ON THE ERGODIC DECOMPOSITION FOR A COCYCLE

BY

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Abstract. Let $(X,\mathcal{X},\mu,\tau)$ be an ergodic dynamical system and $\varphi$ be a measurable map from $X$ to a locally compact second countable group $G$ with left Haar measure $m_G$. We consider the map $\tau_\varphi$ defined on $X \times G$ by $\tau_\varphi : (x,g) \mapsto (\tau x, \varphi(x)g)$ and the cocycle $(\varphi_n)_{n \in \mathbb{Z}}$ generated by $\varphi$.

Using a characterization of the ergodic invariant measures for $\tau_\varphi$, we give the form of the ergodic decomposition of $\mu(dx) \otimes m_G(dg)$ or more generally of the $\tau_\varphi$-invariant measures $\mu_\chi(dx) \otimes \chi(g)m_G(dg)$, where $\mu_\chi(dx)$ is $\chi \circ \varphi$-conformal for an exponential $\chi$ on $G$.

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1. INTRODUCTION

We consider a dynamical system \((X, \mathfrak{X}, \mu, \tau)\), where \((X, \mathfrak{X})\) is a standard Borel space, \(\mu\) a \(\sigma\)-finite measure on \(\mathfrak{X}\), and \(\tau\) an invertible measurable transformation on \(X\) such that \(\mu\) is quasi-invariant and ergodic for the action of \(\tau\).

Let \(G\) be a locally compact second countable (lcsc) group. We denote by \(\mathfrak{B}_G\) the \(\sigma\)-algebra of its Borel sets, \(m_G(dg)\) (or simply \(dg\)) a left Haar measure on \(G\), and \(e\) its identity element.

Let \(\varphi\) be a measurable function on \(X\) taking its values in \(G\) and \(\tau\varphi\) the map on \(X \times G\) (skew product) defined by
\[
(1) \quad \tau\varphi : (x, g) \mapsto (\tau x, \varphi(x)g).
\]
The corresponding \(G\)-valued cocycle \((\varphi_n)_{n \in \mathbb{Z}}\) over \((X, \mu, \tau)\) (denoted also \((\varphi, \tau)\)) is
\[
\varphi_n(x) = \begin{cases} 
\varphi(\tau^{n-1}x) \cdots \varphi(x) & \text{for } n > 0, \\
 e & \text{for } n = 0, \\
\varphi(\tau^n x)^{-1} \cdots \varphi(\tau^{-1}x)^{-1} & \text{for } n < 0.
\end{cases}
\]
If \(\mu\) is \(\tau\)-invariant, the map \(\tau\varphi\) leaves invariant the product measure \(\lambda_1 := \mu \otimes m_G\). The cocycle \((\varphi_n)\) can be seen as a stationary walk in \(G\) over the dynamical system \((X, \mu, \tau)\).

More generally, let \(\chi\) be an exponential on \(G\), i.e. a continuous map from \(G\) to \([0, +\infty[\) such that \(\chi(g g') = \chi(g) \chi(g')\) for all \(g, g' \in G\). If \(\mu_\chi\) is a \(\chi \circ \varphi\)-conformal \(\sigma\)-finite measure on \(X\), i.e. such that
\[
(2) \quad (\tau \mu_\chi)(dx) = \chi(\varphi(\tau^{-1}x)) \mu_\chi(dx),
\]
then the measure \(\lambda_\chi(dx, dg) := \mu_\chi(dx) \otimes \chi(g) m_G(dg)\) (sometimes called Maharam measure) is a \(\sigma\)-finite measure on \(X \times G\) which is \(\tau\varphi\)-invariant.

The study of cocycles was the subject of many papers since K. Schmidt ([Sc77]) and J. Feldman and C. C. Moore ([FeMo77]). There has recently been a new interest in the invariant measures for skew products (cf. [ANSS02], [Sa04], [LeSa07]).

Our main goal is to give the precise form of the ergodic decomposition (for the skew product \(\tau\varphi\)) of the measures \(\lambda_\chi\) on \(X \times G\). In the first section we give the statement of the results on this ergodic decomposition, then some consequences in terms of regularity, boundedness and essential values of the cocycle \((\varphi_n)_{n \in \mathbb{Z}}\). The following sections are devoted to the proof of the main results. We also discuss a conjugacy equation for the closed subgroups of \(G\) which arises in the ergodic decomposition. In the appendix, we recall and specify some results on ergodic decompositions and regular conditional probabilities.
2. STATEMENT OF THE MAIN RESULTS

2.1. Ergodic decomposition. Before we state the main results, we recall some facts about a topology on the set $\mathcal{F}(G)$ of closed subsets of $G$ and give some notations.

A topology on $\mathcal{F}(G)$. Let $G$ be a lcsc group. We equip the set $\mathcal{F}(G)$ of closed subsets of $G$ with the so-called Chabauty’s topology [Ch50]. In this topology the open sets are defined by

$$U(O, C) = \{ S \in \mathcal{F}(G) : \forall U \in O, S \cap U \neq \emptyset \text{ and } S \cap C = \emptyset \},$$

where $O$ is a finite family of open sets of $G$, and $C$ is a compact subset of $G$.

It can be shown that a sequence $(F_n)$ of closed subsets of $G$ converges to a closed subset $F$ in Chabauty’s topology if and only if the following two properties are satisfied:

- Let $\xi : \mathbb{N} \to \mathbb{N}$ be an increasing sequence and let $(g_n)_{n \in \mathbb{N}}$ be a sequence such that $g_n \in F_{\xi(n)}$ for every $n \geq 0$. If $(g_n)_{n \in \mathbb{N}}$ converges to $g \in G$, then the limit $g$ is in $F$.
- Each $g \in F$ is the limit of a sequence $(g_n)_{n \in \mathbb{N}}$ with $g_n \in F_n$ for every $n \geq 0$.

The Borel structure associated to this topology is generated by the sets

$$\{ S \in \mathcal{F}(G) : S \subseteq F \}$$

where $F \in \mathcal{F}(G)$. The lcsc group $G$ is metrizable. We denote by $d$ a metric on $G$ which defines the topology of $G$. For any dense sequence $(g_n)_{n \in \mathbb{N}}$ of elements of $G$, the family of continuous functions $\{d(g_n, \cdot), n \in \mathbb{N}\}$ separates the points of $\mathcal{F}(G)$ (see [AuMo66, Ch. II, Section 2]).

Notations

Notations 2.1.1. For a locally compact second countable group $H$, we denote by $m_H(d_\gamma)$ (or simply $d_\gamma$) a left Haar measure on the Borel sets of $H$, and by $\delta_u$ the Dirac measure at a point $u \in H$. The identity element is denoted by $e$.

If $\rho_1$ and $\rho_2$ are positive measures on the Borel subsets of $H$, we denote by $\rho_1 \ast \rho_2$ their convolution (i.e. the image of the product measure $\rho_1 \otimes \rho_2$ under the map $(g, g') \in H \times H \mapsto gg' \in H$).

As in the introduction, we consider a measurable map $\varphi$ from $X$ to $G$ and the skew product $\tau_{\varphi}$ defined by (1). Let $\lambda$ be a $\tau_{\varphi}$-quasi-invariant positive measure on $X \times G$. We denote by $\mathcal{J}$ or $\mathcal{J}_{\varphi}$ the $\sigma$-algebra of $\tau_{\varphi}$-invariant subsets. We are interested in the $\mathcal{X} \times \mathcal{B}_G$-measurable functions on $X \times G$ which are invariant under the map $\tau_{\varphi}$.

The following remark is useful. If $f$ is $\tau_{\varphi}$-invariant $\lambda$-a.e., then there is a $\tau_{\varphi}$-invariant function $g$ such that $f = g \lambda$-a.e. Therefore it is enough to consider functions which are everywhere $\tau_{\varphi}$-invariant.
Recall that two $G$-valued cocycles $(\varphi, \tau)$ and $(\psi, \tau)$ over the dynamical system $(X, \mu, \tau)$ are $\mu$-cohomologous if there is a measurable map $u : X \to G$ such that

$$\varphi(x) = u(\tau x) \psi(x)(u(x))^{-1}$$

for $\mu$-a.e. $x$.

The function $u$ in (3) is called the transfer function. We write $\varphi \sim^{(u, \mu)} \psi$ when (3) is satisfied. A cocycle $(\varphi, \tau)$ is a $\mu$-coboundary if it is $\mu$-cohomologous to the constant function $\psi \equiv e$.

**Notations 2.1.2.** In what follows, we consider a $\tau\varphi$-invariant measure $\lambda_\chi$ of the form $\lambda_\chi = \mu_\chi \otimes (\chi m_G)$, where $\chi$ is an exponential on $G$, and $\mu_\chi$ is a $\sigma$-finite measure which is $\chi \circ \varphi$-conformal and $\tau$-ergodic on $X$. When $\chi \equiv 1$, the measure $\mu_\chi$ is $\tau$-invariant.

Once and for all we choose a measurable positive function $h$ on $X \times G$ such that

$$\int_{X \times G} h(x, g) \mu_\chi(dx) \chi(g) m_G(dg) = 1.$$  

The existence of $h$ results from the facts that $\mu_\chi$ is $\sigma$-finite on $X$ and that $G$ is a lcsc group.

Let $P^h$ be a regular conditional probability with respect to the probability measure $h\lambda_\chi$ and the $\sigma$-algebra $\mathcal{J}$ of $\tau\varphi$-invariant subsets (i.e. $P^h$ is a transition probability on $X \times G$ such that, for every nonnegative measurable function $f$ on $X \times G$, $P^h f$ is a version of the conditional expectation $\mathbb{E}_{h\lambda_\chi}[f | J]$).

We define a positive kernel $M^h$ on $X \times G$ by

$$M^h f(x, g) = P^h(f/h)(x, g)$$

for any measurable nonnegative function $f$ on $X \times G$.

If we replace $h$ by another density $h'$, we have

$$M^{h'}((x, g), \cdot) = P^h(h/h')(x, g)M^h((x, g), \cdot).$$

For $\lambda_\chi$-a.e. $(x, g) \in X \times G$, the positive measure $M^h((x, g), \cdot)$ on $X \times G$ is $\tau\varphi$-invariant ergodic (see the appendix).

**Statement of the main result.** The formula $\mathbb{E}_{h\lambda_\chi}[\cdot] = \mathbb{E}_{h\lambda_\chi}[\mathbb{E}_{h\lambda_\chi}[\cdot | J]]$ can be written

$$\lambda_\chi(dy, dt) = \int_{X \times G} M^h((x, g), (dy, dt)) h(x, g) \lambda_\chi(dx, dg),$$

which represents a decomposition of $\lambda_\chi$ into $\tau\varphi$-ergodic components. Our goal is to give a precise description of these ergodic components. This is the content of the following theorem:
Theorem 2.1.3 (Ergodic decomposition of $\lambda_X$). (i) There exist:

- a family $(\mu_x)_{x \in X}$ of $\sigma$-finite $\tau$-quasi-invariant measures on $X$ defining a $\sigma$-finite positive kernel from $(X, \mathcal{F})$ to $(X, \mathcal{X})$ (i.e., for every $x \in X$, $\mu_x$ is a $\sigma$-finite positive measure on $\mathcal{X}$ and for every $A \in \mathcal{X}$ the map $x \mapsto \mu_x(A) \in [0, +\infty]$ is $\mathcal{X}$-measurable),
- a family $(H_x)_{x \in X}$ of closed amenable subgroups of $G$ such that the map $x \mapsto H_x$ from $X$ to $\mathcal{F}(G)$ is measurable,

without distinguishing action of $\gamma$-invariance.

(ii) When there exist a fixed closed subgroup $H$ of $G$ and a measurable map $a : X \to G$ such that $H_x = a(x)H(a(x))^{-1}$ for $\mu_X$-a.e. $x \in X$ (which is the case when $G$ is a nilpotent connected Lie group (Theorem 5.1.1)), the ergodic measures can be written, with $\hat{\chi}_x(\gamma) := \chi_x(a_x^{-1}a_x^{-1}),$

\begin{equation}
M^h f(x,g) = \int_X \left( \int_H f(y, u_x(y)a_x(\gamma(a_x))^{-1}g) \hat{\chi}_x(\gamma) d\gamma \right) \mu_x(dy).
\end{equation}
H of Theorem 2.1.3 are conjugate to \( \mu \tau \) if there exist a closed subgroup \( H \) such that the cocycle \( \psi \) such that the cocycle

\[
\psi: (x, h) \mapsto \tau x, \psi(x)h
\]

\( \in \) \( G \) is ergodic for the product measure \( \mu \otimes (\chi m_H) \).

The proof of Theorem 2.2.2 will be given in Section 4.

2.2. Notion of regularity for a cocycle

Regularly

**Definition 2.2.1.** We say that the cocycle defined by \( \varphi \) is \( \mu_\chi \)-regular if there exist a closed subgroup \( H \) of \( G \) and a measurable map \( u: X \to G \) such that the cocycle \( \psi := (u \circ \tau)^{-1} \varphi u \) takes \( \mu_\chi \)-a.e. its values in \( H \) and \( \tau \psi: (x, h) \mapsto (\tau x, \psi(x)h) \) is ergodic for the product measure \( \mu_\chi \otimes (\chi m_H) \).

The measure \((\chi \circ u)\mu_\chi \otimes \chi m_H \) is \( \tau \psi \)-invariant. In the regular case we have a “good” ergodic decomposition of \( \mu_\chi \otimes (\chi dg) \), and the subgroups \( H_x \) of Theorem 2.1.3 are conjugate to \( H: H_x = u(x)H(u(x))^{-1} \).

**Theorem 2.2.2.** (i) For \( x_0 \in X \), the set \( \{ x \in X : \mu_x \sim \mu_{x_0} \} \) is measurable and has zero or full \( \mu_\chi \)-measure.

(ii) Assume that the cocycle \((\varphi, \tau) \) is \( \mu_\chi \)-regular. Then every measurable \( \tau \varphi \)-invariant function \( f \) can be written \( f(x, g) = F_f((u(x))^{-1} g) \), \( \mu_\chi \otimes m_G \)-a.e., where \( F_f \) is a left \( H \)-invariant function on \( G \). The ergodic components of \( \lambda_\chi \) (see (10)) can be written

\[
M_h f(x, g) = \frac{\int_X (\int_H f(y, u(y)\gamma(u(x))^{-1} g)\chi(\gamma) d\gamma) \chi(u(y)) \mu_\chi(dy)}{\int_X (\int_H h(y, u(y)\gamma(u(x))^{-1} g)\chi(\gamma) d\gamma) \chi(u(y)) \mu_\chi(dy)}.
\]

In other words, \( H_x = u(x)H(u(x))^{-1} \) and \( \chi_x(\gamma) = \chi(u(x)\gamma(u(x))^{-1}) \). We can take \( u_x(y) = u(y)(u(x))^{-1} \) and \( \mu_x(dy) = \chi(u(y)) \mu_\chi(dy) \).

(iii) Assume that the cocycle \((\varphi, \tau) \) is not \( \mu_\chi \)-regular. Then for \( \mu_\chi \)-a.e. \( x \), the measures \( \mu_x \) of the ergodic decomposition of \( \mu_\chi \otimes (\chi m_G) \) are singular with respect to \( \mu_\chi \). There are uncountably many of them pairwise mutually singular. If \( G \) is abelian and \( \mu_\chi \) is finite, then, for \( \mu_\chi \)-a.e. \( x \in X \), the measure \( \mu_x \) is infinite.

