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ON ENVELOPING SEMIGROUPS OF ALMOST ONE-TO-ONE EXTENSIONS OF MINIMAL GROUP ROTATIONS

BY

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Abstract. We consider a class of symbolic systems over a finite alphabet which are minimal almost one-to-one extensions of rotations of compact metric monothetic groups and provide computations of their enveloping semigroups that highlight their algebraic structure. We describe the set of idempotents of these semigroups and introduce a classification that can help distinguish between certain such systems having zero topological entropy.

1. Introduction. The notion of an almost one-to-one extension over a dynamical system is important to topological dynamics as it constitutes, topologically, a relatively simple extension of a given system (because it is one-to-one on a subset that is a dense G_δ set), yet the degree of freedom afforded by such extensions may be so large that the arising systems may exhibit quite a different dynamical behavior from the base system. A good example of this phenomenon can be found in the survey [D] devoted to Toeplitz flows, which are almost one-to-one extensions of minimal rotations on odometers.

Similarly, the notion of the enveloping semigroup, introduced by Ellis (see [E]), proved a handy tool in topological dynamics as it opened up new vistas of exploration in the field. Dynamical properties of a given system can now be viewed through the lenses of algebraic structure carried by its enveloping semigroup. Nevertheless, due to their uwieldiness rarely do we encounter direct calculations of these objects. Some examples of enveloping semigroups can be found, however, in the survey [G], which also contains an overview of important developments in the general theory of enveloping semigroups.

The aim of this paper is to provide characterization of enveloping semigroups for a large subclass of minimal almost one-to-one symbolic extensions (over a finite alphabet K) which are rotations of infinite compact metric monothetic groups, as the more tangible form of these objects will enhance

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our understanding of the mechanics behind the enveloping semigroup machinery.

The structure of this paper is as follows. Section 2 presents the notation and terminology used throughout the paper. In Section 3 we recall a standard construction of almost one-to-one extensions of dynamical systems. Section 4 serves as an introduction to Sections 5 and 6, in which we tackle the problem of describing the algebraic structure of enveloping semigroups of minimal almost one-to-one symbolic extensions of rotations of compact, metric, monothetic groups. We also introduce a classification of such systems based on ‘complexity’ of their enveloping semigroups. In Section 7 we analyze the relationship between this classification and the notion of topological entropy.

2. Notation and terminology. By a (*topological*) *dynamical system* we mean a pair (X, T), where X is a compact metric space, and T is a continuous map from X to itself inducing an $\mathbb{N}$-action on X, where $\mathbb{N} = \{0, 1, 2, \dots\}$.

A system (X, T) is *minimal* if X does not contain any proper, non-empty, closed and invariant subset A (i.e., a subset satisfying $T(A) \subseteq A$).

Let $x \in X$. The *orbit* of x is the set of iterates of x under T, i.e., the set $\{T^n(x) : n \in \mathbb{N}\}$.

Given two systems (X, T) and (Y, S) we say that (X, T) is an *extension* of (Y, S), or equivalently that (Y, S) is a *factor* of (X, T), if there exists a continuous surjection $\pi : X \to Y$ such that

$$\pi T = S\pi.$$

If an extension is given, then by *fibers* we will mean preimages of single points under π.

An extension is called *almost one-to-one* if the set of points having one-point preimages is residual (contains a dense G_δ) in Y. In the case of (X, T) being minimal it is enough to verify that a one-point fiber exists.

The *enveloping semigroup* of a dynamical system (X, T), denoted by $E(X, T)$, is defined to be the closure of the set $\{T^n : n \in \mathbb{N}\}$ in the space X^X equipped with the product topology. It is known that $E(X, T)$ is a compact *left topological semigroup*, i.e., a semigroup in which the multiplication $(x, y) \mapsto xy$ is continuous in the left variable (we follow the terminology of [R]).

We deal mainly in the framework of *symbolic flows*, also called *subshifts*, i.e., shift invariant closed subsets of $\Sigma^{\mathbb{N}}$ (where the set Σ, called *alphabet*, is finite) along with the action induced by the left shift σ.

3. Standard construction. In this section we recall a standard method of constructing minimal almost one-to-one extensions (cf. [DD]).

Let G be an infinite compact metric monothetic group with the neutral element $\mathbf{0}$ and a selected topological generator $\mathbf{1}$. If we write n in place of $n\mathbf{1}$ (where $n\mathbf{1} = \mathbf{0} + \mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}$ with $\mathbf{1}$ added n times) we obtain an injective embedding of $\mathbb{N}$ into G.

