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THE TYPE SET FOR HOMOGENEOUS SINGULAR MEASURES ON $\mathbb{R}^3$ OF POLYNOMIAL TYPE

BY

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Abstract. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$, let μ be the Borel measure on $\mathbb{R}^3$ defined by $\mu(E) = \int_D \chi_E(x, \varphi(x))\, dx$ with $D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ and let T_μ be the convolution operator with the measure μ. Let $\varphi = \varphi_1^{e_1} \cdots \varphi_n^{e_n}$ be the decomposition of φ into irreducible factors. We show that if $e_i \neq m/2$ for each φ_i of degree 1, then the type set $E_\mu := \{(1/p, 1/q) \in [0,1] \times [0,1] : \|T_\mu\|_{p,q} < \infty\}$ can be explicitly described as a closed polygonal region.

1. Introduction. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$ and let $D = \{y \in \mathbb{R}^2 : |y| \leq 1\}$. Let μ be the Borel measure on $\mathbb{R}^3$ given by

$$\mu(E) = \int_D \chi_E(y, \varphi(y))\, dy \tag{1.1}$$

and let T_μ be the operator defined, for $f \in S(\mathbb{R}^3)$, by $T_\mu f = \mu * f$. Let E_μ be the set of pairs $(1/p, 1/q) \in [0,1] \times [0,1]$ such that there exists a positive constant c satisfying $\|Tf\|_q \leq c\|f\|_p$ for all $f \in S(\mathbb{R}^3)$, where the L^p spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^3$. For $(1/p, 1/q) \in E_\mu$, T can be extended to a bounded operator, still denoted by T, from $L^p(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$.

If $\det \varphi''(y)$ is not identically zero and if it vanishes somewhere on $\mathbb{R}^2 - \{0\}$, the set of points y where it vanishes is a finite union of lines $L_1, \ldots, L_k$ through the origin. For each $j = 1, \ldots, k$, let α_j be the vanishing order of $\det \varphi''(y)$ along a transversal direction to L_j, at any point of L_j. As remarked in [2], α_j is independent of the point and of the transversal direction chosen. Let

$$\widetilde{m} = \max\{m, \alpha_1 + 2, \ldots, \alpha_k + 2\}. \tag{1.2}$$

For $s \geq 1$, let Σ_s and $\Sigma_s^{\#}$ be the closed polygonal regions with vertices at

$$(0,0), \quad (1,1), \quad \left(\frac{s+1}{s+2}, \frac{s-1}{s+2}\right), \quad \left(\frac{3}{s+2}, \frac{1}{s+2}\right)$$

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and at

$$(0,0), \quad (1,1), \quad \left(\frac{s}{s+1}, \frac{s-1}{s+1}\right), \quad \left(\frac{2}{s+1}, \frac{1}{s+1}\right)$$

respectively.

Our aim in this paper is to prove the following

THEOREM 1. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$. Let $\varphi = \varphi_1^{e_1} \cdots \varphi_n^{e_n}$ be a decomposition of φ into irreducible factors with $\varphi_i \nmid \varphi_j$ for $i \neq j$. Assume that $e_i \neq m/2$ for each φ_i of degree* 1.

(i) *If* $\det \varphi''(y) \equiv 0$ *then* $E_\mu = \Sigma_m^{\#}$.
(ii) *If* $\det \varphi''(y)$ *vanishes at most at* $y = 0$ *then* $E_\mu = \Sigma_m$.
(iii) *If* $\det \varphi''(y)$ *is not identically zero and if it vanishes somewhere in* $\mathbb{R}^2 - \{0\}$ *then* $E_\mu = \Sigma_{\widetilde{m}}$, *with* $\widetilde{m}$ *defined by* (1.2).

L^p improving properties of convolution operators with singular measures supported on hypersurfaces in $\mathbb{R}^n$ have been widely studied in [3], [5], [7], [8]. In particular, in [5], the type set is studied under our present hypothesis, but the endpoint problem is left open there. Our proof of Theorem 1 will be based on a suitable adaptation of arguments due to M. Christ, developed in [1], where he studies the type set associated to the two-dimensional measure supported on a parabola.

Throughout this paper c will denote a positive constant, not the same at each occurrence.

