\lambda_\chi$.

We denote by $P^h$ a regular conditional probability with respect to $h\lambda_\chi$ and the $\sigma$-algebra $\mathfrak{J}$ of $\tau_\varphi$-invariant sets, and by $M^h$ the positive kernel on $X \times G$ defined for any measurable nonnegative function $f$ on $X \times G$ by

$$\forall (x, g) \in X \times G, \qquad M^h f(x, g) = P^h(f/h)(x, g).$$

We have

$$\lambda_\chi(dy, dt) = \int_{X \times G} M^h((x, g), (dy, dt))\, h(x, g)\, \lambda_\chi(dx, dg). \tag{25}$$

For $\lambda_\chi$-a.e. $(x,g) \in X \times G$, the probability measure $P^h((x,g),\cdot)$ is $\tau_\varphi$-ergodic (Theorem 7.4.5) (i.e. for all $A \in \mathfrak{J}$, $P^h((x,g),A) = 0$ or $1$). Moreover, according to (49) of Lemma 7.2.1, we have

$$\tau_\varphi P^h((x,g),(dy,dt)) = \frac{h\circ\tau_\varphi^{-1}(y,t)}{h(y,t)}\, P^h((x,g),(dy,dt)), \tag{26}$$

which is equivalent to

$$\tau_\varphi M^h((x,g),(dy,dt)) = M^h((x,g),(dy,dt)). \tag{27}$$

We write

$$P^h((x,g),(dy,dt)) = \rho((x,g),dy)\, Q((x,g,y),dt),$$

where $\rho$ is a transition probability from $(X\times G, \mathfrak{X}\otimes\mathfrak{B}_G)$ to $(X,\mathfrak{X})$, and $Q$ a transition probability from $(X\times G\times X, \mathfrak{X}\otimes\mathfrak{B}_G\otimes\mathfrak{X})$ to $(G,\mathfrak{B})$. We also introduce the notations

$$\nu_{(x,g)}(dy) := \rho((x,g),dy) \quad\text{and}\quad N_{(x,g)}(y,dt) := Q((x,g,y),dt).$$

Let $(x,g)\in X\times G$. The probability measure $\nu_{(x,g)}$ is uniquely determined by $\nu_{(x,g)}(A) = P^h((x,g),A\times G)$ for any $A\in\mathfrak{X}$. The family of probability measures $\{N_{(x,g)}(y,\cdot) : y\in X\}$ is determined up to a set of $\nu_{(x,g)}$-measure zero. If we consider on the probability space $(X\times G, \mathfrak{X}\times\mathfrak{B}_G, P^h((x,g),\cdot))$ the projections $U$ and $V$ on $X$ and $G$, then $\nu_{(x,g)}$ is the law of $U$ and $N_{(x,g)}$ is a version of the conditional law of $V$ with respect to $U$.

The kernel $M^h$ can then be written

$$\begin{aligned} M^h((x,g),(dy,dt)) &= \rho((x,g),dy)\,\tilde{Q}((x,g,y),dt) \\ &= \nu_{(x,g)}(dy)\,\tilde{N}_{(x,g)}(y,dt), \end{aligned} \tag{28}$$

where $\tilde{Q}((x,g,y),dt) = \tilde{N}_{(x,g)}(y,dt) = h(y,t)^{-1}N_{(x,g)}(y,dt)$ is a positive kernel from $(X\times G\times X, \mathfrak{X}\times\mathfrak{B}_G\times\mathfrak{X})$ to $(G,\mathfrak{B}_G)$.

Let $f$ be a measurable positive $\mu_\chi$-integrable function on $X$, and $K$ be a compact subset of $G$. We know that

$$\begin{aligned} \int_{X\times G} \Big[\int_X f(y)\tilde{N}_{(x,g)}(y,K)\,\nu_{(x,g)}(dy)\Big] h(x,g)\,\lambda_\chi(dx,dg) \\ = \int_{X\times G} f(x)1_K(g)\,\lambda_\chi(dx,dg) < +\infty. \end{aligned}$$

Therefore, for $\lambda_\chi$-a.e. $(x,g)$, we have $\tilde{N}_{(x,g)}(y,K) < +\infty$ for $\nu_{(x,g)}$-a.e. $y$.

Let $(K_n)_{n\ge0}$ be the sequence of compact subsets of $G$ such that $\bigcup_{n\in\mathbb{N}} K_n = G$. For $\lambda_\chi$-a.e. $(x,g)$, we have, for $\nu_{(x,g)}$-a.e. $y$ and all $n\ge0$, $\tilde{N}_{(x,g)}(y,K_n) < +\infty$, i.e. $\tilde{N}_{(x,g)}(y,\cdot)$ is a Radon measure on $G$.

After a modification of $P^h$ on a set of $\lambda_\chi$-measure zero followed, for any $(x,g)\in X\times G$, by a modification of the family of positive measures $\{\tilde{N}_{(x,g)}(y,\cdot) : y\in X\}$ on a set of $\nu_{(x,g)}$-measure zero, we can assume that:

*For every $(x,g) \in X \times G$, the positive measure $M^h((x,g),\cdot)$ is $\tau_\varphi$-invariant ergodic and, for every $y \in X$, $\tilde{N}_{(x,g)}(y,\cdot)$ is a Radon measure on $G$.*

*Explicit form of the ergodic decomposition.* According to Theorem 3.1.1, the $\tau_\varphi$-invariant ergodic measure $M^h((x,g),\,\cdot\,)$ can be written, up to a multiplicative constant,

$$M^h((x,g),(dy,d\gamma)) = \tilde{\mu}_{(x,g)}(dy) \times [\delta_{v_{(x,g)}(y)} * (\chi_{(x,g)}(\gamma)\, m_{H_{(x,g)}}(d\gamma))], \tag{29}$$

where $H_{(x,g)}$ is a closed subgroup of $G$, $\chi_{(x,g)}$ an exponential on $H_{(x,g)}$, $v_{(x,g)}$ a measurable map from $X$ to $G$, and $\tilde{\mu}_{(x,g)}$ a positive $\sigma$-finite measure on $X$, equivalent to the probability measure $\nu_{(x,g)}$, such that

$$\tau_\varphi(\tilde{\mu}_{(x,g)})(dy) = \chi(\varphi_{v_{(x,g)}}(\tau^{-1}y))\, \tilde{\mu}_{(x,g)}(dy), \tag{30}$$

where

$$\varphi_{v_{(x,g)}}(y) := (v_{(x,g)}(\tau y))^{-1}\, \varphi(y)\ v_{(x,g)}(y) \in H_{(x,g)} \tag{31}$$

for $\tilde{\mu}_{(x,g)}$-a.e. $y \in X$.

For $t \in G$ and $f$ defined on $X \times G$, let $R_t(f)(x,g) := f(x,gt)$. From Lemma 7.2.1 it follows that, for every $t \in G$, every nonnegative measurable function $f$ on $X \times G$, and $\lambda_\chi$-a.e. $(x,g) \in X \times G$,

$$M^h(R_t(f))(x,g) = P^h(R_t h/h)(x,g) M^h(f)(x,gt). \tag{32}$$

Let $c_{(x,g),t}$ be defined by

$$c_{(x,g),t} = P^h(R_t h/h)(x,g). \tag{33}$$

From (32), we have

$$\begin{aligned}\tilde{\mu}_{(x,g)}(dy) &\times [\delta_{v_{(x,g)}}(y) * (\chi_{(x,g)}(\gamma)\, m_{H_{(x,g)}}(d\gamma)) * \delta_t] \\ &= c_{(x,g),t}\, \tilde{\mu}_{(x,gt)}(dy) \times [\delta_{v_{(x,gt)}}(y) * (\chi_{(x,gt)}(\gamma)\, m_{H_{(x,gt)}}(d\gamma))].\end{aligned}$$

Using Fubini's theorem and the separability of the $\sigma$-algebra $\mathfrak{X} \times \mathfrak{B}_G$, it follows that, for $\lambda_\chi$-a.e. $(x,g) \in X \times G$ and $m_G$-a.e. $t \in G$,

$$R_t(M^h((x,g),\,\cdot\,)) = P^h(R_t h/h)(x,g) M^h((x,gt),\,\cdot\,)$$

and therefore

$$R_{g^{-1}}(M^h((x,g),\,\cdot\,)) = P^h(R_t h/h)(x,g) R_{(gt)^{-1}}(M^h((x,gt),\,\cdot\,)).$$

This implies that, for $\lambda_\chi$-a.e. $(x,g) \in X \times G$, the measure $M^h((x,g),(dy,dt))$ is equal, up to a multiplicative positive constant $c(x,g)$, to a fixed measure which has the form

$$\tilde{\mu}_x(dy)\, [\delta_{v_x(y)} * (\chi_x \tilde{m}_{H_x}) * \delta_g](dt),$$

where $\tilde{m}_{H_x}$ is a left Haar measure on $H_x$ (we will later change $\tilde{m}_{H_x}$ to $m_{H_x}$ by multiplying it by a factor).

