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HITTING HALF-SPACES OR SPHERES BY ORNSTEIN–UHLENBECK TYPE DIFFUSIONS

BY

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Abstract. The purpose of the paper is to provide a general method for computing the hitting distributions of some regular subsets D for Ornstein–Uhlenbeck type operators of the form $\frac{1}{2}\Delta + F \cdot \nabla$, with F bounded and orthogonal to the boundary of D. As an important application we obtain integral representations of the Poisson kernel for a half-space and balls for hyperbolic Brownian motion and for the classical Ornstein–Uhlenbeck process. The method developed in this paper is based on stochastic calculus and on the skew product representation of multidimensional Brownian motion and yields more complete results than those based on the Feynman–Kac technique.

1. Introduction. A detailed knowledge of the hitting distribution (equivalently: harmonic measure) of a domain for a diffusion with a given generator $\mathcal{A}$ is fundamental for solving many potential-theoretic problems, e.g. the Dirichlet problem for a domain or the Harnack inequality or even the boundary Harnack inequality for harmonic functions with respect to $\mathcal{A}$.

In this paper we compute the hitting distributions of some subsets D for operators of the form $\frac{1}{2}\Delta + F \cdot \nabla$ on subsets of $\mathbb{R}^n$. It is worth pointing out that even in the case of the classical Ornstein–Uhlenbeck diffusion explicit formulas for half-spaces or balls were obtained only quite recently (see [2] and [14]). Although the inspiration for our work comes from the paper [10], where the potential theory for bounded sets D and the operators

$$\frac{1}{2}\Delta + F \cdot \nabla \tag{1.1}$$

was established, the purpose as well as most of the technical tools are here different: instead of setting up a general theory, we focus on providing explicit formulas for hitting distributions for some important operators of the above type and sets D. The importance of explicit formulas is highlighted e.g. in the recent papers [7] and [16], where the precise asymptotics for the Poisson kernel for Bessel diffusions was obtained. Throughout the paper we assume

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that the vector field F in (1.1) is bounded and orthogonal to the boundary of D. The method developed here is based on stochastic calculus and Girsanov's theorem and consists in computing various integral functionals of Brownian motion and representing them in terms of special functions.

We provide a closed formula for the density function of the hitting ditribution, i.e. the Poisson kernel of a half-space or a ball for hyperbolic Brownian motion or for the classical Ornstein–Uhlenbeck process. The importance of hyperbolic Brownian motion stems from the fact that it is the canonical diffusion on hyperbolic spaces; it also has some important applications in risk theory in financial mathematics (see [11] and [20]). Explicit integral representations are crucial in obtaining estimates of the Poisson kernel and of the Green function [4], [8]. In these papers, the main tool was the Feynman–Kac formula, applied to describe the distribution of a stopped multiplicative functional. The present approach, based on methods related to Girsanov's theorem, enables us to obtain representation formulas for the Poisson kernel, different from those mentioned above. The advantage of this approach is seen in Theorem 4, where we obtain the precise asymptotics of the Poisson kernel for large values of parameters. Another result worth mentioning is Theorem 6, where we provide a convenient representation of the Poisson kernel of a ball. Also the formula for the Poisson kernel of a ball for the classical Ornstein–Uhlenbeck diffusion is more complete than the one obtained in [14] (as a series representation only).

The paper is organized as follows. In Section 2 we provide a general framework for the next sections. Throughout the paper we assume that in (1.1) we deal with a potential vector field F on D, orthogonal to the boundary. Under this assumption, with the aid of stochastic calculus and Girsanov's theorem, we establish a general formula for the harmonic measure of the set D (Theorem 2).

In Section 3 we provide a closed formula for the Poisson kernel $P(x_n, y)$ of a half-space for the hyperbolic Brownian motion on the real hyperbolic space $\mathbb{H}^n$ (Theorem 3) and provide an asymptotic formula for $P(x_n, y)$ (Theorem 4). In Section 4 we provide an integral representation of the Poisson kernel for concentric balls for hyperbolic Brownian motion on the ball model $\mathbb{D}^n$. We remark here that a similar representation from [5] depends on additional conjectures on the zeros of some hypergeometric functions which so far remain unsettled. The important tool here, as well as in the next section, is the skew-product representation of the n-dimensional Brownian motion. In Section 5 we provide an integral representation for the Poisson kernel of a ball for the classical Ornstein–Uhlenbeck process (Theorem 7). In Appendix we collect some useful information on Bessel functions, hypergeometric and Legendre functions and on the skew-product of n-dimensional Brownian motion.

2. Change of measure due to Girsanov's theorem

Notation. For $n > 2$ we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, $\langle x, y\rangle$ denotes the standard inner product of $x, y \in \mathbb{R}^n$, and by $|x|$ we denote the Euclidean length of a vector $x \in \mathbb{R}^n$. The ball with center at zero and radius r is written as $B_r = \{x \in \mathbb{R}^n : |x| < r\}$; its boundary, which is the $(n-1)$-dimensional sphere, is denoted by $S_r^{n-1} = \{x \in \mathbb{R}^n : |x| = r\}$ and the spherical measure on S_r^{n-1} is denoted by σ_r^{n-1}. Furthermore, we write $f(x) \sim g(x)$ as $x \to b$ when $\lim_{x\to b} f(x)/g(x) = 1$. If for two functions f and g there exist constants c_1, c_2 such that $c_1 < f(x)/g(x) < c_2$ for every $x \in D$ we will write briefly $f \approx g, x \in D$.

Throughout the paper D will stand for a domain in $\mathbb{R}^n$ with a smooth, connected boundary ∂D, and F will be a bounded vector field which is defined on an open Lipschitz set U containing D. We assume that F is continuously differentiable up to the boundary of U and continuously vanishes on the boundary of U. We further assume that $F = \nabla V$ on D for a scalar valued function V and call the function V a *potential* (and F a *potential vector field* on D). We set $F = V \equiv 0$ on the complement of U. We say that the vector field F is *orthogonal to the boundary* ∂D if for every differentiable curve $\Gamma : [0, 1) \to \partial D$ we have $F(\Gamma(s)) \cdot \Gamma'(s) = 0$ for every $s \in [0, 1)$.

LEMMA 2.1. *Under the above assumptions, if the potential vector field F is orthogonal to the boundary of the set D then the potential function V determined, up to a constant, by the equation*

$$\nabla V(x) = F(x)$$

is constant on the boundary ∂D.

Proof. Fix $x_0 \in \partial D$. The potential V is given by the curve integral

$$V(x) = \int_\gamma F(r)\, dr + V(x_0),$$

where γ is an arbitrary continuously differentiable path beginning at x_0 and ending at x. For $x \in \partial D$ we choose γ to follow the boundary of the set D, i.e. $\gamma : [0, 1] \to \partial D$, $\gamma(0) = x_0$, $\gamma(1) = x$. Then

$$V(x) - V(x_0) = \int_0^1 F(\gamma(s)) \cdot \gamma'(s)\, ds = 0.$$

Since ∂D is connected, we obtain the conclusion. ∎

Throughout the paper we work within the framework of the canonical representation of processes, i.e. our basic probability space is the space of all continuous $\mathbb{R}^n$-valued functions defined on $[0, \infty)$ with appropriate σ-fields (see [17]). The standard n-dimensional Brownian motion is denoted by $W(t) = (W_1(t), \ldots, W_n(t))$.

Define a process X by the SDE

$$dX(t) = dW(t) + F(X(t))dt\,, \tag{2.1}$$

under the conditions specified above. Then X is a local diffusion on U with generator $L = \frac{1}{2}\Delta + F(x)\cdot\nabla$. Since the field F is bounded, X can be defined as a local semimartingale (see e.g. [17]). Let τ be the first exit time of the trajectory from the set D. The harmonic measure w^x on ∂D is defined as the distribution of $X(\tau)$ under the distribution P^x of the process X starting at $x \in D$. We define a local martingale M by the formula

$$M(t) = \int_0^t F(W(s)) \cdot dW(s). \tag{2.2}$$

Its quadratic variation is then given by the formula

$$\langle M\rangle(t) = \int_0^t |F(W(s))|^2\, ds.$$

We further define the basic object of our study, namely

$$N(t) = \exp\Big(M(t) - \frac{1}{2}\langle M\rangle(t)\Big). \tag{2.3}$$

We now provide the basic formula for the harmonic measure of the process defined by (2.1) under some additional conditions.

THEOREM 2.2. *Under the conditions stated above, suppose that X is the process defined by the system* (2.1). *Assume additionally that:*

(i) *The vector field F is potential and orthogonal to the boundary of D.*
(ii) *For every $t > 0$,*

$$\mathbb{E}^x[\exp(\langle M\rangle(t\wedge\tau))] < \infty. \tag{2.4}$$

(iii)

$$\{N(t\wedge\tau)\}_{t>0} \text{ is uniformly integrable.} \tag{2.5}$$

Then for $x \in D$ the harmonic measure w^x has the density function

$$w^x(dz) = e^{V(\partial D) - V(x)}\mathbb{E}^x[e^{-\frac{1}{2}\int_0^\tau (|\nabla V(W(s))|^2 + \Delta V(W(s)))\,ds};\, W(\tau)\in dz],$$

where V is the potential function of the field F and $V(\partial D)$ is its value on ∂D.