Let K be a compact space, and let $f : \mathbb{N} \to K$ be a function continuous with respect to the topology in $\mathbb{N}$ inherited from G. Let F denote the closure of the graph of f in $G \times K$. By continuity, the sections

$$F_g = \{k \in K : \langle g, k\rangle \in F\}$$

are singletons for $g = n$. Moreover, by a standard argument, they are singletons for g's in a set of type G_δ, and thus, since $\mathbb{N}$ lies densely in G, the singletons appear over a residual (i.e., dense G_δ) set C_f. If we assign to each $g \in C_f$ the unique $k \in F_g$, then we obtain an extension of the function f to C_f, which is continuous in the relative topology on C_f. From now on f denotes this extension. The complement D_f of C_f is exactly the set of points to which f cannot be extended continuously, hence we will call it the *set of discontinuities of* f. The set D_f is of first category, and it is easily seen that if K is a finite discrete space, then D_f is closed.

REMARK 3.1. The above observation is one of the main reasons why we are particularly interested in the case of K being a finite discrete space. Indeed, without assuming it the set D_f need not be closed, which implies that the set D_τ (see Definition 4.4) need not be a subset of D_f (the property that $D_\tau \subset D_f$ is further exploited in the paper). Moreover, the proof of the fundamental Proposition 5.2 (and also Theorem 5.3) relies on the fact that K is finite.

Therefore, even though the construction of this section holds for any compact metric space K, from now on we assume that K is a finite discrete space.

DEFINITION 3.2. The function f is said to be *invariant under the rotation by* $h \in G \setminus \{\mathbf{0}\}$ if the condition

$$F_{g+h} = F_g$$

holds for every $g \in G$. If there is no such h, we say that f is *invariant under no non-trivial rotations.*

Finally, define the system (X_f, σ) to be the shift orbit closure of the element $(f(n))_{n\in\mathbb{N}}$ in the space $K^{\mathbb{N}}$, and let R be the rotation on G given by the formula $R(g) = g + \mathbf{1}$. Note that the system (G, R) is clearly minimal.

The following theorem, whose proof can be found in [DD], characterizes minimal almost one-to-one extensions of (G, R).

THEOREM 3.3. *Let* (G, R) *and* (X_f, σ) *be as defined above, and let* (X, T) *be a dynamical system. Then* (X, T) *is a minimal almost one-to-one exten-*

sion of (G, R) if and only if it is topologically isomorphic to (X_f, σ), where f is invariant under no non-trivial rotations.

4. Setup. Using the notation from Section 3, we define X_F as the collection of all K-valued sequences $x = (x_n)$ for which there exists g satisfying, for each $n \in \mathbb{N}$, the condition

$$x_n \in F_{g+n}.$$

Since F is closed, one can easily verify that X_F is closed and shift-invariant. By a standard argument, uniqueness of the above g for every x is equivalent to the condition that f is invariant under no non-trivial rotations, and in that case the system (X_F, σ) (with the shift map σ) is an extension of the rotation on G, which is also almost one-to-one, but not necessarily minimal. Clearly, X_F contains the almost one-to-one minimal extension X_f of G (cf. Section 3).

From now on we assume that f is invariant under no non-trivial rotations (so that (X_F, σ) is a subshift and an extension of the rotation on G). We will denote by π the factor map from X_F onto G, and, given $x \in X_F$, we will often write g_x instead of $\pi(x)$. We are going to describe the enveloping semigroup of the system (X_F, σ)—the enveloping semigroup of (X_f, σ) will then be obtained by a straightforward restriction, i.e., we are going to apply the restriction map $\phi : E(X_F, \sigma) \to E(X_f, \sigma)$, defined for $\tau \in E(X_F, \sigma)$ by

$$\phi(\tau) = \tau|_{X_f},$$

which is a left continuous semigroup homomorphism (note that ϕ need not be an isomorphism).

Let $E(X_F, \sigma)$ be the enveloping semigroup of the dynamical system (X_F, σ). Given $\tau \in E(X_F, \sigma)$, there exists a net $(n_\alpha)_{\alpha \in \mathcal{A}}$, where $\mathcal{A}$ is some directed family of indices, such that $\sigma^{n_\alpha} \to \tau$ in $E(X_F, \sigma)$, i.e., for all $x \in X_F$, we have

$$\tau(x) = \lim_\alpha \sigma^{n_\alpha}(x).$$

Definition 4.1. Let $\mathcal{N}_\tau$ denote the collection of all nets (n_α) for which the net (σ^{n_α}) converges to τ in $E(X_F, \sigma)$.