The proof of Theorem 2.2.2 will be given in Section 4.

Examples of nonregular cocycles over rotations were given by Lemańczyk in [Le95]. In Remark 5.2.2, we give an example of a nonregular cocycle over a rotation which is the difference \( 1_{[0, \beta]} - 1_{[0, \beta]}(\cdot + r) \) for some \( \beta \) and \( r \) on the circle.

**Boundedness.** In the proposition below, we discuss the boundedness of the map \( u \) and of the cocycle \((\varphi_n) \). The notations are those of Theorem 2.1.3.
As the group $G$ is lcsc, we can write $G = \bigcup_n U_n$ for an increasing sequence of open sets such that $K_n = \overline{U}_n$ is compact. Consequently, $G = \bigcup_{n \in \mathbb{N}} K_n$ and for any compact subset $K$ of $G$ there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$.

**Lemma 2.2.3.** (i) Let $u$ be the measurable map from $X \times X$ to $G$ defined in Theorem 2.1.3. For any compact subset $K$ of $G$ we define the following subset of $X$:

\[ X_K = \{ x \in X : u_x(y)H_x \subseteq KH_x \text{ for } \mu_x\text{-a.e. } y \in X \} \]

\[ = \{ x \in X : \supp(u_x(\mu_x)) \subseteq KH_x \}. \]

Then $X_K$ is measurable and $x \in X_K \Rightarrow \tau x \in X_{K(\varphi(x))^{-1}}$.

The set $\bigcup_{n \in \mathbb{N}} X_{Kn}$ is a $\tau$-invariant measurable subset of $X$ and (by ergodicity of $\mu_x$) has zero or full $\mu_x$-measure.

(ii) If there exists a compact subset $K$ of $G$ such that $\mu_x(X_K) > 0$, then $\bigcup_{n \in \mathbb{N}} X_{Kn}$ has full $\mu_x$-measure. In this case, we can replace the measurable map $u$ by another measurable map $u$ satisfying, for any $n \in \mathbb{N}$,

\[ (14) \quad \text{for } \mu_x\text{-a.e. } x \in X_n = X_{Kn} \setminus X_{Kn-1}, \quad u_x(y) \in K_n \text{ for } \mu_x\text{-a.e. } y \in X. \]

(iii) In particular, the set $\{ x : G/H_x \text{ is compact} \}$ is measurable and has zero or full measure. If this set has full measure, we are in the above situation.

**Proof.** (i) If $K$ is a fixed compact set in $G$, the map $F \mapsto KF$ from the set $\mathcal{F}(G)$ of closed subsets of $G$ into itself is continuous. Since $x \mapsto H_x$ is measurable, the map $x \mapsto KH_x$ is measurable. In Section 3, we will see that the map $(x, y) \in X \times X \mapsto u(x, y)H_x \in \mathcal{F}(G)$ is measurable. We also know that, for any $g \in G$, the map $F \in \mathcal{F}(G) \mapsto d(g, F) \in \mathbb{R}_+$ is continuous. It follows that the set $\{ (x, y) \in X \times X : d(g, KH_x) \leq d(g, u(x, y)H_x) \}$ is measurable. Let $(g_n)_{n \in \mathbb{N}}$ be a dense sequence in $G$. Then we have

\[ X_K = \{ x \in X : \forall n \in \mathbb{N}, \quad \nu(x,e)(\{ y \in X : d(g_n, KH_x) \leq d(g_n, u(x, y)H_x) \}) = 1 \}. \]

This shows that $X_K$ is measurable.

From the formulas (4) and (8) of Theorem 2.1.3, we obtain $x \in X_K \Rightarrow \tau x \in X_{K(\varphi(x))^{-1}}$. Since for any compact subset $K$ of $G$, there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$, we deduce that the measurable set $\bigcup_{n \in \mathbb{N}} X_{Kn}$ is $\tau$-invariant and (by ergodicity of $\mu_x$) has zero or full measure.

(ii) If $\mu_x(X_K) > 0$ for some compact subset $K$ of $G$, then the same argument shows that $\bigcup_{n \in \mathbb{N}} X_{Kn}$ has full $\mu_x$-measure. The last assertion follows from the construction of $u$ (cf. Lemma 7.1.1).

(iii) We have

\[ \{ x \in X : G/H_x \text{ is compact} \} = \bigcup_{n \in \mathbb{N}} \{ x \in X : K_nH_x = G \}, \]
which shows that the set is measurable. By the conjugacy relation (4) this set is \( \tau \)-invariant and (ergodicity of \( \mu_\chi \)) has zero or full \( \mu_\chi \)-measure. In the last case, for \( \mu_\chi \)-a.e. \( x \in X \), we have \( \bigcup_{n \in \mathbb{N}} K_n H_x = G \), which implies that \( \bigcup_{n \in \mathbb{N}} X K_n \) has full \( \mu_\chi \)-measure.

**Proposition 2.2.4.** (i) Assume that the measure \( \mu_\chi \) in the basis is a finite measure and that there exists a compact subset \( K \) of \( G \) such that \( \mu_\chi(K) > 0 \). If the map \( u \) satisfies the boundedness condition (14), then the measures \( \mu_x \) are finite for \( \mu_\chi \)-a.e. \( x \in X \).

(ii) Assume that \( G \) is abelian and there exists a compact subset \( K \) of \( G \) such that \( \mu_\chi(K) > 0 \). Then the cocycle is regular.

(iii) Assume that \( G \) is abelian, \( \mu_\chi \) finite and \( \tau \) conservative for \( \mu_\chi \). If \( \mu_\chi(\{x \in X : \hat{\mu}_x(X) < +\infty\}) > 0 \), where

\[
\hat{\mu}_x(dy) := (\chi(u_x(y)))^{-1} \mu_x(dy),
\]

then the cocycle is regular.

(iv) Assume that \( \tau \) is conservative for \( \mu_\chi \). If the cocycle \( (\varphi_n) \) is \( \mu_\chi \)-bounded (i.e. there exists a compact subset \( K \) of \( G \) such that \( \varphi_n(x) \in K \) for \( \mu_\chi \)-a.e. \( x \in X \) and all \( n \geq 0 \)), then \( H_x \) is a compact subgroup of \( G \) and the cocycle is cohomologous with a bounded transfer function to a cocycle taking its values in a compact subgroup of \( G \).

**Proof.** Let \( r \) be a positive continuous function on \( G \) with \( \int_G r(t) \chi(t) dt = 1 \). For any compact subset \( K \) of \( G \), we set \( r_K(g) := \min_{u \in K} r(ug) > 0 \). For all measurable nonnegative functions \( f \) on \( X \), we have (cf. (34))

\[
M^h(f \otimes r)(x, g) = c(x, g) \int_X f(y) \left( \int_{H_x} r(u_x(y) \gamma g) \chi_x(\gamma) m_{H_x} (d\gamma) \right) \mu_x(dy)
\]

\[
\geq c(x, g) \int_X f(y) \left( \int_{H_x} 1_K(u_x(y)) r(u_x(y) \gamma g) \chi_x(\gamma) m_{H_x} (d\gamma) \right) \mu_x(dy)
\]

\[
= c(x, g) \left( \int_{H_x} r_K(\gamma g) \chi_x(\gamma) m_{H_x} (d\gamma) \right) \int_X f(y) 1_K(u_x(y)) \mu_x(dy)
\]

and therefore

\[
(16) \quad \mu_\chi(f) = \lambda_\chi(f \otimes r) \geq \int_X \Psi_K(x) \left( \int_X f(y) 1_K(u_x(y)) \mu_x(dy) \right) \mu_\chi(dx),
\]

where \( \Psi_K(x) := \int_G c(x, g) (\int_{H_x} r_K(\gamma g) \chi_x(\gamma) m_{H_x} (d\gamma)) h(x, g) \chi(g) dg > 0 \).

(i) Under the assumptions of the first assertion, we have from (14) and (16), for each \( n \in \mathbb{N} \),

\[
\mu_\chi(f) \geq \int_{X_n} \Psi_{K_n}(x) \mu_x(f) \mu_\chi(dx),
\]

and taking \( f = 1_X \), we find that \( \mu_x(X) < +\infty \) for \( \mu_\chi \)-a.e. \( x \in X_n \), hence for \( \mu_\chi \)-a.e. \( x \in X \) since \( \bigcup_n X_n \) has full measure in \( X \).
From the conformal property (17), it follows that
\( \xi \) measurable function
\( \tau \mu_x(dy) = \chi(\varphi(\tau^{-1}y)) \mu_x(dy) \). One easily sees that the measures \( \tilde{\mu}_x(dy) \) defined by (15) satisfy, as the measure \( \mu_\chi \), the conformal property
\[
\tau \tilde{\mu}_x(dy) = \chi(\varphi(\tau^{-1}y)) \tilde{\mu}_x(dy).
\]

By (16) we have, for any \( n \in \mathbb{N} \),
\[
\mu_\chi(f) \geq \int \Phi_{K_n}(x) \tilde{\mu}_x(f) \mu_\chi(dx),
\]
where \( \Phi_{K_n}(x) = \Psi_{K_n}(x) \inf_{u \in K_n} \chi(u) \).
This implies that, for any \( B \in \mathcal{X} \) with \( B \subset X_n \), there exists a nonnegative measurable function \( \xi_B \) on \( X \) such that
\[
\int 1_B(x) \Phi_{K_n}(x) \tilde{\mu}_x(dy) \mu_\chi(dx) = \xi_B(y) \mu_\chi(dy).
\]

From the conformal property (17), it follows that \( \xi_B \circ \tau^{-1} = \xi_B \), \( \mu_\chi \) a.e. As \( \mu_\chi \) is \( \tau \)-ergodic, \( \xi_B \) is \( \mu_\chi \) a.e. equal to a constant \( \nu(B) \). The map \( B \mapsto \nu(B) \) defines a positive measure \( \nu \) on \( (X_n, X_n \cap \mathcal{X}) \) absolutely continuous with respect to the measure \( \mu_\chi \). Therefore there exists a measurable nonnegative function \( \xi \) on \( X \) such that
\[
\int 1_B(x) \Phi_{K_n}(x) \tilde{\mu}_x(dy) \mu_\chi(dx) = \nu(B) \mu_\chi(dy) = \left( \int 1_B(x) \xi(x) \mu_\chi(dx) \right) \mu_\chi(dy)
\]
and, for \( \mu_\chi \)-a.e. \( x \in X_n \),
\[
\xi(x) \mu_\chi(dy) = \Phi_{K_n}(x) \tilde{\mu}_x(dy).
\]

As \( \bigcup_{n \in \mathbb{N}} X_n \) is of full measure, by gluing the \( \Phi_{K_n} \), we obtain a function \( \Phi \) such that, for \( \mu_\chi \)-a.e. \( x \in X \),
\[
\xi(x) \mu_\chi(dy) = \Phi(x) \tilde{\mu}_x(dy).
\]
This shows the regularity of the cocycle.

(iii) We set \( X_0 = \{ x \in X : \tilde{\mu}_x(X) < +\infty \} \). For \( x \in X_0 \), we denote by \( \hat{\mu}_x \) the probability measure \( \tilde{\mu}_x/\tilde{\mu}_x(X) \). From (16), for any compact subset \( K \) of \( G \), we have
\[
\mu_\chi(f) \geq \int_{X_0} \Phi_K(x) \left( \int_X f(y) 1_{K(u_x(y))} \tilde{\mu}_x(dy) \right) \mu_\chi(dx),
\]
where \( \Phi_K(x) := \Psi_K(x) \inf_{u \in K} \chi(u) \tilde{\mu}_x(X) \).

Let \( h_1 \) be a positive bounded measurable function on \( X \). We know that \( \tau \) is conservative, i.e. \( \mu_\chi(\{ \sum_{k \geq 0} h_1 \circ \tau^k < +\infty \}) = 0 \). From (19), it follows that, for \( \mu_\chi \)-a.e. \( x \in X_0 \),
\[
\forall n \in \mathbb{N}, \quad \hat{\mu}_x \left( \left\{ \sum_{k \geq 0} h_1 \circ \tau^k < +\infty \right\} \cap \{ u_x \in K_n \} \right) = 0.
\]
As \( n \to +\infty \), from the monotone convergence theorem, we obtain, for \( \mu_\chi \)-a.e. \( x \in X \),

\[
\hat{\mu}_x \left( \left\{ \sum_{k \geq 0} h_1 \circ \tau^k < +\infty \right\} \right) = 0.
\]

Since \( h_1 \) is bounded and thus \( \hat{\mu}_x \)-integrable for \( x \in X_0 \), we deduce that, for \( \mu_\chi \)-a.e. \( x \in X_0 \), \( \tau \) is conservative for \( \hat{\mu}_x \). Replacing \( X_0 \) by \( X_0 \cap \{ x \in X : \hat{\mu}_x (\{ \sum_{k \geq 0} h_1 \circ \tau^k < +\infty \}) = 0 \} \) we can assume that, for any \( x \in X_0 \), \( \tau \) is conservative for \( \hat{\mu}_x \).

From (19), there exists a measurable \([0,1]\)-valued function \( \xi_K \) such that

\[
\xi_K(y) \mu_\chi(dy) = \int_{X_0} \Phi_K(x) \mu_\chi(dx) \leq \int_{X_0} \Phi_K(x) \hat{\mu}_x(dy) \mu_\chi(dx).
\]

Consequently, there exists a measurable \([0,1]\)-valued function \( \psi_K \) such that

\[
\xi_K(y) \mu_\chi(dy) = \psi_K(y) \int_{X_0} \Phi_K(x) \hat{\mu}_x(dy) \mu_\chi(dx).
\]

Since the measures \( \mu_\chi \) and \( \hat{\mu}_x \) have the same conformal property (cf. (17)), we have

\[
\sum_{k=0}^{n-1} T^k \xi_K(y) \mu_\chi(dy) = \int_{X_0} \Phi_K(x) \sum_{k=0}^{n-1} T^k \psi_K(y) \hat{\mu}_x(dy) \mu_\chi(dx)
\]

where \( T \) is the operator defined by

\[
T f(y) = f \circ \tau^{-1}(y) \chi(\varphi(\tau^{-1}y)).
\]

As \( \tau \) is conservative for \( \mu_\chi \) and for \( \hat{\mu}_x \), \( x \in X_0 \), by Hurewicz’s ergodic theorem, for any bounded measurable function \( f \) on \( X \), the sequence of functions

\[
\left( \sum_{k=0}^{n-1} T^k f / \sum_{k=0}^{n-1} T^k 1 \right)_{n \in \mathbb{N}}
\]

converges \( \mu_\chi \)-a.e. to \( \mu_\chi(f) \) and converges \( \hat{\mu}_x \)-a.e. to \( \hat{\mu}_x(f) \), for \( x \in X_0 \). As the sequence of functions is bounded and the measures are finite, these convergences also hold in \( L^1 \)-norm.