Proof. According to (2.2) the process

$$F(W(t))\cdot dW(t) - \frac{1}{2}|F(W(t))|^2 dt = dM(t) - \frac{1}{2}d\langle M\rangle(t)$$

is a local semimartingale. Writing, as in (2.3),

$$N(t) = \exp\Big(M(t) - \frac{1}{2}\langle M\rangle(t)\Big)$$

we deduce, as an application of Itô's formula, that $N(t)$ is a local martingale. If we define a measure Q^x by

$$\left.\frac{dQ^x}{dP^x}\right|_{\mathcal{F}_{t\wedge\tau}} = N(t\wedge\tau),$$

then, as a consequence of Girsanov's theorem, (W, Q^x) and (X, P^x) are different descriptions of the same process, up to time τ (see [10]). Consequently, for a continuous bounded function f defined on $\mathbb{R}^n$ we obtain

$$\mathbb{E}^x f(X(t\wedge\tau)) = \mathbb{E}^x[N(t\wedge\tau);\ f(W(t\wedge\tau))].$$

Now, the condition (2.5) shows that the expression on the right-hand side converges to $\mathbb{E}^x[N(\tau); f(W(\tau))]$ as $t\to\infty$. The left-hand side converges to $\mathbb{E}^x f(X(\tau))$, by the continuity of the process X. This indicates that indeed w^x has a density given by

$$w^x(dz) = \mathbb{E}^x[N(\tau);\ W(\tau)\in dz]. \tag{2.6}$$

We now provide a further description of the function w^x. Recall that F is the potential of the vector field B. Define

$$Z(t) = V(W(t)).$$

Applying the Itô formula we see that

$$\begin{aligned} Z(t) - Z(0) &= \int_0^t \nabla V(W(s))\cdot dW(s) + \frac{1}{2}\int_0^t \Delta V(W(s))\,ds \\ &= M(t) - \frac{1}{2}\langle M\rangle(t) + \frac{1}{2}\langle M\rangle(t) + \frac{1}{2}\int_0^t \Delta V(W(s))\,ds. \end{aligned}$$

Consequently,

$$\begin{aligned} N(t) &= \exp\left(M(t) - \frac{1}{2}\langle M\rangle(t)\right) \\ &= \exp\left(Z(t) - Z(0) - \frac{1}{2}\int_0^t [|\nabla V(W(s))|^2 + \Delta V(W(s))]\,ds\right). \end{aligned} \tag{2.7}$$

Observe that $Y(\tau) = V(W(\tau)) = V(\partial D)$ since the vector field F is orthogonal to ∂D. Hence, stopping at $t\wedge\tau$ and taking expectation we get, when $t\to\infty$,

$$w^x(dz) = e^{V(\partial D)-V(x)}\mathbb{E}^x[e^{-\frac{1}{2}\int_0^\tau [|\nabla V(W(s))|^2+\Delta V(W(s))]\,ds};\ W(\tau)\in dz].$$

This, together with (2.6), finishes the proof. ∎

3. Harmonic measure of a hyperbolic horocycle in $\mathbb{H}^n$. For every $a>0$ we define $H_a = \{x\in\mathbb{R}^n : x_n > a\}$. In this section we consider the harmonic measure ω_a^x of the set H_a for the operator

$$\Delta_{\mathrm{LB}} = x_n^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - (n-2)x_n \frac{\partial}{\partial x_n}, \qquad n \geq 2.$$

The motivation for studying this operator comes from hyperbolic geometry. More precisely, this is the Laplace–Beltrami operator associated with the Riemannian metric in the half-space model $\mathbb{H}^n$ of real n-dimensional hyperbolic space. From the geometric point of view, the set H_a is the interior of the hyperbolic horocycle $\partial H_a = \{x \in \mathbb{R}^n : x_n = a\}$. Let $(B_i(t))_{i=1}^n$ be the n-dimensional Brownian motion on $\mathbb{R}^n$ with generator $\frac{d^2}{dx^2}$ (and not $\frac{1}{2}\frac{d^2}{dx^2}$), i.e. the variance $E^0 B_i^2(t)$ is $2t$. Then the Brownian motion on $\mathbb{H}^n$, $Y = (Y_i)_{i=1}^n$, can be described by the following system of stochastic differential equations:

$$\begin{cases} dY_1(t) = Y_n(t)dB_1(t), \\ dY_2(t) = Y_n(t)dB_2(t), \\ \cdots \\ dY_n(t) = Y_n(t)dB_n(t) - (n-2)Y_n(t)dt. \end{cases} \tag{3.1}$$

By the Itô formula the generator of the solution of this system is Δ_{LB}. The Laplace–Beltrami operator can be rewritten in the form $\Delta_{\mathrm{LB}} = 2x_n^2 L_1$, where

$$L_1 = \frac{1}{2}\Delta + F_1(x) \cdot \nabla, \tag{3.2}$$

with $F_1(x) = (0, \ldots, 0, (2-n)/(2x_n))$. Now, we make a change of time. Namely, we write

$$A(u) = \int_0^u Y_n^2(s)\, ds,$$

and

$$\sigma(t) = \inf\{u > 0 : A(u) > t\}.$$

If we now write

$$\tilde{B}_k(t) = \int_0^{\sigma(t/2)} Y_n(s)\, dB_k(s), \qquad k = 1, \ldots, n,$$

then $\tilde{B}_k$ are martingales with mutual variations $\langle \tilde{B}_k, \tilde{B}_l \rangle(t) = \delta(k,l)t$, $k, l = 1, \ldots, n$, so $\tilde{B} = (\tilde{B}_k)$ is the standard n-dimensional Brownian motion. Substituting

$$\tilde{Y}_k(t) = Y_k(\sigma(t/2)), \qquad k = 1, \ldots, n,$$

we find that (3.1) transforms into the following system of SDEs:

$$
(3.3)\qquad \begin{cases} d\tilde{Y}_1(t) = d\tilde{B}_1(t), \\ d\tilde{Y}_2(t) = d\tilde{B}_2(t), \\ \dots \\ d\tilde{Y}_n(t) = Z_n(t)d\tilde{B}_n(t) - (n-2)\dfrac{dt}{2Z_n(t)}. \end{cases}
$$

Again, by Itô's formula, L_1 is the generator of the process $\tilde{Y} = (\tilde{Y}_k)$. Since the change of time does not affect the exit place, the harmonic measures of the operators Δ_{LB} and L_1 are the same.

The potential of the vector field F_1 is given by $V_1(x) = (2-n)\ln(x_n)/2$. Moreover, it is easy to check that the vector field F_1 is orthogonal to ∂H_a. Using (2.2) and (2.7), we obtain

$$
M(t) = \frac{2-n}{2}\int_0^t \frac{dW_n(s)}{W_n(s)}, \qquad \langle M\rangle(t) = \left(\frac{n-2}{2}\right)^2 \int_0^t \frac{ds}{W_n^2(s)},
$$

$$
N(t) = \left(\frac{W_n(0)}{W_n(t)}\right)^{(n-2)/2} \exp\left(-\frac{n(n-2)}{8}\int_0^t \frac{ds}{W_n^2(s)}\right),
$$

where $W(t) = (W_1(t),\dots,W_n(t))$ denotes the standard Brownian motion in $\mathbb{R}^n$ starting from $W(0) = x$. If we put $\tau_a = \inf\{t > 0 : W(t) \notin H_a\}$ we obtain

$$
\begin{aligned}
\mathbb{E}^x \exp(\langle M\rangle(t\wedge\tau_a)) &= \mathbb{E}^x \exp\left(\frac{(n-2)^2}{4}\int_0^{t\wedge\tau_a} \frac{ds}{W_n^2(s)}\right) \\
&\le \mathbb{E}^x \exp\left(\frac{(n-2)^2}{4}\int_0^{t\wedge\tau_a} \frac{ds}{a^2}\right) < \infty.
\end{aligned}
$$

Moreover,

$$
N(t\wedge\tau_a) \le \left(\frac{W_n(0)}{W_n(t\wedge\tau_a)}\right)^{(n-2)/2} \le \left(\frac{x_n}{a}\right)^{(n-2)/2}, \qquad t \ge 0.
$$

Now, the results of Theorem 2.2 imply that

$$
(3.4)\quad \omega_a^x(dy) = \left(\frac{x_n}{a}\right)^{(n-2)/2} \mathbb{E}^x\left[\exp\left(-\frac{n(n-2)}{8}\int_0^{\tau_a} \frac{ds}{W_n(s)^2}\right); W(\tau_a) \in dy\right].
$$

The above formula enables us to find the density function $P_a(x,y)$, $x \in H_a$, $y \in \partial H_a$, of the measure $\omega_a^x(dy)$ with respect to the Lebesgue measure on ∂H_a. The scaling property of n-dimensional Brownian motion implies the scaling property for the Poisson kernels,

$$
P_a(x,y) = a^{1-n} P_1(x/a, y/a), \qquad x \in H_a,\ y \in \partial H_a.
$$

Moreover, the Brownian motion $W(t)$ and the set H_1 are invariant under translations $(\tilde{x}, x_n) \mapsto (\tilde{x} + b, x_n + b)$, where $b \in \mathbb{R}^{n-1}$. Consequently,

$P_1(x,y) = P_1((0,x_n),(\tilde{y}-\tilde{x},1))$ for all $x \in H_1$ and $y \in \partial H_1$. We will use these properties in further considerations to simplify the notation. We will write

$$P(x_n,y) = P_1((0,x_n),(y,1)), \quad y \in \mathbb{R}^{n-1}, \quad \text{and} \quad \tau = \tau_1.$$

THEOREM 3.1. *For every $x_n > 1$ and $y \in \mathbb{R}^{n-1}$ we have*

(3.5)

$$P(x_n,y) = \frac{1}{2^{\nu-1}\pi^{\nu+1}} \frac{x_n^\nu}{|y|^{\nu-1}} \int_0^\infty \frac{J_\nu(t)Y_\nu(tx_n) - J_\nu(tx_n)Y_\nu(t)}{J_\nu^2(t)+Y_\nu^2(t)} t^\nu K_{\nu-1}(t|y|)\,dt,$$

where $\nu = (n-1)/2$.

