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# PROBABILITY MEASURES CORRESPONDING TO AVAL NUMBERS

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**Abstract.** We describe the class of probability measures whose moments are given in terms of the Aval numbers. They are expressed as the multiplicative free convolution of measures corresponding to the ballot numbers $\frac{m-k}{m+k}\binom{m+k}{m}$.

**Introduction.** Aval [3] introduced multivariate Fuss–Catalan numbers (1.3), which count $p$-Raney sequences (or, equivalently, Dyck paths or trees) of a special type. They decompose the Fuss–Catalan numbers in an analogous way as the *ballot numbers* $B(m,k) := \frac{m-k}{m+k}\binom{m+k}{k}$ decompose the Catalan numbers $\frac{1}{2m+1}\binom{2m+1}{m}$.

The aim of this paper is to prove that the generating function (2.1) for these numbers, with nonnegative parameters $a_0, \ldots, a_{p-1}$, is the moment generating function of a certain probability measure $\mu$ on $\mathbb{R}$. It turns out that $\mu$ can be represented as the multiplicative free convolution

$$\mu(a_0) \boxtimes \mu(a_1) \boxtimes \cdots \boxtimes \mu(a_{p-1}),$$

where $\mu(a)$ is given by (3.4) and corresponds to the classical ballot numbers. It is worth mentioning here that for a real parameter $p \geq 1$ the Fuss–Catalan numbers $\frac{1}{mp+1}\binom{mp+1}{m}$, $m = 0, 1, \ldots$, constitute the moment sequence of the probability measure $\mu(1)^{\boxtimes(p-1)}$, the multiplicative free power of the *Marchenko–Pastur distribution* (see [4, 9, 10]).

The paper is organized as follows. First we reprove the result of Aval: the formula (1.3) for the number of elements in the class $\mathcal{F}_m(k_0, \ldots, k_{p-1})$ of Raney sequences. Next we find the generating function for these numbers, with parameters $a_0, \ldots, a_{p-1} \in \mathbb{R}$ (Theorem 2.1). In Section 3 we characterize those probability measures which correspond to $p = 1$, i.e. the case of the ballot numbers. Finally we briefly recall the notion of multiplicative free convolution and apply it to the proof of our main theorem.

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**1. Raney sequences.** Let $p \geq 1$ be a fixed integer. A *$p$-Raney sequence* is a sequence $\mathbf{x} = (x_1, \ldots, x_N)$ such that $x_i \in \{1, -p\}$ and all the partial sums $S_n(\mathbf{x}) = S_n := x_1 + \cdots + x_n$, $0 \leq n \leq N$, are nonnegative. For $0 \leq n \leq N$ define $P_n := (n, S_n) \in \mathbb{Z}^2$. Then $\mathbf{x}$ can be identified with the path $(P_0, P_1, \ldots, P_N)$.

Denote by $\mathcal{F}$ the class of those $p$-Raney sequences $(x_1, \ldots, x_N)$ for which $x_N = 1$ and $p$ divides $S_N$, and by $\mathcal{G}$ the class of those for which $S_N = 0$ (in particular here $p+1$ divides $N$). By convention we assume that the empty sequence $\emptyset$ belongs to $\mathcal{F}$ as well as to $\mathcal{G}$, so that $\mathcal{F} \cap \mathcal{G} = \{\emptyset\}$. For $\mathbf{x} \in \mathcal{F} \cup \mathcal{G}$ we have $|\{k : x_k = 1\}| = mp$ for some integer $m \geq 0$ and then we put $\|\mathbf{x}\| := m$. Define

$$\mathcal{F}_m := \{\mathbf{x} \in \mathcal{F} : \|\mathbf{x}\| = m\} \quad \text{and} \quad \mathcal{G}_m := \{\mathbf{y} \in \mathcal{G} : \|\mathbf{y}\| = m\}.$$

Then the map $(x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_M)$, where $M < N$ is such that $x_M = 1$, $x_{M+1} = x_{M+2} = \cdots = x_N = -p$ (for $N = 0$ we put $M = 0$) is a bijection $\mathcal{G}_m \to \mathcal{F}_m$ and elements of these sets are counted by the *Fuss–Catalan numbers*:

$$|\mathcal{F}_m| = |\mathcal{G}_m| = \frac{1}{m(p+1)+1}\binom{m(p+1)+1}{m} \tag{1.1}$$

(see [8]). Now we are going to decompose $\mathcal{F}_m$ and $\mathcal{G}_m$ into smaller sets.

For $r \in \{0, 1, \ldots, p-1\}$ and for a $p$-Raney sequence $\mathbf{x} = (x_1, \ldots, x_N)$ we define

$$|\mathbf{x}|_r := |\{k : x_k = -p \text{ and } p \text{ divides } S_k - r\}|.$$