2. Preliminaries and statement of auxiliary results. If $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and if $V \subset \mathbb{R}^2$ is a measurable set, let $\mu_{V,\psi}$ be the measure defined as μ, but with V and ψ instead of D and φ respectively. Let $T_{V,\psi}$ be the convolution operator with the measure $\mu_{V,\psi}$ and let $E_{V,\psi}$ be the associated type set. Finally, let $T_{V,\psi}^*$ be the adjoint operator of $T_{V,\psi}$.

REMARK 1. (i) A computation shows that $(T_{V,\psi}^* f)^\vee = \mu_{V,\psi} * (f^\vee)$, $f \in S(\mathbb{R}^3)$, where $f^\vee(x) = f(-x)$. Thus $E_{V,\psi}$ is symmetric with respect to the non-principal diagonal $1/p + 1/q = 1$. Also, from the Riesz–Thorin theorem (as stated in [9]), $E_{V,\psi}$ is a convex set. If V has finite Lebesgue measure $|V|$ we also have $\|T_{V,\psi} f\|_p \leq |V| \, \|f\|_p$ for $1 \leq p \leq \infty$; thus in this case the closed segment with endpoints $(0,0)$ and $(1,1)$ is contained in $E_{V,\psi}$.

(ii) If $S \in \mathrm{GL}(2,\mathbb{R})$ then, for $f \in S(\mathbb{R}^3)$,

$$\mu_{V,\psi\circ S} * f = |\det S|^{-1} (\mu_{S(V),\psi} * (f \circ (S^{-1} \otimes \mathrm{Id}))) \circ (S \otimes \mathrm{Id}).$$

where Id is the identity map on $\dot{\mathbb{R}}$. This fact implies that $E_{V,\psi\circ S} = E_{S(V),\psi}$.

Let α be the order of the zero of the function $y_2 \mapsto \det \varphi''(1, y_2)$ at $y_2 = 0$, with the convention that $\alpha = 0$ if $\det \varphi''(1,0) \neq 0$ and that $\alpha = \infty$ if

$\det \varphi''(1, y_2)$ vanishes identically (i.e., by the homogeneity of φ, if $\det \varphi''(y)$ vanishes identically on $\mathbb{R}^2$).

The following result is proved in [2, Lemmas 2.2 and 2.4].

LEMMA 1. *Let* $a_0, \ldots, a_m \in \mathbb{R}$ *and let* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$\varphi(y_1, y_2) = \sum_{0 \le j \le m} a_j y_1^{m-j} {y_2}^j. \tag{2.1}$$

Set $l = \min\{j \in \{0, 1, \ldots, m\} : a_j \neq 0\}$.

(i) *If* $l = 0$ *and*
$$\frac{a_s}{a_0} = \binom{m}{s} m^{-s} \left(\frac{a_1}{a_0}\right)^s \quad \text{for } s = 0, 1, \ldots, r \text{ with } 1 \le r \le m-1,$$
$$\frac{a_{r+1}}{a_0} \neq \binom{m}{r+1} m^{-r-1} \left(\frac{a_1}{a_0}\right)^{r+1},$$
then $\alpha = r - 1$.

(ii) *If* $l = 0$ *and* $a_s/a_0 = \binom{m}{s} m^{-s} (a_1/a_0)^s$ *for* $s = 0, 1, \ldots, m$, *then* $\alpha = \infty$.

(iii) *If* $1 \le l \le m - 1$ *then* $\alpha = 2l - 2$.

(iv) *If* $l = m$ *then* $\alpha = \infty$.

For $\delta > 0$ let

$$V_\delta = D \cap \{y = (y_1, y_2) \in \mathbb{R}^2 : |y_2| \le \delta |y_1|\}. \tag{2.2}$$

Our first step will be to study $E_{V_\delta,\varphi}$ for δ positive and small enough. The following well known result gives some necessary conditions on p, q in order that $(1/p, 1/q) \in E_{V_\delta,\varphi}$.

LEMMA 2. *If* $(1/p, 1/q) \in E_{V_\delta,\varphi}$ *then*

$$\frac{1}{q} \le \frac{1}{p}, \qquad \frac{1}{q} \ge \frac{3}{p} - 2, \qquad \frac{1}{q} \ge \frac{4}{p}, \qquad \frac{1}{q} \ge \frac{1}{p} - \frac{2}{m+2}.$$

Proof. For the first condition see e.g. [7], the second one is proved in [6], the third condition follows from the second by symmetry, and for the fourth see the proof of Proposition 2.2 in [7]. ∎

The following lemma provides an additional restriction.