Proof. Observe that the integral appearing in (3.4) as well as the hitting time τ depend only on the last coordinate of the Brownian motion $W(t) = (\tilde{W}(t), W_n(t))$. Since the processes $\tilde{W}(t)$ and $W_n(t)$ are independent we obtain

$$(3.6) \quad \omega_1^x(dy) = x_n^{(n-2)/2} \int_0^\infty \mathbb{E}^0[\tilde{W}(s) \in dy]\mathbb{E}^{x_n}\Big[e^{-\frac{n(n-2)}{8}\int_0^\tau \frac{ds}{W_n(s)^2}}; \tau \in ds\Big]$$
$$= x_n^{(n-2)/2}\Big(\int_0^\infty \frac{\exp(-|y|^2/(2s))}{(2\pi s)^{(n-1)/2}}\,\mu_{x_n}(ds)\Big)dy,$$

where

$$\mu_{x_n}(ds) = \mathbb{E}^{x_n}\Big[\exp\Big(-\frac{n(n-2)}{8}\int_0^\tau \frac{ds}{W_n(s)^2}\Big); \tau \in ds\Big].$$

The Laplace transform of μ_{x_n} is given by

$$\mathcal{L}\mu_{x_n}(w) = \mathbb{E}^{x_n} \exp\Big(-\frac{n(n-2)}{8}\int_0^\tau \frac{ds}{W_n(s)^2} - w\int_0^\tau ds\Big) = E^{x_n} e_q(\tau),$$

where $w \geq 0$ and $q(x) = -n(n-2)/(8x^2) - w$. The function $\varphi(x_n) = E^{x_n}e_q(\tau)$ is a gauge function for an appropriate Schrödinger operator based on the generator of $W_n(t)$. Consequently, φ is a bounded solution of the equation

$$\frac{1}{2}\varphi''(x) - \Big(\frac{n(n-2)}{8x^2} + w\Big)\phi(x) = 0, \qquad x \geq 1,$$

such that $\varphi(1) = 1$. Making the substitution $\sqrt{x}\,\psi(x\sqrt{2w}) = \varphi(x)$ we reduce the above equation to

$$2x^2w\psi''(x\sqrt{2w}) + x\sqrt{2w}\,\psi'(x\sqrt{2w}) - \Big(\frac{(n-1)^2}{4} + 2wx^2\Big)\psi(x\sqrt{2w}) = 0,$$

which is the modified Bessel equation (6.5) with $\nu = (n-1)/2$. Taking into account the general form of solutions of (6.5), the boundary condition and boundedness of φ we arrive at

$$\mathcal{L}\mu_{x_n}(w) = \sqrt{x_n}\,\frac{K_\nu(x_n\sqrt{2w})}{K_\nu(\sqrt{2w})}, \quad w \geq 0. \tag{3.7}$$

Since square root can be extended to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ and the modified Bessel function K_ν has no zeros in the positive half-plane $\operatorname{Re} w \geq 0$, the Laplace transform $\mathcal{L}\mu_{x_n}(w)$ can also be extended to an analytic function on $\mathbb{C} \setminus (-\infty, 0]$. Moreover, using the asymptotic expansion (6.6) we obtain

$$\begin{aligned}|\mathcal{L}\mu_{x_n}(w)| &\leq \left| e^{-(x_n-1)\sqrt{2w}}\,\frac{1+E(x_n\sqrt{2w})}{1+E(\sqrt{2w})}\right| \\ &\leq 2\exp\left(-(x_n-1)\sqrt{2|w|}\cos\frac{\arg w}{2}\right)\end{aligned} \tag{3.8}$$

for every $w \in \mathbb{C} \setminus (-\infty, 0]$ such that $|w|$ is large enough. Note that here $\arg w \in [-\pi, \pi]$. In particular, $\mathcal{L}\mu_{x_n}(w)$ is bounded for $|w| \geq 1$. These properties of $\mathcal{L}\mu_{x_n}$ and its analytic continuation guarantee that we can apply the inverse Laplace transform to (3.7) (see [13, Theorem 8.5, p. 267]). More precisely, there exists a density function of μ_{x_n} with respect to Lebesgue measure on $(0, \infty)$ given by the inversion formula

$$\mu_{x_n}(s) = \frac{1}{2\pi i}\lim_{r\to\infty}\int_{1-ir}^{1+ir} \mathcal{L}\mu_{x_n}(w)e^{sw}\,dw.$$

To compute the limit we integrate the function $f_s(w) = \mathcal{L}\mu_{x_n}(w)e^{sw}$ over a rectangular contour surrounding the branch-cut of f_s which is the negative real axis. Let Γ be the positively oriented contour consisting of four horizontal segments $\gamma_1 = [-r+i/r, i/r]$, $\gamma_2 = [-r-i/r, -i/r]$, $\gamma_3 = [-r+ir, 1+ir]$, $\gamma_4 = [-r-ir, 1-ir]$, three vertical segments $\gamma_5 = [-r+i/r, -r+ir]$, $\gamma_6 = [-r-i/r, -r-ir]$, $\gamma_7 = [1-ir, 1+ir]$, and a semi-circle $\gamma_8 = \{|w| = 1/r, \operatorname{Re} w > 0\}$. The formula (6.7) implies that f_s is bounded for small w such that $\operatorname{Re} w > 0$. Consequently, the integral over γ_8 tends to zero when $r \to \infty$. The boundedness of $\mathcal{L}\mu_{x_n}(w)$ for large w implies that for $r \geq 1$ and every $s > 0$ we have

$$\left|\Big(\int_{\gamma_5} + \int_{\gamma_6}\Big) f_s(w)\,dw\right| \leq 2\sup_{|w|\geq 1}\mathcal{L}\mu_{x_n}(w)\,re^{-rs} \to 0$$

as $r \to \infty$. Finally, using (3.8), we obtain

$$\left|\Big(\int_{\gamma_3} + \int_{\gamma_4}\Big) f_s(w)\,dw\right| \leq 4\exp(-(x_n-1)\cos(3\pi/8)\sqrt{r})\int_{-1}^{\infty} e^{-su}\,du \to 0$$

as $r \to \infty$. The Cauchy theorem together with the previous considerations and (6.8) gives

$$\begin{aligned}\mu_{x_n}(s) &= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{1-ir}^{1+ir}\mathcal{L}\mu_{x_n}(w)e^{sw}\,dw = \frac{1}{2\pi i}\lim_{r\to\infty}\Big(\int_{\gamma_1}+\int_{\gamma_2}\Big)f_s(w)\,dw\\
&= \frac{\sqrt{x_n}}{2\pi i}\int_0^\infty\left[\frac{K_\nu(-i\sqrt{2t}\,x_n)}{K_\nu(-i\sqrt{2t})}-\frac{K_\nu(i\sqrt{2t}\,x_n)}{K_\nu(i\sqrt{2t})}\right]e^{-st}\,dt\\
&= -\frac{\sqrt{x_n}}{\pi}\int_0^\infty \mathrm{Im}\bigg(\frac{K_\nu(i\sqrt{2t}x_n)}{K_\nu(i\sqrt{2t})}\bigg)e^{-st}\,dt\\
&= -\frac{\sqrt{x_n}}{\pi}\int_0^\infty \mathrm{Im}\bigg(\frac{K_\nu(itx_n)}{K_\nu(it)}\bigg)te^{-st^2/2}\,dt\\
&= \frac{\sqrt{x_n}}{\pi}\int_0^\infty \frac{J_\nu(t)Y_\nu(tx_n)-J_\nu(tx_n)Y_\nu(t)}{J_\nu^2(t)+Y_\nu^2(t)}te^{-st^2/2}\,dt.\end{aligned}$$

From (6.3) and (6.4) it is easy to see that $J_\nu^2(t)+Y_\nu^2(t)\sim t^{-1}\vee t^{-2\nu}$ and

$$\left|\frac{J_\nu(t)Y_\nu(tx_n)-J_\nu(tx_n)Y_\nu(t)}{J_\nu^2(t)+Y_\nu^2(t)}\right|\le C(1+t),\qquad t>0,$$

for some constant $C=C(x_n)>0$. With the use of (6.9) we verify

$$\begin{aligned}\int_0^\infty \frac{|J_\nu(t)Y_\nu(tx_n)-J_\nu(tx_n)Y_\nu(t)|}{J_\nu^2(t)+Y_\nu^2(t)}t\bigg(\int_0^\infty \frac{e^{-|y|^2/(2s)}e^{-st^2/2}ds}{(2\pi s)^{(n-1)/2}}\bigg)\,dt&\\
\le C\int_0^\infty (t+1)t^\nu K_{\nu-1}(t|y|)\,dt.&\end{aligned}$$

The last integral is finite by (6.6) and (6.7). Consequently, by Fubini's theorem we obtain

$$\begin{aligned}P(x_n,y) &= \frac{x_n^\nu}{\pi}\int_0^\infty \frac{J_\nu(t)Y_\nu(tx_n)-J_\nu(tx_n)Y_\nu(t)}{J_\nu^2(t)+Y_\nu^2(t)}t\bigg[\int_0^\infty \frac{e^{-|y|^2/(2s)}e^{-st^2/2}ds}{(2\pi s)^{(n-1)/2}}\bigg]\,dt\\
&= \frac{1}{2^{\nu-1}\pi^{\nu+1}}\,\frac{x_n^\nu}{|y|^{\nu-1}}\int_0^\infty \frac{J_\nu(t)Y_\nu(tx_n)-J_\nu(tx_n)Y_\nu(t)}{J_\nu^2(t)+Y_\nu^2(t)}\,t^\nu K_{\nu-1}(t|y|)\,dt. \ \blacksquare\end{aligned}$$

The integral formula of Theorem 3.1 can be used to obtain the asymptotics of the Poisson kernel $P(x_n,y)$ as well as its sharp bounds for small x_n and large $|y|$. Note that the results of the next theorem cover those obtained in [4] (see Theorems 4.9 and 4.10 there; compare also with Theorem 5.3 of [9]). Moreover, the formula (3.5) allows us to omit laborious and sophisticated computations used in [4] to examine the behavior of $P(x_n,y)$ when $|y|$ tends to infinity. Our approach is simpler and gives more general results.