Therefore, for any bounded measurable function \( f \),

\[
\int_X f(y) \sum_{k=0}^{n-1} T^k \xi_K(y) \mu_\chi(dy) \xrightarrow{n \to +\infty} \mu_\chi(\xi_K) \mu_\chi(f),
\]

and for \( \mu_\chi \)-a.e. \( x \in X \),

\[
\alpha_n(x) = \int_X f(y) \sum_{k=0}^{n-1} T^k \psi_K(y) \hat{\mu}_x(dy) \xrightarrow{n \to +\infty} \hat{\mu}_x(\psi_K) \hat{\mu}_x(f).
\]
The inequality (19) shows that $\Phi_K$ is $\mu_\chi$-integrable. Moreover, the sequence of functions $(\alpha_n)$ is bounded. By the dominated convergence theorem,

$$\int_{X_0} \Phi_K(x) \alpha_n(x) \mu_\chi(dx) \xrightarrow{n \to +\infty} \int_{X_0} \Phi_K(x) \hat{\mu}_x(dy) \mu_\chi(dx).$$

We deduce that

$$\mu_\chi(dy) = \int_{X_0} \hat{\Phi}_K(x) \hat{\mu}_x(dy) \mu_\chi(dx),$$

where $\hat{\Phi}_K(x) = \Phi_K(x) \hat{\mu}_x(\psi_K)/\mu_\chi(\xi_K)$.

Now, as above, for any $B \in \mathcal{X}$ with $B \subset X_0$, there exists a nonnegative measurable function $\xi_B$ such that

$$\xi_B(y) \mu_\chi(dy) = \int_B \hat{\Phi}_K(x) \hat{\mu}_x(dy) \mu_\chi(dx).$$

From the conformal property (17), it follows that $\xi_B \circ \tau^{-1} = \xi_B$, $\mu_\chi$-a.e. With the same argument as in (ii), since $\mu_\chi$ is $\tau$-ergodic, $\xi_B$ is $\mu_\chi$-a.e. equal to $\nu(B)$, where $\nu$ is a positive measure on $(X_n, X_n \cap \mathcal{X})$ absolutely continuous with respect to $\mu_\chi$. Therefore there exists a measurable nonnegative function $\xi$ on $X$ such that

$$\int 1_B(x) \hat{\Phi}_K(x) \hat{\mu}_x(dy) \mu_\chi(dx) = \nu(B) \mu_\chi(dy) = \left( \int_B \xi(x) \mu_\chi(dx) \right) \mu_\chi(dy)$$

and, for $\mu_\chi$-a.e. $x \in X_0$,

$$\xi(x) \mu_\chi(dy) = \hat{\Phi}_K(x) \hat{\mu}_x(dy).$$

This shows the regularity of the cocycle.

(iv) Now assume that $\tau$ is conservative for $\mu_\chi$ and that there exists a compact subset $K$ such that, for $\mu_\chi$-a.e. $x \in X$, $\varphi_n(x) \in K$ for every $n \in \mathbb{N}$.

For any nonnegative measurable function $f$ on $X$ with $\mu_\chi(f) \in [0, +\infty[$, we have

$$\sum_{n \geq 0} f(\tau^n x)1_K(\varphi_n(x)) = \sum_{n \geq 0} f(\tau^n x) = +\infty \quad \mu_\chi\text{-a.e.}$$

Hence $\tau_\varphi$ is conservative for $\lambda_\chi$. We deduce that, for $x \in X_0$ where $X_0$ is a set of full $\mu_\chi$-measure, and any $g \in G$, $\tau_\varphi$ is conservative for $M^{h)((x, g), \cdot)$.

We take $x \in X_0$. Let $s \in \text{supp}(u_x(\mu_\chi))$ and $t \in H_x$. Then, for any neighborhoods $V$ and $W$ of $s$ and $t$, for $\mu_\chi$-a.e. $y \in X$, $\sum_{n \geq 0} 1_V(\tau^n y)1_W(\varphi_n(y)) = +\infty$. From the inclusion

$$u_x(\tau^n y)\psi_n(y) = \varphi_n(y)u_x(y) \subset Ku_x(y) \quad \text{for } \mu_\chi\text{-a.e. } y \in X,$$

it follows that $st \in Ku_x(y)$ for $\mu_\chi$-a.e. $x \in X$ and $\mu_x$-a.e. $y \in X$. Taking a fixed $s$ and a dense sequence $(t_n)$ in $H_x$, we infer that $t_n \in s^{-1}Ku_x(y)$ for
\(\mu_\chi\text{-a.e. } x \in X \) and \(\mu_x\text{-a.e. } y \in X\), for \(n \geq 0\). Therefore \(H_x \subset s^{-1}Ku_x(y)\) is a compact subgroup of \(G\) and, with a similar argument, \(\text{supp}(u_x(\mu_x)) \subset Ku_x(y)H_x\) for \(\mu_\chi\text{-a.e. } x \in X \) and \(\mu_x\text{-a.e. } y \in X\). This implies that, for \(\mu_\chi\text{-a.e. } x \in X\), there exists a compact subset \(K_x\) of \(G\) such that \(\text{supp}(u_x(\mu_x)) \subset K_xH_x\). Since any compact subset \(K\) of \(G\) satisfies \(K \subset K_n\) for \(n\) large enough, we deduce that \(\bigcup_{n \in \mathbb{N}} X_{K_n}\) has full \(\mu_\chi\)-measure. So we can assume that \(u\) satisfies the boundedness condition (14) (cf. Lemma 2.2.3).

By (16) we have, for any \(n \in \mathbb{N}\),

\[
(20) \quad \mu_\chi(f) \geq \int_{X_n} \Psi_{K_n}(x) \mu_x(f) \mu_\chi(dx).
\]

This implies that there exists a \([0, 1]\)-valued measurable function \(\xi\) such that

\[
\int_{X_n} \Psi_{K_n}(x) \mu_x(dy) \mu_\chi(dx) = \xi(y) \mu_\chi(dy).
\]

Observe that for any \(x \in X\), the exponential \(\chi_x\) on the compact group is trivial and consequently the measures \(\mu_x, x \in X\), are \(\tau\)-invariant.

From the conformal property (17), it follows that \(\xi \circ \tau^{-1} d\tau \mu_\chi/d\mu_\chi = \xi\), \(\mu_\chi\text{-a.e.}\). This shows that the measure \(\xi \mu_\chi\) is \(\tau\)-invariant. Moreover, \(\{\xi > 0\}\) is \(\mu_\chi\text{-a.e.}\) \(\tau\)-invariant and therefore has full \(\mu_\chi\)-measure.

For any \(B \in \mathcal{F}\) with \(B \subset X_n\), there exists a \([0, 1]\)-valued measurable function \(\xi_B\) such that

\[
(21) \quad \int_B \Psi_{K_n}(x) \mu_x(dy) \mu_\chi(dx) = \xi_B(y) \xi(y) \mu_\chi(dy).
\]

From the conformal property (17), it follows that \(\xi_B \circ \tau^{-1} = \xi_B\), \(\mu_\chi\text{-a.e.}\). As in (ii) and (iii), \(\xi_B\) is \(\mu_\chi\text{-a.e.}\) equal to \(\nu(B)\), where \(\nu\) is a positive measure on \((X_n, X_n \cap \mathcal{F})\) absolutely continuous with respect to \(\mu_\chi\). Therefore there exists a measurable nonnegative function \(\psi\) on \(X\) such that

\[
\int_B \Psi_{K_n}(x) \mu_x(dy) \mu_\chi(dx) = \nu(B) \xi(y) \mu_\chi(dy) = \left( \int_B \psi(x) \mu_\chi(dx) \right) \xi(y) \mu_\chi(dy)
\]

and, for \(\mu_\chi\text{-a.e. } x \in X_n\),

\[
\psi(x) \xi(y) \mu_\chi(dy) = \Psi_{K_n}(x) \mu_x(dy).
\]

This shows the regularity of the cocycle hence the last assertion of (iv).

Remark. If \(G\) is a compact group, then it is well known that every \(G\)-valued cocycle \(\varphi\) is regular and therefore cohomologous to a cocycle \(\psi\) taking its values in a compact subgroup \(K\) of \(G\) such that \(\mu \otimes m_K\) is ergodic for \(\tau_\psi\) (cf. [PaPo97], [Pa97] for the regularity of the cohomology when \(G\) is compact and the cocycle \(\varphi\) is Hölderian over a subshift of finite type).

See also [AaWe00] for results under the assumption of tightness for the cocycle \((\varphi_n)\).
2.3. Essential values and periods of invariant functions. The notion of essential values was introduced by K. Schmidt [Sc77] and J. Feldman and C. C. Moore [FeMo77]. See also [Sc75], [Sc79], [Sc81], [Aa97]. The results in this section, except Proposition 2.3.6, are not new, at least when $\mu$ is $\tau$-invariant. For the sake of completeness, we will give proofs. Note that we are here in the more general case of a quasi-invariant measure.

DEFINITIONS 2.3.1. Let $\mu$ be a $\tau$-quasi-invariant conservative measure on $X$. An element $a \in G \cup \{\infty\}$ is an essential value of the cocycle $(\varphi, \tau)$ (with respect to $\mu$) if, for every neighborhood $V$ of $a$, and for every subset $B$ such that $\mu(B) > 0$, there is $n \in \mathbb{Z}$ such that

$$\mu(B \cap \tau^{-n}B \cap \{x : \varphi_n(x) \in V\}) > 0.$$ 

We denote by $\mathcal{E}(\varphi)$ the set of essential values of the cocycle $(\varphi, \tau)$ and by $\mathcal{E}(\varphi) = \overline{\mathcal{E}(\varphi)} \cap G$ the set of finite essential values.

Let $B$ be a measurable set of positive $\mu$-measure. Let $\tau_B$ be the induced transformation on $B$ and $\varphi^B(x) := \varphi_n(x)$, where $n(x) = n_B(x) := \inf\{j \geq 1 : \tau^jx \in B\}$ for $x \in B$. The “induced” cocycle is given, for $n \geq 1$, by $\varphi^B_n(x) := \varphi^B(x)\varphi^B(\tau_Bx)\cdots\varphi^B(\tau_B^{n-1}x)$.

Equivalently to Definition 2.3.1, an element $a \in G \cup \{\infty\}$ is an essential value of the cocycle $(\varphi, \tau)$ if and only if, for every subset $B$ such that $\mu(B) > 0$, and for any neighborhood $V$ of $a$, $\mu(\{x : \varphi^B_n(x) \in V\}) > 0$ for some $n \in \mathbb{Z}$.

PROPOSITION 2.3.2. Assume that $\tau$ is conservative for $\mu_\chi$. If $\infty \notin \mathcal{E}(\varphi)$, then $\varphi$ is cohomologous to a cocycle taking its values in a compact subgroup of $G$. When $G$ is abelian we have $\mathcal{E}(\varphi) = \{e\}$ if and only if $\varphi$ is a coboundary.

Proof. If $\infty \notin \mathcal{E}(\varphi)$, then there is $B$ with $\mu_\chi(B) > 0$ such that $(\varphi^B_n)_{n \in \mathbb{Z}}$ is a bounded sequence. This implies that $\varphi^B$ is $\tau_B$-cohomologous to a cocycle taking values in a compact subgroup of $G$ (cf. Proposition 2.2.4), i.e. there are measurable maps $\zeta^B$ from $B$ to $G$ and $\psi^B$ from $B$ to a compact subgroup of $G$ such that

$$\varphi^B = (\zeta^B \circ \tau_B)\psi^B(\zeta^B)^{-1}. \quad (22)$$

By ergodicity and conservativity of $(X, \mu_\chi, \tau)$, for $\mu_\chi$-a.e. $y \in X$ there are a unique $x \in B$ and an integer $k$ with $0 \leq k < n_B(x)$ such that $y = \tau^kx$. We define $\zeta$ on $X$ by taking, for $y = \tau^kx$ and $0 \leq k < n_B(x)$,

$$\zeta(y) = \varphi_k(x)\zeta^B(x)(\psi(y))^{-1}$$

with $\psi(y) = e$ if $k < n_B(x) - 1$, and $\psi(y) = \psi^B(x)$ for $k = n_B(x) - 1$.

For $0 \leq k < n_B(x) - 1$, the cocycle relation is clearly satisfied by construction. For $k = n_B(x) - 1$, it results from the cocycle relation (22) for the induced cocycle.
Now we consider the abelian case. Let us show that if $\mathcal{E}(\varphi) = \{e\}$ then $\varphi$ is a coboundary. From the first assertion we know that the cocycle is cohomologous to a cocycle $\psi$ taking values in a compact subgroup $K$ of $G$. The set of essential values is the same for $\phi$ and $\psi$ (see below). As $\tau_\psi$ is ergodic conservative and $\mathcal{E}(\psi) = \{e\}$, one has $K = \{e\}$. ■

We now consider, as in Theorem 2.1.3, a measure $\lambda_\chi$.

**Notation 2.3.3.** Let $\mathcal{P}(\varphi)$ be the closed subgroup of $G$ of left periods of the $\tau_\varphi$-invariant measurable functions, i.e. the subgroup of elements $\gamma \in G$ such that, for every $\tau_\varphi$-invariant function $f$, $f(x, \gamma g) = f(x, g)$ for $\lambda_\chi$-a.e. $(x, g) \in X \times G$.

Note that we should write $\mathcal{P}(\varphi, \mu_\chi)$, since $\mathcal{P}(\varphi)$ and $\mathcal{E}(\varphi)$ depend on the measure $\mu_\chi$. We will show that $\mathcal{P}(\varphi) = \mathcal{E}(\varphi)$ by using the following lemma from [ArNgOs].

Let $(Y, \rho)$ be a complete separable metric space with a continuous action $(g, y) \mapsto g.y$ of a group $G$ on it. Let $f$ be a measurable map from $X$ to $Y$. Given a $G$-valued cocycle $\varphi$, we say that $f$ is $(\varphi, \tau)$-invariant if $f(\tau x) = \varphi(x).f(x)$, $\mu$-a.e.

**Lemma 2.3.4 ([ArNgOs]).** If $f$ is $(\varphi, \tau)$-invariant, then $a.f(x) = f(x)$, $\mu$-a.e. for all $a \in \mathcal{E}(\varphi)$.