We say that $\mathbf{x}$ is *of type* $(k_0, \ldots, k_{p-1})$, denoted $\mathbf{t}(\mathbf{x}) := (k_0, \ldots, k_{p-1})$, if $|\mathbf{x}|_0 = k_0, \ldots, |\mathbf{x}|_{p-1} = k_{p-1}$. By $\mathcal{F}_m(k_0, \ldots, k_{p-1})$ (resp. $\mathcal{G}_m(k_0, \ldots, k_{p-1})$) we will denote the set of all $\mathbf{x}$ in $\mathcal{F}_m$ (resp. $\mathcal{G}_m$) such that $\mathbf{t}(\mathbf{x}) = (k_0, \ldots, k_{p-1})$. In particular, if $k_0 + \cdots + k_{p-1} = m \geq 1$ then $\mathcal{F}_m(k_0, \ldots, k_{p-1}) = \emptyset$. Hence we have

$$\mathcal{F} = \{\emptyset\} \,\dot{\cup} \dot{\bigcup_{\substack{m \geq 1 \\ k_0, \ldots, k_{p-1} \geq 0 \\ k_0 + \cdots + k_{p-1} < m}}} \mathcal{F}_m(k_0, \ldots, k_{p-1}),$$

$$\mathcal{G} = \{\emptyset\} \,\dot{\cup} \dot{\bigcup_{\substack{m \geq 1 \\ k_0, \ldots, k_{p-1} \geq 0 \\ k_0 + \cdots + k_{p-1} = m}}} \mathcal{G}_m(k_0, \ldots, k_{p-1}).$$

Now we are going to count the elements in these sets. In particular we will reprove Proposition 2.4 from [3].

LEMMA 1.1. *For $m \geq 0$ and $k_0, \ldots, k_{p-1} \geq 0$ with $k_0 + \cdots + k_{p-1} < m+1$, we have*

$$|\mathcal{F}_{m+1}(k_0,\ldots,k_{p-1})| = \sum_{\substack{0\le j_0\le k_0\\ \cdots \\ 0\le j_{p-1}\le k_{p-1}}} |\mathcal{F}_m(j_0,\ldots,j_{p-1})|.$$

*Proof.* For $\mathbf{x}=(x_1,\ldots,x_N)\in\mathcal{F}_{m+1}(k_0,\ldots,k_{p-1})$ put $T\mathbf{x}:=(x_1,\ldots,x_M)$ where $M$ is unique such that $x_M = 1$ and $|\{j : 1 \le j \le M, x_j = 1\}| = mp$. For $m = 0$ put $M = 0$. Then $T$ is a bijection from $\mathcal{F}_{m+1}(k_0,\ldots,k_{p-1})$ onto the disjoint union of the sets $\mathcal{F}_m(j_0,\ldots,j_{p-1})$, where $0 \le j_0 \le k_0,\ldots,0 \le j_{p-1} \le k_{p-1}$. ■

We will apply the following elementary identity:

$$\sum_{i=0}^{k}(c-i)\binom{m-1+i}{i} = \left(c-\frac{m\,k}{m+1}\right)\binom{m+k}{k}, \tag{1.2}$$

for $k \ge 0$, $m \ge 1$ and $c \in \mathbb{R}$.

THEOREM 1.2. *For $m \ge 1$ and for $k_0,\ldots,k_{p-1} \ge 0$ with $k_0+\cdots+k_{p-1} < m$, we have*

$$|\mathcal{F}_m(k_0,\ldots,k_{p-1})| = \frac{m-\sum_{i=0}^{p-1}k_i}{m}\prod_{i=0}^{p-1}\binom{m-1+k_i}{k_i}. \tag{1.3}$$

Theorem 1.2 was proved in [3] by combinatorial means; here we follow Remark 2.6 in [3]. Note that for $p = 1$ we get the classical ballot numbers.

*Proof.* Put $e_0(0,\ldots,0) := 1$, and for $m \ge 1$ and $k_0,\ldots,k_{p-1} \ge 0$ with $k_0+\cdots+k_{p-1} \le m$, define

$$e_m(k_0,\ldots,k_{p-1}) := \frac{m-\sum_{i=0}^{p-1}k_i}{m}\prod_{i=0}^{p-1}\binom{m-1+k_i}{k_i}.$$

In view of Lemma 1.1, it is sufficient to prove that if $m \ge 0$, $k_0,\ldots,k_{p-1} \ge 0$ and $k_0+\cdots+k_{p-1} < m+1$ then

$$\sum_{\substack{0\le j_0\le k_0\\ \cdots \\ 0\le j_{p-1}\le k_{p-1}}} e_m(j_0,\ldots,j_{p-1}) = e_{m+1}(k_0,\ldots,k_{p-1}).$$

This is true for $m = 0$ because

$$e_1(0,\ldots,0) = e_0(0,\ldots,0) = 1.$$

Now assume that $m \ge 1$. Applying (1.2) consecutively $p$ times we get

$$\sum_{j_{p-1}=0}^{k_{p-1}} \cdots \sum_{j_0=0}^{k_0} e_m(j_0,\ldots,j_{p-1})$$
$$= \sum_{j_{p-1}=0}^{k_{p-1}} \cdots \sum_{j_1=0}^{k_1} \frac{m-j_1-\ldots-j_{p-1}-\frac{m\,k_0}{m+1}}{m}\binom{m+k_0}{k_0}\prod_{i=1}^{p-1}\binom{m-1+j_i}{j_i}$$
$$\cdots = \frac{m-\frac{m\,k_0}{m+1}-\ldots-\frac{m\,k_{p-1}}{m+1}}{m}\prod_{i=0}^{p-1}\binom{m+k_i}{k_i} = e_{m+1}(k_0,\ldots,k_{p-1}),$$

which concludes the proof. ■

THEOREM 1.3. *For $m \geq 1$ and $k_0,\ldots,k_{p-1} \geq 0$ with $k_0+\cdots+k_{p-1}=m$, we have*

$$|\mathcal{G}_m(k_0,k_1,\ldots,k_{p-1})| = \frac{1}{m}\binom{m+k_0}{k_0-1}\prod_{i=1}^{p-1}\binom{m+k_i-1}{k_i}. \tag{1.4}$$