THEOREM 3.2. *For every $x_0 \geq 1$ we have*

$$P(x_n, y) \sim \frac{\Gamma(n/2)}{2^{n-2}\pi^{n/2}} \sum_{k=0}^{n-2} x_0^k \cdot \frac{x_n - 1}{|y|^{2n-2}}, \qquad x_n \to x_0,\ |y| \to \infty. \tag{3.9}$$

Moreover, for every $y_0 > 0$ we have

$$P(x_n, y) \sim \frac{c(y_0)}{\pi^{(n+3)/2} 2^{(n-5)/2}} \cdot (x_n - 1), \quad x_n \to 1,\ |y| \to y_0, \tag{3.10}$$

where

$$c(y_0) = |y_0|^{(1-n)/2} \int_0^\infty \frac{s^\nu K_{\nu-1}(sy_0)\, ds}{J_\nu^2(s) + Y_\nu^2(s)}.$$

Proof. Making the substitution $t|y| = s$ in (3.5) we can rewrite the Poisson kernel in the following way:

$$\frac{P(x_n, y)|y|^{4\nu}}{x_n - 1} = \frac{x_n^\nu}{\pi^{\nu+1} 2^{\nu-1}} \int_0^\infty g_\nu\left(x_n, \frac{s}{|y|}\right) s^{3\nu} K_{\nu-1}(s)\, ds,$$

where

$$g_\nu(x, t) = \frac{1}{t^{2\nu}(x-1)} \frac{J_\nu(t) Y_\nu(xt) - J_\nu(tx) Y_\nu(t)}{J_\nu^2(t) + Y_\nu^2(t)}, \qquad x > 1,\ t > 0.$$

Since $s^{3\nu} K_{\nu-1}(s)$ is integrable on $[0, \infty)$ and (6.13) gives boundedness of $|g_\nu(x_n, s/|y|)|$ for $x_n < R$, we can apply the Lebesgue dominated convergence theorem to get

$$\begin{aligned} \lim_{(x_n, |y|) \to (1, \infty)} \frac{P(x_n, y)|y|^{4\nu}}{x_n - 1} &= \frac{1}{\pi^\nu 2^{3\nu-2} \Gamma^2(\nu)} \int_0^\infty s^{3\nu} K_{\nu-1}(s)\, ds \\ &= \frac{2\Gamma(2\nu)\Gamma(\nu+1)}{\pi^\nu \Gamma^2(\nu)} \end{aligned}$$

and

$$\begin{aligned} \lim_{(x_n, |y|) \to (x_0, \infty)} \frac{P(x_n, y)|y|^{4\nu}}{x_n - 1} &= \frac{x_0}{\pi^{\nu+1} 2^{\nu-1}} \int_0^\infty \frac{\pi(x_0^{2\nu} - 1) s^{3\nu} K_{\nu-1}(s)\, ds}{2^{2\nu}(x_0 - 1)\Gamma(\nu)\Gamma(\nu+1) x_0^\nu} \\ &= \frac{(x_0^{2\nu} - 1)\Gamma(2\nu)}{\pi^\nu (x_0 - 1)\Gamma(\nu) x_0^{\nu-1}} = \frac{(\sum_{k=0}^{n-2} x_0^k)\Gamma(2\nu)}{\pi^\nu x_0^{\nu-1} \Gamma(\nu)} \end{aligned}$$

whenever $x_0 > 1$. Here we used formula (6.11) from Lemma 6.1 and relation (6.10). The duplication formula for the gamma function gives (3.9). In the same way, using (6.11), we get

$$\lim_{(x_n, |y|) \to (1, y_0)} \frac{P(x_n, y)|y|^{4\nu}}{x - 1} = \frac{1}{\pi^{\nu+1} 2^{\nu-1}} \int_0^\infty \frac{2s^\nu |y_0|^{2\nu} K_{\nu-1}(s)\, ds}{\pi(J_\nu^2(s/|y_0|) + Y_\nu^2(s/|y_0|))}.$$

which, by substituting $s = ty_0$, proves (3.10). ∎

As a consequence of Theorem 3.2 we obtain the following sharp bounds for the Poisson kernel for small x_n and large $|y|$. Similar results have recently been obtained in [7, Theorem 7]. Those results are more general (there is no restriction on x_n and $|y|$), but the methods of proof are much more complicated.

COROLLARY 3.3. *We have*

$$P(x_n, y) \approx \frac{x_n - 1}{|y|^{2n-2}}, \qquad 1 < x_n \le 2,\ |y| \ge 1.$$

Proof. Existence and positivity of the limits proved in Theorem 3.2 imply that for every $x \in [1,2]$ there exist $\varepsilon_x > 0$ and $Y_x > 0$ and strictly positive constants $c_1(x), c_2(x)$ such that

$$c_1(x)\frac{x' - 1}{|y|^{2n-2}} \le P(x', y) \le c_2(x)\frac{x' - 1}{|y|^{2n-2}}$$

for every $x' \in [1,2]$ and $y \in \mathbb{R}^{n-1}$ satisfying $|x - x'| < \varepsilon_x$ and $|y| > Y_x$. Since the family of intervals $(x - \varepsilon_x, x + \varepsilon_x)$ is an open cover of $[1,2]$, we can choose a finite subcover $\{(x_i - \varepsilon_{x_i}, x_i + \varepsilon_{x_i}) : i = 1, \ldots, m\}$. Putting $Y = \max\{Y_{x_i} : i = 1, \ldots, m\}$, $c_1 = \min\{c_1(x_i) : i = 1, \ldots, m\}$ and $c_2 = \max\{c_2(x_i) : i = 1, \ldots, m\}$ we get

$$c_1 \frac{x_n - 1}{|y|^{2n-2}} \le P(x_n, y) \le c_2 \frac{x_n - 1}{|y|^{2n-2}} \tag{3.11}$$

for every $x_n \in [1,2]$ and $|y| > Y$. Observe that formula (3.6) implies positivity of $P(x_n, y)$. Moreover, by (3.10), the function $P(x_n, y)|y|^{2n-2}/(x_n - 1)$ can be continuously extended to a strictly positive function on the compact set $[1,2] \times [1, Y]$ and consequently (3.11) is also true for $|y| \in [1, Y]$ (with possibly different constants c_1 and c_2). ∎

4. Harmonic measure of hyperbolic balls in $\mathbb{D}^n$. In this section we consider the harmonic measure of balls associated with the operator

$$\Delta_{\mathrm{LB}} = \frac{(1 - |x|^2)^2}{4} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + (n-2)\frac{1 - |x|^2}{2} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}. \tag{4.1}$$

This operator appears naturally as the Laplace–Beltrami operator on the ball model of the real hyperbolic space $\mathbb{D}^n$ (see [5] for more details).

In particular, if $B = (B_k)$ is the standard n-dimensional Brownian motion then the system of SDEs

$$\frac{dY_k(t)}{1 - |Y(t)|^2} = dB_k(t) + 2(n-2)Y_k(t)dt, \qquad k = 1, \ldots, n,$$

describes a diffusion with values in $\mathbb{D}^n$ with generator $2\Delta_{\mathrm{LB}}$. As in the case of the half-space model, we perform a change of time defined by

$$A(u) = \int_0^u (1 - |X(s)|^2)\, ds \quad \text{and} \quad \sigma(t) = \inf\{u;\, A(u) > t\}.$$

Then the process defined by

$$\tilde{Y}_k(t) = Y_k(\sigma(t))$$

is a diffusion with values in $\mathbb{D}^n$ with generator L_2.

As in the case of $\mathbb{H}^n$ in Section 3, the harmonic measures of the operators Δ_{LB} and L_2 are the same.

We now consider the harmonic measure ω_r^x of the ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $r < 1$, supported on the boundary of B_r which is the sphere S_r^{n-1}. We denote by $P_r(x, y)$ the Poisson kernel of B_r, i.e. the density of the measure ω_r^x with respect to the $(n-1)$-dimensional spherical measure σ_r^{n-1}. As in the previous section, we can write the Laplace–Beltrami operator in the form $\frac{(1-|x|^2)^2}{2} L_2$, where

$$L_2 = \frac{1}{2}\Delta + F_2(x) \cdot \nabla \quad \text{with} \quad F_2(x) = \frac{n-2}{1-|x|^2}(x_1, \ldots, x_n).$$

The positivity of the factor $(1-|x|^2)^2/2$ implies that the harmonic functions on the ball B_r for the operators (4.1) and L_2 are exactly the same and consequently the harmonic measures coincide.

Moreover, the vector field F_2 is orthogonal to S_r^{n-1} and its potential function is $V_2(x) = \frac{2-n}{2} \ln(1 - |x|^2)$. Denote by $\tau_r = \inf\{t > 0 : W(t) \notin B_r\}$ the first exit time of the Brownian motion $W(t)$ from the ball B_r. Note that τ_r depends only on the Euclidean norm of W.

The martingale M related to the vector field F_2 and its quadratic variation are

$$M(t) = (n-2) \int_0^t \frac{W_i(s)\, dW_i(s)}{1 - |W(s)|^2}, \quad \langle M \rangle(t) = (n-2)^2 \int_0^t \frac{|W(s)|^2\, ds}{(1 - |W(s)|^2)^2}.$$

Observe that the condition (2.4) is fulfilled in this case, since

$$\mathbb{E}^x \exp\biggl[(n-2)^2 \int_0^{t\wedge\tau_r} \frac{|W(s)|^2\, ds}{(1-|W(s)|^2)^2}\biggr] \le \mathbb{E}^x \exp\biggl[(n-2)^2\, \frac{r^2(t\wedge\tau_r)}{(1-r^2)^2}\biggr],$$

and the last expression is finite. By (2.7), the kernel $N(t)$ is of the form

$$N(t) = \biggl(\frac{1 - |W(0)|^2}{1 - |W(t)|^2}\biggr)^{(n-2)/2} \exp\biggl(-\frac{n(n-2)}{2} \int_0^t \frac{ds}{(1-|W(s)|^2)^2}\biggr),$$

and it is now evident that $\{N(t \wedge \tau_r)\}_{t>0}$ is uniformly bounded in t so the condition (2.5) holds. Applying Theorem 2.2 we obtain

$$(4.2)\qquad w_r^x(dy) = \left(\frac{1-|x|^2}{1-r^2}\right)^{(n-2)/2} \times \mathbb{E}^x\left[\exp\left(-\frac{n(n-2)}{2}\int_0^{\tau_r}\frac{ds}{(1-|W(s)|^2)^2}\right);\ W(\tau_r)\in dy\right],$$