Now, $P^h(1)(x,g) = M^h(h)(x,g) = 1$ for $\lambda_\chi$-a.e. $(x,g) \in X\times G$. Therefore

$$(c(x,g))^{-1} = \int_X \Big( \int_{H_x} h(y, v_x(y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma)\Big)\, \tilde\mu_x(dy) \tag{34}$$

and, for $\lambda_\chi$-a.e. $(x,g) \in X \times G$ and every measurable nonnegative function $f$ on $X \times G$,

$$M^h(f)(x,g) = \frac{\int_X (\int_{H_x} f(y, v_x(y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma))\, \tilde\mu_x(dy)}{\int_X (\int_{H_x} h(y, v_x(y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma))\, \tilde\mu_x(dy)}. \tag{35}$$

Now we carry out the suitable modifications in order to obtain the desired properties of measurability for the decomposition.

*Measurability.* We can specify the decomposition of $M^h$ given in (28). We have

$$\begin{aligned}\nu_{(x,g)}(dy) &= P^h((x,g), dy \times G)\\ &= c(x,g)\Big(\int_{H_x} h(y, v_x(y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma)\Big)\, \tilde\mu_x(dy)\end{aligned} \tag{36}$$

and

$$\tilde N_{(x,g)}(y,dt) = \Big(\int_{H_x} h(y, v_x(y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma)\Big)^{-1} (\delta_{v_x(y)} * (\chi_x\, \tilde m_{H_x}) * \delta_g)(dt).$$

The closed set $v_x(y)H_x$ is the support $S(x,y)$ of the probability measure $Q((x,e,y),\cdot) = N_{(x,e)}(y,\cdot)$ on $G$, and $H_x$ is the support of the probability measure $\hat Q((x,e,y),\cdot) * Q((x,e,y),\cdot)$, where $\hat Q((x,e,y),\cdot)$ is the image of the positive measure $Q((x,e,y),\cdot)$ under the transformation $t \mapsto t^{-1}$ of $G$. It follows that the maps $x \in X \mapsto H_x \in \mathcal F(G)$ and $(x,y) \in X \times X \mapsto v_x(y)H_x \in \mathcal F(G)$ are measurable. For instance, the last property follows from the fact that, for any closed subset $F$ of $G$, we have

$$\{(x,y) \in X \times X : v_x(y)H_x \subset F\} = \{(x,e,y) : Q((x,e,y), F^c) = 0\}.$$

From Lemma 7.1.1 we can find a measurable map $u : X\times X \to G$ such that, for any $(x,y) \in X \times X$, $u(x,y) \in S(x,y)$. Then $v_x(y)H_x = u(x,y)H_x$ and, for any nonnegative measurable function $f$ on $X \times G$,

$$\begin{aligned}&\int_{H_x} f(y, v_x(y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma)\\ &\qquad = \chi_x^{-1}((u(x,y))^{-1}\, v_x(y)) \int_{H_x} f(y, u(x,y)\gamma g)\chi_x(\gamma)\, \tilde m_{H_x}(d\gamma).\end{aligned}$$

As

$$\delta_{(u(x,y))^{-1}\, v_x(y)} * (\chi_x \tilde m_{H_x}) = \chi_x^{-1}((u(x,y))^{-1}\, v_x(y))(\chi_x \tilde m_{H_x}),$$

the positive kernel $R((x,g,y),dt) = \delta_{(u(x,y))^{-1}} * \widetilde N_{x,g}(y,dt) * \delta_{g^{-1}}$ from $X\times X$

to $G$ is equal to

$$\Big(\int_{H_x} h(y,u(x,y)\gamma g)\chi_x(\gamma)\,\tilde{m}_{H_x}(d\gamma)\Big)^{-1}\chi_x(t)\,\tilde{m}_{H_x}(dt).$$

Denoting by $U$ the closed unit ball in $G$ centered at $e$, we have

$$\Big(\int_{H_x} h(y,u(x,y)\gamma g)\chi_x(\gamma)\,\tilde{m}_{H_x}(d\gamma)\Big)^{-1}\int_{H_x\cap U}\chi_x(t)\,\tilde{m}_{H_x}(dt)=R((x,g,y),U)>0$$

and, for any $\gamma\in H_x$,

$$\chi_x(\gamma)=\frac{R((x,e,y),\gamma U)}{R((x,e,y),U)},$$

which proves that there exists a measurable map $\eta: X\times G\to\mathbb{R}_+^*$ such that $\chi_x(\gamma)=\eta(x,\gamma)$ for $\mu_\chi$-a.e. $x\in X$ and all $\gamma\in H_x$.

We also have

$$\frac{\tilde{m}_{H_x}(dt)}{\int_{H_x\cap U}\chi_x(\gamma)\,\tilde{m}_{H_x}(d\gamma)}=\frac{R((x,e,y),dt)}{R((x,e,y),t\,U)}$$

which shows that the left member defines a positive kernel from $X$ to $G$. We observe that the left member is the unique left Haar measure, denoted by $m_{H_x}$, of $H_x$ such that

$$\int_{H_x\cap U}\chi_x(\gamma)\,m_{H_x}(d\gamma)=1.$$

Finally, we obtain

$$M^h((x,g),dy,dt)=R((x,g,y),U)\,\nu_{(x,g)}(dy)\,(\delta_{u(x,y)}*(\chi_x m_{H_x})*\delta_g)(dt)$$

and

$$\begin{aligned}&R((x,g,y),U)\,\nu_{(x,g)}(dy)\\&\qquad=c(x,g)\chi_x((u(x,y))^{-1}v_x(y))\Big(\int_{H_x\cap U}\chi_x(t)\,\tilde{m}_{H_x}(dt)\Big)\,\tilde{\mu}_x(dy).\end{aligned}$$

We deduce that

$$\chi_x((u(x,y))^{-1}\,v_x(y))\,\tilde{\mu}_x(dy)=d(x)\,\mu_x(dy)$$

with

$$(d(x))^{-1}=c(x,e)\Big(\int_{H_x\cap U}\chi_x(t)\,\tilde{m}_{H_x}(dt)\Big),$$

$$\mu_x(dy)=R((x,e,y),U)\,\nu_{(x,e)}(dy).$$

We observe that $(\mu_x(dy))_{x\in X}$ is a positive kernel on $(X,\mathfrak{X})$.

The formula (35) can be written

$$M^h(f)(x,g)=\frac{\int_X(\int_{H_x}f(y,u(x,y)\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma))\,\mu_x(dy)}{\int_X(\int_{H_x}h(y,u(x,y)\gamma g)\chi_x(\gamma)\,m_{H_x}(d\gamma))\,\mu_x(dy)}. \tag{37}$$

For every $(x,g)\in X\times G$, we choose the expression (37) for $M^h((x,g),\cdot)$.

*Proof of the relations* (4) *to* (9). The equality of measures $\tau_\varphi(M^h((x,g),(dy,dt)) = M^h((x,g),(dy,dt))$ is equivalent to

$$(\tau\mu_x)(dy)\,(\delta_{\varphi(\tau^{-1}y)} * \delta_{u_x(\tau^{-1}y)} * (\chi_x\, m_{H_x}) * \delta_g)(dt) \\ = \mu_x(dy)\ (\delta_{u_x(y)} * (\chi_x\, m_{H_x}) * \delta_g)(dt),$$

which leads to

$$\varphi(\tau^{-1}y)u_x(\tau^{-1}y)H_x = u_x(y)H_x \quad \text{ for } \mu_x\text{-a.e. } x \in X$$

and

$$(\tau\mu_x)(dy) = \chi_x((u_x(y))^{-1}\varphi(\tau^{-1}y)u_x(\tau^{-1}y))\,\mu_x(dy);$$

hence the relations (5) and (6) follow.

The equality $M^h((x,g),\cdot) = M^h(\tau_\varphi(x,g),\cdot)$ is equivalent to $\nu_{(x,g)} = \nu_{\tau_\varphi(x,g)}$ and $\tilde{N}_{(x,g)}(y,\cdot) = \tilde{N}_{\tau_\varphi(x,g)}(y,\cdot)$ for $\nu_{(x,g)}$-a.e. $y \in X$.

The equality $\tilde{N}_{(x,g)}(y,\cdot) = \tilde{N}_{\tau_\varphi(x,g)}(y,\cdot)$ is equivalent to the following conditions:

$$u_x(y)H_x = u_{\tau x}(y)H_{\tau x}\varphi(x)$$

(equality of the supports), which implies

$$\zeta_x(y) = (u_x(y))^{-1}u_{\tau x}(y)\varphi(x) \in H_x, \tag{38}$$

$$H_{\tau x} = \varphi(x)H_x(\varphi(x))^{-1}, \tag{39}$$

and therefore

$$\chi_{\tau x}(\varphi(x)\zeta_x(y)(\varphi(x))^{-1})\,\delta_{u_{\tau x}}(y) * (\chi_{\tau x} m_{H_{\tau x}}) * \delta_{\varphi(x)} \\ = \delta_{u_x(y)} * (\chi_{\tau x}(\varphi(x)\cdot(\varphi(x))^{-1})m_{H_x}$$

where $\hat{m}_{H_x} = \delta_{\varphi(x)} * m_{H_{\tau x}} * \delta_{(\varphi(x))^{-1}}$ is a left Haar measure on $H_x$.