*Proof.* We define $\Lambda : \mathcal{G} \to \mathcal{F}$ by putting $\Lambda(x_1,\ldots,x_N) := (x_1,\ldots,x_M)$ whenever $x_M = 1$ and $x_{M+1} = x_{M+2} = \cdots = x_N = -p$. For $N = 0$ we put $M = 0$. Note that for $m \geq 1$ and $k_0,\ldots,k_{p-1} \geq 0$ with $k_0+k_1+\cdots+k_{p-1} = m$, $\Lambda$ establishes a 1-1 correspondence between the class $\mathcal{G}_m(k_0,k_1,\ldots,k_{p-1})$ and the disjoint union $\bigcup_{k=0}^{k_0-1}\mathcal{F}_m(k,k_1,\ldots,k_{p-1})$. Now it remains to yoke (1.2) together with (1.3). ■

**2. Generating functions.** Fix parameters $a_0,\ldots,a_{p-1} \in \mathbb{R}$. The aim of this section is to describe the generating functions

$$F(z) = 1+\sum_{m=1}^{\infty} z^m \sum_{\substack{k_0,k_1,\ldots,k_{p-1}\geq 0\\ k_0+k_1+\cdots+k_{p-1}<m}} |\mathcal{F}_m(k_0,\ldots,k_{p-1})|a_0^{k_0}a_1^{k_1}\ldots a_{p-1}^{k_{p-1}}, \tag{2.1}$$

$$G(z) = 1+\sum_{m=1}^{\infty} z^m \sum_{\substack{k_0,k_1,\ldots,k_{p-1}\geq 0\\ k_0+k_1+\cdots+k_{p-1}=m}} |\mathcal{G}_m(k_0,\ldots,k_{p-1})|a_0^{k_0}a_1^{k_1}\ldots a_{p-1}^{k_{p-1}}. \tag{2.2}$$

We will use ideas from the proof of Proposition 2.7 in [3].

First we introduce some notation. For $0 \leq r \leq p-1$ define

$$\mathbf{a}_{(r)} := (a_r,a_{r+1},\ldots,a_{p-1},a_0,\ldots,a_{r-1})$$

and for $\mathbf{k} = (k_0,\ldots,k_{p-1})$ put

$$\mathbf{a}_{(r)}^{\mathbf{k}} := a_r^{k_0}a_{r+1}^{k_1}\ldots a_{p-1}^{k_{p-r-1}}a_0^{k_{p-r}}\ldots a_{r-1}^{k_{p-1}}.$$

Now define auxiliary functions

$$G_r(z) := \sum_{\mathbf{y}\in\mathcal{G}} z^{\|\mathbf{y}\|}\mathbf{a}_{(r)}^{\mathbf{t}(\mathbf{y})}, \tag{2.3}$$

in particular $G_0 = G$.

Fix $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{F} \setminus \{\emptyset\}$. For $0 \le r \le p-1$ we put $\alpha_r := \max\{n : S_n(\mathbf{x}) = r\}$. Then we decompose

$$\mathbf{x} = (\mathbf{x}_0, 1, \mathbf{x}_1, 1, \ldots, 1, \mathbf{x}_p)$$

where

$$\mathbf{x}_r := (x_{\alpha_{r-1}+2}, x_{\alpha_{r-1}+3}, \ldots, x_{\alpha_r})$$

under the convention that $\alpha_{-1} := -1$, $\alpha_p := N$. Then $\mathbf{x}_0, \ldots, \mathbf{x}_{p-1} \in \mathcal{G}$, $\mathbf{x}_p \in \mathcal{F}$ and we have

$$\|\mathbf{x}\| = 1 + \|\mathbf{x}_0\| + \cdots + \|\mathbf{x}_{p-1}\| + \|\mathbf{x}_p\|$$

and

$$z^{\|\mathbf{x}\|}\mathbf{a}^{t(\mathbf{x})} = z \cdot z^{\|\mathbf{x}_0\|}\mathbf{a}_{(0)}^{t(\mathbf{x}_0)} \ldots z^{\|\mathbf{x}_{p-1}\|}\mathbf{a}_{(p-1)}^{t(\mathbf{x}_{p-1})} z^{\|\mathbf{x}_p\|}\mathbf{a}_{(0)}^{t(\mathbf{x}_p)}, \tag{2.4}$$

because the level of a given step of $\mathbf{x}_r$ in the whole $\mathbf{x}$ is equal to its level within $\mathbf{x}_r$ plus $r$, i.e.

$$S_k(\mathbf{x}_r) + r = S_{\alpha_{r-1}+1+k}(\mathbf{x}).$$

Note that the map $\mathbf{x} \mapsto (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_p)$ is a bijection from $\mathcal{F} \setminus \{\emptyset\}$ onto $\mathcal{G}^p \times \mathcal{F}$.

Similarly, we decompose every $\mathbf{y} \in \mathcal{G} \setminus \{\emptyset\}$ as $\mathbf{y} = (\mathbf{y}_0, 1, \mathbf{y}_1, 1, \ldots, 1, \mathbf{y}_p, -p)$, with $\mathbf{y}_0, \ldots, \mathbf{y}_p \in \mathcal{G}$, and

$$z^{\|\mathbf{y}\|}\mathbf{a}^{t(\mathbf{y})} = za_0 \cdot z^{\|\mathbf{y}_0\|}\mathbf{a}_{(0)}^{t(\mathbf{y}_0)} \ldots z^{\|\mathbf{y}_{p-1}\|}\mathbf{a}_{(p-1)}^{t(\mathbf{y}_{p-1})} z^{\|\mathbf{y}_p\|}\mathbf{a}_{(0)}^{t(\mathbf{y}_p)}, \tag{2.5}$$

which leads to a bijection $\mathcal{G} \setminus \{\emptyset\} \to \mathcal{G}^{p+1}$.