From now on we assume that $x \neq 0$. For $x = 0$, from the rotational invariance of the Laplace–Beltrami operator, we easily see that ω_r^x is just $\sigma_r^{n-1}/\sigma_r^{n-1}(S_r^{n-1})$. Recall the skew-product representation of the Brownian motion,

$$W(t) = R^{(\nu)}(t)\Theta(A^{(\nu)}(t)),$$

where $R^{(\nu)}$ is the Bessel process with index $\nu = n/2-1$ starting from $|x|$ and Θ is a spherical Brownian motion on S_1^{n-1} independent of $R^{(\nu)}$ (see Appendix). Using the fact that τ_r depends only on $R^{(\nu)}$ we find that $W(\tau_r) = R^{(\nu)}(\tau_r)\Theta(A^{(\nu)}(\tau_r))$, where Θ is independent of $R^{(\nu)}(\tau_r)$ and $A^{(\nu)}(\tau_r)$. Applying this decomposition to formula (4.2) we get

$$\omega_r^x(dy) = \left(\frac{1-|x|^2}{1-r^2}\right)^{(n-2)/2} \times \int_0^\infty P^{x/|x|}(\Theta_t\in dy)\mathbb{E}^{|x|}\left[\exp\left(\int_0^{\tau_r} q(R_s^{(\nu)})\,ds\right);\ A^{(\nu)}(\tau_r)\in dt\right],$$

where $q(y) = -\frac{n(n-2)}{2(1-y^2)^2}$. Rotational invariance of spherical Brownian motion implies that the harmonic measure ω_r^x is symmetric with respect to the x axis. As a consequence, its density $P_r(x,y)$ depends only on the cosine of the angle between the starting point x and the point y, i.e.

$$P_r(x,y) = \tilde{P}_r\left(x, \frac{\langle x,y\rangle}{|x|\,|y|}\right).$$

If we consider sets of the form $A = \{\eta\in S_1^{n-1} : \langle x,\eta\rangle/|x| \in (a,b)\}$, where $-1 < a < b < 1$, we get (for the definition of the process S see Appendix)

$$\begin{aligned}P^{x/|x|}(\Theta_t\in A) &= P^1(S_t\in(a,b))\\ &= \int_a^b p_t^S(1,z)\,m(dz) = 2\int_a^b p_t^S(1,z)(1-z^2)^{(n-3)/2}\,dz,\end{aligned}$$

where p_t^S is defined in (6.23). On the other hand, using spherical coordinates we obtain

$$
\begin{aligned}
\omega_r^x(rA) &= \int\limits_{rA} P_r(x,y)\, d\sigma_r^{n-1}(y) \\
&= r^{n-1}\sigma_r^{n-2}(S_1^{n-2}) \int\limits_{\cos\phi\in(a,b)} \tilde{P}_r(x,\cos\phi)\sin^{n-2}\phi\, d\phi \\
&= \frac{n\pi^{(n-1)/2} r^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)} \int\limits_a^b \tilde{P}_r(x,z)(1-z^2)^{(n-3)/2}\, dz.
\end{aligned}
$$

Comparing both sides we infer that the Poisson kernel $P_r(x,y)$ equals

$$
\frac{\Gamma\left(2\frac{n+1}{2}\right)}{\pi^{(n-1)/2} n r^{n-1}} \left(\frac{1-|x|^2}{1-r^2}\right)^{(n-2)/2} \int\limits_0^\infty p_t^S\left(1, \frac{\langle x,y\rangle}{|x|\,|y|}\right) \mu_{|x|}(dt), \tag{4.3}
$$

where

$$
\mu_y(dt) = \mathbb{E}^y\left[\exp\left(\int\limits_0^{\tau_r} q(R^{(\nu)}(s))\, ds\right);\ A^{(\nu)}(\tau_r)\in dt\right], \qquad y\in(0,r].
$$

The formula for p_t^S can be computed from the appropriate formula for the transition density function for Θ, which is given in terms of spherical harmonics, and that approach leads to the series representation for $P_r(x,y)$ presented in [5]. However, we want to compute the Laplace transform of p_t^S which is the so called λ-Green function of the process S,

$$
G_\lambda(x,1) = \int\limits_0^\infty e^{-\lambda t} p_t^S(1,x)\, dt, \qquad x\in(-1,1),
$$

and we do this directly. From the general theory (see for example [3, Chapter II] for a short résumé) the function G_λ is described by solutions of the second-order differential equation

$$
\frac{1-x^2}{2}u''(x) - \frac{n-1}{2}xu'(x) = \lambda u(x), \qquad x\in(-1,1). \tag{4.4}
$$

Note that the expression on the left-hand side is just $\mathcal{G}u(x)$, where $\mathcal{G}$ is the generator of S described in (6.22). More precisely, we have

$$
G_\lambda(x,1) = \frac{\varphi_\lambda(1)\psi_\lambda(x)}{w_\lambda}, \qquad x\in(-1,1),
$$

where φ_λ is a decreasing and ψ_λ is an increasing solution of (4.4) such that $\varphi_\lambda^-(1)=0$ and $\psi_\lambda^+(-1)=0$. The boundary conditions for the derivatives follow from the fact that the non-singular points -1 and 1 are reflecting. Here f^+ and f^- denote the right and left derivatives with respect to the speed function $s(x)$. The Wronskian w_λ is given by

$$
w_\lambda = \psi_\lambda^-(x)\varphi_\lambda(x) - \psi_\lambda(x)\varphi_\lambda^-(x)
$$

and it does not depend on x. Putting $x=1$ in the above formula and using the boundary conditions we obtain

$$G_\lambda(x,1) = \frac{\varphi_\lambda(1)\psi_\lambda(x)}{\psi_\lambda^-(1)\varphi_\lambda(1) - \psi_\lambda(1)\varphi_\lambda^-(1)} = \frac{\psi_\lambda(x)}{\psi_\lambda^-(1)}, \qquad x \in (-1,1).$$

This implies that $G_\lambda(x,1)$ is uniquely described as a solution of (4.4) such that $u^+(-1) = 0$ and $u^-(1) = 1$. Making the substitution $u(x) = f(z)$ with $z = (1+x)/2$ in equation (4.4) we reduce it to the hypergeometric equation

$$z(1-z)f''(z) + \left(\frac{n-1}{2} - (n-1)z\right)f'(z) - 2\lambda f(z) = 0,$$

i.e. equation (6.14) with $\alpha = \frac{n-2}{2} - A(\lambda)$, $\beta = \frac{n-2}{2} + A(\lambda)$, $\gamma = \frac{n-1}{2}$, $A(\lambda) = \frac{1}{2}\sqrt{(n-2)^2 - 8\lambda}$. Consider the function

$$h_\lambda(x) = {}_2F_1\left(\frac{n-2}{2} - A(\lambda), \frac{n-2}{2} + A(\lambda); \frac{n-1}{2}; \frac{1+x}{2}\right).$$

The above computation implies that h_λ is a solution of (4.4). Using (6.16) and (6.17) we compute the derivative of this function with respect to the scale function $s(x)$ in the following way:

$$\begin{aligned}\frac{d}{dx}h_\lambda(x) &= \frac{2\lambda}{n-1}\,{}_2F_1\left(\frac{n}{2} - A(\lambda), \frac{n}{2} + A(\lambda); \frac{n+1}{2}; \frac{1+x}{2}\right)\\ &= \frac{2^{(n+1)/2}\lambda}{n-1}\left(\frac{1+x}{1-x^2}\right)^{(n-1)/2} {}_2F_1\left(\frac{1}{2} + A(\lambda), \frac{1}{2} - A(\lambda); \frac{n+1}{2}; \frac{1+x}{2}\right).\end{aligned}$$

The first equality and the fact that the hypergeometric function ${}_2F_1$ is equal to 1 at zero imply $h_\lambda^+(-1) = 0$. Using the second equality and (6.15) we obtain

$$\begin{aligned}h_\lambda^-(1) &= \frac{2^n\lambda}{n-1}\,{}_2F_1\left(\frac{1}{2} + A(\lambda), \frac{1}{2} - A(\lambda); \frac{n+1}{2}; 1\right)\\ &= \frac{2^{n-1}\lambda\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2} - A(\lambda)\right)\Gamma\left(\frac{n-2}{2} + A(\lambda)\right)}.\end{aligned}$$

Moreover, using once again (6.17) and the definition (6.19) we can express h_λ in terms of the Legendre function of the first kind:

$$\begin{aligned}h_\lambda(x) &= \left(\frac{1-x}{2}\right)^{(3-n)/2} {}_2F_1\left(\frac{1}{2} - A(\lambda), \frac{1}{2} + A(\lambda); \frac{n-1}{2}; \frac{1+x}{2}\right)\\ &= \frac{(1-x^2)^{(3-n)/4}}{2^{(3-n)/2}}\Gamma\left(\frac{n-1}{2}\right)P^{(3-n)/2}_{A(\lambda)-1/2}(-x).\end{aligned}$$

Finally we have just shown that

$$G_\lambda(x,1) = B_n(\lambda)(1-x^2)^{(3-n)/4}P^{(3-n)/2}_{A(\lambda)-1/2}(-x),$$

where

$$B_n(\lambda) = \frac{\Gamma\left(\frac{n-2}{2} - A(\lambda)\right)\Gamma\left(\frac{n-2}{2} + A(\lambda)\right)}{2^{(n+1)/2}\lambda\Gamma\left(\frac{n-1}{2}\right)}.$$