Let $\tau \in E(X_F, \sigma)$, and choose $(n_\alpha) \in \mathcal{N}_\tau$. Since

$$\pi\tau(x) = \lim_\alpha \pi\sigma^{n_\alpha}(x) = \lim_\alpha R^{n_\alpha}(\pi(x)) = g_x + \lim_\alpha n_\alpha,$$

we can see that, as (n_α) ranges over $\mathcal{N}_\tau$, the nets (n_α) are convergent in G, and their common limit depends only on τ.

Definition 4.2. Let

$$g_\tau := \lim_\alpha n_\alpha,$$

where (n_α) is any given element of $\mathcal{N}_\tau$.

With every $\tau \in E(X_F, \sigma)$ we may now associate the element $g_\tau \in G$. Furthermore, it is easy to see that the mapping $\tau \mapsto g_\tau$ thereby induced is onto.

Recall that if (A_α) is a net of sets, then

$$\liminf_\alpha A_\alpha = \bigcup_\beta \bigcap_{\alpha>\beta} A_\alpha,$$

i.e., $\liminf_\alpha A_\alpha$ is the collection of points belonging to 'almost all' sets A_α (for all indices α larger than some β).

Definition 4.3. Given $\tau \in E(X_F, \sigma)$ and $(n_\alpha) \in \mathcal{N}_\tau$, let

$$D_{(n_\alpha)} = g_\tau + \liminf_\alpha (D_f - n_\alpha).$$

Observe that the set $D_{(n_\alpha)}$ consists of those points $g \in G$ for which the points $g - g_\tau + n_\alpha$ belong to D_f for 'almost all' indices α.

Definition 4.4. Let $\tau \in E(X_F, \sigma)$. Define

$$D_\tau = \bigcap_{(n_\alpha)\in\mathcal{N}_\tau} D_{(n_\alpha)}, \qquad C_\tau = D_f \setminus D_\tau.$$

Since D_f is closed and $(g_\tau - n_\alpha) \to \mathbf{0}$ in G, it is immediate to see that $D_\tau \subseteq D_f$. Hence, C_τ can be thought of as the *complement* of D_τ within D_f.

We end this section with yet another bit of preparatory notation. For $A \subseteq G$ we denote by S_A the collection of all functions $s : A \to K$ whose graph is contained in F. If A, B are disjoint and $s \in S_A$, $t \in S_B$, then $s \cup t$ denotes the function in $S_{A\cup B}$ obtained by uniting the graphs of s and t. The largest collection of the above kind is S_G, which consists of all possible 'prolongations' of f to all of G maintaining the graph within F. Notice that $s \mapsto f \cup s|_{D_f}$ establishes a 1-1 correspondence between S_{D_f} and S_G.

5. Enveloping semigroup—a convenient representation. The strategy of describing the enveloping semigroup $E(X_F, \sigma)$ is as follows: we will show that each $\tau \in E(X_F, \sigma)$ determines and is determined by two essential objects, g_τ in G and a mapping $\bar{h}_\tau : S_G \to S_G$.

Further, we will show that all the images $\bar{h}_\tau(s)$ (as s ranges over S_G) differ only on the set D_τ. Since it is obvious that they may differ at most on D_f, what we claim here is basically that they do not differ on C_τ. Moreover, we will show that C_τ is the maximal set (contained in D_f) with this property. In other words, we will in fact associate with τ a quadruple $\langle g_\tau, D_\tau, s_\tau, h_\tau\rangle$, where $s_\tau \in S_{C_\tau}$, $h_\tau : S_{D_f} \to S_{D_\tau}$ so that

$$\bar{h}_\tau(s) = f \cup s_\tau \cup h_\tau(s|_{D_f}).$$

We proceed to prove what we claim above.

Definition 5.1. Let $g \in G$, $\tau \in E(X_F, \sigma)$, and $s \in S_G$. Let x be the element of X_F such that $x(n) = s(g - g_\tau + n)$ for any $n \in \mathbb{N}$. Define a map $\bar{h}_\tau$ (with domain S_G) via the formula

$$\bar{h}_\tau(s)(g) = \tau(x)(0).$$

The following proposition establishes fundamental properties of the mapping $\bar{h}_\tau$.

