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# CIRCULAR CONE AND ITS GAUSS MAP

BY

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**Abstract.** The family of cones is one of typical models of non-cylindrical ruled surfaces. Among them, the circular cones are unique in the sense that their Gauss map satisfies a partial differential equation similar, though not identical, to one characterizing the so-called 1-type submanifolds. Specifically, for the Gauss map $G$ of a circular cone, one has $\Delta G = f(G + C)$, where $\Delta$ is the Laplacian operator, $f$ is a non-zero function and $C$ is a constant vector. We prove that circular cones are characterized by being the only non-cylindrical ruled surfaces with $\Delta G = f(G + C)$ for a nonzero constant vector $C$.

**1. Introduction.** Since the late 1970s, much work has been done on finite type immersions in Euclidean and pseudo-Euclidean spaces. An immersion $x$ of a manifold $M$ into a Euclidean space $\mathbb{E}^m$ is said to be of *finite type* if it can be expressed as

$$x = x_0 + x_1 + \cdots + x_k$$

for some positive integer $k$, where $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. Here $\Delta$ is the Laplacian operator defined on $M$. If each $x_i$ is non-zero somewhere, and $\lambda_i$ are all different, $x$ is said to be of *k-type*. Minimal submanifolds of Euclidean spaces or minimal submanifolds of spheres are of the simplest finite type, i.e., 1-type. The references [c1, c2] list many papers dealing with finite type immersions from various points of view.

The notion of finite type immersion naturally extends to smooth functions on submanifolds of Euclidean spaces or pseudo-Euclidean spaces. The most natural among them is the Gauss map of a hypersurface.

B.-Y. Chen and P. Piccini ([cp]) initiated the study of Gauss maps of finite type, classifying compact surfaces with 1-type Gauss map, that is, $\Delta G = \lambda(G + C)$, where $C$ is a constant vector and $\lambda \in \mathbb{R}$. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss map ([bb, cck, ck, cky, ky1, ky2, ky3]).

However, some submanifolds such as helicoids in $\mathbb{E}^3$ satisfy $\Delta G = f(G + C)$ for some smooth function $f$ and a constant vector $C$, i.e., its

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parametrization $x = x(u, v)$ defined by

$$x(u, v) = (u \cos v, u \sin v, hv), \qquad h \neq 0,$$

has the Gauss map

$$G = \frac{1}{\sqrt{h^2 + u^2}}(h \sin v, -h \cos v, u).$$

Its Laplacian $\Delta G$ is given by

$$\Delta G = \frac{2h^2}{(h^2 + u^2)^2} G.$$

On the other hand, the circular cone $C_a$ with parametrization

$$x(u, v) = (v \cos u, v \sin u, av), \qquad a \geq 0,$$

has the Gauss map

$$G = \frac{1}{\sqrt{1 + a^2}}(a \cos u, a \sin u, -1).$$

Consequently, its Laplacian $\Delta G$ satisfies

$$\Delta G = \frac{1}{v^2}\Big(G + \Big(0, 0, \frac{1}{\sqrt{1 + a^2}}\Big)\Big).$$

Based on these examples, we define

Definition 1.1. An oriented $(m-1)$-dimensional submanifold of the Euclidean space $\mathbb{E}^m$ is said to have *pointwise* 1-*type Gauss map* if it satisfies the condition

$$\Delta G = f(G + C),$$

where $f$ is a non-zero smooth function on $M$ and $C$ is some constant vector. In particular, if $C$ is zero, the Gauss map $G$ is said to be of *the first kind*. Otherwise, it is said to be of *the second kind* ([cck, ck, cky, ky2]).

In the present paper, we completely classify non-cylindrical ruled surfaces in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind. If $f$ is not constant, the surface is said to be *proper*. So, a non-proper pointwise 1-type Gauss map is just of the ordinary 1-type.

Throughout this paper, we assume that all the geometric objects are smooth and all surfaces are connected unless otherwise mentioned.