**Proof.** $(Y, \rho)$ being a separable metric space, the set

$$X_f := \{x \in X : \mu(\{x' \in X : \rho(f(x'), f(x)) < \varepsilon\}) > 0 \text{ for every } \varepsilon > 0\}$$

has full $\mu$-measure since it contains $f^{-1}(\text{supp } f(\mu))$. Let $x \in X_f$ and $a \in \mathcal{E}(\varphi)$. Let $\varepsilon > 0$ be arbitrary. Then the subset $E_x = \{x' : \rho(f(x'), f(x)) < \varepsilon\}$ has positive $\mu$-measure. Since $a \in \mathcal{E}(\varphi)$, for every $\varepsilon_1 > 0$ there exist $x_1 \in E_x$ and $n \in \mathbb{Z}$ such that $\tau^n x_1 \in E_x$ and $d(a, \varphi_n(x_1)) < \varepsilon_1$, where $d$ is a distance on $G$. By the invariance of $f$ we have

$$\rho(a.f(x), f(x)) \leq \rho(a.f(x), a.f(x_1)) + \rho(a.f(x_1), \varphi_n(x_1).f(x_1))$$

$$+ \rho(f(\tau^n x_1), f(x)).$$

Since $\varepsilon$ and $\varepsilon_1$ are arbitrary and the action of $G$ is continuous, we get $\rho(a.f(x), f(x)) = 0$. ■

**Proposition 2.3.5.** $\mathcal{E}(\varphi) = \mathcal{P}(\varphi)$.

**Proof.** If $a \notin \mathcal{E}(\varphi)$, there are a subset $A$ with $\mu(A) > 0$ and a neighborhood $V$ of $e$ such that

$$A \cap \tau^{-n} A \cap \{\varphi_n \in aVV^{-1}\} = \emptyset, \quad \forall n \in \mathbb{Z}.$$

This implies that $a$ is not a period of the $\tau_\varphi$-invariant set $B = \bigcup_{n \in \mathbb{Z}} \tau_\varphi^n (A \times V)$. Conversely, let $h$ be a strictly positive function on $G$ such that $\int h(g) \mu_G(dg) = 1$. We apply Lemma 2.3.4 to the $G$-space $Y$ of real
measurable functions on \( G \), with the metric defined by \( \rho(f_1, f_2) = \| f_1 - f_2 \|_1 h \) \( dm_G \). A function on \( X \times G \) can be viewed as a function on \( X \) taking its values in \( Y \). By Lemma 2.3.4, if a function \( f \) on \( X \times G \) is \( \tau_\varphi \)-invariant, then every element of \( \mathcal{E}(\varphi) \) is a period for \( f \). □

The proposition shows that \( \mathcal{E}(\varphi) = G \) if and only if \( \lambda_X \) is ergodic for \( \tau_\varphi \).

With the notations of Theorem 2.1.3, we have:

**Proposition 2.3.6.** An element \( \gamma \) in \( G \) belongs to \( \mathcal{P}(\varphi) \) if and only if \( \gamma \) belongs to \( H_x \) for \( \mu_X \)-a.e. \( x \in X \). In the abelian case, \( \mathcal{P}(\varphi) \) (and therefore \( \mathcal{E}(\varphi) \)) coincides with the subgroup \( H \).

**Proof.** For \( (x, g) \in X \times G \), we set (cf. (34))

\[
c(x, g) = \left( \int_X \left( \int_{H_x} h(y, u_x(y)\gamma g)\chi_x(\gamma) d\gamma \right) \mu_x(dy) \right)^{-1}.
\]

According to Theorem 2.1.3, we have

\[
\gamma \in \mathcal{P}(\varphi) \iff M^h((x, \gamma g), \cdot) = M^h((x, g), \cdot) \text{ for } \lambda_X\text{-a.e. } (x, g) \in X \times G.
\]

For \( \lambda_X\text{-a.e. } (x, g) \in X \times G \), the right member is equivalent to

\[
c(x, g) \mu_x(dy) \delta_{u_x(y)}(\chi_xm_{H_x}) * \delta_g = c(x, \gamma g) \mu_x(dy) \delta_{u_x(y)}(\chi_xm_{H_x}) * \delta_{\gamma g},
\]

that is, for \( \mu_x\)-a.e. \( y \in X \),

\[
c(x, g) \delta_{u_x(y)}(\chi_xm_{H_x}) * \delta_{\gamma} = c(x, \gamma g) \delta_{u_x(y)}(\chi_xm_{H_x}).
\]

The equality of the supports of these measures implies \( H_x\gamma = H_x \) for \( \mu_X\)-a.e. \( x \in X \). Hence the result. □

**Abelian groups.** If \( \varphi \) and \( \psi \) are two cohomologous cocycles, \( \varphi (u, \mu) \sim \psi \), then \( f \) is \( \tau_\varphi \)-invariant if and only if \( \tilde{f} \) is \( \tau_\psi \)-invariant, where \( \tilde{f}(x, g) = f(x, u(x)g) \).

If \( G \) is abelian, this implies that \( \mathcal{P}(\varphi) = \mathcal{P}(\psi) \), so that two cohomologous cocycles have the same set of essential values. This is false in the nonabelian case (cf. [ArNgOs]).

When \( G \) is abelian, the cocycle \( \tilde{\varphi} := \varphi \mod \mathcal{E}(\varphi) \) satisfies \( \mathcal{E}(\tilde{\varphi}) = \{0\} \). If \( \tilde{\mathcal{E}}(\tilde{\varphi}) = \{0\} \), then by 2.3.2, \( \varphi \) is \( \mu_X \)-cohomologous to a cocycle taking its values in \( \mathcal{E}(\varphi) \). Therefore the regularity of the cocycle is equivalent to \( \tilde{\mathcal{E}}(\tilde{\varphi}) = \{0\} \). This last property, for an invariant measure, corresponds to the definition of regularity given by K. Schmidt for a cocycle (defined for a group action) taking its values in an abelian group.

If \( G/\mathcal{E}(\varphi) \) is compact, then \( \tilde{\mathcal{E}}(\tilde{\varphi}) = \{0\} \) and \( \varphi \) is regular. In particular, this is the case when \( G = \mathbb{R} \) and \( \mathcal{E}(\varphi) \neq \{0\} \).

Note that if \( \varphi \) is cohomologous to \( \varphi_1 \) and to \( \varphi_2 \), two functions with values respectively in closed subgroups whose intersection reduces to the identity element \( e \) of \( G \), then \( \mathcal{E}(\varphi) = \{e\} \).
For instance, if $\varphi$ is a $\mathbb{Z}$-valued cocycle such that there is $s \notin \mathbb{Q}$ for which the multiplicative equation $e^{2\pi is\varphi} = \psi/\psi \circ \tau$ has a measurable solution $\psi$, then either $\varphi$ is a coboundary or the cocycle $\varphi$ is not regular. We will use this remark to give an example of a nonregular cocycle in Section 5.

3. PROOF OF THEOREM 2.3.1

3.1. Characterization of the $\tau_{\varphi}$-invariant ergodic measures. The key tool in the proof of Theorem 2.1.3 is the following result:

**Theorem 3.1.1 ([Ra07]).** Let $\lambda$ be a $\tau_{\varphi}$-invariant ergodic measure of the form $\lambda(dy, dg) = \mu(dy) N(y, dg)$, where $\mu$ is a probability measure on $X$ and $N$ is a positive Radon kernel (i.e. such that, for every $y \in X$, $N(y, dg)$ is a positive Radon measure on the Borel subsets of $G$ and, for every Borel set $B$ in $G$, the map $y \mapsto N(y, B)$ is measurable).

Then there exist a closed subgroup $H$ of $G$ and a measurable map $u$ from $X$ to $G$ such that:

- $\varphi_u(y) := (u(\tau y))^{-1} \varphi(y) u(y) \in H$ for $\mu$-a.e. $y \in X$;
- the measure $\tilde{\lambda}$ that is the image of $\lambda$ under the map $(y, g) \mapsto (y, (u(y))^{-1} g)$ is a $\tau_{\varphi_u}$-invariant ergodic measure with support $X \times H$ and has the form

$$(23) \quad \tilde{\lambda}(dy, dh) = \tilde{\mu}(dy) \chi(h) dh,$$

where $\chi$ is an exponential on $H$ and $\tilde{\mu}$ a positive $\sigma$-finite measure, equivalent to $\mu$ such that

$$(24) \quad \tau \tilde{\mu}(dy) = \chi(\varphi_u(\tau^{-1} y)) \tilde{\mu}(dy).$$

If $H = G$, then $u(y) \equiv e$,

$$\lambda(dy, dg) = \tilde{\mu}(dy) \chi(g) dg, \quad \tau \tilde{\mu}(dy) = \chi(\varphi(\tau^{-1} y)) \tilde{\mu}(dy).$$

3.2. Ergodic decomposition of $\lambda_\chi$

**Abstract ergodic decomposition.** Let $h$ be a positive measurable function on $X \times G$ such that $\lambda_\chi(h) = 1$ (cf. 2.1.2). We apply the results of the appendix to the Borel standard space $(X \times G, \mathcal{X} \times \mathcal{B}_G)$ and to the probability measure $h \lambda_\chi$.

We denote by $P^h$ a regular conditional probability with respect to $h \lambda_\chi$ and the $\sigma$-algebra $\mathfrak{J}$ of $\tau_{\varphi}$-invariant sets, and by $M^h$ the positive kernel on $X \times G$ defined for any measurable nonnegative function $f$ on $X \times G$ by

$$\forall (x, g) \in X \times G, \quad M^h f(x, g) = P^h(f/h)(x, g).$$

We have

$$(25) \quad \lambda_\chi(dy, dt) = \int_{X \times G} M^h((x, g), (dy, dt)) h(x, g) \lambda_\chi(dx, dg).$$
For \( \lambda_X \)-a.e. \( (x, g) \in X \times G \), the probability measure \( P^h((x, g), \cdot) \) is \( \tau_\varphi \)-ergodic (Theorem 7.4.5) (i.e. for all \( A \in \mathfrak{F} \), \( P^h((x, g), A) = 0 \) or 1). Moreover, according to (49) of Lemma 7.2.1, we have

\[
\tau_\varphi P^h((x, g), (dy, dt)) = \frac{h \circ \tau_\varphi^{-1}(y, t)}{h(y, t)} P^h((x, g), (dy, dt)),
\]

which is equivalent to

\[
\tau_\varphi M^h((x, g), (dy, dt)) = M^h((x, g), (dy, dt)).
\]

We write

\[
P^h((x, g), (dy, dt)) = \rho((x, g), dy) Q((x, g, y), dt),
\]

where \( \rho \) is a transition probability from \( (X \times G, \mathfrak{F} \otimes \mathfrak{B}_G) \) to \( (X, \mathfrak{F}) \), and \( Q \) a transition probability from \( (X \times G \times X, \mathfrak{F} \otimes \mathfrak{B}_G \otimes \mathfrak{F}) \) to \( (G, \mathfrak{B}) \). We also introduce the notations

\[
\nu_{(x, g)}(dy) := \rho((x, g), dy) \quad \text{and} \quad N_{(x, g)}(y, dt) := Q((x, g, y), dt).
\]

Let \( (x, g) \in X \times G \). The probability measure \( \nu_{(x, g)} \) is uniquely determined by \( \nu_{(x, g)}(A) = P^h((x, g), A \times G) \) for any \( A \in \mathfrak{F} \). The family of probability measures \( \{N_{(x, g)}(y, \cdot) : y \in X\} \) is determined up to a set of \( \nu_{(x, g)} \)-measure zero. If we consider on the probability space \( (X \times G, \mathfrak{F} \otimes \mathfrak{B}_G, P^h((x, g), \cdot)) \) the projections \( U \) and \( V \) on \( X \) and \( G \), then \( \nu_{(x, g)} \) is the law of \( U \) and \( N_{(x, g)} \) is a version of the conditional law of \( V \) with respect to \( U \).

The kernel \( M^h \) can then be written

\[
M^h((x, g), (dy, dt)) = \rho((x, g), dy) \tilde{Q}((x, g, y), dt)
\]

\[
= \nu_{(x, g)}(dy) \tilde{N}_{(x, g)}(y, dt),
\]

where \( \tilde{Q}((x, g, y), dt) = \tilde{N}_{(x, g)}(y, dt) = h(y, t)^{-1}N_{(x, g)}(y, dt) \) is a positive kernel from \( (X \times G \times X, \mathfrak{F} \otimes \mathfrak{B}_G \times \mathfrak{F}) \) to \( (G, \mathfrak{B}_G) \).

Let \( f \) be a measurable positive \( \mu_X \)-integrable function on \( X \), and \( K \) be a compact subset of \( G \). We know that

\[
\int_{X \times G} \left[ \int_X f(y) \tilde{N}_{(x, g)}(y, K) \nu_{(x, g)}(dy) \right] h(x, g) \lambda_X(dx, dg) = \int_{X \times G} f(x) 1_K(g) \lambda_X(dx, dg) < +\infty.
\]

Therefore, for \( \lambda_X \)-a.e. \( (x, g) \), we have \( \tilde{N}_{(x, g)}(y, K) < +\infty \) for \( \nu_{(x, g)} \)-a.e. \( y \).

Let \( (K_n)_{n \geq 0} \) be the sequence of compact subsets of \( G \) such that \( \bigcup_{n \in \mathbb{N}} K_n = G \). For \( \lambda_X \)-a.e. \( (x, g) \), we have, for \( \nu_{(x, g)} \)-a.e. \( y \) and all \( n \geq 0 \), \( \tilde{N}_{(x, g)}(y, K_n) < +\infty \), i.e. \( \tilde{N}_{(x, g)}(y, \cdot) \) is a Radon measure on \( G \).

After a modification of \( P^h \) on a set of \( \lambda_X \)-measure zero followed, for any \( (x, g) \in X \times G \), by a modification of the family of positive measures \( \{\tilde{N}_{(x, g)}(y, \cdot) : y \in X\} \) on a set of \( \nu_{(x, g)} \)-measure zero, we can assume that:
For every $(x, g) \in X \times G$, the positive measure $M^h((x, g), \cdot)$ is $\tau_{x,g}$-invariant ergodic and, for every $y \in X$, $\tilde{N}_{(x,g)}(y, \cdot)$ is a Radon measure on $G$.

Explicit form of the ergodic decomposition. According to Theorem 3.1.1, the $\tau_{x,g}$-invariant ergodic measure $M^h((x, g), \cdot)$ can be written, up to a multiplicative constant,

$$(29) \quad M^h((x, g), (dy, d\gamma)) = \tilde{\mu}(x, g)(dy) \times [\delta_{v(x, g)}(y) * (\chi(x, g) \gamma) m_H((x, g)(d\gamma))],$$

where $H_{(x, g)}$ is a closed subgroup of $G$, $\chi(x, g)$ an exponential on $H((x, g), v(x, g))$ a measurable map from $X$ to $G$, and $\tilde{\mu}(x, g)$ a positive $\sigma$-finite measure on $X$, equivalent to the probability measure $\nu(x, g)$, such that

$$(30) \quad \tau_{x,g}(\tilde{\mu}(x, g))(dy) = \chi(\varphi_{v(x, g)}(\tau^{-1}y)) \tilde{\mu}(x, g)(dy),$$

where

$$(31) \quad \varphi_{v(x, g)}(y) := (v(x, g)(\tau y))^{-1} \varphi(y) \nu(x, g)(y) \in H_{(x, g)}$$

for $\tilde{\mu}(x, g)$-a.e. $y \in X$.