We write $\hat{m}_{H_x} = d(x)m_{H_x}$ for a constant $d(x)$ depending on $x$ and we obtain for any $\gamma \in H_x$,

$$\chi_x(\gamma) = \chi_{\tau x}(\varphi(x)\gamma(\varphi(x))^{-1})$$

and

$$\chi_x(\zeta_x(y)) \int_{H_{\tau x}} h(y, u_{\tau x}\gamma\varphi(x)g)\chi_{\tau x}(\gamma)\,d\gamma = d(x) \int_{H_x} h(y, u_x(y)\gamma g)\chi_x(\gamma)\,d\gamma.$$

Then the equality $\nu_{(x,g)} = \nu_{\tau_\varphi(x,g)}$ is equivalent to

$$\tilde{\mu}_{\tau x}(dy) = c(x)\chi_x(\zeta_x(y))\,\tilde{\mu}_x(dy)$$

for a constant $c(x)$ depending on $x$.

This yields the relations (4), (7), (8), (9).

The ergodicity of the cocycle $\varphi_{u_x}$ on $H_x$ over the $\sigma$-finite ergodic measure $\mu_x$ implies that $H_x$ is amenable [Zi78].

The first assertion of Theorem 2.1.3 is proved.

*Assertions* (ii) *and* (iii) *of Theorem* 2.1.3. (a) We suppose that the subgroups $H_x$ are conjugate to a fixed closed subgroup $H$ (cf. Theorem 5.1.1 for the nilpotent connected Lie group case), i.e. there exists a measurable map $a : X \to G$ such that $H_x = a(x)H(a(x))^{-1}$.

Let $x \in X$. The element $a(x)$ is defined modulo the normalizer of $H$. The element $\psi(x) := (a(\tau x))^{-1}\varphi(x)a(x)$ is in the normalizer of $H$ and we have

$$(a(x))^{-1}(u_x(y))^{-1}u_{\tau x}(y)\varphi(x)a(x) \in H.$$

The ergodic components applied to a function $f$ can be written

$$\begin{aligned}(40)\qquad & M^h f(x,g) \\ &= \frac{\int_X(\int_H f(y, u_x(y)a(x)\gamma(a(x))^{-1}g)\chi_x(a(x)\gamma(a(x))^{-1})\,d\gamma)\,\mu_x(dy)}{\int_X(\int_H h(y, u_x(y)a(x)\gamma(a(x))^{-1}g)\chi_x(a(x)\gamma(a(x))^{-1})\,d\gamma)\,\mu_x(dy)}.\end{aligned}$$

We have $\chi_{\tau x}(a(\tau x)\gamma(a(\tau x))^{-1}) = \chi_x(a(x)(\psi(x))^{-1}\gamma\psi(x)(a(x))^{-1})$. Setting $\tilde{\chi}_x(\gamma) := \chi_x(a(x)\gamma(a(x))^{-1})$, we have $\tilde{\chi}_{\tau x}(\gamma) = \tilde{\chi}_x((\psi(x))^{-1}\gamma\psi(x))$.

(b) *Abelian groups.* If $G$ is abelian, we have $H_{\tau(x)} = H_x$ for $\mu_\chi$-a.e. $x \in X$. Since the map $x \in X \mapsto H_x \in \mathcal{F}(G)$ is measurable and Chabauty's topology countably separates the points, there exists a closed subgroup $H$ of $G$ such that $H_x = H$ for $\mu_\chi$-a.e. $x \in X$.

For every $\gamma \in H$, we have $\lambda_\chi(R_\gamma(f)) = \chi^{-1}(\gamma)\lambda_\chi(f)$ and, for $\lambda_\chi$-a.e. $(x,g) \in X \times G$, $M^h R_\gamma(f)(x,g) = \chi_x^{-1}(\gamma)M^h f(x,g)$. For $f = h$, it follows that

$$\forall \gamma \in H, \qquad \chi(\gamma) = \int\limits_{X\times G} \chi_x(\gamma)h(x,g)\,\lambda_\chi(dx,dg),$$

and therefore $\chi_x = \chi$ for $\mu_\chi$-a.e. $x \in X$.

The ergodic component of $\lambda_\chi$ applied to a function $f$ can be written

$$(41)\qquad M^h f(x,g) = \frac{\int_X(\int_H f(y, u_x(y)\gamma g)\chi(\gamma)\,d\gamma)\,\mu_x(dy)}{\int_X(\int_H h(y, u_x(y)\gamma g)\chi(\gamma)\,d\gamma)\,\mu_x(dy)}.$$

This completes the proof of Theorem 2.1.3. ∎

## 4. PROOF OF THEOREM 2.2.2

**4.1. Lemmas.** For the proof of Theorem 2.2.2, we begin with a lemma which allows us to compare the ergodic components.

LEMMA 4.1.1. (i) *Let $\varphi$ be a cocycle with values in a closed subgroup $H_1$ of $G$, and $\mu_1 \otimes m_{H_1}$ be a $\tau_\varphi$-quasi-invariant positive measure. Suppose that the measure $\mu_1 \otimes m_{H_1}$ is $\tau_\varphi$-ergodic and that $\varphi$ is $\mu_1$-cohomologous to a cocycle $\psi$ with values in a closed subgroup $H_2$ of $G$, with transfer function $u$. Then there exists $g_0 \in G$ such that, for $\mu_1$-a.e. $x \in X$,*

$$u(x)H_2 = g_0H_2 \quad \textit{and} \quad H_1 \subset u(x)H_2(u(x))^{-1} = g_0H_2g_0^{-1}.$$

(ii) *Assume in addition that there exists a positive $\tau_\psi$-quasi-invariant measure $\mu_2 \otimes m_{H_2}$ with $\mu_2 \sim \mu_1$ which is $\tau_\varphi$-ergodic. Then there exists $g_0 \in G$ such that*

$$H_1 u(x) = H_1 g_0, \quad u(x) H_2 = g_0 H_2 \quad \textit{and} \quad g_0^{-1} H_1 g_0 = H_2.$$

(iii) *Assume in addition that $\mu_1$ [resp. $\mu_2$] is $\chi_1 \circ \tau_\varphi$-conformal [resp. $\chi_2 \circ \tau_\psi$-conformal] for an exponential $\chi_1$ on $H_1$ [resp. $\chi_2$ on $H_2$]. Then*

- $\chi_1(\gamma) = \chi_2(g_0^{-1}\gamma g_0)$ *for $\mu_1$-a.e. $x \in X$ and every $\gamma \in H_1$,*
- $\chi_1(u(x)\, g_0^{-1}) = \chi_2(g_0^{-1} u(x))$ *for $\mu_1$-a.e. $x \in X$,*
- $\mu_2(dx) = \chi_1(u(x) g_0^{-1})\mu_1(dx)$ *up to a multiplicative constant.*

*The $\tau_\varphi$-invariant ergodic measure $\mu_2 \otimes (\delta_{u(x)} * (\chi_2\, m_{H_2}))$ is equal to $\mu_1 \otimes ((\chi_1\, m_{H_1}) * \delta_{g_0})$ up to a multiplicative constant.*

*Proof.* (i) For every continuous left $H_2$-invariant function $F$ on $G$ and every $g \in G$ the function $f^g(x,t) = F((u(x))^{-1} t g)$ is $\tau_\varphi$-invariant. This function is therefore $\mu_1 \otimes m_{H_1}$-a.e. constant. Applying Fubini's theorem and the continuity of $F$, it follows that, for $\mu_1$-a.e. $x \in X$ and any $g \in G$, the function $t \in H_1 \mapsto F((u(x))^{-1} t u(x) g)$ is constant and therefore equal to $F(g)$, its value for $t = e$. Consequently, $(u(x))^{-1} H_1 u(x) \subset H_2$.

Since $\varphi$ [resp. $\psi$] takes values in $H_1$ [resp. $H_2$], the above inclusion implies that, for $\mu_1$-a.e. $x \in X$, $(u(\tau x))^{-1}\, u(x) \in H_2$. Therefore $u(\tau x) H_2 = u(x) H_2$. By ergodicity of $(\mu_1, \tau)$, we deduce the existence of $g_0 \in G$ such that $u(x) H_2 = g_0 H_2$ for $\mu_1$-a.e. $x \in X$.

(ii) The cocycle $\psi$ is $\mu_2$-cohomologous to the cocycle $\varphi$, via the map $x \in X \mapsto (u(x))^{-1} \in G$. Then the second statement is a consequence of the first one.