LEMMA 3. *Let* φ *and* l *be as in Lemma* 1. *If* $(1/p, 1/q) \in E_{V_\delta,\varphi}$ *then*

$$\frac{1}{q} \ge \frac{1}{p} - \frac{1}{l+1}.$$

Proof. We have

$$\varphi(y_1, y_2) = y_2^l P(y_1, y_2) \quad \text{where} \quad P(y_1, y_2) = \sum_{l \le j \le m} a_j y_1^{m-j} {y_2}^{j-l}$$

with $a_l \neq 0$. For $\delta > 0$, let $I_\delta = [-2, 2] \times [-2\delta, 2\delta] \times [-2M\delta^l, 2M\delta^l]$ where $M = \|P\|_{L^\infty(D)}$, let $f = \chi_{I_\delta}$, let $A_\delta = [-1, 1] \times [-\delta, \delta] \times [-M\delta^l, M\delta^l]$ and let $J_\delta = [-1,1] \times [-\delta,\delta]$. For $x \in A_\delta$ and $y \in J_\delta$ we have

$$(x_1 - y_1, x_2 - y_2, x_3 - \varphi(y_1,y_2)) \in I_\delta$$

and so $(\mu * f)(x) \ge c\delta$ with c independent of δ. Now if $(1/p,1/q) \in E_\mu$ then

$$c\delta^{1+(l+1)/q} \le \|\mu * f\|_q \le c'\|f\|_p = c''\delta^{(l+1)/p}$$

for all $0 < \delta < 1$ and so $1 + (l+1)/q - (l+1)/p \ge 0$. ∎

The next section will be devoted to the proof of the following two propositions:

PROPOSITION 1. *Let φ, l and α be as in Lemma* 1. *For δ positive and small enough we have*:

(i) *If* $1 \le l < m/2$ *then* $E_{V_\delta,\varphi} = \Sigma_m$.
(ii) *If* $m/2 < l < m$, *then* $E_{V_\delta,\varphi} = \Sigma_{2l}$.
(iii) *If* $l = 0$ *and* $\alpha < m-2$ *then* $E_{V_\delta,\varphi} = \Sigma_m$.

PROPOSITION 2. *Let V be a closed connected cone with vertex at the origin. Assume that* $\det\varphi''(y) \ne 0$ *for all* $y \in V - \{0\}$. *Then* $E_{D\cap V,\varphi} = \Sigma_m$.

3. Proofs of Propositions 1 and 2. For $k \in \mathbb{N}$, let

$$I_k = \{(y_1,y_2) \in \mathbb{R}^2 : 1/2 \le |y_1| \le 1,\ 2^{-k-1} \le |y_2| \le 2^{-k}\}, \tag{3.1}$$

$$\widetilde{I}_k = \{(y_1,y_2) \in \mathbb{R}^2 : 1/4 \le |y_1| \le 2,\ 2^{-k-2} \le |y_2| \le 2^{-k+1}\}, \tag{3.2}$$

$$\Delta_k = \bigcup_{j=0}^{\infty} 2^{-j} I_k, \qquad \widetilde{\Delta}_k = \bigcup_{j=0}^{\infty} 2^{-j} \widetilde{I}_k. \tag{3.3}$$

Let φ, l, r and α be as in Lemma 1. Then $\varphi(y_1,y_2) = y_2^l P(y_1,y_2)$ where P is a homogeneous polynomial function of degree $m-l$ such that $P(1,0) \ne 0$. Since $y_2 \mapsto \det\varphi''(1,y_2)$ has a zero of order α at $y_2 = 0$, it follows that if $\alpha < \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$c_1|y_2|^\alpha \le |\det\varphi''(y)| \le c_2|y_2|^\alpha \tag{3.4}$$

for all $y = (y_1,y_2) \in \widetilde{I}_k$ and $k \ge k_0$. For $k \in \mathbb{N}$ let $\varphi_k : \widetilde{I}_{k_0} \to \mathbb{R}$ be defined by

$$\varphi_k(y_1,y_2) = y_2^l P(y_1, 2^{k_0-k}y_2). \tag{3.5}$$

For $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ and $t > 0$, let

$$t \circ x = (tx_1, tx_2, t^m x_3), \qquad t \bullet x = (x_1, tx_2, t^l x_3).$$