For $t \in G$ and $f$ defined on $X \times G$, let $R_t(f)(x, g) := f(x, gt)$. From Lemma 7.2.1 it follows that, for every $t \in G$, every nonnegative measurable function $f$ on $X \times G$, and $\lambda_{x}$-a.e. $(x, g) \in X \times G$,

$$(32) \quad M^h(R_t(f))(x, g) = P^h(R_t h/h)(x, g) M^h(f)(x, gt).$$

Let $c(x, g, t)$ be defined by

$$(33) \quad c(x, g, t) = P^h(R_t h/h)(x, g).$$

From (32), we have

$$\tilde{\mu}(x, g)(dy) \times [\delta_{v(x, g)}(y) * (\chi(x, g) \gamma) m_H((x, g)(d\gamma)) \delta_t] = c(x, g, t) \tilde{\mu}(x, gt)(dy) \times [\delta_{v(x, gt)}(y) * (\chi(x, gt) \gamma) m_H((x, gt)(d\gamma))].$$

Using Fubini’s theorem and the separability of the $\sigma$-algebra $\mathcal{X} \times \mathcal{B}_G$, it follows that, for $\lambda_{x}$-a.e. $(x, g) \in X \times G$ and $m_G$-a.e. $t \in G$,

$$R_t(M^h((x, g), \cdot)) = P^h(R_t h/h)(x, g) M^h((x, gt), \cdot)$$

and therefore

$$R_{g^{-1}}(M^h((x, g), \cdot)) = P^h(R_t h/h)(x, g) R_{(gt)^{-1}}(M^h((x, gt), \cdot)).$$

This implies that, for $\lambda_{x}$-a.e. $(x, g) \in X \times G$, the measure $M^h((x, g), (dy, dt))$ is equal, up to a multiplicative positive constant $c(x, g)$, to a fixed measure which has the form

$$\tilde{\mu}_x(dy) [\delta_{v_x}(y) * (\chi_x \tilde{m}_{H_x}) * \delta_y](dt),$$

where $\tilde{m}_{H_x}$ is a left Haar measure on $H_x$ (we will later change $\tilde{m}_{H_x}$ to $m_{H_x}$ by multiplying it by a factor).
Now, \( P^h(1)(x, g) = M^h(h)(x, g) = 1 \) for \( \lambda_x \)-a.e. \((x, g) \in X \times G \). Therefore
\[
(c(x, g))^{-1} = \int_X \left( \int_{H_x} h(y, v_x(y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) \right) \tilde{\mu}_x(dy)
\]
and, for \( \lambda_x \)-a.e. \((x, g) \in X \times G \) and every measurable nonnegative function \( f \) on \( X \times G \),
\[
M^h(f)(x, g) = \frac{\int_X (\int_{H_x} f(y, v_x(y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma)) \tilde{\mu}_x(dy)}{\int_X (\int_{H_x} h(y, v_x(y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma)) \tilde{\mu}_x(dy)}.
\]

Now we carry out the suitable modifications in order to obtain the desired properties of measurability for the decomposition.

**Measurability.** We can specify the decomposition of \( M^h \) given in (28). We have
\[
\nu_{(x, g)}(dy) = P^h((x, g), dy \times G) = c(x, g) \left( \int_{H_x} h(y, v_x(y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) \right) \tilde{\mu}_x(dy)
\]
and
\[
\tilde{N}_{(x, g)}(y, dt) = \left( \int_{H_x} h(y, v_x(y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) \right)^{-1} (\delta_{v_x(y)g} \ast (\chi_x \tilde{m}_{H_x} \ast \delta_g))(dt).
\]
The closed set \( v_x(y)H_x \) is the support \( S(x, y) \) of the probability measure \( Q((x, e, y), \cdot) = N_{(x,e)}(y, \cdot) \) on \( G \), and \( H_x \) is the support of the probability measure \( \hat{Q}((x, e, y), \cdot) \ast Q((x, e, y), \cdot) \), where \( \hat{Q}((x, e, y), \cdot) \) is the image of the positive measure \( Q((x, e, y), \cdot) \) under the transformation \( t \mapsto t^{-1} \) of \( G \). It follows that the maps \( x \in X \mapsto H_x \in \mathcal{F}(G) \) and \( (x, y) \in X \times X \mapsto v_x(y)H_x \in \mathcal{F}(G) \) are measurable. For instance, the last property follows from the fact that, for any closed subset \( F \) of \( G \), we have
\[
\{(x, y) \in X \times X : v_x(y)H_x \subset F\} = \{(x, e, y) : Q((x, e, y), F^c) = 0\}.
\]
From Lemma 7.1.1 we can find a measurable map \( u : X \times X \to G \) such that, for any \((x, y) \in X \times X \), \( u(x, y) \in S(x, y) \). Then \( v_x(y)H_x = u(x, y)H_x \) and, for any nonnegative measurable function \( f \) on \( X \times G \),
\[
\int_{H_x} f(y, v_x(y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) = \chi_x^{-1}((u(x, y))^{-1} v_x(y)) \int_{H_x} f(y, u(x, y)g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma).
\]
As
\[
\delta_{u(x,y)^{-1}v_x(y)} \ast (\chi_x \tilde{m}_{H_x}) = \chi_x^{-1}((u(x, y))^{-1} v_x(y)) (\chi_x \tilde{m}_{H_x}),
\]
the positive kernel \( R((x, g, y), dt) = \delta_{(u(x,y))^{-1} \tilde{N}_{x,g}(y, dt)} \ast \delta_g^{-1} \) from \( X \times X \)
to \( G \) is equal to
\[
\left( \int_{H_x} h(y, u(x, y)\gamma g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) \right)^{-1} \chi_x(t) \tilde{m}_{H_x}(dt).
\]

Denoting by \( U \) the closed unit ball in \( G \) centered at \( e \), we have
\[
\left( \int_{H_x} h(y, u(x, y)\gamma g) \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) \right)^{-1} \int_{H_x \cap U} \chi_x(t) \tilde{m}_{H_x}(dt) = R((x, g, y), U) > 0
\]
and, for any \( \gamma \in H_x \),
\[
\chi_x(\gamma) = \frac{R((x, e, y), \gamma U)}{R((x, e, y), U)},
\]
which proves that there exists a measurable map \( \eta : X \times G \to \mathbb{R}_+^* \) such that \( \chi_x(\gamma) = \eta(x, \gamma) \) for \( \mu_x \)-a.e. \( x \in X \) and all \( \gamma \in H_x \).

We also have
\[
\int_{H_x \cap U} \chi_x(\gamma) \tilde{m}_{H_x}(d\gamma) = R((x, e, y), du) + \int_{H_x \cap U} \chi_x(t) \tilde{m}_{H_x}(dt)
\]
which shows that the left member defines a positive kernel from \( X \) to \( G \).

We observe that the left member is the unique left Haar measure, denoted by \( m_{H_x} \), of \( H_x \) such that
\[
\int_{H_x \cap U} \chi_x(\gamma) m_{H_x}(d\gamma) = 1.
\]

Finally, we obtain
\[
M^h((x, g), dy, dt) = R((x, g, y), U) \nu_{(x, g)}(dy) (\delta_{u(x, y)} * (\chi_x m_{H_x}) * \delta_g)(dt)
\]
and
\[
R((x, g, y), U) \nu_{(x, g)}(dy) = c(x, g) \chi_x((u(x, y))^{-1} v_x(y)) \left( \int_{H_x \cap U} \chi_x(t) \tilde{m}_{H_x}(dt) \right) \tilde{\mu}_x(dy).
\]

We deduce that
\[
\chi_x((u(x, y))^{-1} v_x(y)) \tilde{\mu}_x(dy) = d(x) \mu_x(dy)
\]
with
\[
(d(x))^{-1} = c(x, e) \left( \int_{H_x \cap U} \chi_x(t) \tilde{m}_{H_x}(dt) \right),
\]
\[
\mu_x(dy) = R((x, e, y), U) \nu_{(x, e)}(dy).
\]

We observe that \( (\mu_x(dy))_{x \in X} \) is a positive kernel on \( (X, \mathcal{X}) \).

The formula (35) can be written
\[
M^h(f)(x, g) = \frac{\chi_x(\left( \int_{H_x} f(y, u(x, y)\gamma g) \chi_x(\gamma) m_{H_x}(d\gamma) \right) \mu_x(dy))}{\chi_x(\left( \int_{H_x} h(y, u(x, y)\gamma g) \chi_x(\gamma) m_{H_x}(d\gamma) \right) \mu_x(dy))}.
\]
For every \( (x, g) \in X \times G \), we choose the expression (37) for \( M^h((x, g), \cdot) \).
Proof of the relations (4) to (9). The equality of measures $\tau_\varphi(M^h((x, g), (dy, dt))) = M^h((x, g), (dy, dt))$ is equivalent to
\[
(\tau \mu_x)(dy) (\delta_{\varphi(\tau^{-1}y)} * \delta_{u_x(\tau^{-1}y)} * (\chi_x m_{H_x}) * \delta_g)(dt)
= \mu_x(dy) (\delta_{u_x(y)} * (\chi_x m_{H_x}) * \delta_g)(dt),
\]
which leads to
\[
\varphi(\tau^{-1}y) u_x(\tau^{-1}y) H_x = u_x(y) H_x \quad \text{for } \mu_x\text{-a.e. } x \in X
\]
and
\[
(\tau \mu_x)(dy) = \chi_x((u_x(y))^{-1} \varphi(\tau^{-1}y) u_x(\tau^{-1}y)) \mu_x(dy);
\]
hence the relations (5) and (6) follow.

The equality $M^h((x, g), \cdot) = M^h(\tau_\varphi(x, g), \cdot)$ is equivalent to $\nu_{(x, g)} = \nu_{\tau_\varphi(x, g)}$ and $\tilde{N}_{(x, g)}(y, \cdot) = \tilde{N}_{\tau_\varphi(x, g)}(y, \cdot)$ for $\nu_{(x, g)}$-a.e. $y \in X$.

The equality $\tilde{N}_{(x, g)}(y, \cdot) = \tilde{N}_{\tau_\varphi(x, g)}(y, \cdot)$ is equivalent to the following conditions:
\[
u_x(y) H_x = u_{\tau_x}(y) H_{\tau_x} \varphi(x)\]
(equality of the supports), which implies
\[
\zeta_x(y) = (u_x(y))^{-1} u_{\tau_x}(y) \varphi(x) \in H_x,
\]
\[
H_{\tau_x} = \varphi(x) H_x (\varphi(x))^{-1},
\]
and therefore
\[
\chi_{\tau_x}(\varphi(x) \zeta_x(y) (\varphi(x))^{-1}) \delta_{u_{\tau_x}}(y) * (\chi_{\tau_x} m_{H_{\tau_x}}) * \delta_{\varphi(x)}
= \delta_{u_x(y)} * (\chi_{\tau_x}(\varphi(x) \cdot (\varphi(x))^{-1}) m_{H_x}
\]
where $\tilde{m}_{H_x} = \delta_{\varphi(x)} * m_{H_{\tau_x}} * \delta_{(\varphi(x))^{-1}}$ is a left Haar measure on $H_x$.

We write $\tilde{m}_{H_x} = d(x) m_{H_x}$ for a constant $d(x)$ depending on $x$ and we obtain for any $\gamma \in H_x$,
\[
\chi_x(\gamma) = \chi_{\tau_x}(\varphi(x) \gamma (\varphi(x))^{-1})
\]
and
\[
\chi_x(\zeta_x(y)) \int_{H_{\tau_x}} h(y, u_{\tau_x} \gamma \varphi(x) g) \chi_{\tau_x}(\gamma) d\gamma = d(x) \int_{H_x} h(y, u_x(y) \gamma g) \chi_x(\gamma) d\gamma.
\]
Then the equality $\nu_{(x, g)} = \nu_{\tau_\varphi(x, g)}$ is equivalent to
\[
\tilde{\mu}_{\tau_x}(dy) = c(x) \chi_x(\zeta_x(y)) \tilde{\mu}_x(dy)
\]
for a constant $c(x)$ depending on $x$.

This yields the relations (4), (7), (8), (9).

The ergodicity of the cocycle $\varphi_{u_x}$ on $H_x$ over the $\sigma$-finite ergodic measure $\mu_x$ implies that $H_x$ is amenable [Zi78].

The first assertion of Theorem 2.1.3 is proved.
Assertions (ii) and (iii) of Theorem 2.1.3. (a) We suppose that the subgroups $H_x$ are conjugate to a fixed closed subgroup $H$ (cf. Theorem 5.1.1 for the nilpotent connected Lie group case), i.e. there exists a measurable map $a : X \to G$ such that $H_x = a(x)H(a(x))^{-1}$.

Let $x \in X$. The element $a(x)$ is defined modulo the normalizer of $H$. The element $\psi(x) := (a(\tau x))^{-1}\varphi(x)a(x)$ is in the normalizer of $H$ and we have

$$(a(x))^{-1}(u_x(y))^{-1}u_{\tau x}(y)\varphi(x)a(x) \in H.$$ 

The ergodic components applied to a function $f$ can be written

$$M^hf(x,g) = \int_X \left( \int_H f(y,u_x(y)a(x)\gamma(a(x))^{-1}g)\chi_x(a(x)\gamma(a(x))^{-1})d\gamma \right) \mu_x(dy).$$

We have $\chi_{\tau x}(a(\tau x)\gamma(a(\tau x))^{-1}) = \chi_x(a(x)(\psi(x))^{-1}\gamma\psi(x)(a(x))^{-1})$. Setting $\tilde{\chi}_x(\gamma) := \chi_x(a(x)\gamma(a(x))^{-1})$, we have $\tilde{\chi}_{\tau x}(\gamma) = \tilde{\chi}_x((\psi(x))^{-1}\gamma\psi(x))$.

(b) Abelian groups. If $G$ is abelian, we have $H_{\tau(x)} = H_x$ for $\mu_x$-a.e. $x \in X$. Since the map $x \in X \mapsto H_x \in \mathcal{F}(G)$ is measurable and Chabauty’s topology countably separates the points, there exists a closed subgroup $H$ of $G$ such that $H_x = H$ for $\mu_x$-a.e. $x \in X$.

For every $\gamma \in H$, we have $\lambda_x(R_{\gamma}(f)) = \chi^{-1}(\gamma)\lambda_x(f)$ and, for $\lambda_x$-a.e. $(x,g) \in X \times G$, $M^hR_{\gamma}(f)(x,g) = \chi_x^{-1}(\gamma)M^hf(x,g)$. For $f = h$, it follows that

$$\forall \gamma \in H, \quad \chi(\gamma) = \int_{X \times G} \chi_x(\gamma)h(x,g)\lambda_x(dx,dg),$$

and therefore $\chi_x = \chi$ for $\mu_x$-a.e. $x \in X$.