Then (2.4) and (2.5) yield

$$F(z) = 1 + \sum_{\mathbf{x} \in \mathcal{F}\setminus\{\emptyset\}} z^{\|\mathbf{x}\|}\mathbf{a}_{(0)}^{\mathbf{t}(\mathbf{x})} = 1 + zG_0(z)G_1(z)\ldots G_{p-1}(z)F(z), \tag{2.6}$$

$$G(z) = 1 + \sum_{\mathbf{y} \in \mathcal{G}\setminus\{\emptyset\}} z^{\|\mathbf{y}\|}\mathbf{a}_{(0)}^{\mathbf{t}(\mathbf{y})} = 1 + a_0 zG_0(z)G_1(z)\ldots G_{p-1}(z)G_0(z). \tag{2.7}$$

In the same way we can proceed with $G_r$, obtaining the equations

$$G_r(z) = 1 + a_r zG_0(z)G_1(z)\ldots G_{p-1}(z)G_r(z). \tag{2.8}$$

Now we are in a position to describe the functions $F$ and $G$.

THEOREM 2.1. *In a neighborhood of* 0 *we have* $F(z) = 1 + \widetilde{F}(z)$ *and* $G(z) = 1 + \widetilde{G}(z)$, *where* $\widetilde{F}$ *is the inverse function of*

$$w \mapsto \frac{w(1 + w - a_0 w)(1 + w - a_1 w)\ldots(1 + w - a_{p-1}w)}{(1+w)^{p+1}}, \tag{2.9}$$

*and* $\widetilde{G}$ *is the inverse function of*

$$(2.10)\quad w \mapsto \frac{w(a_0+a_0w-a_1w)(a_0+a_0w-a_2w)\dots(a_0+a_0w-a_{p-1}w)}{a_0^p(1+w)^{p+1}}$$

*in a neighborhood of* 0.

*Proof.* Put $H(z) := zG_0(z)G_1(z)\dots G_{p-1}(z)$. Then from (2.8),

$$G_r(z) = \frac{1}{1-a_rH(z)}.$$

Taking the product over $0 \le r \le p-1$ and multiplying by $z$ we get

$$(2.11)\qquad H(z) = \frac{z}{(1-a_0H(z))(1-a_1H(z))\dots(1-a_{p-1}H(z))}.$$

Now putting $\widetilde{F} := F-1$, $\widetilde{G} := G-1$ we see from (2.6) and (2.7) that

$$H = \frac{\widetilde{F}}{1+\widetilde{F}},\qquad H = \frac{\widetilde{G}}{a_0+a_0\widetilde{G}}.$$

Substituting to (2.11) we get

$$z = \frac{\widetilde{F}(1+\widetilde{F}-a_0\widetilde{F})(1+\widetilde{F}-a_1\widetilde{F})\dots(1+\widetilde{F}-a_{p-1}\widetilde{F})}{(1+\widetilde{F})^{p+1}}$$

and

$$z = \frac{\widetilde{G}(a_0+a_0\widetilde{G}-a_1\widetilde{G})(a_0+a_0\widetilde{G}-a_2\widetilde{G})\dots(a_0+a_0\widetilde{G}-a_{p-1}\widetilde{G})}{a_0^p(1+\widetilde{G})^{p+1}},$$

which concludes the proof. ∎

Now it is easy to find $F$ for $p=1$:

COROLLARY 2.2. *For $p=1$ and $a_0 := a$ we have*

$$F(z) = \frac{1-2a+\sqrt{1-4az}}{2(1-a-z)} = \frac{2a}{2a-1+\sqrt{1-4az}}.$$

*Proof.* Here $\widetilde{F}$ is the inverse function of $w \mapsto w(1+w-aw)(1+w)^{-2}$, which implies that $F(z) = \widetilde{F}(z)+1$ satisfies the quadratic equation

$$(2.12)\qquad (F(z)-1)(F(z)-aF(z)+a) = zF(z)^2,$$

with $F(0)=1$, from which we find $F(z)$. ∎

We can also compute the Taylor expansion of the powers $F(z)^s$:

THEOREM 2.3. *For $s \in \mathbb{R}$ we have*

$$F(z)^s = 1 + \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{k=1}^{m} k\binom{s-1+k}{k} \sum_{\substack{k_0,\dots,k_{p-1}\ge 0\\ k_0+\dots+k_{p-1}+k=m}} \prod_{i=0}^{p-1}\binom{m-1+k_i}{k_i} a_0^{k_0}a_1^{k_1}\dots a_{p-1}^{k_{p-1}}.$$

*Proof.* For $\mathbf{x} \in \mathcal{F}$, with $\|\mathbf{x}\| = m$ and $\mathbf{t}(\mathbf{x}) = (k_0, \dots, k_{p-1})$ we define

$$\phi(\mathbf{x}) := m - (k_0 + \cdots + k_{p-1}),$$

in particular $\phi(\emptyset) = 0$. Now we observe that if $\mathbf{x}_1, \dots, \mathbf{x}_s \in \mathcal{F}$ then the concatenation $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_s)$ is also in $\mathcal{F}$ and we have $\mathbf{t}(\mathbf{x}) = \mathbf{t}(\mathbf{x}_1) + \cdots + \mathbf{t}(\mathbf{x}_s)$ (as $p$-vectors), and $\|\mathbf{x}\| = \|\mathbf{x}_1\| + \cdots + \|\mathbf{x}_s\|$.