Proposition 5.2. *If $s \in S_G$, then also $\bar{h}_\tau(s) \in S_G$. Moreover, as s ranges over S_G, the functions $\bar{h}_\tau(s)$ differ only on the set D_τ, which is the smallest set with this property.*

Proof. Let $\tau \in E(X_F, \sigma)$, and let $(n_\alpha) \in \mathcal{N}_\tau$ (so that $n_\alpha \to g_\tau$ in G). For given $g \in G$ and $s \in S_G$ we have

$$\bar{h}_\tau(s)(g) = \tau(x)(0) = \lim_\alpha x(n_\alpha) = \lim_\alpha s(g - g_\tau + n_\alpha). \tag{5.1}$$

Since the pairs

$$\langle g - g_\tau + n_\alpha, s(g - g_\tau + n_\alpha)\rangle$$

are in F, it follows that the limiting pair

$$\langle g, \tau(x)(0)\rangle$$

is in F as well. This shows that the graph of $\bar{h}_\tau(s)$ (as a function of g) is contained in F, i.e., $\bar{h}_\tau(s) \in S_G$.

Suppose that $g \notin D_\tau$ (i.e., $g \in C_\tau \cup C_f$). By Definition 4.4 there exists a net $(n_\alpha) \in \mathcal{N}_\tau$ such that $g \notin D_{(n_\alpha)}$. This means that there exists a subnet $(n_{\alpha'})$ with the property that

$$g - g_\tau + n_{\alpha'} \in C_f$$

for all α'. Since s is just f on C_f, we get

$$\bar{h}_\tau(s)(g) = \lim_{\alpha'} s(g - g_\tau + n_{\alpha'}) = \lim_{\alpha'} f(g - g_\tau + n_{\alpha'}),$$

which shows that in this case $\bar{h}_\tau(s)(g)$ does not depend on s. Furthermore, if $g \in C_f$, then $\bar{h}_\tau(s)(g) = f(g)$, by the continuity of f on C_f.

In the last part of the proof we are going to show that if $g \in D_\tau$, then $\bar{h}_\tau(s)(g)$ does depend on s.

Fix $s \in S_G$, and choose $s' \in S_G$ in such a way that

$$s'(g) \neq s(g),$$

for every $g \in D_f$. This can be done since, for any such g, the set F_g of all admissible values of functions belonging to S_G has at least two elements (cf. Section 3).

Assume that $g \in D_\tau$, so that, for a given net $(n_\alpha) \in \mathcal{N}_\tau$, we also have $g \in D_{(n_\alpha)}$. Then, by the definition of $D_{(n_\alpha)}$, the points $g - g_\tau + n_\alpha$ are in D_f

for almost all α. Clearly, for such α,

$$s'(g - g_\tau + n_\alpha) \neq s(g - g_\tau + n_\alpha).$$

Since both s and s' assume values in a finite discrete space K, it follows from (5.1) that

$$\bar{h}_\tau(s)(g) \neq \bar{h}_\tau(s')(g). \blacksquare$$

The last part of the above proof allows us to gain a better understanding of the sets D_τ and $D_{(n_\alpha)}$. Indeed, what we have actually shown is that if g is a given point in $D_{(n_\alpha)}$, where (n_α) is any given element of $\mathcal{N}_\tau$, then, as s ranges over S_G, the functions $\bar{h}_\tau(s)$ differ at g. But then $g \notin C_\tau$, hence $g \in D_\tau$, and it follows that $D_\tau \supseteq D_{(n_\alpha)}$. Combining the latter statement with Definition 4.4 we obtain

THEOREM 5.3. *Let τ be in $E(X_F, \sigma)$. If $(n_\alpha) \in \mathcal{N}_\tau$, then*

$$D_\tau = D_{(n_\alpha)}.$$

Also, as seen below, the proof of Proposition 5.2 sheds some light on how the functions $\bar{h}_\tau(s)$ behave on G.

COROLLARY 5.4. *Given $\tau \in E(X_F, \sigma)$, we may associate with it a quadruple $\langle g_\tau, D_\tau, s_\tau, h_\tau \rangle$, where $s_\tau \in S_{C_\tau}$ and $h_\tau : S_{D_f} \to S_{D_\tau}$ so that*

$$\bar{h}_\tau(s) = f \cup s_\tau \cup h_\tau(s|_{D_f})$$

for $s \in S_G$.

Proof. The splitting comes from considering the behavior of $\bar{h}_\tau(s)$ on the disjoint union of three sets (which exhaust G): C_f, C_τ and D_τ. ■

REMARK 5.5. If $D_\tau = \emptyset$, then $h_\tau(s|_{D_f})$ disappears and

$$\bar{h}_\tau(s) = f \cup s_\tau,$$

so that $\bar{h}_\tau(s)$ does not depend on s at all.