The ergodic component of $\lambda_x$ applied to a function $f$ can be written

$$M^hf(x,g) = \frac{\int_X \left( \int_H f(y,u_x(y)\gamma g)\chi(\gamma)\gamma d\gamma \right) \mu_x(dy)}{\int_X \left( \int_H h(y,u_x(y)\gamma g)\chi(\gamma)\gamma d\gamma \right) \mu_x(dy)}.$$

This completes the proof of Theorem 2.1.3. □

4. PROOF OF THEOREM 2.2.2

4.1. Lemmas. For the proof of Theorem 2.2.2, we begin with a lemma which allows us to compare the ergodic components.

Lemma 4.1.1. (i) Let $\varphi$ be a cocycle with values in a closed subgroup $H_1$ of $G$, and $\mu_1 \otimes m_{H_1}$ be a $\tau_\varphi$-quasi-invariant positive measure. Suppose that the measure $\mu_1 \otimes m_{H_1}$ is $\tau_\varphi$-ergodic and that $\varphi$ is $\mu_1$-cohomologous to a cocycle $\psi$ with values in a closed subgroup $H_2$ of $G$, with transfer function $u$. Then there exists $g_0 \in G$ such that, for $\mu_1$-a.e. $x \in X$,

$$u(x)H_2 = g_0H_2 \quad \text{and} \quad H_1 \subset u(x)H_2(u(x))^{-1} = g_0H_2g_0^{-1}.$$
(ii) Assume in addition that there exists a positive $\tau_\psi$-quasi-invariant measure $\mu_2 \otimes m_{H_2}$ with $\mu_2 \sim \mu_1$ which is $\tau_\varphi$-ergodic. Then there exists $g_0 \in G$ such that
\[ H_1 u(x) = H_1 g_0, \quad u(x) H_2 = g_0 H_2 \quad \text{and} \quad g_0^{-1} H_1 g_0 = H_2. \]

(iii) Assume in addition that $\mu_1$ [resp. $\mu_2$] is $\chi_1 \circ \tau_\varphi$-conformal [resp. $\chi_2 \circ \tau_\psi$-conformal] for an exponential $\chi_1$ on $H_1$ [resp. $\chi_2$ on $H_2$]. Then
\begin{itemize}
  \item $\chi_1(\gamma) = \chi_2(g_0^{-1} g_0)$ for $\mu_1$-a.e. $x \in X$ and every $\gamma \in H_1$,
  \item $\chi_1(u(x) g_0^{-1}) = \chi_2(g_0^{-1} u(x))$ for $\mu_1$-a.e. $x \in X$,
  \item $\mu_2(dx) = \chi_2(u(x) g_0^{-1}) \mu_1(dx)$ up to a multiplicative constant.
\end{itemize}

The $\tau_\varphi$-invariant ergodic measure $\mu_2 \otimes (\delta_{u(x)} \ast (\chi_2 m_{H_2}))$ is equal to $\mu_1 \otimes ((\chi_1 m_{H_1}) \ast \delta_{g_0})$ up to a multiplicative constant.

Proof. (i) For every continuous left $H_2$-invariant function $F$ on $G$ and every $g \in G$ the function $f^g(x,t) = F((u(x))^{-1} t g)$ is $\tau_\varphi$-invariant. This function is therefore $\mu_1 \otimes m_{H_1}$-a.e. constant. Applying Fubini’s theorem and the continuity of $F$, it follows that, for $\mu_1$-a.e. $x \in X$ and any $g \in G$, the function $t \in H_1 \mapsto F((u(x))^{-1} t u(x) g)$ is constant and therefore equal to $F(g)$, its value for $t = e$. Consequently, $(u(x))^{-1} H_1 u(x) \subset H_2$.

Since $\varphi$ [resp. $\psi$] takes values in $H_1$ [resp. $H_2$], the above inclusion implies that, for $\mu_1$-a.e. $x \in X$, $(u(\tau x))^{-1} u(x) \in H_2$. Therefore $u(\tau x) H_2 = u(x) H_2$.

By ergodicity of $(\mu_1, \tau)$, we deduce the existence of $g_0 \in G$ such that $u(x) H_2 = g_0 H_2$ for $\mu_1$-a.e. $x \in X$.

(ii) The cocycle $\psi$ is $\mu_2$-cohomologous to the cocycle $\varphi$, via the map $x \in X \mapsto (u(x))^{-1} \in G$. Then the second statement is a consequence of the first one.

(iii) Set $\mu_2 = \beta \mu_1$ where $\beta$ is a positive function on $X$. By the conformal property of the measure it follows that, for $\mu_1$-a.e. $x \in X$,
\[ \chi_2(\psi(x)) = \frac{\beta(x)}{\beta(\tau x)} \chi_1(\varphi(x)). \]

From (ii), this equality can be written
\[ \frac{\chi_2((u(\tau x))^{-1} g_0)}{\chi_2((u(x))^{-1} g_0)} \chi_2((u(x))^{-1} \varphi(x) u(x)) = \frac{\beta(x)}{\beta(\tau x)} \chi_1(\varphi(x)). \]

For any $x \in X$, we consider the exponential $\tilde{\chi}_x$ on $H_1$ and the function $f$ on $X$, defined by
\[ \tilde{\chi}_x(t) = \frac{\chi_2((u(x))^{-1} t u(x))}{\chi_1(t)} \quad \text{and} \quad f(x) = \beta(x) \chi_2((u(x))^{-1} g_0). \]

We observe that $\tilde{\chi}_{\tau x}(t) = \tilde{\chi}_x(t)$ for any $t \in H_1$, and the positive function $(x, t) \mapsto f(x) \chi_x(t)$ on $X \times H$ is $\tau_\varphi$-invariant. It follows that this function is
constant $\mu_1 \otimes m_{H_1}$-a.e. Hence for $\mu_1$-a.e. $x \in X$,
- $\chi_2((u(x))^{-1}tu(x)) = \chi_1(t)$ for every $t \in H_1$,
- $\beta(x)$ is equal to $\chi_2(g_0^{-1}u(x)) = \chi_1(u(x)g_0^{-1})$ up to a multiplicative constant.

**Corollary 4.1.2.** Let $\mu_x(dy) \otimes (\delta_{u_x(y)} * (\chi_x m_{H_x}))(dt)$ and $\mu_{x'}(dy) \otimes (\delta_{u_{x'}(y)} * (\chi_{x'} m_{H_{x'}}))(dt)$ be two ergodic components of $\lambda_X$. Then either
- the measures $\mu_x$ and $\mu_{x'}$ on $X$ are mutually singular, or
- there is $g_{x',x} \in G$ such that, for every $g \in G$,
  \[ \mu_{x'}(dy) \otimes (\delta_{u_{x'}(y)} * (\chi_{x'} m_{H_{x'}})) = \mu_x(dy) \otimes (\delta_{u_x(y)} * (\chi_x m_{H_x})) \ast g_{x',x}. \]

Hence $P^h((x,g),\cdot) = P^h((x',g',xg),\cdot)$.

**Proof.** For a $G$-valued cocycle $\varphi$ and a measurable map $u$ from $X$ to $G$, we denote by $\varphi_u$ the cocycle $\varphi_u(y) := (u(\tau y))^{-1}\varphi(y)u(y)$ for $y \in X$.

The values of the cocycles $\varphi_{u_x}$ and $\varphi_{u_{x'}}$ are respectively in $H_x$ and $H_{x'}$. The measures $\mu_x \otimes (\chi_x m_{H_x})$ and $\mu_{x'} \otimes (\chi_{x'} m_{H_{x'}})$ are respectively $\tau_{\varphi_{u_x}}$-invariant ergodic and $\tau_{\varphi_{u_{x'}}}$-invariant ergodic, and $\varphi_{u_{x'}}^{-1}u_{x'} \sim u_x \varphi_{u_x}$.

The result follows from the previous lemma.

**4.2. Proof of Theorem 2.2.2.** (i) Let $x_0 \in X$. From Corollary 4.1.2, for any $x \in X$, if the measure $\mu_x$ is equivalent to $\mu_{x_0}$ then there is $g_x \in G$ such that $P^h((x,e),\cdot) = P^h((x_0,e),\cdot) \ast \delta_{g_x}$, and consequently, with the notations of Subsection 3.2 (cf. (36)), we have $\nu_{(x,e)} = \nu_{(x_0,e)}$. Conversely, the equality $\nu_{(x,e)} = \nu_{(x_0,e)}$ implies the equivalence of the measures $\mu_x$ and $\mu_{x_0}$.

The $\sigma$-algebra $X \times B(G)$ is separable, i.e. generated by a countable sub-algebra $A$. We deduce the equality of sets
\[ \{x \in X : \mu_x \sim \mu_{x_0}\} = \{x \in X : \nu_{(x,e)} = \nu_{(x_0,e)}\} \]
\[ = \{x \in X : \forall A \in A, \nu_{(x,e)}(A) = \nu_{(x_0,e)}(A)\}, \]
which proves that $\{x \in X : \mu_x \sim \mu_{x_0}\}$ is measurable. Since, for any $x \in X$, $\mu_x \sim \mu_{\tau x}$, this set is $\tau$-invariant and therefore (by ergodicity of $\mu_X$) has zero or full measure.

(ii) Assume that the cocycle is regular. Then every measurable $\tau_{\psi}$-invariant function $f$ is $\mu_X \otimes m_{H}$-a.e. constant. The function $F(g) := \|f(\cdot, \cdot; g)\|_{L_X(H \otimes m_H)}$ is left $H$-invariant on $G$ and we have, for every $g \in G$,
\[ f(x, \gamma g) = F(g) \quad \text{for } \mu_X \otimes m_{H}-a.e. \ (x, \gamma) \in X \times H. \]

The first statement of (ii) follows from the fact that $f$ is a measurable $\tau_{\psi}$-invariant function if and only if the function $\tilde{f}(x, g) = f(x, u(x)g)$ is $\tau_{\psi}$-invariant.
We consider the bijective map $\theta_u$ from $X \times G$ onto itself defined by $\theta_u(x, g) = (x, u(x)g)$ for $(x, g) \in X \times G$. A measurable nonnegative function $f$ on $X \times G$ is $\tau_\varphi$-invariant if and only if $f \circ \theta_u$ is $\tau_\psi$-invariant. If $\mathfrak{J} = \mathfrak{J}_\varphi$ is the $\sigma$-algebra of $\tau_\varphi$-invariant subsets of $X \times G$ then $\theta_u \mathfrak{J}_\varphi$ is the $\sigma$-algebra $\mathfrak{J}_\psi$ of $\tau_\psi$-invariant subsets of $X \times G$. From Lemma 7.2.1 we have, for any nonnegative measurable function $f$ on $X \times G$ and $\lambda_\chi$-a.e. $(x, g) \in X \times G$,

$$E_{h \lambda_\chi}[f | \mathfrak{J}_\varphi](x, g) = \frac{\mathbb{E}_{h \lambda_\chi}[f \circ \theta_u h \circ u \chi | \mathfrak{J}_\psi] \circ \theta_u(x, g)}{\mathbb{E}_{h \lambda_\chi}[h \circ u \chi | \mathfrak{J}_\psi] \circ \theta_u(x, g)}.$$  

Any nonnegative measurable $\tau_\psi$-invariant function is $\mu_\chi \otimes m_H$-a.e. constant. Hence, for any nonnegative measurable function $f$ and $\lambda_\chi$-a.e. $(x, g) \in X \times G$, we have

$$E_{h \lambda_\chi}[f | \mathfrak{J}_\psi](x, g) = \frac{\int_{X \times H} f(y, \gamma g) h(y, \gamma g) \, d\gamma \, \mu_\chi(dy)}{\int_{X \times H} h(y, \gamma g) \, d\gamma \, \mu_\chi(dy)}.$$

From (42) it follows that

$$M^h f(x, g) = E_{h \lambda_\chi}[h f | \mathfrak{J}_\varphi](x, g) = \frac{\int_X \int_H f(y, u(y) \gamma(u(x))^{-1} g) \chi(u(y)) \, d\gamma \, \mu_\chi(dy)}{\int_X \int_H h(y, u(y) \gamma(u(x))^{-1} g) \chi(u(y)) \, d\gamma \, \mu_\chi(dy)}.$$

(iii) If there exists some $x$ such that $\mu_x \sim \mu_\chi$, then the reduction of the cocycle given by (8) is “global” $\mu_\chi$-a.e.: there exists a measurable function $u$ and a closed subgroup $H$ such that the cocycle is cohomologous to an ergodic cocycle with values in $H$ and it is regular.

If there are a countable number of different equivalence classes among the measures $\mu_x$, $x \in X$, then by (i), for $\mu_\chi$-a.e. $x$, all the measures $\mu_x$ are equivalent and this equivalence class is that of $\mu_\chi$.

The last assertion of (iii) follows from assertion (iii) of Proposition 2.2.4. ■

5. ON THE EQUATION $H_{\tau x} = \varphi(x)H_x(\varphi(x))^{-1}$

In Theorem 2.1.3 we encounter a measurable family of subgroups $H_x$ such that the following conjugacy equation holds:

$$H_{\tau x} = \varphi(x)H_x(\varphi(x))^{-1} \quad \text{for } \mu_\chi\text{-a.e. } x \in X.$$  

For this conjugacy problem, see [GoSi99].

5.1. Nilpotent groups. When $G$ is a nilpotent connected Lie group, the subgroups $H_x$ are conjugate to a fixed subgroup $H$.

**Theorem 5.1.1 ([GoSi99]).** Assume $G$ is a nilpotent connected Lie group. If $(H_x)$ is a measurable family of subgroups such that (43) holds $\mu$-a.e., where $\mu$ is a $\sigma$-finite measure which is quasi-invariant and ergodic for
\[ \tau, \text{ then there is a fixed closed subgroup } H \text{ and a measurable map } x \mapsto a(x) \text{ from } X \text{ into } G \text{ such that for } \mu_X\text{-a.e. } x \in X, \]
\[ H_x = a(x)H(a(x))^{-1}. \]

**Proof.** We equip the set \( \mathcal{F}(G) \) of closed subsets of \( G \) with Chabauty’s topology (cf. Section 2).

We know that the map \( x \in X \mapsto H_x \in \mathcal{F}(G) \) is measurable. For any \( F \in \mathcal{F}(G) \), we have
\[ \{ x \in X : \{ gH_x g^{-1} : g \in G \} \subset F \} = \left\{ x \in X : H_x \subset \bigcap_{g \in G} g^{-1} F g \right\}. \]

It follows that the map \( x \in X \mapsto \{ gH_x g^{-1} : g \in G \} \in \mathcal{F}(G) \) is measurable.

We denote by \( \mathfrak{g} \) the Lie algebra of \( G \) and denote by \( \text{ad} \) the adjoint representation of \( \mathfrak{g} \) (i.e. for any \( (X,Y) \in \mathfrak{g}^2 \), \( (\text{ad} X)(Y) = [X,Y] \)). We denote by \( \exp : \mathfrak{g} \to G \) the exponential map and by \( \text{Ad} \) the adjoint representation of \( G \) on \( \mathfrak{g} \). We have
\[ g \exp X g^{-1} = \exp(\text{Ad} g(X)), \quad \forall g \in G, \forall X \in \mathfrak{g}, \]
\[ \text{Ad}(\exp Y) = \text{Exp}(\text{ad} Y) = \sum_{k \in \mathbb{N}} \frac{(\text{ad} Y)^k}{k!}, \quad \forall Y \in \mathfrak{g}. \]

**First case.** Assume that \( G \) is a connected and simply connected nilpotent Lie group. For \( \mu \)-a.e. \( x \in X \), we have
\[ \{ gH_x g^{-1} : g \in G \} = \{ gH_{\tau x} g^{-1} : g \in G \}. \]

Since the points of \( \mathcal{F}(G) \) are separated by a countable family of continuous functions, there exists a closed subgroup \( H \) of \( G \) such that, for \( \mu \)-a.e. \( x \in X \),
\[ \{ gH_x g^{-1} : g \in G \} = \{ gH_{\tau x} g^{-1} : g \in G \}. \]

Now, from the proposition below, this equality implies that the two open dense subsets \( \{ gH_x g^{-1} : g \in G \} \) and \( \{ gHg^{-1} : g \in G \} \) of \( \{ gHg^{-1} : g \in G \} \) are not disjoint. Therefore the two \( G \)-orbits coincide. Hence the result.

**Second case.** Assume that \( G \) is a connected nilpotent Lie group. Let \( f : \tilde{G} \to G \) be a group cover of \( G \) with \( \tilde{G} \) connected and simply connected (see [Ho65, Ch. IV, Theorems 2.2 and 3.2]).