The second part of formula (4.3) relates to the measure μ_y. Observe that μ_y depends only on the Bessel process $R^{(\nu)}$, which is a one-dimensional diffusion. For every $w \geq 0$ the Laplace transform $\mathcal{L}\mu_y(w)$ is

$$\mathbb{E}^y \exp\left(-\frac{n(n-2)}{2}\int_0^{\tau_r} \frac{ds}{(1-(R^{(\nu)}(s))^2)^2} - w\int_0^{\tau_r} \frac{ds}{(R^{(\nu)}(s))^2}\right) = \mathbb{E}^y e_g(\tau_r),$$

where $g(y) = -\frac{n(n-2)}{2(1-y^2)^2} - \frac{w}{y^2}$. The function $\varphi(y) = \mathbb{E}^y e_g(\tau_r)$ is by definition the gauge function for the Schrödinger operator based on the generator of the process $R^{(\nu)}$ and the non-positive potential g. From the Feynman–Kac formula, φ is a solution of the Schrödinger equation. Using (6.21) we find that φ is a bounded solution to the second-order differential equation

$$\frac{1}{2}\varphi''(y) + \frac{n-1}{2y}\varphi'(y) - \left(\frac{n(n-2)}{2(1-y^2)^2} + \frac{w}{y^2}\right)\varphi(y) = 0, \tag{4.5}$$

where $y \in [0, r)$, $w \geq 0$. The corresponding boundary condition is $\varphi(r) = 1$. Substituting $\varphi(y) = y^{1-n/2}\psi\left(\frac{1+y^2}{1-y^2}\right)$ we obtain

$$\begin{aligned}
\varphi'(y) &= \left(1 - \frac{n}{2}\right)y^{-n/2}\psi\left(\frac{1+y^2}{1-y^2}\right) + \frac{4y^{2-n/2}}{(1-y^2)^2}\psi'\left(\frac{1+y^2}{1-y^2}\right),\\
\varphi''(y) &= \frac{n}{2}\left(\frac{n}{2} - 1\right)y^{-1-n/2}\psi\left(\frac{1+y^2}{1-y^2}\right) + \frac{16y^{3-n/2}}{(1-y^2)^4}\psi''\left(\frac{1+y^2}{1-y^2}\right)\\
&\quad + \frac{4y^{1-n/2}}{(1-y^2)^3}(3 - n + (n+1)y^2)\psi'\left(\frac{1+y^2}{1-y^2}\right).
\end{aligned}$$

Inserting the above formulas into the differential equation (4.5) and dividing both sides by $-\frac{2y^{1-n/2}}{(1-y^2)^2}$ gives

$$\begin{aligned}
0 &= \frac{-4y^2}{(1-y^2)^2}\psi''\left(\frac{1+y^2}{1-y^2}\right) - \frac{2(1-y^4)}{(1-y^2)^2}\psi'\left(\frac{1+y^2}{1-y^2}\right)\\
&\quad + \left(\frac{(1-y^2)^2}{4y^2}\left[\frac{(n-2)^2}{4} + 2w\right] + \frac{n(n-2)}{4}\right)\psi\left(\frac{1+y^2}{1-y^2}\right).
\end{aligned}$$

Moreover, putting $z = \frac{1+y^2}{1-y^2}$ and using the equality $1 - \left(\frac{1+y^2}{1-y^2}\right)^2 = \frac{-4y^2}{(1-y^2)^2}$ leads to the following differential equation for ψ:

$$(1-z^2)\psi''(z) - 2z\psi'(z) + \left(\nu(\nu+1) - \frac{A(-w)^2}{1-z^2}\right)\psi(z) = 0, \quad z \geq 1,$$

with $\nu = n/2 - 1$ and $A(-w) = \frac{1}{2}\sqrt{(n-2)^2 + 8w}$. This is the Legendre differential equation (6.18). Thus, the general solution of (4.5) is given by

$$\varphi(y) = c_1 y^{1-n/2} P_\nu^{-A(-w)}\left(\frac{1+y^2}{1-y^2}\right) + c_2 y^{1-n/2} Q_\nu^{-A(-w)}\left(\frac{1+y^2}{1-y^2}\right),$$

where $y \in [0, r]$ and c_1, c_2 are absolute constants. Using (6.19) and (6.20) one can easily check that the function $y^{-\nu} P_\nu^{-A(-w)}\left(\frac{1+y^2}{1-y^2}\right)$ is bounded on the interval $\left[1, \frac{1+r^2}{1-r^2}\right)$ in contrast to the function $y^{-\nu} Q_\nu^{-A(-w)}\left(\frac{1+y^2}{1-y^2}\right)$, which is unbounded in the neighborhood of 1. Thus $c_2 = 0$ and the boundary condition $\varphi(r) = 1$ gives

$$\mathcal{L}\mu_{|x|}(w) = \left(\frac{r}{|x|}\right)^{n/2-1} \frac{P_\nu^{-A(-w)}\left(\frac{1+|x|^2}{1-|x|^2}\right)}{P_\nu^{-A(-w)}\left(\frac{1+r^2}{1-r^2}\right)}, \qquad |x| \le r,\ w \ge 0. \tag{4.6}$$

Now observe that for every complex number w such that $\mathrm{Re}(w) > -\nu^2/2 = -(n-2)^2/8$,

$$\begin{aligned} |\mathcal{L}\mu_{|x|}(w)| &\le \mathbb{E}^{|x|}\left[e^{-\frac{n(n-2)}{2}\int_0^{\tau_r} \frac{ds}{(1-(R_s^{(\nu)})^2)^2} - \mathrm{Re}(w)\int_0^{\tau_r} \frac{ds}{(R_s^{(\nu)})^2}}\right] \\ &\le \mathbb{E}^{|x|} \exp\left(\frac{(n-2)^2}{8}\int_0^{\tau_r} \frac{ds}{(R_s^{(\nu)})^2}\right) \\ &= \mathbb{E}^{|x|} \exp\left(\frac{\nu^2}{2}\int_0^{\tau_r} \frac{ds}{(R_s^{(\nu)})^2}\right) = \left(\frac{r}{x}\right)^\nu. \end{aligned}$$

The last equality follows from (see [3, 2.20.4, p. 407])

$$\mathbf{P}_x^{(\nu)}\left[\int_0^{\tau_r} \frac{ds}{(R^{(\nu)}(s))^2} \in dy\right] = \left(\frac{r}{x}\right)^\nu \frac{\ln(r/x)}{\sqrt{2\pi}\, y^{3/2}} e^{-\frac{\nu^2 y}{2} - \frac{\ln^2(r/x)}{2y}}\, dy$$

for $y > 0$. In particular $\mathcal{L}\mu_{|x|}(-\nu^2/2)$ is finite. This implies that the formula

$$\mathcal{L}\mu_{|x|}(w) = \int_0^\infty e^{-wt}\, \mu_{|x|}(dt)$$

defines a holomorphic function in the complex half-plane $\mathrm{Re}(w) > -v^2/2$. Moreover, for $|z| < 1$ the function ${}_2F_1(\alpha, \beta; \gamma; z)/\Gamma(\alpha)$ is analytic on $\mathbb{C}$ as a function of α. Using this fact and the representation of P_ν^μ in terms of the hypergeometric function ${}_2F_1$ we deduce that the right-hand side of (4.6) is a meromorphic function of w in the half-plane $\mathrm{Re}(w) > -\nu^2/2$. In fact, (4.6) implies that the ratio of the Legendre functions is analytic for $\mathrm{Re}(w) > -\nu^2/2$ and consequently the denominator has no zeros in this region (compare this result with Conjecture 5.2 in [5]). Moreover, we have just proved that (4.6) holds whenever $\mathrm{Re}(w) > -\nu^2/2$.

Now let $c = -(n-2)^2/16$. We have

$$\int_0^\infty p_t^S\Big(1, \frac{\langle x, y\rangle}{|x|\,|y|}\Big)\,\mu_{|x|}(dt) = \int_0^\infty p_t^S\Big(1, \frac{\langle x, y\rangle}{|x|\,|y|}\Big)\Big(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\mathcal{L}\mu_{|x|}(z)\,dz\Big)\,dt$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}\mu_{|x|}(z)\Big(\int_0^\infty e^{zt}p_t^S\Big(1, \frac{\langle x, y\rangle}{|x|\,|y|}\Big)\,dt\Big)\,dz$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}\mu_{|x|}(z)G_z\Big(1, \frac{\langle x, y\rangle}{|x|\,|y|}\Big)\,dt\,dz.$$

Taking into account the previously found formulas for the Laplace transform $\mathcal{L}\mu_{|x|}(z)$ and the Green function $G_z(1, x)$ we finally obtain

THEOREM 4.1. *For every $x \in B_r$, $x \neq 0$ and $y \in S_r^{n-1}$ the Poisson kernel $P_r(x, y)$ is given by the formula*

$$\frac{2\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}nr^{n-1}}\Big(\frac{1-|x|^2}{1-r^2}\,\frac{r}{|x|}\Big)^{\nu}\frac{\sin^{(3-n)/2}\varphi}{2\pi i}$$
$$\times\int_{c-i\infty}^{c+i\infty}\frac{P_\nu^{-A(-z)}\big(\frac{1+|x|^2}{1-|x|^2}\big)}{P_\nu^{-A(-z)}\big(\frac{1+r^2}{1-r^2}\big)}B_n(z)P_{A(z)-1/2}^{(3-n)/2}(-\cos\varphi)\,dz,$$

where

$$A(z) = \frac{1}{2}\sqrt{(n-2)^2 - 8z}, \quad B_n(z) = \frac{\Gamma\big(\frac{n-2}{2} - A(z)\big)\Gamma\big(\frac{n-2}{2} + A(z)\big)}{2^{(n+1)/2}z\Gamma\big(\frac{n-1}{2}\big)},$$

$c = -\nu^2/4$ and φ is the angle between x and y.