**2. Preliminaries.** Let $M$ be an oriented surface in $\mathbb{E}^3$. The map $G : M \to S^2 \subset \mathbb{E}^3$ which sends each point of $M$ to the unit vector normal to $M$ at the point is called the *Gauss map* of the surface $M$. Here $S^2$ is the unit sphere in $\mathbb{E}^3$ centered at the origin. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on $M$, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. The Laplacian $\Delta$ on $M$ is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{\mathcal{G}}\, \tilde{g}^{ij} \frac{\partial}{\partial x^j}\right).$$

In view of the results of [cck, ck, ky2] concerning the mean curvature, we have

LEMMA 2.1. *Let $M$ be a surface in $\mathbb{E}^3$. Then the mean curvature $H$ is constant if and only if the Gauss map $G$ is of pointwise 1-type of the first kind.*

In particular, if $M$ is a ruled surface, the first and the third named authors ([ck]) proved the following theorem:

THEOREM 2.2 ([ck]). *A ruled surface in $\mathbb{E}^3$ with pointwise 1-type Gauss map of the first kind is an open portion of either a circular cylinder or a helicoid.*

Thus, we immediately have

COROLLARY 2.3 ([cky]). *The helicoid is the only ruled surface in $\mathbb{E}^3$ with proper pointwise 1-type Gauss map of the first kind.*

**3. Main results.** Let $M$ be a non-cylindrical ruled surface in $\mathbb{E}^3$ parametrized by $x(s,t) = \alpha(s) + t\beta(s)$, where $\alpha$ is a base curve and $\beta$ a director vector field satisfying $\langle \alpha', \beta\rangle = 0$, $\langle \beta, \beta\rangle = 1$ and $\langle \beta', \beta'\rangle = 1$.

REMARK. A non-cylindrical ruled surface $M$ defined above with $\alpha' = 0$ is a (generalized) cone.

Suppose that $M$ has a pointwise 1-type Gauss map of the second kind, that is, the Gauss map $G$ of $M$ in $\mathbb{E}^3$ satisfies the condition

$$\Delta G = f(G + C)$$

for a non-zero smooth function $f : M \to \mathbb{R}$ and some non-zero constant vector $C$.

We now define the smooth functions $q$, $u$, $v$, $Q$ and $R$ as follows:

$$q = \langle x_s, x_s\rangle, \quad u = \langle \alpha', \beta'\rangle, \quad v = \langle \alpha', \alpha'\rangle,$$
$$Q = \langle \alpha', \beta \times \beta'\rangle, \quad R = \langle \beta'', \beta \times \beta'\rangle.$$

Then $\{\beta, \beta', \beta\times\beta'\}$ forms an orthonormal frame along $\alpha$. In this orthonormal frame, we have

$$\alpha' = u\beta' + Q\beta \times \beta' \quad \text{and} \quad \beta'' = -\beta + R\beta \times \beta'. \tag{3.1}$$

Next, the Gauss map $G$ and the mean curvature $H$ are expressed by

$$G = \frac{x_s \times x_t}{\|x_s \times x_t\|} = q^{-1/2}\{Q\beta' - (u+t)\beta \times \beta'\},$$
$$H = \frac{1}{2}q^{-3/2}(-Rt^2 - (2uR + Q')t - u^2R - uQ' + u'Q - Q^2R).$$

Since $G$ is of pointwise 1-type of the second kind, we have

$$\begin{aligned}\frac{1}{2}q^{-7/2}\{2qB_1\beta + (u+t)A_1\beta' + QA_1\beta \times \beta'\}&\\ + q^{-7/2}D_1\{Q\beta' - (u+t)\beta \times \beta'\}&\\ = q^{-1/2}f\{Q\beta' - (u+t)\beta \times \beta'\} + fC&,\end{aligned} \tag{3.2}$$

where we have put

$$\begin{aligned} A_1 = & -2R't^4 + 2(-Q'' + u'R - 4uR')t^3 \\ & + 2\left(3u'Q' + u''Q - Q^2R' + \frac{3}{2}v'R - 2QQ'R - 3uQ'' - 5u^2R' - vR'\right)t^2 \\ & + 2\left(\frac{3}{2}v'Q' - 3u'^2Q + 3u'Q^2R + 3uu'Q' + 2uu''Q - 2uQ^2R' + 3uv'R \right.\\ & \left. - 4uQQ'R - 2u^2Q'' - u^2u'R - 2u^3R' - vQ'' - 2u'vR - 2uvR'\right)t \\ & - 3u'v'Q + 3v'Q^2R + 3uv'Q' + 3u^2v'R + 2u''vQ - 2vQ^2R' \\ & - 4vQQ'R - 2uvQ'' - 4uu'vR - 2u^2vR', \\ B_1 = & \; Rt^3 + (2Q' + 3uR)t^2 + (-3u'Q + 3Q^2R + 4uQ' + 5u^2R - 2vR)t \\ & - 3uu'Q + 3uQ^2R + 3u^2Q' + 3u^3R - vQ' - 2uvR \end{aligned}$$

and

$$\begin{aligned} D_1 = & \; R^2t^4 + (4uR^2 + 2Q'R)t^3 \\ & + (-2u'QR + 2Q^2R^2 + 6uQ'R + 6u^2R^2 + Q'^2 + 2Q^2)t^2 \\ & + (-4uu'QR + 4uQ^2R^2 + 6u^2Q'R + 4u^3R^2 \\ & - 2u'QQ' + 2Q^2Q'R + 2uQ'^2 + 4uQ^2)t \\ & + (u'Q - Q^2R - uQ' - u^2R)^2 + 2vQ^2. \end{aligned}$$