On the other hand, for given $\mathbf{x} \in \mathcal{F}$ and natural $s$ there are exactly $\binom{s-1+\phi(\mathbf{x})}{\phi(\mathbf{x})}$ ways of decomposing $\mathbf{x}$ into $\mathbf{x}_1, \dots, \mathbf{x}_s$, with $\mathbf{x}_i \in \mathcal{F}$. Indeed, the beginning and the end points of the pieces can only be the points $\gamma_i$, $0 \le i \le \phi(\mathbf{x})$, where

$$\gamma_i := \min\{j \ge 0 : \text{if } k \ge j \text{ then } S_k(\mathbf{x}) \ge ip\}$$

for $0 \le i \le \phi(\mathbf{x})$ (in particular $\gamma_0 = 0$), so this is equivalent to the problem of counting distributions of $s-1$ indistinguishable balls into $\phi(x)+1$ cells. Therefore for natural $s \in \mathbb{N}$ we have

$$\begin{aligned} F(z)^s &= \Big(\sum_{\mathbf{x}\in\mathcal{F}} \mathbf{a}^{\mathbf{t}(\mathbf{x})} z^{\|\mathbf{x}\|}\Big)^s = \sum_{\mathbf{x}_1,\dots,\mathbf{x}_s\in\mathcal{F}} \mathbf{a}^{\mathbf{t}(\mathbf{x}_1)+\cdots+\mathbf{t}(\mathbf{x}_s)} z^{\|\mathbf{x}_1\|+\cdots+\|\mathbf{x}_s\|} \\ &= \sum_{\mathbf{x}\in\mathcal{F}} \binom{s-1+\phi(\mathbf{x})}{\phi(\mathbf{x})} \mathbf{a}^{\mathbf{t}(\mathbf{x})} z^{\|\mathbf{x}\|} \\ &= 1 + \sum_{m=1}^{\infty} z^m \sum_{k=1}^{m} \binom{s-1+k}{k} \sum_{\substack{\mathbf{x}\in\mathcal{F}_m \\ \phi(\mathbf{x})=k}} \mathbf{a}^{\mathbf{t}(\mathbf{x})}, \end{aligned}$$

which, by (1.3), proves the theorem for $s \in \mathbb{N}$.

Now for $s \in \mathbb{R}$ define $c_0(s) := 1$ and for $m \ge 1$ put

$$c_m(s) := \sum_{k=1}^{m} \frac{k}{m} \binom{s-1+k}{k} \sum_{\substack{k_0,\dots,k_{p-1}\ge 0 \\ k_0+\cdots+k_{p-1}+k=m}} \prod_{i=0}^{p-1} \binom{m-1+k_i}{k_i} a_0^{k_0} a_1^{k_1} \dots a_{p-1}^{k_{p-1}}.$$

Then for natural $s$ we have

$$F(z)^s = \sum_{m=0}^{\infty} c_m(s) z^m. \tag{2.13}$$

This implies that for $u, v \in \mathbb{N}$,

$$c_m(u+v) = \sum_{k=0}^{m} c_k(u) c_{m-k}(v). \tag{2.14}$$

Since both sides of (2.14) are polynomials on $u$ and $v$, (2.14) remains true for all $u, v \in \mathbb{R}$ and that in turn proves that (2.13) holds for all $s \in \mathbb{R}$. ∎

**3. Probability measures: the case $p = 1$.** Our purpose now is to prove that if all the parameters $a_j$ are nonnegative then $F$ and $G$ are moment generating functions of certain probability measures. In other words, if $a_0, \dots, a_{p-1} \geq 0$ then both the sequences

$$f_m(a_0, \dots, a_{p-1}) := \sum_{\substack{k_0, k_1, \dots, k_{p-1} \geq 0 \\ k_0 + k_1 + \dots + k_{p-1} \leq m}} |\mathcal{F}_m(k_0, \dots, k_{p-1})| a_0^{k_0} a_1^{k_1} \dots a_{p-1}^{k_{p-1}},$$

$$g_m(a_0, \dots, a_{p-1}) := \sum_{\substack{k_0, k_1, \dots, k_{p-1} \geq 0 \\ k_0 + k_1 + \dots + k_{p-1} = m}} |\mathcal{G}_m(k_0, \dots, k_{p-1})| a_0^{k_0} a_1^{k_1} \dots a_{p-1}^{k_{p-1}}$$

(in particular: $f_0(a_0, \dots, a_{p-1}) = g_0(a_0, \dots, a_{p-1}) = 1$), $m \geq 0$, are positive definite. We conjecture that this condition is also necessary (apart from the trivial case $g_m(0, a_1, \dots, a_{p-1}) = \delta_{0,m}$, $a_i \in \mathbb{R}$).

First we start with the case $p = 1$, so that

$$f_m(a) = \sum_{k=0}^{m} B(m,k) a^k, \tag{3.1}$$

where $B(m,k) := |\mathcal{F}_m(k)|$ are the classical *ballot numbers*: $B(0,0) := 1$ and $B(m,k) = \binom{m+k}{m} \frac{m-k}{m+k}$ for $m \geq 1$, $0 \leq k \leq m$. In particular $f_m(1) = \binom{2m}{m}/(m+1)$, the $m$th Catalan number. These numbers satisfy the recurrence $B(0,0) = 1$ and

$$B(m+1,k) = \sum_{i=0}^{k} B(m,i) \tag{3.2}$$

for $0 \leq k \leq m$, $B(m+1, m+1) = 0$, which is equivalent to the following recurrence for the polynomials $f_m$: $f_0(a) = 1$ and

$$f_{m+1}(a) = \frac{f_m(1) a^{m+1} - f_m(a)}{a - 1} \tag{3.3}$$

for $m \geq 0$.