If \( H \) is a closed subgroup of \( G \) then \( \tilde{H} = f^{-1}(H) \) is a closed subgroup of \( \tilde{G} \). Moreover, the \( G \)-orbit of \( H \) is the image under \( f \) of the \( \tilde{G} \)-orbit of \( \tilde{H} \). The theorem follows from the first case. \( \blacksquare \)

**Proposition 5.1.2.** For any closed subgroup \( H \) of a connected simply connected nilpotent Lie group \( G \), the \( G \)-orbit \( \{ gHg^{-1} : g \in G \} \) of \( H \) is open in its closure.
Proof. We know that the exponential map exp is an analytic diffeomorphism. We set \( \Sigma = \{1, \ldots, \dim(\mathfrak{g})\} \). For any \( p \in \Sigma \), we consider the exterior product \( V_p = \bigwedge_p \mathfrak{g} \) and the corresponding projective space \( \mathbf{P}(V_p) \). We denote by \( \pi_p \) the natural map from \( V_p \setminus \{0\} \) onto \( \mathbf{P}(V_p) \).

To each \( p \)-dimensional subspace \( \mathfrak{u} \) of \( \mathfrak{g} \) we associate the element \( u_\mathfrak{u} = \pi_p(u_1 \wedge \cdots \wedge u_p) \) of \( \mathbf{P}(V_p) \) where \( (u_1, \ldots, u_p) \) is a linear basis of \( \mathfrak{u} \). We denote by \( D_p \) the image in \( \mathbf{P}(V_p) \) of the set of \( p \)-dimensional subspaces of \( \mathfrak{g} \). We consider the disjoint union \( \bigcup_{p \in \Sigma} D_p \), equipped with the following topology. A sequence \( (u_n)_{n\in\mathbb{N}} \) converges to \( x \) if the following two properties are satisfied:

- there exist \( N \in \mathbb{N} \) and \( r \in \Sigma \) such that \( u_n \in D_r \) for \( n \geq N \),
- the sequence \( (u_n)_{n \geq N} \) converges to \( x \) on \( D_r \) for the usual induced topology of \( \mathbf{P}(V_p) \).

One easily sees that a sequence \( (\mathfrak{v}_n) \) of subspaces of \( \mathfrak{g} \) converges in Chabauty’s topology if and only if \( (u_{\mathfrak{v}_n})_{n \in \mathbb{N}} \) converges in \( \bigcup_{p \in \Sigma} D_p \). Hence, the map \( \mathfrak{v} \mapsto u_\mathfrak{v} \) is a homeomorphism from the set of nontrivial subspaces of \( \mathfrak{g} \) onto \( \bigcup_{p \in \Sigma} D_p \).

Let \( H \) be a closed subgroup of \( G \) with Lie algebra \( \mathfrak{h} = \exp^{-1}(H) \). The \( G \)-orbit \( \{gHg^{-1} : g \in G\} \) of \( H \) is identified with the \( \bigwedge_p \text{Ad} G \)-orbit of \( u_\mathfrak{h} \). Now, for a connected simply connected nilpotent Lie group \( G \), we know (see for example [BoSe64]) that this orbit is open in its closure. This yields the result. \( \blacksquare \)

5.2. A counterexample. Let \( G \) be the semidirect product of \( \mathbb{R} \) and \( \mathbb{C}^2 \), with the composition law

\[
(t, z_1, z_2) \ast (t', z'_1, z'_2) = (t + t', z_1 + e^{2\pi it}z'_1, z_2 + e^{2\pi it}z'_2),
\]

where \( \theta \) is a fixed irrational.

The conjugate in \( G \) of \((0, z_1, z_2)\) by \( a = (s, v_1, v_2)\) is

\[
(s, v_1, v_2)(0, z_1, z_2)(s, v_1, v_2)^{-1} = (0, e^{2\pi is}z_1, e^{2\pi is}z_2).
\]

Consider the dynamical system defined by an irrational rotation \( (\tau : x \mapsto x + \alpha \mod 1) \) on \( X = \mathbb{R}/\mathbb{Z} \). Let \( \Phi : X \to G \) be the cocycle defined by \( \Phi(x) = (\varphi(x), 0, 0) \), where \( \varphi \) has its values in \( \mathbb{Z} \).

Let \( H_x := \{0, v z_1, v e^{2\pi iv\psi(x)}z_2 : v \in \mathbb{R}\} \), where \( \psi \) is a function to be defined and \( z_1, z_2 \) are given natural real numbers. Consider the function \( x \mapsto H_x \) with values in the set of closed subgroups of \( G \). It satisfies the conjugacy relation

\[
H_{\tau x} = \Phi(x)H_x(\Phi(x))^{-1}
\]

if \( \varphi \) has integral values and satisfies

\[
\theta \varphi(x) + \psi(x) = \psi(\tau x) \mod 1.
\]
For every $\alpha$ whose partial quotients are not bounded, there are real numbers $\beta$ and $r$ for which the function
\[ \varphi := 1_{[0,\beta]} - 1_{[0,\beta]}(\cdot + r) \]
is not a coboundary and there are irrational values of $s$ such that
\[ e^{2\pi i s(1_{[0,\beta]} - 1_{[0,\beta]}(\cdot + r))} \]
is a multiplicative coboundary (cf. [Co07]).

If we take for $\theta$ one of these values of $s$ and for $\psi$ a function satisfying the multiplicative coboundary equation
\[ e^{2\pi i \theta \varphi} = e^{2\pi i (\psi \circ \tau - \psi)} \],
we get (46).

**Proposition 5.2.1.** For these choices of $\beta$, $r$, $\theta$, $\psi$, there is no subgroup $H$ such that the equation
\[ H x = a(x) H (a(x))^{-1} \]
has a measurable solution $a$.

**Proof.** Suppose that there are a fixed subgroup $H$ and a measurable function $a : X \to G$ such that $H x = a(x) H (a(x))^{-1}$.

According to (44), this is equivalent to the existence of a function $t$ defined on $X$ such that the set
\[ \{(0, ve^{2\pi i t(x)} z_1, ve^{2\pi i (\theta t(x) + \psi(x))} z_2) : v \in \mathbb{R}\} \]
does not depend on $x$. This implies that $t$ and $\psi + \theta t$ have a fixed value mod 1; therefore $\theta(\varphi(x) - t(x) + t(\tau x)) = \theta \varphi(x) + \psi(x) - \psi(\tau x) \mod 1 = 0$. As $\varphi$ and $t - t \circ \tau$ have integral values and $\theta$ is irrational, it follows that
\[ \varphi = t \circ \tau - t, \]
contrary to the fact that $\varphi$ is not a coboundary. $\blacksquare$

**Remark 5.2.2.** By the same arguments it can be shown that the cocycle
\[ 1_{[0,\beta]} - 1_{[0,\beta]}(\cdot + r) \]
is nonregular in the sense of Definition 2.2.1.

6. **COMMENTS**

6.1. **Remarks on transience/recurrence.** The cocycle $(\varphi_n)_{n \in \mathbb{Z}}$ gives the position at time $n$ of the “vertical” coordinate for the iterates $\tau^n \varphi$. If it is recurrent (i.e. if the stationary random walk $(\varphi_n)$ returns infinitely often to any neighborhood of the identity element), the transformation $\tau \varphi$ is conservative.

The ergodicity of the basis implies that the cocycle is either recurrent or transient. When $\varphi$ has its values in $\mathbb{R}$ and is integrable, $(\varphi_n)_{n \in \mathbb{Z}}$ is recurrent if and only if $\mu(\varphi) = 0$.

For every amenable group $G$ and every ergodic system $(X, \mu, \tau)$, there is a measurable ergodic cocycle $(\varphi, \tau)$ over the system, taking its values in $G$, such that $(X \times G, \mu \otimes m_G, \tau \varphi)$ is ergodic (cf. [He79], [GoSi85]). However, a problem is to construct explicitly recurrent cocycles generated by regular functions over particular dynamical systems and to find whether or not they are ergodic.

In the recurrent case, the transformation $\tau \varphi$ is conservative: there is no wandering set $E$ with a positive measure (wandering means that the images
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$\left(\tau^{-k}E, k \in \mathbb{Z}\right)$ are pairwise disjoint). This implies that every subinvariant set is invariant $\mu \otimes m_G$-a.e.

Note that the ergodic decomposition of a recurrent system gives recurrent systems. In particular, the recurrence of the cocycle $(\varphi_n)$ relative to $\mu$ implies (with the notations of 2.1.3) that for $\mu$-a.e. $x$ the cocycle $(\varphi_n)$ is recurrent relative to the measure $\mu_x$, which is infinite if the cocycle is not regular (cf. Theorem 2.2.2).

Assume now that the cocycle is transient. Let $E$ be a wandering set and $h > 0$ be a function on $G$ such that $\int h \, dg = 1$. The series $\sum_{k \in \mathbb{Z}} h(\varphi_k(x)g)$ converges for $\mu \otimes m_G$-a.e. $(x,g)$, according to

$$
\int \sum_{E, k \in \mathbb{Z}} h(\varphi_k(x)g) \, d\mu(x) \, dg = \int h(g) \left( \sum_{E, k \in \mathbb{Z}} 1_E(x, (\varphi_k(x))^{-1}g) \right) \, dg \, d\mu(x)
$$

$$
= \int h(g) \left( \sum_{E, k \in \mathbb{Z}} 1_E(\tau^{-k}x, (\varphi_k(\tau^{-k}x))^{-1}g) \right) \, d\mu(x) \, dg = \int h(g) \, dg = 1.
$$

The function $\tilde{h}(x,g) := \sum_{k \in \mathbb{Z}} h(\varphi_k(x)g)$ is therefore $\tau\varphi$-invariant and finite for $\mu \otimes m_G$-a.e. $(x,g)$.

The subgroups $H_x$ defined in Theorem 2.1.3 reduce to \{e\}, and the ergodic measures are given, up to a multiplicative factor, by $\lambda_{(x,g)}(f) = \sum_{k \in \mathbb{Z}} f(\tau^{-k}x, \varphi_k(x)g)$.

The function $\varphi$ is a coboundary with respect to the $\sigma$-finite measure $\tilde{\mu}_x(dy) := \sum_{k \in \mathbb{Z}} \delta_{\tau^{-k}x}(dy)$ (we get $u_x(y)\varphi(y)(u_x(\tau y))^{-1} = e$ by setting $u_x(y) := \varphi_k(x)$ at the point $y = \tau^{-k}x$). The ergodic decomposition of $\mu(dx) \times dg$ can be written

$$
\int \int_{X \times G} f(x,g) \, d\mu(x) \, dg = \int \int_{X \times G} \left[ (\tilde{h}(x,g))^{-1} \sum_{k \in \mathbb{Z}} f(\tau^{-k}x, \varphi_k(x)g) \right] \, h(g) \, d\mu(x) \, dg.
$$

This shows that, in the transient case, there is no interesting information in the ergodic decomposition. Therefore it is convenient to have examples of recurrent cocycles $(\varphi, \tau)$. A family of such cocycles is provided when the basic system is a rotation on the circle and $\varphi$ is a BV-function with values in $\mathbb{R}^d$. There are also examples over rotations for cocycles taking their values in nilpotent groups (see [Gr05], [Co07]). For rotations, using BV-functions, one can construct conformal probability measures $\mu_\chi$ for which the rotation is conservative. More precisely, if $\varphi$ is a BV-function on the circle with zero integral and $\chi$ is an exponential on $\mathbb{R}$, and if $\tau$ is an ergodic rotation, then there is a unique probability measure $\mu_\chi$ on the circle such that $d(\tau \mu_\chi)/d\mu_\chi = \chi \circ \varphi$ and the corresponding measure $\lambda_\chi$ on $X \times G$ is conservative due to Koksma’s inequality (see for example [CoGu00]).
6.2. Extension to a group action. For simplicity, we have restricted the paper to the framework of a single transformation, but the domain of validity can be extended by taking more generally the action of a countable group $\Gamma$. This gives access to more examples of transient cocycles with a nontrivial ergodic decomposition. Most of the results presented here when $\Gamma = \mathbb{Z}$ are still valid for the action of a countable group $\Gamma$.

Indeed, we can use the result of Theorem 3.1.1, since one can easily extend it from the case of a single invertible transformation to an ergodic group action. Another important point is the ergodicity of the measures given by a regular conditional probability with respect to the $\sigma$-algebra of invariant sets. This point also remains valid (see Remark 7.3.3 at the end of 7.3).

7. APPENDIX

In this appendix, we recall a selection lemma and some results on the conditional expectation and the ergodic decomposition that were used in the previous sections.

7.1. A selection lemma. Let $G$ be a lcsc group. Recall that the set $\mathcal{F}(G)$ of closed subsets of $G$ is equipped with Chabauty’s topology, for which the open sets are defined by

$$U(O, C) = \{S \in \mathcal{F}(G) : \forall U \in O, S \cap U \neq \emptyset \text{ and } S \cap C = \emptyset\},$$

where $O$ is a finite family of open sets of $G$, and $C$ is a compact subset of $G$.

The Borel structure associated to this topology is generated by the sets $\{S \in \mathcal{F}(G) : S \subseteq F\}$ where $F \in \mathcal{F}(G)$ (cf. 2.1). For the sake of completeness, we give a proof of a selection lemma (cf. the theorem of Kuratowski and Ryll-Nardzewski) that was used in Section 3:

**Lemma 7.1.1.** If $t \mapsto F_t$ is a Borel map from a Borel space $(T, \mathcal{T})$ to $\mathcal{F}(G)$, then there exists a Borel map $f$ from $T$ to $G$ such that $f(t) \in F_t$ for each $t \in T$.

**Proof.** Let $K$ be a compact set in $G$. Assume that $F_t \subseteq K$ for $t \in R \subseteq T$, where $R$ is a Borel set in $T$. For every $n \geq 1$, there exists a finite family $(K_{n,i} : i \in I_n)$ of compact sets such that $\text{diam}(K_{n,i}) < 1/n$ and $K \subseteq \bigcup_{j \in I_{n+1}} K_{n+1,j}$.

For a compact set $C$ in $G$, the set $\{t : F_t \cap C \neq \emptyset\}$ is Borel (its complement is the union of the sets $\{t : F_t \subseteq G \setminus U_n\}$, where $U_n$ is a basis of open neighborhoods of $C$). Therefore, for every $n$ and $j$, the set $\{t : F_t \cap K_{n,j} \neq \emptyset\}$ is Borel.
We define \( i_n(t) \) by \( i_n(t) = \inf\{j \in I_n : F_t \cap K_{n,j} \neq \emptyset\} \). The map \( t \mapsto K_{n,i_n(t)} \) is Borel.

Now we define the point \( f(t) \) for \( t \in R \) by

\[
f(t) := \bigcap_{n \geq 1} K_{n,i_n(t)}.
\]

From the condition on the diameters and the compactness of the sets, it follows that \( f(t) \) is well defined for every \( t \in T \).

We have to show that \( f \) is Borel, that is, \( \{t \in T : f(t) \in C\} \) is a Borel set for any closed subset \( C \) in \( G \). Let \( (O_k) \) be a decreasing sequence of open sets such that \( O_{k+1} \subset \overline{O}_{k+1} \subset O_k \) for every \( k \) and \( C = \bigcap_k O_k \). We have

\[
f(t) \in C \iff \bigcap_{n \geq 1} K_{n,i_n(t)} \subset O_k, \forall k \geq 1 \iff \bigcap_{n \geq 1} K_{n,i_n(t)} \subset \overline{O}_k, \forall k \geq 1.
\]

As \( \{t \in T : K_{n,i_n(t)} \subset \overline{O}_k\} \) is Borel for each \( k \), the assertion follows.