(iii) Set $\mu_2 = \beta \mu_1$ where $\beta$ is a positive function on $X$. By the conformal property of the measure it follows that, for $\mu_1$-a.e. $x \in X$,

$$\chi_2(\psi(x)) = \frac{\beta(x)}{\beta(\tau x)}\, \chi_1(\varphi(x)).$$

From (ii), this equality can be written

$$\frac{\chi_2((u(\tau x))^{-1} g_0)}{\chi_2((u(x))^{-1} g_0)}\, \chi_2((u(x))^{-1} \varphi(x) u(x)) = \frac{\beta(x)}{\beta(\tau x)}\, \chi_1(\varphi(x)).$$

For any $x \in X$, we consider the exponential $\widetilde{\chi}_x$ on $H_1$ and the function $f$ on $X$, defined by

$$\widetilde{\chi}_x(t) = \frac{\chi_2((u(x))^{-1} t u(x))}{\chi_1(t)} \quad \text{and} \quad f(x) = \beta(x)\chi_2((u(x))^{-1} g_0).$$

We observe that $\widetilde{\chi}_{\tau x}(t) = \widetilde{\chi}_x(t)$ for any $t \in H_1$, and the positive function $(x,t) \mapsto f(x)\chi_x(t)$ on $X \times H$ is $\tau_\varphi$-invariant. It follows that this function is constant $\mu_1 \otimes m_{H_1}$-a.e. Hence for $\mu_1$-a.e. $x \in X$,

- $\chi_2((u(x))^{-1}tu(x)) = \chi_1(t)$ for every $t \in H_1$,
- $\beta(x)$ is equal to $\chi_2(g_0^{-1}u(x)) = \chi_1(u(x)g_0^{-1})$ up to a multiplicative constant. ■

Corollary 4.1.2. *Let $\mu_x(dy) \otimes (\delta_{u_x(y)} * (\chi_x\, m_{H_x}))(dt)$ and $\mu_{x'}(dy) \otimes (\delta_{u_{x'}(y)} * (\chi_{x'}\, m_{H_{x'}}))(dt)$ be two ergodic components of $\lambda_\chi$. Then either*

- *the measures $\mu_x$ and $\mu_{x'}$ on $X$ are mutually singular, or*
- *there is $g_{x',x} \in G$ such that, for every $g \in G$,*

$$\mu_{x'}(dy) \otimes (\delta_{u_{x'}(y)} * (\chi_{x'} m_{H_{x'}})) = \mu_x(dy) \otimes (\delta_{u_x(y)} * (\chi_x m_{H_x})) * \delta_{g_{x',x}}.$$

*Hence $P^h((x, g), \cdot) = P^h((x', g_{x',x}\, g), \cdot)$.*

*Proof.* For a $G$-valued cocycle $\varphi$ and a measurable map $u$ from $X$ to $G$, we denote by $\varphi_u$ the cocycle $\varphi_u(y) := (u(\tau y))^{-1}\varphi(y)u(y)$ for $y \in X$.

The values of the cocycles $\varphi_{u_x}$ and $\varphi_{u_{x'}}$ are respectively in $H_x$ and $H_{x'}$. The measures $\mu_{\chi_x} \otimes (\chi_x m_{H_x})$ and $\mu_{x'} \otimes (\chi_{x'} m_{H_{x'}})$ are respectively $\tau_{\varphi_{u_x}}$-invariant ergodic and $\tau_{\varphi_{u_{x'}}}$-invariant ergodic, and $\varphi_{u_{x'}} \overset{(u_x)^{-1}u_{x'}}{\sim} \varphi_{u_x}$.

The result follows from the previous lemma. ■

**4.2. Proof of Theorem 2.2.2.** (i) Let $x_0 \in X$. From Corollary 4.1.2, for any $x \in X$, if the measure $\mu_x$ is equivalent to $\mu_{x_0}$ then there is $g_x \in G$ such that $P^h((x, e), \cdot) = P^h((x_0, e), \cdot) * \delta_{g_x}$, and consequently, with the notations of Subsection 3.2 (cf. (36)), we have $\nu_{(x,e)} = \nu_{(x_0,e)}$. Conversely, the equality $\nu_{(x,e)} = \nu_{(x_0,e)}$ implies the equivalence of the measures $\mu_x$ and $\mu_{x_0}$.

The $\sigma$-algebra $\mathfrak{X} \times \mathfrak{B}(G)$ is separable, i.e. generated by a countable subalgebra $\mathcal{A}$. We deduce the equality of sets

$$\begin{aligned}\{x \in X : \mu_x \sim \mu_{x_0}\} &= \{x \in X : \nu_{(x,e)} = \nu_{(x_0,e)}\} \\ &= \{x \in X : \forall A \in \mathcal{A},\ \nu_{(x,e)}(A) = \nu_{(x_0,e)}(A)\},\end{aligned}$$

which proves that $\{x \in X : \mu_x \sim \mu_{x_0}\}$ is measurable. Since, for any $x \in X$, $\mu_x \sim \mu_{\tau x}$, this set is $\tau$-invariant and therefore (by ergodicity of $\mu_\chi$) has zero or full measure.

(ii) Assume that the cocycle is regular. Then every measurable $\tau_\psi$-invariant function $f$ is $\mu_\chi \otimes m_H$-a.e. constant. The function $F(g) := \|f(\cdot, \cdot\, g)\|_{\mathbb{L}^\infty(X\times H, \mu_\chi \otimes m_H)}$ is left $H$-invariant on $G$ and we have, for every $g \in G$,

$$f(x, \gamma g) = F(g) \quad \text{for } \mu_\chi \otimes m_H\text{-a.e. } (x, \gamma) \in X \times H.$$

The first statement of (ii) follows from the fact that $f$ is a measurable $\tau_\varphi$-invariant function if and only if the function $\tilde{f}(x, g) = f(x, u(x)g)$ is $\tau_\psi$-invariant.

We consider the bijective map $\theta_u$ from $X \times G$ onto itself defined by $\theta_u(x,g) = (x, u(x)g)$ for $(x,g) \in X \times G$. A measurable nonnegative function $f$ on $X \times G$ is $\tau_\varphi$-invariant if and only if $f \circ \theta_u$ is $\tau_\psi$-invariant. If $\mathfrak{J} = \mathfrak{J}_\varphi$ is the $\sigma$-algebra of $\tau_\varphi$-invariant subsets of $X \times G$ then $\theta_u \mathfrak{J}_\varphi$ is the $\sigma$-algebra $\mathfrak{J}_\psi$ of $\tau_\psi$-invariant subsets of $X \times G$. From Lemma 7.2.1 we have, for any nonnegative measurable function $f$ on $X \times G$ and $\lambda_\chi$-a.e. $(x,g) \in X \times G$,

$$\mathbb{E}_{h\lambda_\chi}[f \mid \mathfrak{J}_\varphi](x,g) = \frac{\mathbb{E}_{h\lambda_\chi}\big[f \circ \theta_u \,\frac{h \circ \theta_u}{h}\, \chi \circ u \,\big|\, \mathfrak{J}_\psi\big] \circ \theta_u(x,g)}{\mathbb{E}_{h\lambda_\chi}\big[\frac{h \circ \theta_u}{h}\, \chi \circ u \,\big|\, \mathfrak{J}_\psi\big] \circ \theta_u(x,g)}. \tag{42}$$

Any nonnegative measurable $\tau_\psi$-invariant function is $\mu_\chi \otimes m_H$-a.e. constant. Hence, for any nonnegative measurable function $f$ and $\lambda_\chi$-a.e. $(x,g) \in X \times G$, we have

$$\mathbb{E}_{h\lambda_\chi}[f \mid \mathfrak{J}_\psi](x,g) = \frac{\int_{X \times H} f(y, \gamma g) h(y, \gamma g)\, d\gamma\, \mu_\chi(dy)}{\int_{X \times H} h(y, \gamma g)\, d\gamma\, \mu_\chi(dy)}.$$

From (42) it follows that

$$\begin{aligned} M^h f(x,g) &= \mathbb{E}_{h\lambda_\chi}[hf \mid \mathfrak{J}_\varphi](x,g) \\ &= \frac{\int_X \int_H f(y, u(y)\gamma(u(x))^{-1}g)\chi(u(y))\, d\gamma\, \mu_\chi(dy)}{\int_X \int_H h(y, u(y)\gamma(u(x))^{-1}g)\chi(u(y))\, d\gamma\, \mu_\chi(dy)}. \end{aligned}$$

(iii) If there exists some $x$ such that $\mu_x \sim \mu_\chi$, then the reduction of the cocycle given by (8) is "global" $\mu_\chi$-a.e.: there exists a measurable function $u$ and a closed subgroup $H$ such that the cocycle is cohomologous to an ergodic cocycle with values in $H$ and it is regular.

If there are a countable number of different equivalence classes among the measures $\mu_x$, $x \in X$, then by (i), for $\mu_\chi$-a.e. $x$, all the measures $\mu_x$ are equivalent and this equivalence class is that of $\mu_\chi$.

The last assertion of (iii) follows from assertion (iii) of Proposition 2.2.4. ∎

## 5. ON THE EQUATION $H_{\tau x} = \varphi(x) H_x (\varphi(x))^{-1}$

In Theorem 2.1.3 we encounter a measurable family of subgroups $H_x$ such that the following conjugacy equation holds:

$$H_{\tau x} = \varphi(x) H_x (\varphi(x))^{-1} \quad \text{for } \mu_\chi\text{-a.e. } x \in X. \tag{43}$$

For this conjugacy problem, see [GoSi99].