If we take the inner product of equation (3.2) with $\beta, \beta'$ and $\beta \times \beta'$ successively, we get

$$f\langle C, \beta\rangle = q^{-5/2}B_1, \tag{3.3}$$
$$f(\langle C, \beta'\rangle + q^{-1/2}Q) = \frac{1}{2}q^{-7/2}(u+t)A_1 + q^{-7/2}QD_1$$

and

$$f(\langle C, \beta \times \beta' \rangle - q^{-1/2}(u+t)) = \frac{1}{2} q^{-7/2} Q A_1 - q^{-7/2}(u+t) D_1,$$

respectively. Combining the above equations, we have

$$4qQ^2B_1^2 - \{(u+t)\lambda A_1 + 2\lambda Q D_1 - 2qB_1\mu\}^2 = 0, \tag{3.4}$$

$$4q(u+t)^2B_1^2 - \{\lambda Q A_1 - 2\lambda(u+t)D_1 - 2qB_1\nu\}^2 = 0, \tag{3.5}$$

$$qA_1^2 - \{(u+t)A_1\nu + 2D_1Q\nu - QA_1\mu + 2(u+t)D_1\mu\}^2 = 0, \tag{3.6}$$

where we have put $\lambda = \langle C, \beta \rangle$, $\mu = \langle C, \beta' \rangle$ and $\nu = \langle C, \beta \times \beta' \rangle$.

Differentiating the constant vector $C = \lambda\beta + \mu\beta' + \nu\beta \times \beta'$ with respect to the parameter $s$ yields

$$\lambda' = \mu, \qquad \nu' + \mu R = 0 \quad \text{and} \quad \lambda + \mu' - \nu R = 0. \tag{3.7}$$

On the other hand, (3.4) is a polynomial in $t$ with functions of $s$ as coefficients. Thus, the leading coefficient must be zero, i.e., $\mu R + \lambda R' = 0$. Making use of (3.7), we see that $\lambda R$ is a constant. Since (3.5) is an identity, the coefficient of the term $t^{10}$ in (3.5) must be zero, which yields

$$R^2 = R^2(\lambda R + \nu)^2. \tag{3.8}$$

Now, we have two possible cases.

Case 1: *$R$ is not identically zero on $M$.* Consider the open set $\mathbf{U} = \{p \in M \mid R(p) \neq 0\}$. Suppose $\mathbf{U} \neq \emptyset$. Then (3.8) implies that $\nu$ is constant on a component $\mathbf{U}_0$ on $\mathbf{U}$ since $\lambda R$ is a constant. From (3.7), we also see that $\mu = 0$ on $\mathbf{U}_0$. Thus (3.7) shows that $\lambda$ is a constant on $\mathbf{U}_0$ and so is $R$ on $\mathbf{U}_0$. By continuity, $R$ is a non-zero constant on $M$. Using (3.8), we see that $\nu$ is a constant on $M$ since $\lambda R$ is a constant. Hence, $\lambda$ is also a constant on $M$. As $\nu$ is constant, (3.7) implies $\mu = 0$ and so $\lambda = \nu R$.

Moreover, combining equations (3.4) and (3.6) with the above results, we obtain

$$R^2A_1^2 - 4Q^2B_1^2 = 0. \tag{3.9}$$

Also, the leading coefficient of (3.9) is zero, i.e.,

$$Q^2 - (Q'' - u'R)^2 = 0. \tag{3.10}$$

Suppose $Q$ is not identically zero. Without loss of generality, we may assume $Q = Q'' - u'R$. Comparing the leading coefficients of equations (3.5) and (3.6) after substituting $Q = Q'' - u'R$ in them, we have a contradiction. Therefore, $Q \equiv 0$. Consequently, $u$ is constant.