For $f : \mathbb{R}^3 \to \mathbb{R}$ let

$$(t \circ f)(x) = f(t \circ x), \qquad (t \bullet f)(x) = f(t \bullet x).$$

A computation, using the homogeneity of φ, shows that

$$T_{\Delta_k,\varphi}f(x) = 2^{k_0-k}T_{\Delta_{k_0},\varphi_k}(2^{k_0-k}\bullet f)(2^{k-k_0}\bullet x),$$
$$T_{2^{-j}I_{k_0},\varphi_k}f(x) = 2^{-2j}T_{I_{k_0},\varphi_k}(2^{-j}\circ f)(2^j\circ x),$$

for all $f\in S(\mathbb{R}^3)$ and for all $j,k\in\mathbb{N}$. These identities imply that there exists $c>0$ such that

$$\|T_{\Delta_k,\varphi}\|_{p,q} \le c2^{-k(1+\frac{l+1}{q}-\frac{l+1}{p})}\|T_{\Delta_{k_0},\varphi_k}\|_{p,q}, \tag{3.6}$$

$$\|T_{2^{-j}I_{k_0},\varphi_k}\|_{p,q} \le c2^{-j(2+\frac{m+2}{q}-\frac{m+2}{p})}\|T_{I_{k_0},\varphi_k}\|_{p,q} \tag{3.7}$$

for all $j,k\in\mathbb{N}$. Let A be the intersection point of the lines $1/q=3/p-2$ and $1/q=1/p-2/(m+2)$ and let B the intersection point of the lines $1/q=3/p-2$ and $1/q=1/p-1/(l+1)$. Let p_A, q_A and p_B,q_B be defined by $A=(1/p_A,1/q_A)$ and $B=(1/p_B,1/q_B)$. Then

$$\left(\frac{1}{p_A},\frac{1}{q_A}\right)=\left(\frac{m+1}{m+2},\frac{m-1}{m+2}\right),\qquad \left(\frac{1}{p_B},\frac{1}{q_B}\right)=\left(\frac{2l+1}{2l+2},\frac{2l-1}{2l+2}\right).$$

Remark 2. If $1\le l<m/2$ then $1+(l+1)/q_A-(l+1)/p_A>0$. Let $\Delta=\bigcup_{k\ge k_0}\Delta_k$. From (3.6) we will obtain $\|T_{\Delta,\varphi}\|_{p_A,q_A}\le c$, once we have proved that

$$\sup_{k\ge k_0}\|T_{\Delta_{k_0},\varphi_k}\|_{p_A,q_A}<\infty.$$

This last inequality will follow from an adaptation of Christ's argument (see [1]) that, in our case, involves a Littlewood–Paley decomposition of the operator $T_{\Delta_{k_0},\varphi_k}$.

Let $\theta\in C_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp}(\theta)\subset\widetilde{I}_{k_0}$, $\theta\equiv 1$ on I_{k_0} and $0\le\theta\le 1$. We observe that $1\le\sum_{j\in\mathbb{Z}}\theta(2^jy)\le 3$ for $y\in\Delta_{k_0}$. For $j\in\mathbb{N}\cup\{0\}$ and $k\ge k_0$, let $\mu_{j,k}$ be the measure defined by

$$\mu_{j,k}(E)=\int\chi_E(y,\varphi_k(y))\theta(2^jy)\,dy \tag{3.8}$$

with φ_k defined by (3.5), and let $T_{j,k}$ be the convolution operator with the measure $\mu_{j,k}$. Then, for $0\le f\in S(\mathbb{R}^3)$, $T_{\Delta_{k_0},\varphi_k}f\le\sum_{j\ge 0}T_{j,k}f$.