**5.1. Nilpotent groups.** When $G$ is a nilpotent connected Lie group, the subgroups $H_x$ are conjugate to a fixed subgroup $H$.

THEOREM 5.1.1 ([GoSi99]). *Assume $G$ is a nilpotent connected Lie group. If $(H_x)$ is a measurable family of subgroups such that* (43) *holds $\mu$-a.e., where $\mu$ is a $\sigma$-finite measure which is quasi-invariant and ergodic for*

*$\tau$, then there is a fixed closed subgroup $H$ and a measurable map $x \mapsto a(x)$ from $X$ into $G$ such that for $\mu_{\mathcal{X}}$-a.e. $x \in X$,*

$$H_x = a(x)H(a(x))^{-1}.$$

*Proof.* We equip the set $\mathcal{F}(G)$ of closed subsets of $G$ with Chabauty's topology (cf. Section 2).

We know that the map $x \in X \mapsto H_x \in \mathcal{F}(G)$ is measurable. For any $F \in \mathcal{F}(G)$, we have

$$\{x \in X : \overline{\{gH_xg^{-1} : g \in G\}} \subset F\} = \Big\{x \in X : H_x \subset \bigcap_{g\in G} g^{-1}Fg\Big\}.$$

It follows that the map $x \in X \mapsto \overline{\{gH_xg^{-1} : g \in G\}} \in \mathcal{F}(G)$ is measurable. We denote by $\mathfrak{g}$ the Lie algebra of $G$ and denote by ad the adjoint representation of $\mathfrak{g}$ (i.e. for any $(X,Y) \in \mathfrak{g}^2, (\operatorname{ad} X)(Y) = [X,Y]$). We denote by $\exp : \mathfrak{g} \to G$ the exponential map and by Ad the adjoint representation of $G$ on $\mathfrak{g}$. We have

$$g \exp X g^{-1} = \exp(\operatorname{Ad} g(X)), \quad \forall g \in G, \forall X \in \mathfrak{g},$$
$$\operatorname{Ad}(\exp Y) = \operatorname{Exp}(\operatorname{ad} Y) = \sum_{k\in\mathbb{N}} \frac{(\operatorname{ad} Y)^k}{k!}, \quad \forall Y \in \mathfrak{g}.$$

*First case.* Assume that $G$ is a connected and simply connected nilpotent Lie group. For $\mu$-a.e. $x \in X$, we have

$$\overline{\{gH_xg^{-1} : g \in G\}} = \overline{\{gH_{\tau x}g^{-1} : g \in G\}}.$$

Since the points of $\mathcal{F}(G)$ are separated by a countable family of continuous functions, there exists a closed subgroup $H$ of $G$ such that, for $\mu$-a.e. $x \in X$,

$$\overline{\{gH_xg^{-1} : g \in G\}} = \overline{\{gHg^{-1} : g \in G\}}.$$

Now, from the proposition below, this equality implies that the two open dense subsets $\{gH_xg^{-1} : g \in G\}$ and $\{gHg^{-1} : g \in G\}$ of $\overline{\{gHg^{-1} : g \in G\}}$ are not disjoint. Therefore the two $G$-orbits coincide. Hence the result.

*Second case.* Assume that $G$ is a connected nilpotent Lie group. Let $f : \tilde{G} \to G$ be a group cover of $G$ with $\tilde{G}$ connected and simply connected (see [Ho65, Ch. IV, Theorems 2.2 and 3.2]).

If $H$ is a closed subgroup of $G$ then $\tilde{H} = f^{-1}(H)$ is a closed subgroup of $\tilde{G}$. Moreover, the $G$-orbit of $H$ is the image under $f$ of the $\tilde{G}$-orbit of $\tilde{H}$. The theorem follows from the first case. ∎

PROPOSITION 5.1.2. *For any closed subgroup $H$ of a connected simply connected nilpotent Lie group $G$, the $G$-orbit $\{gHg^{-1} : g \in G\}$ of $H$ is open in its closure.*

*Proof.* We know that the exponential map exp is an analytic diffeomorphism. We set $\Sigma = \{1, \ldots, \dim(\mathfrak{g})\}$. For any $p \in \Sigma$, we consider the exterior product $V_p = \bigwedge_p \mathfrak{g}$ and the corresponding projective space $\mathbf{P}(V_p)$. We denote by $\pi_p$ the natural map from $V_p \setminus \{0\}$ onto $\mathbf{P}(V_p)$.

To each $p$-dimensional subspace $\mathfrak{v}$ of $\mathfrak{g}$ we associate the element $u_{\mathfrak{v}} = \pi_p(u_1 \wedge \cdots \wedge u_p)$ of $\mathbf{P}(V_p)$ where $(u_1, \ldots, u_p)$ is a linear basis of $\mathfrak{v}$. We denote by $\mathcal{D}_p$ the image in $\mathbf{P}(V_p)$ of the set of $p$-dimensional subspaces of $\mathfrak{g}$. We consider the disjoint union $\bigcup_{p\in\Sigma} \mathcal{D}_p$ equipped with the following topology. A sequence $(u_n)_{n\in\mathbb{N}}$ converges to $x$ if the following two properties are satisfied:

- there exist $N \in \mathbb{N}$ and $r \in \Sigma$ such that $u_n \in \mathcal{D}_r$ for $n \geq N$,
- the sequence $(u_n)_{n\geq N}$ converges to $x$ on $\mathcal{D}_r$ for the usual induced topology of $\mathbf{P}(V_p)$.

One easily sees that a sequence $(\mathfrak{v}_n)$ of subspaces of $\mathfrak{g}$ converges in Chabauty's topology if and only if $(u_{\mathfrak{v}_n})_{n\in\mathbb{N}}$ converges in $\bigcup_{p\in\Sigma} \mathcal{D}_p$. Hence, the map $\mathfrak{v} \mapsto u_{\mathfrak{v}}$ is a homeomorphism from the set of nontrivial subspaces of $\mathfrak{g}$ onto $\bigcup_{p\in\Sigma} \mathcal{D}_p$.

Let $H$ be a closed subgroup of $G$ with Lie algebra $\mathfrak{h} = \exp^{-1}(H)$. The $G$-orbit $\{gHg^{-1} : g \in G\}$ of $H$ is identified with the $\bigwedge_p \operatorname{Ad} G$-orbit of $u_{\mathfrak{h}}$. Now, for a connected simply connected nilpotent Lie group $G$, we know (see for example [BoSe64]) that this orbit is open in its closure. This yields the result. ∎

**5.2. A counterexample.** Let $G$ be the semidirect product of $\mathbb{R}$ and $\mathbb{C}^2$, with the composition law

$$(t, z_1, z_2) * (t', z_1', z_2') = (t + t', z_1 + e^{2\pi it} z_1', z_2 + e^{2\pi\theta it} z_2'),$$

where $\theta$ is a fixed irrational.

The conjugate in $G$ of $(0, z_1, z_2)$ by $a = (s, v_1, v_2)$ is

$$(s, v_1, v_2)(0, z_1, z_2)(s, v_1, v_2)^{-1} = (0, e^{2\pi is} z_1, e^{2\pi\theta is} z_2). \tag{44}$$

Consider the dynamical system defined by an irrational rotation ($\tau : x \to x + \alpha \bmod 1$) on $X = \mathbb{R}/\mathbb{Z}$. Let $\Phi : X \to G$ be the cocycle defined by $\Phi(x) = (\varphi(x), 0, 0)$, where $\varphi$ has its values in $\mathbb{Z}$.

Let $H_x := \{(0, vz_1, ve^{2\pi i\psi(x)} z_2) : v \in \mathbb{R}\}$, where $\psi$ is a function to be defined and $z_1$, $z_2$ are given natural real numbers. Consider the function $x \mapsto H_x$ with values in the set of closed subgroups of $G$. It satisfies the conjugacy relation

$$H_{\tau x} = \Phi(x) H_x (\Phi(x))^{-1} \tag{45}$$

if $\varphi$ has integral values and satisfies

$$\theta\varphi(x) + \psi(x) = \psi(\tau x) \bmod 1. \tag{46}$$

For every $\alpha$ whose partial quotients are not bounded, there are real numbers $\beta$ and $r$ for which the function

$$\varphi := 1_{[0,\beta]} - 1_{[0,\beta]}(. + r)$$

is not a coboundary and there are irrational values of $s$ such that $e^{2\pi i s(1_{[0,\beta]} - 1_{[0,\beta]}(\cdot + r))}$ is a multiplicative coboundary (cf. [Co07]).

If we take for $\theta$ one of these values of $s$ and for $\psi$ a function satisfying the multiplicative coboundary equation $e^{2\pi i \theta\varphi} = e^{2\pi i(\psi\circ\tau - \psi)}$, we get (46).