5. Harmonic measure of the Ornstein–Uhlenbeck process. As in the previous section, for fixed $r > 0$ we denote $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. Consider the vector field $F_3(x) = \lambda x$, where $\lambda > 0$. It is a potential vector field with potential function $V_3(x) = \lambda|x|^2/2$ and as in the previous cases it is orthogonal to the boundary of B_r. The corresponding martingale M and its quadratic variation are

$$M(t) = \lambda\sum_{i=1}^n\int_0^t W_i(s)\,dW_i(s), \quad \langle M\rangle(t) = \int_0^t \lambda^2|W(s)|^2\,ds.$$

The validity of (2.4) in this case follows from

$$\mathbb{E}^x[\exp\langle M\rangle(t\wedge\tau_r)] = \mathbb{E}^x\Big[\exp\int_0^{t\wedge\tau_r}\lambda^2|W(s)|^2\,ds\Big]$$
$$\le \mathbb{E}^x[\exp[\lambda^2r^2(t\wedge\tau_r)]] < \infty,$$

where $\tau_r = \inf\{t > 0 : W(t) \notin B_r\}$. Since

$$N(t) = \exp\frac{\lambda(|W(t)|^2 - |W(0)|^2)}{2}\left[\exp\left(-\frac{1}{2}\int_0^{\tau}(\lambda^2|W(s)|^2 + 2n\lambda)\,ds\right)\right],$$

all the assumptions of Theorem 2.2 are satisfied, and consequently the harmonic measure $w_r^x(dy)$ of B_r for the operator

$$L_3 = \frac{1}{2}\Delta + \lambda x\cdot\nabla$$

is

$$w_r^x(dy) = \exp\frac{\lambda(r^2 - |x|^2)}{2}\mathbb{E}^x[e^{-\frac{1}{2}\int_0^{\tau_r}(\lambda^2|W(s)|^2+2n\lambda)\,ds};\ W(\tau_r)\in dy].$$

Computations in this case mimic those in the previous section so we omit some details and present only a sketch. For $x \neq 0$ the skew-product representation of the Brownian motion allows us to write

$$\omega_r^x(dy) = e^{\frac{\lambda}{2}(r^2-|x|^2)}\int_0^\infty P^{x/|x|}(\Theta_t \in dy)$$
$$\times\ \mathbb{E}^{|x|}\left[\exp\left(\int_0^{\tau_r} q(R^{(\nu)}(s))\,ds\right);\ \int_0^{\tau_r}\frac{ds}{(R^{(\nu)}(s))^2}\in dt\right],$$

where $q(y) = -\frac{\lambda^2}{2}|x|^2 - n\lambda$, and consequently the Poisson kernel $P_r(x,y)$ is given by

$$P_r(x,y) = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n-1)/2}nr^{n-1}}e^{\frac{\lambda}{2}(r^2-|x|^2)}\int_0^\infty p_t^S\left(1, \frac{\langle x,y\rangle}{|x|\,|y|}\right)\mu_{|x|}(dt),$$

where

$$\mu_y(dt) = \mathbb{E}^y\left[\exp\left(\int_0^{\tau_r} q(R_s^{(\nu)})ds\right);\ \int_0^{\tau_r}\frac{ds}{(R_s^{(\nu)})^2}\in dt\right],\qquad y\in(0,r].$$

As previously, the Laplace transform $\mathcal{L}\mu_y(w)$ given by

$$\mathbb{E}^y\exp\int_0^{\tau_r}\left(-\frac{\lambda^2}{2}(R_s^{(\nu)})^2 - n\lambda - \frac{w}{(R_s^{(\nu)})^2}\right)ds = \mathbb{E}^y e_g(\tau_r), \tag{5.1}$$

where $g(y) = -\frac{\lambda^2}{2}y^2 - n\lambda - \frac{w}{y^2}$, can be identified (by applying the Feynman–Kac formula) as a bounded solution of the Schrödinger equation

$$\frac{1}{2}\varphi''(y) + \frac{n-1}{2y}\varphi'(y) - \left(\frac{\lambda^2}{2}y^2 + n\lambda + \frac{w}{y^2}\right)\varphi(y) = 0,\quad y\in(0,r),\ w\ge 0,$$

with the boundary condition $\varphi(r) = 1$. Setting $\varphi(y) = y^{-n/2} f(\lambda y^2)$ we reduce this equation to the Whittaker equation

$$f''(x) + f(x)\left[-\frac{1}{4} - \frac{n}{2x} - \left(\frac{n(n-4)}{16} + \frac{w}{2}\right)\frac{1}{x^2}\right] = 0,$$

with parameters $k = -n/2$ and $\mu = \sqrt{(n-2)^2 + 8w}/4$. Consequently,

$$\varphi(y) = y^{-n/2}[c_1 M(k,\mu,\lambda y^2) + c_2 W(k,\mu,\lambda y^2)],$$

where M and W are Whittaker functions (see [1, 13.1.32, 13.1.33, p. 505]). The boundedness of φ implies that $c_2 = 0$, and the boundary condition $\varphi(r) = 1$ gives

$$\mathcal{L}\mu_y(w) = \left(\frac{r}{y}\right)^{n/2} \frac{M(k,\mu,\lambda y^2)}{M(k,\mu,\lambda r^2)}. \tag{5.2}$$

If we look at (5.1), the probabilistic definition of $\mathcal{L}\mu_y(w)$, the same argument as previously gives that $\mathcal{L}\mu_y(w)$ can be extended to a holomorphic function in the complex half-plane $\mathrm{Re}(w) > -(n-2)^2/8$. Since the Whittaker functions are well defined in this region, formula (5.2) is also satisfied there. As before, we use the Laplace inverse formula and for $c = -(n-2)^2/16$ we obtain

$$\begin{aligned}\int_0^\infty p_t^S\left(1, \frac{\langle x,y\rangle}{|x|\,|y|}\right)\mu_{|x|}(dt) &= \int_0^\infty p_t^S\left(1, \frac{\langle x,y\rangle}{|x|\,|y|}\right)\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\mathcal{L}\mu_{|x|}(z)\,dz\right)dt\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathcal{L}\mu_{|x|}(z)\left(\int_0^\infty e^{zt}p_t^S\left(1, \frac{\langle x,y\rangle}{|x|\,|y|}\right)dt\right)dz\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathcal{L}\mu_{|x|}(z)G_z\left(1, \frac{\langle x,y\rangle}{|x|\,|y|}\right)dt\,dz.\end{aligned}$$

THEOREM 5.1. *For every $x \in B_r$, $x \neq 0$ and $y \in S_r^{n-1}$ the Poisson kernel $P_r(x,y)$ is given by the formula*

$$\begin{aligned}&\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n-1)/2} n r^{n-1}}\left(\frac{1-|x|^2}{1-r^2}\right)^{(n-2)/2}\left(\frac{r}{|x|}\right)^{n/2}\frac{\sin^{(3-n)/2}\varphi}{2\pi i}\\ &\qquad\times\int_{c-i\infty}^{c+i\infty}\frac{M(k,\mu,\lambda y^2)}{M(k,\mu,\lambda r^2)}B_n(z)P^{(3-n)/2}_{A(z)-1/2}(-\cos\varphi)\,dz,\end{aligned}$$

where φ is the angle between x and y, $A(z) = \frac{1}{2}\sqrt{(n-2)^2 - 8z}$ and

$$B_n(z) = \frac{\Gamma\left(\frac{n-2}{2} - A(z)\right)\Gamma\left(\frac{n-2}{2} + A(z)\right)}{2^{(n+1)/2} z\Gamma\left(\frac{n-1}{2}\right)}.$$

6. Appendix. For the convenience of the reader we collect here basic information about Bessel functions, hypergeometric functions and other special functions appearing throughout the paper. We mainly follow the exposition given in [1] and [12], where we refer the reader for more details (see also [19] and [15]).

6.1. Bessel functions. The Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ are independent solutions of the Bessel equation

$$z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0, \qquad \nu \in \mathbb{R}.$$

The Wronskian of the pair $(J_\nu(z), Y_\nu(z))$ is equal to $2/(\pi z)$ (see [19, p. 113]). The derivatives of the Bessel functions can be expressed by these functions in the following way:

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x), \qquad x > 0, \tag{6.1}$$

$$Y'_\nu(x) = Y_{\nu-1}(x) - \frac{\nu}{x} Y_\nu(x), \qquad x > 0. \tag{6.2}$$

For every $\nu > 0$ we have (see [19, 5.16, pp. 134–135])

$$J_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(\nu+1)}, \quad Y_\nu(x) \sim -\frac{2^\nu \Gamma(\nu)}{\pi} \frac{1}{x^\nu}, \quad x \to 0^+, \tag{6.3}$$

$$\begin{aligned} J_\nu(x) &\sim \sqrt{\frac{2}{x\pi}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \\ Y_\nu(x) &\sim \sqrt{\frac{2}{x\pi}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad x \to \infty. \end{aligned} \tag{6.4}$$

The modified Bessel functions $K_\nu(z)$ are independent solutions to the modified Bessel equation

$$z^2 y''(z) + zy'(z) - (\nu^2 + z^2)y(z) = 0. \tag{6.5}$$

The following asymptotic expansion holds (see [15, 8.451, (6)]):

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}(1 + E(z)), \quad |E(z)| = O(|z|^{-1}) \text{ as } |z| \to \infty, \tag{6.6}$$

whenever $|\arg z| \le 3\pi/2$. The behavior of K_ν near zero is described by (see [1, 9.6.9])

$$K_\nu(z) \approx \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu}, \qquad \operatorname{Re} z > 0. \tag{6.7}$$

The connection between modified Bessel functions of purely imaginary argument and Bessel functions is given by

$$K_\nu(ix) = -\frac{i\pi}{2} e^{-i\nu\pi/2}(J_\nu(x) - iY_\nu(x)), \qquad x > 0. \tag{6.8}$$

Finally, we recall the integral representation of K_ν (see [15, 8.432, (7)]):

$$K_\vartheta(z) = \frac{z^\vartheta}{2}\int_0^\infty \exp\left(-\frac{t+z^2/t}{2}\right)t^{-\vartheta-1}dt, \qquad z>0,\ \vartheta\in\mathbb{R}, \tag{6.9}$$

as well as the formula ([15, 6.561, formula 16, p. 676])

$$\int_0^\infty x^\mu K_\nu(x)\,dx = 2^{\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right), \tag{6.10}$$

where $\mu+1>\nu>0$.

For every $\nu>1/2$ we introduce the following function of two variables:

$$g_\nu(x,t) = \frac{1}{t^{2\nu}(x-1)}\frac{J_\nu(t)Y_\nu(tx)-J_\nu(tx)Y_\nu(t)}{J_\nu^2(t)+Y_\nu^2(t)}, \qquad x>1,\ t>0.$$

It is obvious that g_ν is a continuous function on $(1,\infty)\times\mathbb{R}_+$. However, most crucial for the considerations in Section 3 are the following asymptotic properties of g_ν.