As above we obtain, for $0\le f\in S(\mathbb{R}^3)$,

$$T_{I_k,\varphi}f(x)\le 2^{k_0-k}T_{0,k}(2^{k_0-k}\bullet f)(2^{k-k_0}\bullet x),$$

and so we have

$$\|T_{I_k,\varphi}\|_{p,q}\le c2^{-k(1+\frac{l+1}{q}-\frac{l+1}{p})}\|T_{0,k}\|_{p,q}. \tag{3.9}$$

Let $R=\bigcup_{k\ge k_0}I_k$. Then, as in (3.7), we have

$$\|T_{2^{-j}R,\varphi}\|_{p,q}\le c2^{-j(2+\frac{m+2}{q}-\frac{m+2}{p})}\|T_{R,\varphi}\|_{p,q}. \tag{3.10}$$

Lemma 4. *Suppose that* $1\le l<m$. *Then there exists* $c>0$ *such that* $\|T_{0,k}\|_{p,q}\le c$ *for* $k\ge k_0$, $1/q=3/p-2$ *and* $3/4\le 1/p\le 1$.

Proof. A computation shows that

$$\det \varphi_k''(y_1, y_2) = 2^{2(k-k_0)(l-1)} \det \varphi''(y_1, 2^{k_0-k} y_2) \tag{3.11}$$

for all $y=(y_1,y_2)\in \widetilde{I}_{k_0}$. So, since $\alpha=2l-2$, (3.4) yields $c_1 \le |\det \varphi_k''(y)| \le c_2$ for all $y \in \widetilde{I}_{k_0}$ and $k \ge k_0$. For $\xi \in \mathbb{R}^3$ and $k \in \mathbb{N}$, we have

$$(\mu_{0,k})^\wedge(\xi) = \int_{\mathbb{R}^2} e^{-i(\xi_1 y_1 + \xi_2 y_2 + \xi_3 \varphi_k(y_1,y_2))} \theta(y)\, dy, \tag{3.12}$$

and we can apply [10, Proposition 6, p. 344] to deduce that there exists $c > 0$ such that

$$|(\mu_{0,k})^\wedge(\xi)| \le c(1+|\xi_3|)^{-1} \tag{3.13}$$

for all $k \ge k_0$ and $\xi \in \mathbb{R}^3$.

Now, the complex interpolation theorem (as stated, e.g., in [11, p. 205]) implies that there exists $c > 0$ such that $\|T_{0,k}\|_{4/3,4} \le c$ for all $k \ge k_0$. Indeed, for $\operatorname{Re} z > 0$ and $t \in \mathbb{R}$, we consider the fractional integration kernel $J_z(t) = 2^{-z/2}(\Gamma(z/2))^{-1}|t|^{z-1}$ and its analytic extension to $z \in \mathbb{C}$. In particular, we have $\widehat{J_z} = J_{1-z}$, also $J_0 = c\delta$ where δ denotes the Dirac distribution at the origin. For $-1 \le \operatorname{Re} z \le 1$, let $U_z f = f * \mu_{0,k} * (\delta \otimes \delta \otimes J_z)$. For $\operatorname{Re} z = 1$ a brief computation shows that $\|U_z\|_{1,\infty} \le c(z)$, and for $\operatorname{Re} z = -1$, from (3.13) we obtain $\|U_z\|_{2,2} \le c_1(z)$, for some constants $c(z)$ and $c_1(z)$ that satisfy the hypothesis of the complex interpolation theorem. So $\|T_{0,k}\|_{4/3,4} \le c$. Since we also have $\|T_{0,k}\|_{1,1} \le c2^{-k_0}$, the lemma follows. ∎

LEMMA 5. *Suppose that $l=0$ and $0 \le \alpha < m-2$. Then $\|T_{R,\varphi}\|_{p_A,q_A} < \infty$.*

Proof. From (3.11) and (3.4) there exists a positive constant c_{α,k_0} such that $|\det \varphi_k''(y_1,y_2)| \ge c_{\alpha,k_0} 2^{-k(\alpha+2)}$ for all $(y_1,y_2) \in \widetilde{I}_{k_0}$ and $k \ge k_0$. Then we obtain, as in Lemma 4,

$$|(\mu_{0,k})^\wedge(\xi)| \le c'_{\alpha,k_0} 2^{k(\alpha+2)/2}(1+|\xi_3|)^{-1}$$