REMARK 5.6. In the case $\tau = \sigma^n$ the quadruple $\langle g_\tau, D_\tau, s_\tau, h_\tau \rangle$ is particularly easy to describe. Indeed, we leave it to the reader to check that

$$g_\tau = n, \quad D_\tau = D_f, \quad s_\tau = \emptyset \quad \text{and} \quad h_\tau = \mathrm{id}|_{D_f}.$$

PROPOSITION 5.7. *Let $\tau \in E(X_F, \sigma)$. The assignment*

$$\tau \mapsto \langle g_\tau, D_\tau, s_\tau, h_\tau \rangle$$

is injective.

Proof. Let τ and τ' be elements of $E(X_F, \sigma)$ such that

$$\langle g_\tau, D_\tau, s_\tau, h_\tau \rangle = \langle g_{\tau'}, D_{\tau'}, s_{\tau'}, h_{\tau'} \rangle.$$

Pick an $x \in X_F$, and put $g = g_x + g_\tau$. Let s be chosen in such a way that $\tau(x)(0) = \bar{h}_\tau(s)(g)$. It suffices to show that

$$\tau(x)(n) = \bar{h}_\tau(s)(g + n)$$

for all $n \in \mathbb{N}$, since then, by assumption, $\bar{h}_\tau = \bar{h}_{\tau'}$ and we get

$$\tau(x)(n) = \bar{h}_\tau(s)(g+n) = \bar{h}_{\tau'}(s)(g+n) = \tau'(x)(n).$$

To this end, fix $n > 0$, and set $x' = \sigma^n(x)$. Then

$$\tau(x)(n) = \tau(x')(0) = \bar{h}_\tau(s)(g_{x'} + g_\tau) = \bar{h}_\tau(s)(g_x + n + g_\tau) = h_\tau(s)(g+n),$$

since $g_{x'} = g_x + n$. ■

We have proved that the quadruple $\langle g_\tau, D_\tau, s_\tau, h_\tau\rangle$ determines τ as an element of the enveloping semigroup of (X_F, σ) (and all the more as an element of the enveloping semgiroup of (X_f, σ)).

6. Enveloping semigroup—the composition rule. In this section we are going to analyze how the quadruples introduced in the previous section behave under the composition of their corresponding τ's. Let $\tau_1, \tau_2 \in E(X_F, \sigma)$, and let $\tau = \tau_2 \circ \tau_1$. We want to describe the quadruple $\langle g_\tau, D_\tau, s_\tau, h_\tau\rangle$ in terms of τ_1, τ_2 (preferably in terms of the quadruples $\langle g_{\tau_1}, D_{\tau_1}, s_{\tau_1}, h_{\tau_1}\rangle$ and $\langle g_{\tau_2}, D_{\tau_2}, s_{\tau_2}, h_{\tau_2}\rangle$).

A moment's thought will convince the reader that

$$g_\tau = g_{\tau_2} + g_{\tau_1}.$$

Now, we focus our attention on finding D_τ.

THEOREM 6.1. *Given any $(n_\alpha) \in \mathcal{N}_{\tau_2}$, we have*

$$D_\tau = g_{\tau_2} + \liminf_\alpha (D_{\tau_1} - n_\alpha).$$

Proof. Note that D_τ is precisely the set of elements g such that if x projects to $g - g_\tau$ and is determined by a function s then $\tau(x)(0)$ depends on the choice of s (i.e., cannot be determined without referring to s). For that it must be, that no matter what net (n_α) in $\mathcal{N}_{\tau_2}$ we choose, for almost all α the value $\tau_1(x)(n_\alpha)$ cannot be determined without knowing the function s (otherwise we could determine the limit value without referring to s). But the latter means precisely that, for almost all α, $g - g_{\tau_2} + n_\alpha$ falls into D_{τ_1}. This is equivalent to saying that $g \in g_{\tau_2} + (D_{\tau_1} - n_\alpha)$ for such α, i.e.,

$$g \in g_{\tau_2} + \liminf_\alpha (D_{\tau_1} - n_\alpha),$$

which proves the claim. ■

COROLLARY 6.2. *The set D_τ is a subset of $D_{\tau_2} \cap \overline{D}_{\tau_1}$.*

Proof. First, since $D_{\tau_1} \subseteq D_f$, we get

$$D_\tau = g_{\tau_2} + \liminf_\alpha (D_{\tau_1} - n_\alpha) \subseteq g_{\tau_2} + \liminf_\alpha (D_f - n_\alpha) = D_{\tau_2}.$$

Second, note that

$$D_\tau = \liminf_\alpha (D_{\tau_1} + (g_{\tau_2} - n_\alpha)) \subseteq \overline{D}_{\tau_1},$$

since $g_{\tau_2} - n_\alpha \to \mathbf{0}$ in G. ■

It would be desirable to describe D_τ solely in terms of D_{τ_1} and D_{τ_2} (for example as their intersection), but it seems that the involvement of nets is inevitable.