PROPOSITION 5.2.1. *For these choices of $\beta$, $r$, $\theta$, $\psi$, there is no subgroup $H$ such that the equation $H_x = a(x)H(a(x))^{-1}$ has a measurable solution $a$.*

*Proof.* Suppose that there are a fixed subgroup $H$ and a measurable function $a : X \to G$ such that $H_x = a(x)H(a(x))^{-1}$.

According to (44), this is equivalent to the existence of a function $t$ defined on $X$ such that the set

$$\{(0, ve^{2\pi i t(x)} z_1, ve^{2\pi i(\theta t(x) + \psi(x))} z_2) : v \in \mathbb{R}\}$$

does not depend on $x$. This implies that $t$ and $\psi + \theta t$ have a fixed value mod 1; therefore $\theta(\varphi(x) - t(x) + t(\tau x)) = \theta\varphi(x) + \psi(x) - \psi(\tau x) \bmod 1 = 0$. As $\varphi$ and $t - t\circ\tau$ have integral values and $\theta$ is irrational, it follows that $\varphi = t\circ\tau - t$, contrary to the fact that $\varphi$ is not a coboundary. ∎

REMARK 5.2.2. By the same arguments it can be shown that the cocycle $1_{[0,\beta]} - 1_{[0,\beta]}(. + r))$ is nonregular in the sense of Definition 2.2.1.

## 6. COMMENTS

**6.1. Remarks on transience/recurrence.** The cocycle $(\varphi_n)_{n\in\mathbb{Z}}$ gives the position at time $n$ of the "vertical" coordinate for the iterates $\tau_\varphi^n$. If it is *recurrent* (i.e. if the stationary random walk $(\varphi_n)$ returns infinitely often to any neighborhood of the identity element), the transformation $\tau_\varphi$ is conservative.

The ergodicity of the basis implies that the cocycle is either recurrent or transient. When $\varphi$ has its values in $\mathbb{R}$ and is integrable, $(\varphi_n)_{n\in\mathbb{Z}}$ is recurrent if and only if $\mu(\varphi) = 0$.

For every amenable group $G$ and every ergodic system $(X, \mu, \tau)$, there is a measurable ergodic cocycle $(\varphi, \tau)$ over the system, taking its values in $G$, such that $(X \times G, \mu \otimes m_G, \tau_\varphi)$ is ergodic (cf. [He79], [GoSi85]). However, a problem is to construct explicitly recurrent cocycles generated by regular functions over particular dynamical systems and to find whether or not they are ergodic.

In the recurrent case, the transformation $\tau_\varphi$ is *conservative*: there is no wandering set $E$ with a positive measure (wandering means that the images

$(\tau_\varphi^{-k}E, k \in \mathbb{Z})$ are pairwise disjoint). This implies that every subinvariant set is invariant $\mu \otimes m_G$-a.e.

Note that the ergodic decomposition of a recurrent system gives recurrent systems. In particular, the recurrence of the cocycle $(\varphi_n)$ relative to $\mu$ implies (with the notations of 2.1.3) that for $\mu$-a.e. $x$ the cocycle $(\varphi_n)$ is recurrent relative to the measure $\mu_x$, which is infinite if the cocycle is not regular (cf. Theorem 2.2.2).

Assume now that the cocycle is transient. Let $E$ be a wandering set and $h > 0$ be a function on $G$ such that $\int h\, dg = 1$. The series $\sum_{k\in\mathbb{Z}} h(\varphi_k(x)g)$ converges for $\mu \otimes m_G$-a.e. $(x, g)$, according to

$$\int_E \sum_{k\in\mathbb{Z}} h(\varphi_k(x)g)\, d\mu(x)\, dg = \int_E h(g)\Big(\sum_{k\in\mathbb{Z}} 1_E(x, (\varphi_k(x))^{-1}g)\Big)\, dg\, d\mu(x)$$
$$= \int_E h(g)\Big(\sum_{k\in\mathbb{Z}} 1_E(\tau^{-k}x, (\varphi_k(\tau^{-k}x))^{-1}g)\Big)\, d\mu(x)\, dg = \int h(g)\, dg = 1.$$

The function $\tilde{h}(x, g) := \sum_{k\in\mathbb{Z}} h(\varphi_k(x)g)$ is therefore $\tau_\varphi$-invariant and finite for $\mu \otimes m_G$-a.e. $(x, g)$.

The subgroups $H_x$ defined in Theorem 2.1.3 reduce to $\{e\}$, and the ergodic measures are given, up to a multiplicative factor, by $\lambda_{(x,g)}(f) = \sum_{k\in\mathbb{Z}} f(\tau^k x, \varphi_k(x)g)$.

The function $\varphi$ is a coboundary with respect to the $\sigma$-finite measure $\tilde{\mu}_x(dy) := \sum_{k\in\mathbb{Z}} \delta_{\tau^k x}(dy)$ (we get $u_x(y)\varphi(y)(u_x(\tau y))^{-1} = e$ by setting $u_x(y) := \varphi_k(x)$ at the point $y = \tau^k x$). The ergodic decomposition of $\mu(dx) \times dg$ can be written

$$\int_X \int_G f(x, g)\, d\mu(x)\, dg = \int_X \int_G \Big[(\tilde{h}(x, g))^{-1} \sum_{k\in\mathbb{Z}} f(\tau^k x, \varphi_k(x)g)\Big] h(g)\, d\mu(x)\, dg.$$

This shows that, in the transient case, there is no interesting information in the ergodic decomposition. Therefore it is convenient to have examples of recurrent cocycles $(\varphi, \tau)$. A family of such cocyles is provided when the basic system is a rotation on the circle and $\varphi$ is a BV-function with values in $\mathbb{R}^d$. There are also examples over rotations for cocycles taking their values in nilpotent groups (see [Gr05], [Co07]). For rotations, using BV-functions, one can construct conformal probability measures $\mu_\chi$ for which the rotation is conservative. More precisely, if $\varphi$ is a BV-function on the circle with zero integral and $\chi$ is an exponential on $\mathbb{R}$, and if $\tau$ is an ergodic rotation, then there is a unique probability measure $\mu_\chi$ on the circle such that $d(\tau\mu_\chi)/d\mu_\chi = \chi \circ \varphi$ and the corresponding measure $\lambda_\chi$ on $X \times G$ is conservative due to Koksma's inequality (see for example [CoGu00]).

**6.2. Extension to a group action.** For simplicity, we have restricted the paper to the framework of a single transformation, but the domain of validity can be extended by taking more generally the action of a countable group $\Gamma$. This gives access to more examples of transient cocycles with a nontrivial ergodic decomposition. Most of the results presented here when $\Gamma = \mathbb{Z}$ are still valid for the action of a countable group $\Gamma$.

Indeed, we can use the result of Theorem 3.1.1, since one can easily extend it from the case of a single invertible transformation to an ergodic group action. Another important point is the ergodicity of the measures given by a regular conditional probability with respect to the $\sigma$-algebra of invariant sets. This point also remains valid (see Remark 7.3.3 at the end of 7.3).

## 7. APPENDIX

In this appendix, we recall a selection lemma and some results on the conditional expectation and the ergodic decomposition that were used in the previous sections.

**7.1. A selection lemma.** Let $G$ be a lcsc group. Recall that the set $\mathcal{F}(G)$ of closed subsets of $G$ is equipped with *Chabauty's topology*, for which the open sets are defined by

$$U(\mathcal{O}, C) = \{S \in \mathcal{F}(G) : \forall U \ \in \mathcal{O},\ S \cap U \neq \emptyset \text{ and } S \cap C = \emptyset\},$$

where $\mathcal{O}$ is a finite family of open sets of $G$, and $C$ is a compact subset of $G$.

The Borel structure associated to this topology is generated by the sets $\{S \in \mathcal{F}(G) : S \subseteq F\}$ where $F \in \mathcal{F}(G)$ (cf. 2.1). For the sake of completeness, we give a proof of a selection lemma (cf. the theorem of Kuratowski and Ryll-Nardzewski) that was used in Section 3:

LEMMA 7.1.1. *If $t \mapsto F_t$ is a Borel map from a Borel space $(T, \mathcal{T})$ to $\mathcal{F}(G)$, then there exists a Borel map $f$ from $T$ to $G$ such that $f(t) \in F_t$ for each $t \in T$.*

*Proof.* Let $K$ be a compact set in $G$. Assume that $F_t \subset K$ for $t \in R \subset T$, where $R$ is a Borel set in $T$. For every $n \geq 1$, there exists a finite family $(K_{n,i}, i \in I_n)$ of compact sets such that $\operatorname{diam}(K_{n,i}) < 1/n$ and $K \subset \bigcup_{j \in I_{n+1}} K_{n+1,j}$.

For a compact set $C$ in $G$, the set $\{t : F_t \cap C \neq \emptyset\}$ is Borel (its complement is the union of the sets $\{t : F_t \subset G \setminus U_n\}$, where $U_n$ is a basis of open neighborhoods of $C$). Therefore, for every $n$ and $j$, the set $\{t : F_t \cap K_{n,j} \neq \emptyset\}$ is Borel.

We define $i_n(t)$ by $i_n(t) = \inf\{j \in I_n : F_t \cap K_{n,j} \neq \emptyset\}$. The map $t \mapsto K_{n,i_n(t)}$ is Borel.