and so, by complex interpolation,

$$\|T_{0,k}\|_{4/3,4} \le c2^{k(\alpha+2)/4}$$

for some $c > 0$ and all $k \ge k_0$. From (3.9), we get

$$\|T_{I_k,\varphi}\|_{4/3,4} \le c2^{k\alpha/4}.$$

On the other hand,

$$\|T_{I_k,\varphi}\|_{1,1} \le c2^{-k}.$$

Now $(1/p_A, 1/q_A) = \tau(3/4,1/4) + (1-\tau)(1,1)$ with $\tau = 4/(m+4) < 4/(\alpha+4)$, so the Riesz–Thorin theorem gives $\|T_{R,\varphi}\|_{p_A,q_A} \le \sum_{k \ge k_0} \|T_{I_k,\varphi}\|_{p_A,q_A} < \infty$. ∎

For $\eta \in \mathbb{R}^3$ and $y = (y_1, y_2)$, let

$$\Phi_{k,\eta}(y) := y_1\eta_1 + y_2\eta_2 + \varphi_k(y_1, y_2)\eta_3.$$

We will need the following

LEMMA 6. *Let φ and l be as in Lemma 1 and suppose $l \geq 1$. Then there exist positive constants c_1, c_2, c_3 and $k_1 \in \mathbb{N}$ such that, if we set*

$$C_0 = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : c_1|\xi_3| < |(\xi_1, \xi_2)| < c_2|\xi_3|\},$$

then:

(i) *For $k \geq k_1$, $y \in \operatorname{supp}\theta$ and $\eta = (\eta_1, \eta_2, \eta_3) \notin C_0$ such that $|\eta| = 1$ and $|\eta_2| \leq |\eta_1|$ we have $|D_1\Phi_{k,\eta}(y)| \geq c_3$.*

(ii) *For $k \geq k_1$, $y \in \operatorname{supp}\theta$ and $\eta = (\eta_1, \eta_2, \eta_3) \notin C_0$ such that $|\eta| = 1$ and $|\eta_1| \leq |\eta_2|$ we have $|D_2\Phi_{k,\eta}(y)| \geq c_3$.*

Proof. We write $\varphi(y_1, y_2) = y_2^l P(y_1, y_2)$ with P as in Lemma 3. To see (i), we observe that $D_1P(1, 0) \neq 0$. Then there exist constants M_1, M_2 such that $0 < M_1 \leq |D_1P(y_1, 2^{k_0-k}y_2)| \leq M_2$ for $(y_1, y_2) \in \operatorname{supp}\theta$ and k large enough.

For $\eta \notin C_0$ we have either $c_1|\eta_3| \geq |(\eta_1, \eta_2)|$ or $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$. If the first inequality holds, we obtain, for $(y_1, y_2) \in \operatorname{supp}\theta$ and k large enough,

$$\begin{aligned}|D_1\Phi_{k,\eta}(y_1, y_2)| &= |\eta_1 + y_2^l D_1P(y_1, 2^{k_0-k}y_2)\eta_3| \\ &\geq |y_2^l D_1P(y_1, 2^{k_0-k}y_2)|\,|\eta_3| - |(\eta_1, \eta_2)| \\ &\geq (2^{-(k_0+2)l}M_1 - c_1)|\eta_3| \geq (2^{-(k_0+2)l}M_1 - c_1)(1 + c_1^2)^{-1/2}.\end{aligned}$$

The last inequality follows because $c_1|\eta_3| \geq |(\eta_1, \eta_2)|$ and $|\eta| = 1$. If $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$, a similar computation gives, for $(y_1, y_2) \in \operatorname{supp}\theta$ and all k large enough,

$$|D_1\Phi_{k,\eta}(y_1, y_2)| \geq (2^{-1} - 2^{-(k_0-1)l}c_2^{-1}M_2)(1 + c_2^{-2})^{-1/2}.$$

So (i) holds if we choose $c_1 \leq 2^{-(k_0+2)l-1}M_1$ and $c_2 \geq 4M_2 2^{-(k_0-1)l}$.