Now define a one-parameter family of measures: $\mu(0) := \delta_1$ and for $a > 0$,

$$\mu(a) := \frac{\sqrt{4ax - x^2}}{2\pi x(1 - x + ax)} \chi_{[0,4a]} dx + \mathrm{atom}(a) \tag{3.4}$$

where

$$\mathrm{atom}(a) = \begin{cases} \dfrac{1-2a}{1-a} \delta_{1/(1-a)} & \text{if } 0 < a < 1/2, \\ 0 & \text{if } a \geq 1/2. \end{cases} \tag{3.5}$$

In particular, $\mu(1)$ is called the Marchenko–Pastur distribution.

THEOREM 3.1. *For $a \geq 0$ the moment sequence of $\mu(a)$ is $\{f_m(a)\}_{m=0}^{\infty}$, i.e.*

$$\int_{\mathbb{R}} x^m \, d\mu(a)(x) = f_m(a), \qquad m = 0, 1, 2, \ldots .$$

The proof is based on four elementary lemmas; we will skip the proofs of two of them.

LEMMA 3.2. *For $n = 0, 1, 2, \ldots$ and $c \neq 1$ we have*

$$\frac{u^2}{(u^2+1)^{n+1}(u^2+c)} = \frac{1}{(1-c)(u^2+1)^{n+1}} - \frac{c}{(1-c)^{n+1}(u^2+c)} + \sum_{k=0}^{n-1} \frac{c}{(1-c)^{n+1-k}(u^2+1)^{k+1}}.$$

LEMMA 3.3. *For $n = 0, 1, 2, \ldots$ we have*

$$\int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}} = \binom{2n}{n} \frac{\pi}{2 \cdot 4^n}.$$

LEMMA 3.4. *For $0 < a \neq 1$ and $n = 0, 1, 2, \ldots$ we have*

$$\int_0^{4a} x^n \frac{\sqrt{4ax - x^2}\, dx}{2\pi x(1 - x + ax)} = \binom{2n}{n} \frac{a^n}{2(1-a)} - \frac{|2a-1|}{2(1-a)^{n+1}} + \sum_{k=0}^{n-1} \binom{2k}{k} \frac{(2a-1)^2 a^k}{2(1-a)^{n+1-k}}.$$

*Proof.* Using the third substitution of Euler: $\sqrt{x(4a-x)} =: ux$, we get $x = 4a/(u^2+1)$, $dx = -8au\, dx/(u^2+1)^2$. Putting $c := (2a-1)^2$, applying the previous lemmas and the formula

$$\int_0^{\infty} \frac{dx}{x^2+c} = \frac{\pi}{2\sqrt{c}}, \qquad c > 0,$$

we obtain

$$\int_0^{4a} x^n \frac{\sqrt{4ax - x^2}\, dx}{2\pi x(1 - x + ax)} = \frac{(4a)^{n+1}}{\pi} \int_0^{\infty} \frac{u^2\, du}{(u^2+1)^{n+1}(u^2+(2a-1)^2)} \tag{3.6}$$

$$= (4a)^{n+1} \left[ \binom{2n}{n} \frac{1}{2 \cdot 4^n (1-c)} - \frac{\sqrt{c}}{2(1-c)^{n+1}} + \sum_{k=0}^{n-1} \binom{2k}{k} \frac{c}{2 \cdot 4^k (1-c)^{n+1-k}} \right]$$

$$= \binom{2n}{n} \frac{a^n}{2(1-a)} - \frac{|2a-1|}{2(1-a)^{n+1}} + \sum_{k=0}^{n-1} \binom{2k}{k} \frac{(2a-1)^2 a^k}{2(1-a)^{n+1-k}}. \qquad \blacksquare$$

LEMMA 3.5.
$$\binom{2n}{n}\frac{a^n}{2(1-a)}-\frac{2a-1}{2(1-a)^{n+1}}+\sum_{k=0}^{n-1}\binom{2k}{k}\frac{(2a-1)^2a^k}{2(1-a)^{n+1-k}}=f_n(a).$$

*Proof.* We will prove by induction that
$$\binom{2n}{n}a^n(1-a)^n-(2a-1)+\sum_{k=0}^{n-1}\binom{2k}{k}(2a-1)^2a^k(1-a)^k=2(1-a)^{n+1}f_n(a).$$
It is easy to check this for $n=0$. Now suppose it holds for some $n$. Then the left hand side for $n+1$ equals

$$\begin{aligned}
L_{n+1} &:= \binom{2n+2}{n+1}a^{n+1}(1-a)^{n+1}-(2a-1)+\sum_{k=0}^{n}\binom{2k}{k}(2a-1)^2a^k(1-a)^k\\
&= \binom{2n+2}{n+1}a^{n+1}(1-a)^{n+1}+\binom{2n}{n}(2a-1)^2a^n(1-a)^n\\
&\quad -\binom{2n}{n}a^n(1-a)^n+2(1-a)^{n+1}f_n(a)\\
&= \binom{2n+2}{n+1}a^{n+1}(1-a)^{n+1}-4\binom{2n}{n}a^{n+1}(1-a)^{n+1}+2(1-a)^{n+1}f_n(a)\\
&= \frac{-2}{n+1}\binom{2n}{n}a^{n+1}(1-a)^{n+1}+2(1-a)^{n+1}f_n(a)\\
&= -2(1-a)^{n+1}(f_n(1)a^{n+1}-f_n(a))=2(1-a)^{n+2}f_{n+1}(a)=P_{n+1},
\end{aligned}$$