Now we define the point $f(t)$ for $t \in R$ by

$$f(t) := \bigcap_{n\geq 1} K_{n,i_n(t)}.$$

From the condition on the diameters and the compactness of the sets, it follows that $f(t)$ is well defined for every $t \in T$.

We have to show that $f$ is Borel, that is, $\{t \in T : f(t) \in C\}$ is a Borel set for any closed subset $C$ in $G$. Let $(O_k)$ be a decreasing sequence of open sets such that $O_{k+1} \subset \bar{O}_{k+1} \subset O_k$ for every $k$ and $C = \bigcap_k O_k$. We have

$$f(t) \in C \;\Leftrightarrow\; \bigcap_{n\geq 1} K_{n,i_n(t)} \subset O_k, \forall k \geq 1 \;\Leftrightarrow\; \bigcap_{n\geq 1} K_{n,i_n(t)} \subset \bar{O}_k,\ \forall k \geq 1.$$

As $\{t \in T : K_{n,i_n(t)} \subset \bar{O}_k\}$ is Borel for each $k$, the assertion follows.

Now we construct $f$ on the whole space. For any compact set $K$ in $G$, the map $t \mapsto F_t \cap K$ is Borel, since the map $(F, K) \mapsto F \cap K$ from $\mathcal{F}(G)$ into itself is continuous for a fixed compact set $K$. Let $K_j$ be an increasing sequence of compact sets in $G$ such that $G = \bigcup_j K_j$.

We define $f(t)$ by applying the previous construction to the map $t \mapsto F_t \cap K_1$ on $\{t : F_t \cap K_1 \neq \emptyset\}$, then to $t \mapsto F_t \cap K_2$ on $\{t : F_t \cap K_2 \neq \emptyset\} \cap \{t : F_t \cap K_1 = \emptyset\}$, and so on. ■

## 7.2. A lemma on conditional expectation

LEMMA 7.2.1. *Let $\mathbb{P}$ be a probability measure on a measurable space $(E, \mathcal{F})$ and $h$ a measurable positive function such that $\int h\, d\mathbb{P} = 1$. Then, for every sub-$\sigma$-algebra $\mathcal{B}$ of $\mathcal{F}$ and every measurable nonnegative* (*or $h\mathbb{P}$-integrable*) *function $f$, we have*

$$\mathbb{E}_{h\mathbb{P}}[f \mid \mathcal{B}] = \mathbb{E}_{\mathbb{P}}[fh \mid \mathcal{B}]/\mathbb{E}_{\mathbb{P}}[h \mid \mathcal{B}] \quad \mathbb{P}\text{-}a.e. \tag{47}$$

*If $\theta$ is a bijective bi-measurable map from $E$ onto itself such that $\theta\mathbb{P} \sim \mathbb{P}$, then, for any $\mathbb{P}$-integrable function $f$, we have*

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[f \mid \mathcal{B}] &= \mathbb{E}_{\mathbb{P}}\Bigg[\bigg(\frac{d\theta\mathbb{P}}{d\mathbb{P}}\bigg)^{-1} \circ \theta \,\Bigg|\, \mathcal{B}\Bigg] \mathbb{E}_{\mathbb{P}}\Bigg[f \circ \theta^{-1}\, \frac{d\theta\mathbb{P}}{d\mathbb{P}} \,\Bigg|\, \theta\mathfrak{B}\Bigg] \circ \theta \\ &= \frac{\mathbb{E}_{\mathbb{P}}\big[f \circ \theta^{-1}\, \frac{d\theta\mathbb{P}}{d\mathbb{P}} \mid \theta\mathfrak{B}\big] \circ \theta}{\mathbb{E}_{\mathbb{P}}\big[\frac{d\theta\mathbb{P}}{d\mathbb{P}} \mid \theta\mathfrak{B}\big] \circ \theta}.\end{aligned} \tag{48}$$

*If $\mathbb{P} = h\lambda$, where $\lambda$ is a $\sigma$-finite $\theta$-invariant measure and $\mathcal{B} = \mathfrak{J}$ the $\sigma$-algebra of $\theta$-invariant sets in $\mathfrak{E}$, we have*

$$\mathbb{E}_{h\lambda}[f \circ \theta \mid \mathfrak{J}] = \mathbb{E}_{h\lambda}\bigg[f\, \frac{h \circ \theta^{-1}}{h} \,\bigg|\, \mathfrak{J}\bigg]. \tag{49}$$

*Proof.* We prove only the second assertion. For every bounded $\mathcal{B}$-measurable $\psi$ we have

$$\begin{aligned}
\int \mathbb{E}_{\mathbb{P}}[f\circ\theta \,|\, \mathcal{B}]\psi \, d\mathbb{P} &= \int_E f\circ\theta\,\psi\, d\mathbb{P} = \int_E f\,\psi\circ\theta^{-1}\,\frac{d\theta\mathbb{P}}{d\mathbb{P}}\, d\mathbb{P} \\
&= \int_E \mathbb{E}_{\mathbb{P}}\left[f\,\frac{d\theta\mathbb{P}}{d\mathbb{P}}\,\middle|\,\theta\mathcal{B}\right]\psi\circ\theta^{-1}\, d\mathbb{P} = \int_E \frac{d\theta^{-1}\mathbb{P}}{d\mathbb{P}}\,\mathbb{E}_{\mathbb{P}}\left[f\,\frac{d\theta\mathbb{P}}{d\mathbb{P}}\,\middle|\,\theta\mathfrak{B}\right]\circ\theta\,\psi\, d\mathbb{P} \\
&= \int_E \mathbb{E}_{\mathbb{P}}\left[\frac{d\theta^{-1}\mathbb{P}}{d\mathbb{P}}\,\middle|\,\mathcal{B}\right]\mathbb{E}_{\mathbb{P}}\left[f\,\frac{d\theta\mathbb{P}}{d\mathbb{P}}\,\middle|\,\theta\mathfrak{B}\right]\circ\theta\,\psi\, d\mathbb{P},
\end{aligned}$$

which implies (48):

$$\mathbb{E}_{\mathbb{P}}[f\circ\theta \,|\, \mathcal{B}] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\theta^{-1}\mathbb{P}}{d\mathbb{P}}\,\middle|\,\mathcal{B}\right]\mathbb{E}_{\mathbb{P}}\left[\frac{d\theta\mathbb{P}}{d\mathbb{P}}f\,\middle|\,\theta\mathfrak{B}\right]\circ\theta. \blacksquare$$

### 7.3. Regular conditional probability

Definition 7.3.1. Let $(E, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathfrak{B}$ a sub-$\sigma$-algebra of $\mathcal{F}$. *A regular conditional probability* relative to $\mathcal{B}$ and $\mathbb{P}$ is a map $P$ from $E \times \mathcal{F}$ to $[0, 1]$ such that

- For every $x \in E$, $P(x, \cdot)$ is a probability measure on $\mathcal{F}$.
- For every $A \in \mathcal{F}$, the map $x \in E \mapsto P(x, A)$ is a version of the conditional expectation of $1_A$ with respect to the $\sigma$-algebra $\mathcal{B}$. This map is thus $\mathcal{B}$-measurable and satisfies, for every $\mathfrak{B}$-measurable function $\varphi$,

$$\int_E 1_A(x)\varphi(x)\,\mathbb{P}(dx) = \int_E P(x, A)\varphi(x)\,\mathbb{P}(dx).$$

For every $\mathcal{F}$-measurable nonnegative or bounded function $f$, $Pf$ defined by $Pf(x) := \int_E f(y)\,P(x, dy)$ is then a version of the conditional expectation of $f$ with respect to $\mathcal{B}$.

For the existence of a regular conditional probability, we can refer to the general setting used in Neveu's book ( [Ne64, Corollaire, Proposition V-4-4]):

*In the following, we will assume that there exists an approximating compact class in* $(E, \mathcal{F}, \mathbb{P})$ (*see* [Ne64] *for this notion*) *and that* $\mathcal{F}$ *is generated by a countable family.*

Theorem 7.3.2 ([Ne64]). *For every $\sigma$-algebra $\mathcal{B}$ in $\mathcal{F}$, there exists a regular conditional probability with respect to $\mathcal{B}$.*

This result applied to the product space $(X \times G, \mathfrak{X} \times \mathcal{B}_G)$, the probability $h\lambda$ on $X \times G$, where $h > 0$ on $X \times G$ is such that $\int h(x, g)\,\mu_\chi(dx)\,\chi(g)\,m_G(dg) = 1$, and the sub-$\sigma$-algebra $\mathfrak{J}$ of $\tau_\varphi$-invariant sets (see Notations 2.1.1 and 2.1.2) gives the regular conditional probability $P^h$ used in Section 2.

Now we have to show that the probability measures $P^h((x,g),\,\cdot\,)$ are $\tau_\varphi$-ergodic. For the action of a single transformation, this can be done by applying the ergodic theorem (cf. [Aa97]). For the sake of completeness we give a proof in the last subsection below.