(ii) Since $P(1, 0) \neq 0$, there exist constants M_3, M_4 such that

$$0 < M_3 \leq |lP(y_1, 2^{k_0-k}y_2) + 2^{k_0-k}y_2D_2P(y_1, 2^{k_0-k}y_2)| \leq M_4$$

for all $(y_1, y_2) \in \operatorname{supp}\theta$ and k large enough. Now, if $c_1|\eta_3| \geq |(\eta_1, \eta_2)|$, then

$$\begin{aligned}&|D_2\Phi_{k,\eta}(y_1, y_2)| \\ &\quad = |\eta_2 + [ly_2^{l-1}P(y_1, 2^{k_0-k}y_2) + 2^{k_0-k}y_2^l D_2P(y_1, 2^{k_0-k}y_2)]\eta_3| \\ &\quad \geq (|y_2^{l-1}|M_3 - c_1)|\eta_3| \geq (2^{-(k_0+2)(l-1)}M_3 - c_1)(1 + c_2^{-2})^{-1/2}.\end{aligned}$$

If $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$, a similar computation gives, for $y \in \operatorname{supp}\theta$ and all k

large enough,

$$|D_2\Phi_{k,\eta}(y_1,y_2)| \geq |\eta_2| - M_4|\eta_3| \geq (2^{-1} - M_4c_2^{-1})|(\eta_1,\eta_2)| \geq (2^{-1} - M_4c_2^{-1})(1+c_2^{-2})^{-1/2},$$

and thus (ii) follows if we choose $c_1 < 2^{-(k_0+2)(l-1)}M_3$ and $c_2 > 2M_4$. ■

LEMMA 7. *There exists $k_1 \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ and for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,*

$$\sup_{k \geq k_1} \sup_{\xi \notin C_0} \{|\xi|^N |D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}((\mu_{0,k})^\wedge)(\xi)|\} < \infty.$$

Proof. We observe that

$$D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}((\mu_{0,k})^\wedge)(\xi) = \int_{\mathbb{R}^2} e^{-i|\xi|\Phi_{k,\eta}(y)}\psi_k(y)\,dy$$

with $\psi_k(y) = y_1^{\alpha_1} y_2^{\alpha_2}(\varphi_k(y_1,y_2))^{\alpha_3}\theta(y_1,y_2)$.

For $\xi \notin C_0$, and $|\xi_2| \leq |\xi_1|$, we have

$$\int_{\mathbb{R}^2} e^{-i|\xi|\Phi_{k,\eta}(y)}\psi_k(y)\,dy = \int_{2^{-k_0-2}}^{2^{-k_0+1}} \int_{\mathbb{R}} e^{-i|\xi|\Phi_{k,\eta}(y_1,y_2)}\psi_k(y_1,y_2)\,dy_1\,dy_2,$$

thus, taking into account Lemma 6(i), we can estimate the inner integral following the proof of Proposition 1 in [10, p. 31], to obtain the lemma in this case.

For $\xi \notin C_0$ and $|\xi_1| \leq |\xi_2|$, we consider the other iterated integral and we use Proposition 1 in [10, p. 31] and Lemma 6(ii). ■

REMARK 3. Let C_0 be as in Lemma 6. Then the family of cones $\{2^j \circ C_0\}_{j\in\mathbb{Z}}$ has finite overlapping (i.e., $\#\{j \in \mathbb{Z} : C_0 \cap (2^j \circ C_0) \neq \emptyset\} < \infty$). Enlarging c_2 if necessary, we can construct a homogeneous function of degree zero (with respect to the euclidean dilations on $\mathbb{R}^3$) $m_0 \in C^\infty(\mathbb{R}^3 - \{0\})$ with $\operatorname{supp}(m_0) \subset \overline{C}_0$ and such that the family of functions defined by $m_j(y) = m_0(2^{-j} \circ y)$, $j \in \mathbb{Z}$, is a C^∞ partition of the unity in $\mathbb{R}^3$ minus the subspaces $(\xi_1,\xi_2) = 0$, $\xi_3 = 0$.

Without loss of generality, from now on we suppose $k_1 = k_0$.