which, in view of (3.3), completes the proof. ■

*Proof of Theorem 3.1.* Comparing Lemmas 3.4 and 3.5 we see that if $1/2\le a\ne 1$ then
$$\int_0^{4a}x^n\frac{\sqrt{4ax-x^2}\,dx}{2\pi x(1-x+ax)}=f_n(a),$$
while for $0<a<1/2$,
$$\int_0^{4a}x^n\frac{\sqrt{4ax-x^2}\,dx}{2\pi x(1-x+ax)}+\frac{1-2a}{(1-a)^{n+1}}=f_n(a).$$
For the remaining case $a=1$ we apply (3.6) and Lemma 3.3:

$$\begin{aligned}
\int_0^{4}x^n\frac{\sqrt{4x-x^2}}{2\pi x}\,dx &= \frac{4^{n+1}}{\pi}\int_0^{\infty}\frac{u^2\,du}{(u^2+1)^{n+2}}\\
&= \frac{4^{n+1}}{\pi}\int_0^{\infty}\frac{(u^2+1)-1}{(u^2+1)^{n+2}}\,du=2\binom{2n}{n}-\frac{1}{2}\binom{2n+2}{n+1}=\frac{1}{2n+1}\binom{2n+1}{n}. \ ■
\end{aligned}$$

We can also find the Jacobi parameters of $\mu(a)$:

THEOREM 3.6. *The monic orthogonal polynomials for $\mu(a)$ are given by the following recurrence: $P_0(x) = 1$ and for $m \geq 0$,*

$$xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1}P_{m-1}(x)$$

*(under the convention that $\gamma_{-1} = 0$ and $P_{-1} = 0$), where the Jacobi parameters are*

$$\beta_m = \begin{cases} 1 & \text{if } m = 0, \\ 2a & \text{if } m \geq 1, \end{cases} \qquad \gamma_m = \begin{cases} a & \text{if } m = 0, \\ a^2 & \text{if } m \geq 1. \end{cases}$$

*Proof.* Using the identity

$$\frac{1 - 2az - \sqrt{1-4az}}{2} = \cfrac{a^2z^2}{1 - 2az - \cfrac{1-2az-\sqrt{1-4az}}{2}}$$

we can express $M_{\mu(a)}(z)$ as a continued fraction:

$$M_{\mu(a)}(z) = \frac{2a}{2a - 1 + \sqrt{1-4az}} = \cfrac{1}{1 - z - \cfrac{1-2az-\sqrt{1-4az}}{2a}}$$

$$= \cfrac{1}{1 - z - \cfrac{az^2}{1 - 2az - \cfrac{a^2z^2}{1 - 2az - \cfrac{a^2z^2}{1 - 2az - \cfrac{a^2z^2}{\ddots}}}}},$$

which implies our statement (see [14, 7]). ■

Let us mention that these measures $\mu(a)$ belong to the *free Meixner class* which consists of those probability measures on $\mathbb{R}$ for which the Jacobi sequences $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are constant for $n \geq 1$ (see [1, 2, 6, 12]).

**4. Probability measures: the general case.** In this part we will make use of *free probability*, introduced by Voiculescu (for details we refer to [13, 11, 5]). In this theory by a *noncommutative probability space* we mean a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital $C^*$-algebra and $\phi$ is a *tracial state* on $\mathcal{A}$, i.e. a linear map $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(\mathbf{1}) = 1$ and for every $X, Y \in \mathcal{A}$ we have $\phi(X^*X) \geq 0$ and $\phi(XY) = \phi(YX)$. Every self-adjoint element $X = X^* \in \mathcal{A}$ possesses its *distribution* $\mu_X$, which is a compactly supported probability measure on $\mathbb{R}$ whose moments are $\phi(X^m)$. If in addition $X$ is positive (which means that $X = Z^2$ for some $Z = Z^* \in \mathcal{A}$) then this measure is supported in the positive half-line $[0, \infty)$. A family $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras is said to be *freely independent* if we have $\phi(X_1 \dots X_m) = 0$ whenever $X_1 \in \mathcal{A}_{i_1}, \dots, X_m \in \mathcal{A}_{i_m}$, $i_k \in I$, $i_1 \neq i_2 \neq \dots \neq i_m$ and $\phi(X_1) = \dots = \phi(X_m) = 0$. Elements $Y_1, \dots, Y_n \in \mathcal{A}$ are called *freely independent* if there are free unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that $Y_k \in \mathcal{A}_k$ for $1 \leq k \leq n$.

Now assume that $X = X^*, Y = Y^* \in \mathcal{A}$ are freely independent, with distributions $\mu_X, \mu_Y$. Then it turns out that the distribution of $X+Y$ depends only on $\mu_X$ and $\mu_Y$ and is denoted by $\mu_X \boxplus \mu_Y$. If in addition $X, Y$ are positive, with $X = Z^2$ for some $Z = Z^* \in \mathcal{A}$ then the distribution of $ZYZ$ depends only on $\mu_X$ and $\mu_Y$ and is denoted by $\mu_X \boxtimes \mu_Y$. These operations can be extended to the whole class of probability measures on $\mathbb{R}$, with support in the positive half-line in the case of $\boxtimes$, and are called *additive* and *multiplicative free convolution* respectively (see [5]). Only the latter will be used here.