The following theorem, containing the formulas for s_τ and h_τ, concludes the description of the quadruple $\langle g_\tau, D_\tau, s_\tau, h_\tau\rangle$.

THEOREM 6.3. *Let s_τ and h_τ be such that*

$$\bar{h}_\tau(s) = f \cup s_\tau \cup h_\tau(s|_{D_f}),$$

where $s \in S_G$. Then

$$s_\tau = s_{\tau_2} \cup h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))|_{D_{\tau_2}\setminus D_\tau}$$

and

$$h_\tau(s|_{D_f}) = h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))|_{D_\tau}.$$

Proof. First of all, note that

$$\begin{aligned}\bar{h}_\tau(s) &= (\bar{h}_{\tau_2} \circ \bar{h}_{\tau_1})(s) = \bar{h}_{\tau_2}(f \cup s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))\\ &= f \cup s_{\tau_2} \cup h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f})).\end{aligned}$$

This shows that $\bar{h}_\tau$ is given by $h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))$ on the set D_{τ_2}. However, since $D_\tau \subseteq D_{\tau_2}$ (by Corollary 6.2), the restriction

$$h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))|_{D_{\tau_2}\setminus D_\tau}$$

cannot depend on s (because $\bar{h}_\tau$ does not depend on s outside of D_τ). Therefore, we get

$$s_\tau = s_{\tau_2} \cup h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))|_{D_{\tau_2}\setminus D_\tau}$$

and

$$h_\tau = h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))|_{D_\tau},$$

as claimed. ∎

What is remarkable is that the sets D_τ have the tendency to become smaller and smaller as τ becomes a composition of more and more elements of the enveloping semigroup. This provokes a classification of functions f into the following classes:

DEFINITION 6.4. We say that:

(1) f is of *class* 1 if $D_\tau = \emptyset$ for every 'non-trivial' τ, i.e., τ not being a power of σ. This is equivalent to saying that every orbit in G visits the set D_f at most finitely many times.
(2) f is of *class* 2 if $D_\tau = \emptyset$ for every τ of the form $\tau_2 \circ \tau_1$, where both τ_1 and τ_2 are non-trivial. This is equivalent to saying that although the orbits of some g's may visit D_f infinitely many times, the set T of times when this happens (for a given g) does not contain two infinite

sets A, B such that

$$\forall_{a\in A}\,\exists_{b_a\in B}\,\forall_{b\in B,\,b\geq b_a}\quad a+b\in T.$$

(3) We skip the obvious continuation of further classes, until we reach the terminal *class* ∞ (consisting of all f's which are not of any finite class).

The composition of two elements of the enveloping semigroup (cf. Theorem 6.3) becomes particularly easy to describe in the case where f belongs to one of the first two classes.

PROPOSITION 6.5. *Let $\tau = \tau_2 \circ \tau_1$ be a non-trivial element of $E(X_F, \sigma)$. For $s \in S_G$ we get:*

(1) *If f is of class* 1*, then*
$$\bar{h}_\tau(s) = f \cup s_{\tau_2}.$$

(2) *If f is of class* 2*, then*
$$\bar{h}_\tau(s) = f \cup (s_{\tau_2} \cup h_{\tau_2}(s_{\tau_1} \cup h_{\tau_1}(s|_{D_f}))).$$

In each case, $\bar{h}_\tau$ does not depend on s.

Proof. Since τ is non-trivial, we may assume that τ_2 is also non-trivial. Then (1) follows from the fact that $C_{\tau_2} = D_f$ (and $D_{\tau_2} = \emptyset$), while (2) from the fact that, by assumption, $D_\tau = \emptyset$. ■

In the remainder of this section we provide a handful of facts concerning the enveloping semigroup under study.