Lemma 6.1. *Fix $x_0\ge 1$, $t_0>0$. Then*

$$\lim_{(x,t)\to(x_0,0)} g_\nu(x,t) = \frac{\pi\sum_{k=0}^{n-2}x_0^k}{2^{2\nu}\Gamma(\nu)\Gamma(\nu+1)x_0^\nu}, \tag{6.11}$$

$$\lim_{(x,t)\to(1,t_0)} g_\nu(x,t) = \frac{2}{\pi t_0^{2\nu}(J_\nu^2(t_0)+Y_\nu^2(t_0))}. \tag{6.12}$$

Proof. From the Lagrange theorem, there exist $\theta_1,\theta_2\in(1,x)$, depending on x and such that

$$\begin{aligned}\frac{J_\nu(t)Y_\nu(tx)-J_\nu(tx)Y_\nu(t)}{tx-t} &= J_\nu(t)\frac{Y_\nu(tx)-Y_\nu(t)}{tx-t} - Y_\nu(t)\frac{J_\nu(tx)-J_\nu(t)}{tx-t}\\ &= J_\nu(t)Y_\nu'(t\theta_1) - J_\nu'(t\theta_2)Y_\nu(t).\end{aligned}$$

Obviously, if $x\to 1$ then θ_1, θ_2 also tend to 1. Furthermore, since the Wronskian of $(J_\nu(z),Y_\nu(z))$ is $2/(\pi z)$ we get

$$\lim_{(x,t)\to(1,t_0)}\frac{J_\nu(t)Y_\nu(tx)-J_\nu(tx)Y_\nu(t)}{tx-t} = J_\nu(t_0)Y_\nu'(t_0)-J_\nu'(t_0)Y_\nu(t_0) = \frac{2}{\pi t_0},$$

which proves (6.12). If we use the recurrent formulas for the Bessel function derivatives (6.1) and (6.2) we deduce that $J_\nu(t)Y_\nu'(t\theta_1)-J_\nu'(t\theta_2)Y_\nu(t)$ equals

$$J_\nu(t)Y_{\nu-1}(t\theta_1) - \frac{\nu}{t}\left(\frac{J_\nu(t)Y_\nu(t\theta_1)}{\theta_1} - \frac{J_\nu(t\theta_1)Y_\nu(t)}{\theta_2}\right) - J_{\nu-1}(t\theta_2)Y_\nu(t).$$

Multiplying the last expression by t, letting $(x,t)\to(1,0)$ and using (6.3) it is easy to see that the first two summands tend to zero and the last one tends to $2/\pi$. Since, by (6.3), we have $\lim_{t\to 0^+} t^{2\nu}(J_\nu^2(t)+Y_\nu^2(t)) = 2^{2\nu}\Gamma^2(\nu)/\pi^2$, it follows that

$$\lim_{(x,t)\to(1,0)} g_\nu(x,t) = \frac{\pi}{2^{2\nu-1}\Gamma^2(\nu)},$$

which is (6.11) for $x_0 = 1$. For $x_0 > 1$ relation (6.11) follows directly from (6.3):

$$\lim_{(x,t)\to(x_0,0)} g_\nu(x,t) = \frac{\pi(x_0^\nu - x_0^{-\nu})}{\nu 2^{2\nu}\Gamma^2(\nu)} = \frac{\pi\sum_{k=0}^{n-2} x_0^k}{2^{2\nu}\Gamma(\nu)\Gamma(\nu+1)x_0^\nu}. \ \blacksquare$$

Note that Lemma 6.1 together with (6.3) and (6.4) implies that the function g_ν can be extended to a continuous function on $[1,\infty)\times[0,\infty)$ which is bounded whenever x is bounded, i.e. for every $R > 1$ there exists $C(R) > 0$ such that

$$|g_\nu(x,t)| \le C(R), \quad (x,t) \in [1,R]\times[0,\infty). \tag{6.13}$$

6.2. Hypergeometric functions and Legendre functions. For $\gamma \neq -1,-2,\ldots$ the hypergeometric function is defined by

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta_k)}{(\gamma_k)k!} z^k, \quad |z| < 1.$$

Here $(a)_k = \Gamma(a+k)/\Gamma(a)$. The function ${}_2F_1$ is a solution of the hypergeometric equation

$$z(1-z)u''(z) + [\gamma - (\alpha+\beta+1)z]u'(z) - \alpha\beta u(z) = 0 \tag{6.14}$$

regular at $z = 0$. Whenever $\mathrm{Re}(\gamma-\alpha-\beta) > 0$ we have (see [12, Vol. 1, p. 104, 2.8(46)])

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{6.15}$$

The derivative of ${}_2F_1$ is given by (see [12, Vol. 1, p. 102, 2.8(20)])

$$\frac{d}{dz}\,{}_2F_1(\alpha,\beta;\gamma;z) = \frac{\alpha\beta}{\gamma}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z), \tag{6.16}$$

and the following elementary relation holds (see [12, Vol. 1, p. 105, 2.9(2)])

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z). \tag{6.17}$$

The Legendre functions are solutions of Legendre's differential equation

$$(1-z^2)u''(z) - 2zu'(z) + [a(a+1) - b^2(1-z^2)^{-1}]u(z) = 0. \tag{6.18}$$

By making an appropriate substitution it can be reduced to the hypergeometric equation (6.14) and consequently its solutions are given in terms of the hypergeometric function. More precisely, the Legendre functions of the first and second kind are defined by (see [12, Vol. 1, p. 122, 3.2(3) and p. 143, 3.4(6)])

$$(6.19)\qquad P_a^b(x) = \frac{1}{\Gamma(1-b)}\left(\frac{1+x}{|1-x|}\right)^{b/2} \times {}_2F_1\left(-a, a+1; 1-b; \frac{1-x}{2}\right), \quad x > -1,$$

$$(6.20)\qquad Q_a^b(x) = \frac{e^{bi\pi}\pi^{1/2}}{2^{a+1}x^{a+b+1}}\,\frac{\Gamma(a+b+1)}{\Gamma(a+3/2)}(x^2-1)^{b/2} \times {}_2F_1\left(\frac{a+b}{2}+1, \frac{a+b+1}{2}; a+3/2; \frac{1}{x^2}\right), \quad x > 1,$$

respectively. The functions P_a^b and Q_a^b are independent solutions of (6.18).

6.3. Skew-product representation of Brownian motion. We now introduce the spherical Brownian motion on the unit sphere $S_1^{n-1} \subseteq \mathbb{R}^n$ as a diffusion on S_1^{n-1} with generator being one-half of the Laplace–Beltrami operator $\Delta_{S_1^{n-1}}$ of the manifold S_1^{n-1}. It is well-known that

$$\Delta_{S_1^{n-1}} = (\sin\phi)^{2-n}\frac{\partial}{\partial\phi}\left[(\sin\phi)^{n-2}\,\frac{\partial}{\partial\phi}\right] + (\sin\phi)^{-2}\Delta_{S_1^{n-2}},$$

where ϕ is the angle between the pole and the given point on the sphere and $\Delta_{S_1^1} = \frac{\partial^2}{\partial\phi^2}$. Now, if we consider the action of $\Delta_{S_1^{n-1}}$ on functions depending only on ϕ, this reduces to the generator of the Legendre process LEG(d):

$$\frac{1}{2}\Delta_{S_1^{n-1}} = \frac{1}{2}(\sin\phi)^{2-n}\frac{\partial}{\partial\phi}\left[(\sin\phi)^{n-2}\frac{\partial}{\partial\phi}\right] = \frac{1}{2}\,\frac{\partial^2}{\partial\phi^2} + \frac{n-2}{2}\cot\phi\,\frac{\partial}{\partial\phi}.$$

Changing variable $\cos\phi = t$ we obtain

$$\frac{1-t^2}{2}\,\frac{\partial^2}{\partial t^2} - \frac{n-1}{2}t\frac{\partial}{\partial t}.$$

We now invoke the classical skew-product representation of the n-dimensional Brownian motion (see e.g. [18, (7.15)] stating that it can be represented as the product of $R^{(\nu)} = \{R_t^{(\nu)}; t \geq 0\}$, the Bessel process BES$(n)$, $\nu = n/2-1$, with generator

$$(6.21)\qquad \frac{1}{2}\,\frac{\partial^2}{\partial r^2} + \frac{n-1}{2r}\,\frac{\partial}{\partial r},$$

and an independent spherical Brownian motion $\Theta = \{\Theta(t) : t \geq 0\}$ on S_1^{n-1} with time changed according to the formula

$$A^{(\nu)}(t) = \int_0^t \frac{ds}{(R^{(\nu)}(s))^2}.$$

Moreover, for $x \neq 0$, we introduce a process $S = \{S(t) : t \geq 0\}$ defined by $S(t) = \langle x, \Theta(t)\rangle/|x|$. The process S describes the cosine of the angle between the starting point x and the spherical Brownian motion Θ. Consequently, the cosine between the starting point x and $W(t)$ is just $S(A(t))$. The skew-product representation and the previous considerations imply that S is independent of the Bessel process $R^{(\nu)}$, and the generator of S is given by

$$\mathcal{G} = \frac{1-t^2}{2}\frac{\partial^2}{\partial t^2} - \frac{n-1}{2}t\frac{\partial}{\partial t} \tag{6.22}$$

with domain $D_{\mathcal{G}} = \{u \in \mathcal{C}^2[-1,1] : u'(-1) = u'(1) = 0\}$. Three basic characteristics of the diffusion: the speed measure, the scale function and the killing measure are described by the relations (see also [6]) $m(dx) = 2(1-x^2)^{(n-3)/2}dx$, $s'(x) = (1-x^2)^{(1-n)/2}$, $k(dx) = 0$. Moreover, the points -1 and 1 are non-singular reflecting points. We denote by $p_t^S(x,y)$ the transition density function with respect to the speed measure, i.e.

$$P^x(S(t) \in A) = \int_A p_t^S(x,y)\, m(dx), \quad A \in \mathcal{B}[-1,1]. \tag{6.23}$$

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