Denote by $\mathcal{M}_+^c$ the class of probability measures with compact support contained in $[0, \infty)$, different from $\delta_0$. For $\mu \in \mathcal{M}_+^c$, with moments

$$s_m(\mu) := \int_{\mathbb{R}} t^m \, d\mu(t),$$

and with the *moment generating function*

$$M_\mu(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m = \int_{\mathbb{R}} \frac{d\mu(t)}{1 - tz},$$

we define the *S-transform* $S_\mu(z)$ by the equation

$$M_\mu\left(\frac{z}{1+z} S_\mu(z)\right) = 1 + z$$

on a neighborhood of 0. If $c > 0$ and $\mathbf{D}_c\mu$ denotes the *dilation* of $\mu$, i.e. $\mathbf{D}_c\mu(X) := \mu(c^{-1}X)$, then $M_{\mathbf{D}_c\mu}(z) = M_\mu(cz)$ and $S_{\mathbf{D}_c\mu}(z) = c^{-1} S_\mu(z)$.

For $\mu_1, \mu_2 \in \mathcal{M}_+^c$ the multiplicative free convolution $\mu_1 \boxtimes \mu_2$ is the unique $\mu \in \mathcal{M}_+^c$ which satisfies

$$S_\mu(z) = S_{\mu_1}(z) \cdot S_{\mu_2}(z) \tag{4.1}$$

(see [13, 5]). The operation $\boxtimes$ can be regarded as a free analog of the Mellin convolution (which expresses the distribution of the product of nonnegative independent random variables), but (just as for $\boxplus$) no direct formula is known to compute $\mu_1 \boxtimes \mu_2$ for general $\mu_1, \mu_2 \in \mathcal{M}_+^c$.

Now we can prove

THEOREM 4.1. *Assume that $a_0, \dots, a_{p-1} \geq 0$. Then the function $F$ given by* (2.1) *is the moment generating function of the probability measure*

$$\mu(a_0) \boxtimes \mu(a_1) \boxtimes \dots \boxtimes \mu(a_{p-1}). \tag{4.2}$$

*If moreover* $a_0 > 0$ *then the function* $G$ *given by* (2.2) *is the moment generating function of the probability measure*

$$\mathbf{D}_{a_0}\mu(1) \boxtimes \mu(a_1/a_0) \boxtimes \mu(a_2/a_0) \boxtimes \cdots \boxtimes \mu(a_{p-1}/a_0). \tag{4.3}$$

*If* $a_0 = 0$ *then* $G(z) = 1$ *and it is the moment generating function of* $\delta_0$.

*Proof.* From Corollary 2.2 we have

$$M_{\mu(a)}(z) = \frac{1 - 2a + \sqrt{1 - 4az}}{2(1 - a - z)}$$

and by (2.12) this function satisfies the equation

$$M_{\mu(a)}\left(\frac{z(1 + z - az)}{(1 + z)^2}\right) = 1 + z,$$

which yields

$$S_{\mu(a)}(z) = \frac{1 + z - az}{1 + z}. \tag{4.4}$$

Now, coming to the general case, in view of Theorem 2.1 and formula (4.4) we have

$$F\left(\frac{z}{1 + z} S_{\mu(a_0)}(z) S_{\mu(a_1)}(z) \ldots S_{\mu(a_{p-1})}(z)\right) = 1 + z$$

and

$$G\left(\frac{z}{(1 + z)a_0} S_{\mu(1)}(z) S_{\mu(a_1/a_0)}(z) \ldots S_{\mu(a_{p-1}/a_0)}(z)\right) = 1 + z,$$

which leads to the desired statement. ∎

Remark 4.2. One can check (see Theorem 3.7.3 in [13] or Theorem 6.13 in [5]) that for every $a \geq 0$ the measure $\mu(a)$ is infinitely divisible with respect to the multiplicative free convolution $\boxtimes$, which implies that so are the measures (4.2) and (4.3).

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## REFERENCES

[1] M. Anshelevich, *Free martingale polynomials,* J. Funct. Anal. 201 (2003), 228–261.
[2] M. Anshelevich and W. Młotkowski, *Semigroups of distributions with linear Jacobi parameters*, J. Theoret. Probab. 25 (2012), 1173–1206.
[3] J.-C. Aval, *Multivariate Fuss–Catalan numbers*, Discrete Math. 308 (2008), 4660–4669.
[4] T. Banica, S. T. Belinschi, M. Capitaine and B. Collins, *Free Bessel laws*, Canad. J. Math 63 (2011), 3–37.
[5] H. Bercovici and D. V. Voiculescu, *Free convolutions of measures with unbounded support*, Indiana Univ. Math. J. 42 (1993), 733–773.

[6] M. Bożejko and W. Bryc, *On a class of free Lévy laws related to a regression problem*, J. Funct. Anal. 236 (2006), 59–77.
[7] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Math. 32 (1980), 125–161.
[8] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.
[9] W. Młotkowski, *Fuss–Catalan numbers in noncommutative probability*, Documenta Math. 15 (2010), 939–955.
[10] W. Młotkowski, K. A. Penson and K. Życzkowski, *Densities of the Raney distributions*, arXiv:1211.7259.
[11] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge Univ. Press, 2006.
[12] N. Saitoh and H. Yoshida, *The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory*, Probab. Math. Statist. 21 (2001), 159–170.
[13] D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables*, CRM Monogr. Ser. 1, Amer. Math. Soc., 1992.
[14] H. S. Wall, *Analytic Theory of Continued Fractions*, Chelsea, New York, 1967.

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