PROPOSITION 6.6. *The element $\tau = \langle g_\tau, D_\tau, s_\tau, h_\tau\rangle$ is an idempotent if and only if*
$$g_\tau = \mathbf{0}$$
(*and this is sufficient for class* 1 *functions*),
$$D_{\tau\circ\tau} = D_\tau$$
(*this condition means that every point that visits D_f infinitely many times does it along an IP set*), *and*
$$h_\tau(s_\tau \cup h_\tau(s|_{D_f})) = h_\tau(s|_{D_f})$$
for every $s \in S_G$.

REMARK 6.7. For class 2 functions the condition $D_{\tau\circ\tau} = D_\tau$ implies $D_\tau = \emptyset$, which, together with $g_\tau = \mathbf{0}$, suffices for τ to be an idempotent. By an easy argument, the same holds for functions of any finite class.

PROPOSITION 6.8. *The fiber of any g_0 in the enveloping semigroup* (*i.e., the quadruples with $g_\tau = g_0$*) *contains an element τ for which $D_\tau = \emptyset$. When $g_0 = \mathbf{0}$, such elements are clearly idempotents.*

Proof. This follows easily from the fact that C_f is dense and open, thus we can find a net (n_α) such that $n_\alpha \to g_0$ in G, and, for every $g \in D_f$, the points $g + g_0 - n_\alpha$ belong to C_f for almost all α. ∎

REMARK 6.9. If f is of class 1, then the quadruples of non-trivial elements in $E(X_F, \sigma)$ reduce to pairs $\langle g_\tau, s_\tau \rangle$. In that case the collection of functions s_τ 'paired' with any g is the same. This follows immediately from the composition rule and the fact that we can pass from one g to another by composition. For any fixed function s (appearing as s_τ) the set of pairs $\langle g, s \rangle$ is a subgroup of the enveloping semigroup isomorphic to G. In this manner, the non-trivial part of the enveloping semigroup can be represented as a disjoint union of mutually isomorphic groups.

7. Connections with topological entropy. In the last section we want to discuss the relationship between the classification introduced in Definition 6.4 and the notion of topological entropy of a dynamical system ([W] is a good all-around source of information on topological entropy). We are going to show that if f is of finite class, then the system (X_f, σ) has zero topological entropy. The idea is to show that for such f the set D_f is a subset of Haar measure zero in G. This is accomplished by showing that the set of times when any given point $g \in G$ visits D_f has zero upper density.

Before we embark on proving the aforementioned result we need some preparatory notation and definitions which we copy verbatim from [BF].

DEFINITION 7.1. Let $\{b_n\}_{n=1}^\infty$ be a sequence in $\mathbb{N}$. The set

$$\mathrm{IP}\{b_n\}_{n=1}^\infty = \Big\{\sum_{k=1}^r b_{i_k} : r \in \mathbb{N},\, i_1 < \cdots < i_r\Big\}$$

is called an *IP set* in $\mathbb{N}$.

DEFINITION 7.2. Let $L \in \mathbb{N}$. The *initial L-segment* of $\mathrm{IP}\{b_n\}_{n=1}^\infty$ is the set

$$\mathrm{IP}\{b_n\}_{n=1}^L = \Big\{\sum_{k=1}^r b_{i_k} : r \in \mathbb{N},\, i_1 < \cdots < i_r \le L\Big\}.$$

DEFINITION 7.3. A set $S \subset \mathbb{N}$ *contains a broken* IP *set* if there is a sequence $\{b_n\}_{n=1}^\infty$ in $\mathbb{N}$ such that, for each $L \in \mathbb{N}$, there exists $a_L \in \mathbb{N}$ with

$$a_L + \mathrm{IP}\{b_n\}_{n=1}^L \subset S.$$

DEFINITION 7.4. A set $S \subset \mathbb{N}$ has *positive upper density* if

$$\limsup_{n\to\infty} \frac{\mathrm{card}(S \cap \{1, \ldots, n\})}{n} > 0.$$

The following proposition, relating the latter two notions, is a particularly useful observation made in [BF].

Proposition 7.5. *Every set $S \subset \mathbb{N}$ with positive upper density contains a broken IP set.*

To finish the preparatory stage recall that, given an integer $m \in \mathbb{N}$, the function f is of class m (cf. Definition 6.4) if, for every $\tau \in E(X_F, \sigma)$ of the form $\tau = \tau_m \circ \cdots \circ \tau_1$, we have

$$D_\tau = \emptyset$$

whenever the components $\tau_1, \ldots, \tau_m$ are non-trivial, i.e., none is a power of σ.