Remark 7.3.3. When the action of a single transformation $\tau$ on $X$ is replaced by the Borel action of a countable group, the proof of the ergodicity of $P^h((x,g),\,\cdot\,)$ is more difficult. A reference is [GrSc00].

## 7.4. Ergodic theorem and ergodic decomposition

Notations 7.4.1. Let $\theta$ be a bijective bi-measurable transformation on a measurable space $(E,\mathcal{F})$, $\mu$ a positive $\sigma$-finite $\theta$-quasi-invariant measure, and $\mathfrak{J} := \{B \in \mathcal{F} : \theta^{-1}B = B\}$.

Let $h$ be a measurable function on $E$ such that $h(x) > 0$ and $\mu(h) = 1$. Let $P^h$ be a regular conditional probability with respect to the probability measure $h\mu$ and the $\sigma$-algebra $\mathfrak{J}$ of $\theta$-invariant measurable subsets.

Let $T_h$ be the contraction of $\mathbb{L}^1(E,\mathcal{F},h\mu)$, in duality with the operator of composition with $\theta$ acting on $\mathbb{L}^\infty(E,\mathcal{F},\mu)$, defined by

$$T_h f(x) = f\circ\theta^{-1}(x)\,\frac{d(\theta(h\mu))}{d(h\mu)}(x).$$

Replacing $\theta$ by $\theta^{-1}$ we get the inverse operator $T_h^{-1}$.

Proposition 7.4.2. *For every $f \in \mathbb{L}^1(E,\mathcal{F},h\mu)$ and $\mu$-a.e. $x \in E$,*

$$\sum_{k=-n}^{n} T_h^k f(x) / \sum_{k=-n}^{n} T_h^k 1(x) \xrightarrow{n\to+\infty} \mathbb{E}_{h\mu}[f \mid \mathfrak{J}](x).$$

*Proof.* Applying Hurewicz's ergodic theorem to the contraction $T_h$, we find that the sequence

$$\Big(\sum_{k=0}^{n} T_h^k f(x) / \sum_{k=0}^{n} T_h^k 1(x)\Big)_{n\ge 1}$$

converges $\mu$-a.e., and the same result holds for the contraction $T_h^{-1}$.

On the conservative part $C$ the limit in both directions is equal to $\mathbb{E}_{h\mu}[f \mid \mathfrak{J}](x)$, so that on $C$,

$$\lim_{n\to+\infty}\Big(\sum_{k=-n}^{n} T_h^k f / \sum_{k=-n}^{n} T_h^k 1\Big) = \mathbb{E}_{h\mu}[f \mid \mathfrak{J}] \quad \mu\text{-a.e.}$$

On the dissipative part $D$, the limit is the quotient of the series. For $j,k$ in $\mathbb{Z}$, we have

$$T_h^j f\circ\theta^k = T_h^{j-k} f\,\frac{d(h\mu)}{d(\theta^{-k}(h\mu))} = \frac{T_h^{j-k} f}{T_h^{-k}1}.$$

This implies that on $D$ the quotient of the series is a $\theta$-invariant function and, for every measurable $\theta$-invariant function $\varphi$ which is null on $C$,

$$\int_E \frac{\sum_{k\in\mathbb{Z}} T_h^k f}{\sum_{j\in\mathbb{Z}} T_h^j 1}\,\varphi\,d(h\mu) = \sum_{k\in\mathbb{Z}} \int_E \frac{T_h^k f}{\sum_{j\in\mathbb{Z}} T_h^j 1}\,\varphi\,d(h\mu) = \sum_{k\in\mathbb{Z}} \int_E \frac{f\varphi}{\sum_{j\in\mathbb{Z}} T_h^j 1\circ\theta^k}\,d(h\mu)$$
$$= \sum_{k\in\mathbb{Z}} \int_E \frac{f\varphi}{\sum_{j\in\mathbb{Z}} T_h^j 1}\,T_h^{-k}1\,d(h\mu) = \int_E f\varphi\,d(h\mu).$$

On $D$ the quotient of the series is therefore equal to $\mathbb{E}_{h\mu}[f\,|\,\mathfrak{J}]$. ■

LEMMA 7.4.3. *For $\mu$-a.e. $x \in E$, the measure $\theta P^h(x,\cdot)$ is absolutely continuous with respect to $P^h(x,\cdot)$ and*

$$\frac{d(\theta P^h(x,\cdot))}{dP^h(x,\cdot)} = \frac{d(\theta(h\mu))}{d(h\mu)} = \frac{h\circ\theta^{-1}}{h}\,\frac{d(\theta\mu)}{d\mu}. \tag{50}$$

*Proof.* For a positive $\mathcal{F}$-measurable $f$ and a $\mathfrak{J}$-measurable positive function $\varphi$, we have

$$\int_E f\circ\theta\,\varphi\,d(h\mu) = \int_E f\circ\theta\,\varphi\circ\theta\,d(h\mu) = \int_E f\varphi\,\frac{d(\theta(h\mu))}{d(h\mu)}\,d(h\mu).$$

This shows, $\mu$-a.e.,

$$\theta(P^h)(f) = P^h(f\circ\theta) = \mathbb{E}_{h\mu}[f\circ\theta\,|\,\mathfrak{J}]$$
$$= \mathbb{E}_{h\mu}\bigg[f\,\frac{d(\theta(h\mu))}{d(h\mu)}\,\bigg|\,\mathfrak{J}\bigg] = P^h\bigg(f\,\frac{d(\theta(h\,\mu))}{d(h\,\mu)}\bigg). \ ■$$

We then have:

COROLLARY 7.4.4. *For the elements $x \in E$ for which* (50) *holds, $T_h$ is a positive contraction of $\mathbb{L}^1(E,\mathcal{F},P^h(x,\cdot))$. For every $f \in \mathbb{L}^1(E,\mathcal{F},P^h(x,\cdot))$ and $P^h(x,\cdot)$-a.e. $y \in E$,*

$$\sum_{k=-n}^{n} T_h^k f(y)\Big/\sum_{k=-n}^{n} T_h^k 1(y) \xrightarrow{n\to+\infty} \mathbb{E}_{P^h(x,\cdot)}[f\,|\,\mathfrak{J}](y).$$

THEOREM 7.4.5. *A decomposition of the measure $\mu$ into ergodic components is given by*

$$\mu(dy) = \int_E (h(\cdot))^{-1} P^h(x,\cdot)h(x)\,\mu(dx). \tag{51}$$

*Proof.* The equality is clear. It remains to prove the ergodicity of the probability measures $P^h(x,\cdot)$ for $\mu$-a.e. $x \in E$.

From (51) and Proposition 7.4.2, we have, for every $f \in \mathbb{L}^1(E,\mathcal{F},h\mu)$ and $\mu$-a.e. $x \in E$,

$$\sum_{k=-n}^{n} T_h^k f(y)\Big/\sum_{k=-n}^{n} T_h^k 1(y) \xrightarrow{n\to+\infty} \mathbb{E}_{h\mu}[f\,|\,\mathfrak{J}](y) = P^h f(y)$$

for $P(x,\cdot)$-a.e. $y \in E$. The functions $g = P^h f$ and $g^2 = (P^h f)^2$ are $\mathfrak{J}$-measurable and therefore $P^h$-invariant $\mu$-a.e.: $P^h g(x) = g(x)$ and $P^h g^2(x) = g^2(x)$ for $\mu$-a.e. $x \in E$. By the Cauchy–Schwarz inequality, this implies $g(y) = g(x)$ for $P(x,\cdot)$-a.e. $y \in E$.

Let $\mathcal{F}_0$ be a countable Boole algebra which generates $\mathcal{F}$. For $x \in E$, let $Q_x^h$ be a regular conditional probability with respect to the probability measure $P^h(x,\cdot)$ and the $\sigma$-algebra $\mathfrak{J}$. From the previous property and Corollary 7.4.4, we obtain, for $\mu$-a.e. $x \in E$ and $P^h(x,\cdot)$-a.e. $y \in E$,

$$\forall A \in \mathcal{F}_0, \qquad Q_x^h(y, A) = P^h(x, A),$$

and consequently, for $\mu$-a.e. $x \in E$ and $P^h(x,\cdot)$-a.e. $y \in E$, we have the same property for every $A \in \mathcal{F}$.

For every $I \in \mathfrak{J}$, we know that

$$Q_x^h(y, I) = \mathbb{E}_{P^h(x,\cdot)}[1_I \mid \mathfrak{J}](y) = 1_I(y) \qquad \text{for } P(x,\cdot)\text{-a.e. } y \in E,$$

and therefore as above

$$Q_x^h(y, I) = 1_I(x) \qquad \text{for } P(x,\cdot)\text{-a.e. } y \in E.$$

It follows that, for $\mu$-a.e. $x \in E$,

$$\forall I \in \mathfrak{J}, \qquad P^h(x, I) = Q_x^h(y, I) = 1_I(y) \qquad \text{for } P(x,\cdot)\text{-a.e. } y \in E.$$

This implies the ergodicity of the measures $P^h(x,\cdot)$ for $\mu$-a.e. $x$. ■

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