On the other hand, the second equation of (3.1) implies

$$\beta'''(s) + a^2\beta'(s) = 0, \tag{3.11}$$

where $a = \sqrt{R^2+1}$. Without loss of generality, we may assume that $\beta$ satisfies the initial conditions $\beta(0) = (1,0,0)$, $\beta'(0) = (0,1,0)$, $\beta''(0) =$

$(-1, 0, R)$. Then the unique solution of (3.11) is

$$\beta(s) = \left(1 - \frac{1}{a^2} + \frac{1}{a^2}\cos as, \frac{1}{a}\sin as, \frac{R}{a^2} - \frac{R}{a^2}\cos as\right). \tag{3.12}$$

We easily see that its torsion vanishes and the curvature is the non-zero constant $a$. Since $\beta$ itself is a spherical curve, $M$ is a part of a circular cone. In fact, its parametrization is given by $x(s, \bar{t}) = \bar{t}\beta(s) + \mathbf{C}_0$, where $\bar{t} = u + t$ and $\mathbf{C}_0$ is a constant vector.

Case 2: *$R$ is identically zero on $M$.* From (3.7), $\nu$ is a constant and we also get $\lambda + \lambda'' = 0$. Suppose $\lambda$ is non-trivial. Then we have

$$\lambda(s) = A\sin(s + s_0) \tag{3.13}$$

for some constants $A$ and $s_0$.

If we compute the leading coefficient with respect to $t$ in (3.5) with $R = 0$, we have

$$Q'^2 - \nu^2 Q'^2 = 0.$$

Since $\nu$ is a constant, $Q$ is a constant if $\nu^2 \neq 1$. If $Q = 0$, the mean curvature $H$ vanishes on $M$, that is, $M$ is minimal. This contradicts the hypothesis that the Gauss map is of pointwise 1-type of the second kind. Hence $Q$ is a non-zero constant. Using the leading coefficient of (3.4), we obtain, from (3.7) $\lambda u'' + 3\lambda' u' = 0$. Solving the above equation, we get

$$u' = k_1\lambda^{-3} \tag{3.14}$$

for some constant $k_1$. Also, from the leading coefficient of (3.5), we have

$$9u'^2 = (2\lambda Q - 3\nu u')^2. \tag{3.15}$$

Putting (3.13) and (3.14) into (3.15), we obtain $k_1 = 0$ and $A = 0$, a contradiction. Therefore $\lambda = 0$. From (3.3), we easily get $B_1 = 0$ and thus $u$ is a constant. Hence the mean curvature $H$ vanishes on $M$, a contradiction. Consequently, we have $\nu^2 = 1$.

Next, the leading coefficient of the left hand side of (3.4) gives, by (3.7),

$$\lambda Q'' + 2\lambda' Q' = 0.$$

If $\lambda$ is non-zero, the solution of the above equation is given by

$$Q' = k_2\lambda^{-2} \tag{3.16}$$

for some constant $k_2$. Moreover, we also get

$$\mu Q''(QQ'' + Q'^2 + 2Q^2) = 0,$$

which is derived from the leading coefficient in (3.6) using $\nu^2 = 1$. From (3.7), we easily get $\mu \neq 0$. Hence we have

$$Q''(QQ'' + Q'^2 + 2Q^2) = 0. \tag{3.17}$$

Putting (3.13) and (3.16) into (3.17), we obtain $k_2 = 0$ and so $Q$ is a constant. Since the mean curvature $H$ is non-vanishing, $Q$ is a non-zero constant. Since $\nu^2 = 1$, from the leading coefficient of (3.5), we have

$$\lambda Q - 3\nu u' = 0. \tag{3.18}$$

Putting (3.13) and (3.14) into (3.18), we obtain $\lambda = 0$. In view of those facts, equation (3.3) shows that $B_1 = 0$. This implies that $u' = 0$. Hence, the mean curvature $H$ vanishes, a contradiction. Thus, Case 2 can never occur.

Conversely, as shown in the Introduction, a circular cone has pointwise 1-type Gauss map of the second kind.

Consequently, we have

Main Theorem 3.1 (Characterization). *Let $M$ be a non-cylindrical ruled surface in $\mathbb{E}^3$. Then $M$ has pointwise 1-type Gauss map of the second kind if and only if $M$ is a part of a circular cone.*

Using a result in [cck], we obtain

Corollary 3.2. *Let $M$ be a surface of $\mathbb{E}^3$. Then the following are equivalent:*

(i) *$M$ is a non-cylindrical ruled surface with pointwise 1-type Gauss map of the second kind.*
(ii) *$M$ is a surface of revolution of the polynomial kind with pointwise 1-type Gauss map of the second kind.*
(iii) *$M$ is a part of a circular cone.*

Combining the results in [cky] with the above characterization, we have

Main Theorem 3.3. *Let $M$ be a ruled surface in $\mathbb{E}^3$. If $M$ has pointwise 1-type Gauss map, then $M$ is a part of a plane, a circular cylinder, a helicoid, a cylinder of an infinite type or a circular cone.*

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