By repeatedly applying Theorem 6.1 we can deduce that

$$D_\tau = g_\tau + \liminf_\alpha \ldots \liminf_\alpha (D_f - (n_\alpha^{(1)} + \cdots + n_\alpha^{(m)})),$$

where lim inf is taken m times. This allows us to infer that f is of class m if, for a given element g in G, the set T of times when the orbit of g visits the set D_f does not contain m infinite sets $C_1, \ldots, C_m$ such that

$$(7.1) \quad \exists_{c_1' \in C_1} \forall_{c_1 \in C_1,\, c_1 \geq c_1'} \cdots \exists_{c_m' \in C_m} \forall_{c_m \in C_m,\, c_m \geq c_m'} \quad c_1 + \cdots + c_m \in T,$$

since otherwise the set D_τ will be non-empty.

Now we are in a position to prove the following theorem which will lead to the result alluded to at the beginning of this section.

Theorem 7.6. *Suppose that f is of finite class. For each $g \in G$, denote by T_g the set of times when the orbit of g visits the set D_f. Then, for each $g \in G$, the upper density of the set T_g is zero.*

Proof. First of all, we may assume that for some positive integer m, the function f is of class m.

Suppose that there exists g in G such that the set T_g has positive upper density. We are going to show that there exists $g' \in G$ for which the set $T_{g'}$ contains m infinite subsets $C_1, \ldots, C_m$ satisfying condition (7.1), contrary to the assumption that f is of class m.

By Proposition 7.5 the set T_g contains a broken IP set, i.e., there exists a sequence $\{b_n\}_{n=1}^\infty$ in $\mathbb{N}$ such that, for each $L \in \mathbb{N}$, there exists $a_L \in \mathbb{N}$ with

$$a_L + \mathrm{IP}\{b_n\}_{n=1}^L \subset T_g.$$

This means that

$$\forall_{L \in \mathbb{N}} \quad g + \mathrm{IP}\{b_n\}_{n=1}^L \in D_f - a_L.$$

Choose a subnet (a_{L_α}) such that the net $(\sigma^{a_{L_\alpha}})$ converges in $E(X_F, \sigma)$ to some element τ. Then $g_\tau = \lim_\alpha a_{L_\alpha}$, and we get

$$\forall_{L \in \mathbb{N}} \quad g + \mathrm{IP}\{b_n\}_{n=1}^L \in \liminf_\alpha (D_f - a_{L_\alpha}) = D_\tau - g_\tau \subset D_f - g_\tau.$$

Hence

$$\forall_{L \in \mathbb{N}} \quad (g + g_\tau) + \mathrm{IP}\{b_n\}_{n=1}^L \in D_f.$$

Setting $g' = (g + g_\tau)$ we obtain

$$g' + \mathrm{IP}\{b_n\}_{n=1}^{\infty} \in D_f,$$

i.e.,

$$\mathrm{IP}\{b_n\}_{n=1}^{\infty} \subset T_{g'}.$$

For $k = 0, 1, \dots, m-1$ let

$$C_{k+1} = \mathrm{IP}\{b_{mn+k}\}_{n=1}^{\infty}.$$

These sets are infinite, and since

$$\mathrm{IP}\{b_{mn}\}_{n=1}^{\infty} + \cdots + \mathrm{IP}\{b_{mn+(m-1)}\}_{n=1}^{\infty} \subset \mathrm{IP}\{b_n\}_{n=1}^{\infty},$$

we can see that

$$C_1 + \cdots + C_m \subset T_{g'},$$

thus, these sets satisfy condition (7.1), which leads to a contradiction.

Therefore, for each $g \in G$, the upper density of T_g is zero. ■

COROLLARY 7.7. *Suppose that f is of finite class. Then the topological entropy of the dynamical system (X_f, σ) is zero.*

Proof. Theorem 7.6 shows that every point g visits the set D_f along a sequence of upper density zero. This implies that the Haar measure of D_f is zero, and we can conclude that in this case the system (X_f, σ), given some ergodic measure, is measure-theoretically isomorphic to the underlying rotation on G. It follows that its measure-theoretic entropy is zero. Hence, applying the variational principle, we see that the topological entropy of the system (X_f, σ) must be zero. ■

REMARK 7.8. The above result allows us to distinguish between various systems of the type under consideration with zero topological entropy.

COROLLARY 7.9. *If the dynamical system (X_f, σ) has positive topological entropy, then f is necessarily of class ∞.*

It would be interesting to know whether the implications in these two corollaries could be reversed, since then the division of systems (X_f, σ) into classes would refine the division into systems of positive or zero topological entropy. However, it is not clear whether this has to be the case.

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