Now we construct \( f \) on the whole space. For any compact set \( K \) in \( G \), the map \( t \mapsto F_t \cap K \) is Borel, since the map \( (F,K) \mapsto F \cap K \) from \( \mathcal{F}(G) \) into itself is continuous for a fixed compact set \( K \). Let \( K_j \) be an increasing sequence of compact sets in \( G \) such that \( G = \bigcup_j K_j \).

We define \( f(t) \) by applying the previous construction to the map \( t \mapsto F_t \cap K \) on \( \{t : F_t \cap K \neq \emptyset\} \), then to \( t \mapsto F_t \cap K_2 \) on \( \{t : F_t \cap K_2 \neq \emptyset\} \cap \{t : F_t \cap K_1 = \emptyset\} \), and so on.

### 7.2. A lemma on conditional expectation

**Lemma 7.2.1.** Let \( \mathbb{P} \) be a probability measure on a measurable space \((E, \mathcal{F})\) and \( h \) a measurable positive function such that \( \int h \, d\mathbb{P} = 1 \). Then, for every sub-\( \sigma \)-algebra \( \mathcal{B} \) of \( \mathcal{F} \) and every measurable nonnegative (or \( h\mathbb{P} \)-integrable) function \( f \), we have

\[
\mathbb{E}_{h\mathbb{P}}[f \mid \mathcal{B}] = \mathbb{E}_\mathbb{P}[fh \mid \mathcal{B}] / \mathbb{E}_\mathbb{P}[h \mid \mathcal{B}] \quad \mathbb{P}\text{-a.e.}
\]

If \( \theta \) is a bijective bi-measurable map from \( E \) onto itself such that \( \theta\mathbb{P} \sim \mathbb{P} \), then, for any \( \mathbb{P} \)-integrable function \( f \), we have

\[
\mathbb{E}_\mathbb{P}[f \mid \mathcal{B}] = \mathbb{E}_\mathbb{P}\left[\left(\frac{d\theta\mathbb{P}}{d\mathbb{P}}\right)^{-1} \circ \theta \mid \mathcal{B}\right] \mathbb{E}_\mathbb{P}\left[f \circ \theta^{-1} \frac{d\theta\mathbb{P}}{d\mathbb{P}} \mid \theta\mathcal{B}\right] \circ \theta
\]

\[
= \frac{\mathbb{E}_\mathbb{P}\left[f \circ \theta^{-1} \frac{d\theta\mathbb{P}}{d\mathbb{P}} \mid \theta\mathcal{B}\right] \circ \theta}{\mathbb{E}_\mathbb{P}\left[\frac{d\theta\mathbb{P}}{d\mathbb{P}} \mid \theta\mathcal{B}\right] \circ \theta}.
\]

If \( \mathbb{P} = h\lambda \), where \( \lambda \) is a \( \sigma \)-finite \( \theta \)-invariant measure and \( \mathcal{B} = \mathcal{J} \) the \( \sigma \)-algebra of \( \theta \)-invariant sets in \( \mathcal{E} \), we have

\[
\mathbb{E}_{h\lambda}[f \circ \theta \mid \mathcal{J}] = \mathbb{E}_{h\lambda}\left[f \frac{h \circ \theta^{-1}}{h} \mid \mathcal{J}\right].
\]
Proof. We prove only the second assertion. For every bounded $\mathcal{B}$-measurable $\psi$ we have

$$
\int [E_P[f \circ \theta | \mathcal{B}] \psi dP = \int f \circ \theta \psi dP = \int f \psi \circ \theta^{-1} \frac{d\theta P}{dP} dP
$$

$$
= \int [E_P \left[ \frac{d\theta P}{dP} \right] \theta \mathcal{B}] \psi \circ \theta^{-1} dP = \int \left[ E_P \left[ \frac{d\theta P}{dP} \right] \theta \mathcal{B} \right] \circ \theta \psi dP
$$

$$
= \int E_P \left[ \frac{d\theta^{-1} P}{dP} \theta \mathcal{B} \right] \mathcal{B} = \int E_P \left[ \frac{d\theta P}{dP} f \theta \mathcal{B} \right] \circ \theta dP,
$$

which implies (48):

$$
E_P[f \circ \theta | \mathcal{B}] = E_P \left[ \frac{d\theta^{-1} P}{dP} \theta \mathcal{B} \right] \mathcal{B} E_P \left[ \frac{d\theta P}{dP} f \theta \mathcal{B} \right] \circ \theta.
$$

7.3. Regular conditional probability

Definition 7.3.1. Let $(E, \mathcal{F}, P)$ be a probability space and $\mathfrak{B}$ a sub-$\sigma$-algebra of $\mathcal{F}$. A regular conditional probability relative to $\mathfrak{B}$ and $P$ is a map $P$ from $E \times \mathcal{F}$ to $[0, 1]$ such that

- For every $x \in E$, $P(x, \cdot)$ is a probability measure on $\mathcal{F}$.
- For every $A \in \mathcal{F}$, the map $x \in E \mapsto P(x, A)$ is a version of the conditional expectation of $1_A$ with respect to the $\sigma$-algebra $\mathfrak{B}$. This map is thus $\mathfrak{B}$-measurable and satisfies, for every $\mathfrak{B}$-measurable function $\varphi$,

$$
\int_E 1_A(x) \varphi(x) \, P(dx) = \int_E P(x, A) \varphi(x) \, P(dx).
$$

For every $\mathcal{F}$-measurable nonnegative or bounded function $f$, $Pf$ defined by $Pf(x) := \int_E f(y) \, P(x, dy)$ is then a version of the conditional expectation of $f$ with respect to $\mathfrak{B}$.

For the existence of a regular conditional probability, we can refer to the general setting used in Neveu’s book ([Ne64, Corollaire, Proposition V-4-4]):

In the following, we will assume that there exists an approximating compact class in $(E, \mathcal{F}, P)$ (see [Ne64] for this notion) and that $\mathcal{F}$ is generated by a countable family.

Theorem 7.3.2 ([Ne64]). For every $\sigma$-algebra $\mathfrak{B}$ in $\mathcal{F}$, there exists a regular conditional probability with respect to $\mathfrak{B}$.

This result applied to the product space $(X \times G, \mathcal{X} \times \mathcal{B}_G)$, the probability $h \lambda$ on $X \times G$, where $h > 0$ on $X \times G$ is such that $\int h(x, g) \mu \chi(dx) \chi(g) m_G(dg) = 1$, and the sub-$\sigma$-algebra $\mathfrak{J}$ of $\tau_\varphi$-invariant sets (see Notations 2.1.1 and 2.1.2) gives the regular conditional probability $P^h$ used in Section 2.
Now we have to show that the probability measures $P^h((x,g), \cdot)$ are $\tau_\varphi$-ergodic. For the action of a single transformation, this can be done by applying the ergodic theorem (cf. [Aa97]). For the sake of completeness we give a proof in the last subsection below.

**Remark 7.3.3.** When the action of a single transformation $\tau$ on $X$ is replaced by the Borel action of a countable group, the proof of the ergodicity of $P^h((x,g), \cdot)$ is more difficult. A reference is [GrSc00].

### 7.4. Ergodic theorem and ergodic decomposition

**Notations 7.4.1.** Let $\theta$ be a bijective bi-measurable transformation on a measurable space $(E, \mathcal{F})$, $\mu$ a positive $\sigma$-finite $\theta$-quasi-invariant measure, and $\mathcal{J} := \{B \in \mathcal{F} : \theta^{-1}B = B\}$.

Let $h$ be a measurable function on $E$ such that $h(x) > 0$ and $\mu(h) = 1$. Let $P^h$ be a regular conditional probability with respect to the probability measure $h\mu$ and the $\sigma$-algebra $\mathcal{J}$ of $\theta$-invariant measurable subsets.

Let $T_h$ be the contraction of $L^1(E, \mathcal{F}, h\mu)$, in duality with the operator of composition with $\theta$ acting on $L^\infty(E, \mathcal{F}, \mu)$, defined by

$$T_h f(x) = f \circ \theta(x) \frac{d(\theta(h\mu))}{d(h\mu)}(x).$$

Replacing $\theta$ by $\theta^{-1}$ we get the inverse operator $T_h^{-1}$.

**Proposition 7.4.2.** For every $f \in L^1(E, \mathcal{F}, h\mu)$ and $\mu$-a.e. $x \in E$,

$$\sum_{k=-n}^{n} T_h^k f(x)/ \sum_{k=-n}^{n} T_h^k 1(x) \xrightarrow{n \to +\infty} \mathbb{E}_{h\mu}[f | \mathcal{J}](x).$$

**Proof.** Applying Hurewicz’s ergodic theorem to the contraction $T_h$, we find that the sequence

$$(\sum_{k=0}^{n} T_h^k f(x)/ \sum_{k=0}^{n} T_h^k 1(x))_{n \geq 1}$$

converges $\mu$-a.e., and the same result holds for the contraction $T_h^{-1}$.

On the conservative part $C$ the limit in both directions is equal to $\mathbb{E}_{h\mu}[f | \mathcal{J}](x)$, so that on $C$,

$$\lim_{n \to +\infty} \left( \sum_{k=-n}^{n} T_h^k f/ \sum_{k=-n}^{n} T_h^k 1 \right) = \mathbb{E}_{h\mu}[f | \mathcal{J}] \quad \mu$$.a.e.

On the dissipative part $D$, the limit is the quotient of the series. For $j, k$ in $\mathbb{Z}$, we have

$$T_h^j f \circ \theta^k = T_h^{j-k} f \frac{d(h\mu)}{d(\theta^{-k}(h\mu))} = \frac{T_h^j f}{T_h^{-k} 1}.$$
This implies that on $D$ the quotient of the series is a $\theta$-invariant function
and, for every measurable $\theta$-invariant function $\varphi$ which is null on $C$,

$$\sum_{k \in \mathbb{Z}} T_h^k f \varphi (h \mu) = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} T_h f \varphi \mu j \circ \theta^k = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f \varphi T_h^{-k} \mu j \circ \theta^k = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f \varphi T_h^k \mu j \circ \theta^k = \sum_{k \in \mathbb{Z}} f \varphi \mu j \circ \theta^k = \int f \varphi \mu.$$

On $D$ the quotient of the series is therefore equal to $\mathbb{E}_{h \mu}[f \mid \mathcal{J}]$. 

**Lemma 7.4.3.** For $\mu$-a.e. $x \in E$, the measure $\theta P^h(x, \cdot)$ is absolutely continuous with respect to $P^h(x, \cdot)$ and

$$\frac{d \theta P^h(x, \cdot)}{d P^h(x, \cdot)} = \frac{d \theta(h \mu)}{d(h \mu)} = \frac{h \circ \theta^{-1}}{h} \frac{d \theta(h \mu)}{d(\mu)}.$$

**Proof.** For a positive $\mathcal{F}$-measurable $f$ and a $\mathcal{J}$-measurable positive function $\varphi$, we have

$$\int f \circ \theta \varphi d(h \mu) = \int f \circ \theta \varphi \circ \theta d(h \mu) = \int f \varphi \frac{d \theta(h \mu)}{d(h \mu)} d(h \mu).$$

This shows, $\mu$-a.e.,

$$\theta(P^h)(f) = P^h(f \circ \theta) = \mathbb{E}_{h \mu}[f \circ \theta \mid \mathcal{J}]$$

$$= \mathbb{E}_{h \mu} \left[ f \frac{d \theta(h \mu)}{d(h \mu)} \mid \mathcal{J} \right] = P^h \left( f \frac{d \theta(h \mu)}{d(h \mu)} \right).$$

We then have:

**Corollary 7.4.4.** For the elements $x \in E$ for which (50) holds, $T_h$ is a positive contraction of $L^1(E, \mathcal{F}, P^h(x, \cdot))$. For every $f \in L^1(E, \mathcal{F}, P^h(x, \cdot))$ and $P^h(x, \cdot)$-a.e. $y \in E$,

$$\sum_{k=-n}^{n} T_h^k f(y) / \sum_{k=-n}^{n} T_h^k 1(y) \xrightarrow{n \to +\infty} \mathbb{E}_{P^h(x, \cdot)}[f \mid \mathcal{J}](y).$$

**Theorem 7.4.5.** A decomposition of the measure $\mu$ into ergodic components is given by

$$\mu(dy) = \int (h(\cdot))^{-1} P^h(x, \cdot) h(x) \mu(dx).$$

**Proof.** The equality is clear. It remains to prove the ergodicity of the probability measures $P^h(x, \cdot)$ for $\mu$-a.e. $x \in E$.

From (51) and Proposition 7.4.2, we have, for every $f \in L^1(E, \mathcal{F}, h \mu)$ and $\mu$-a.e. $x \in E$,

$$\sum_{k=-n}^{n} T_h^k f(y) / \sum_{k=-n}^{n} T_h^k 1(y) \xrightarrow{n \to +\infty} \mathbb{E}_{h \mu}[f \mid \mathcal{J}](y) = P^h f(y)$$
for $P(x, \cdot)$-a.e. $y \in E$. The functions $g = P^h f$ and $g^2 = (P^h f)^2$ are $\mathcal{J}$-measurable and therefore $P^h$-invariant $\mu$-a.e.: $P^h g(x) = g(x)$ and $P^h g^2(x) = g^2(x)$ for $\mu$-a.e. $x \in E$. By the Cauchy–Schwarz inequality, this implies $g(y) = g(x)$ for $P(x, \cdot)$-a.e. $y \in E$.

Let $\mathcal{F}_0$ be a countable Boole algebra which generates $\mathcal{F}$. For $x \in E$, let $Q^h_x$ be a regular conditional probability with respect to the probability measure $P^h(x, \cdot)$ and the $\sigma$-algebra $\mathcal{J}$. From the previous property and Corollary 7.4.4, we obtain, for $\mu$-a.e. $x \in E$ and $P^h(x, \cdot)$-a.e. $y \in E$,

$$\forall A \in \mathcal{F}_0, \quad Q^h_x(y, A) = P^h(x, A),$$

and consequently, for $\mu$-a.e. $x \in E$ and $P^h(x, \cdot)$-a.e. $y \in E$, we have the same property for every $A \in \mathcal{F}$.

For every $I \in \mathcal{J}$, we know that

$$Q^h_x(y, I) = \mathbb{E}_{P^h(x, \cdot)}[1_I | \mathcal{J}](y) = 1_I(y) \quad \text{for } P(x, \cdot)$$-

a.e. $y \in E$, and therefore as above

$$Q^h_x(y, I) = 1_I(x) \quad \text{for } P(x, \cdot)$-a.e. $y \in E.$

It follows that, for $\mu$-a.e. $x \in E$,

$$\forall I \in \mathcal{J}, \quad P^h(x, I) = Q^h_x(y, I) = 1_I(y) \quad \text{for } P(x, \cdot)$-a.e. $y \in E.$

This implies the ergodicity of the measures $P^h(x, \cdot)$ for $\mu$-a.e. $x$.

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**REFERENCES**