Let Q_j be the operator with the multiplier m_j, let d_0 be a large constant such that $\widetilde{m}_j := \sum_{|i-j|\leq d_0} m_i$ is identically one on $2^j \circ C_0$ and let $\widetilde{Q}_j = \sum_{|i-j|\leq d_0} Q_i$. Let $h \in C_c^\infty(\mathbb{R}^3)$ be identically one in a neighborhood of the origin. Let $h_j(\xi) = h(2^{-j} \circ \xi)$ and let P_j be the Fourier multiplier operator with the symbol h_j. With these notations, we have the following

LEMMA 8. *Let $\{\sigma_j\}_{j\in\mathbb{N}}$ be a sequence of positive measures on $\mathbb{R}^3$ and let $U_j f = \sigma_j * f$ for $f \in S(\mathbb{R}^3)$. Suppose $1 < p \leq 2$ and $p \leq q < \infty$. If there* *exists* $C > 0$ *such that* $\sup_{j\in\mathbb{N}} \|U_j\|_{p,q} \le C$ *and*

$$\Big\| \sum_{1\le j\le J} U_j(I-P_j)(I-\widetilde{Q}_j)\Big\|_{p,q} \le C, \qquad \Big\| \sum_{1\le j\le J} U_jP_j\Big\|_{p,q} \le C$$

for all $J \in \mathbb{N}$, *then there exists* $\gamma > 0$, *independent of* C, J *and* $\{\sigma_j\}_{j\in\mathbb{N}}$, *such that*

$$\Big\| \sum_{1\le j\le J} U_j\Big\|_{p,q} \le \gamma C.$$

Proof. For $\varepsilon_j = \pm 1$ the operator $\sum_{j\in\mathbb{N}} \varepsilon_j\widetilde{Q}_j$ satisfies the hypothesis of the Marcinkiewicz multiplier theorem ([9, p. 109]), thus $\|\sum_{j\in\mathbb{N}} \varepsilon_j\widetilde{Q}_j\|_{p,p} \le c$ with c independent of $\{\varepsilon_j\}$. As in [9, p. 105] we get the Littlewood–Paley inequality $\|(\sum_{j\in\mathbb{N}} |\widetilde{Q}_jf|^2)^{1/2}\|_p \le c\|f\|_p$ and then the lemma follows as in the proof of Theorem 1 in [1]. ∎

Lemma 9. *There exists a constant* $C > 0$, *independent of* k *and* J, *such that* $\|\sum_{1\le j\le J} T_{j,k}P_j\|_{p_A,q_A} \le C$ *for all* k *large enough.*

Proof. Let $K_{j,k}$ be the kernel of $T_{j,k}P_j$. A computation gives

$$(K_{j,k})^\wedge(\xi) = 2^{-2j}((\mu_{0,k})^\wedge h)(2^{-j}\circ\xi).$$

Thus

$$\sum_{1\le j\le J} |K_{j,k}(\xi)| \le \sum_{j\in\mathbb{N}} 2^{jm}|G_k(2^j\circ\xi)|$$

with G_k defined by $(G_k)^\wedge = (\mu_{0,k})^\wedge h$. Since, by Lemma 7, $(G_k)^\wedge \in S(\mathbb{R}^3)$ with each seminorm bounded on k for $k \ge k_0$, it follows that the same holds for G_k. Proceeding as in [4, Lemma 2.9], we obtain

$$\sum_{j\in\mathbb{N}} |K_{j,k}(\xi)| \le c(\xi_1^{2m} + \xi_2^{2m} + \xi_3^2)^{-1/2}$$

with c independent of k. Since the above majorant belongs to weak $L^{(m+2)/m}$, the lemma follows from Young's weak inequality. ∎

Lemma 10. *There exists a constant* $C > 0$, *independent of* k *and* J, *such that*

$$\Big\| \sum_{1\le j\le J} T_{j,k}(I-P_j)(I-\widetilde{Q}_j)\Big\|_{p_A,q_A} \le C$$

for all k *large enough.*

Proof. The kernel of $T_{j,k}(I-P_j)(I-\widetilde{Q}_j)$ is given by

$$\xi \mapsto 2^{-2j}(\mu_{0,k})^\wedge(1-h)(1-\widetilde{m}_0)(2^{-j}\circ\xi).$$

Observe that, from Lemma 7, we have $(\mu_{0,k})^\wedge(1-h)(1-\widetilde{m}_0) \in S(\mathbb{R}^3)$ with each seminorm bounded on k for $k \ge k_0$. From this fact the assertion follows as in Lemma 9. ∎

Proof of Proposition 1(i). From (3.6) and from Lemmas 8–10 we have, for all $k \geq k_0$,

$$\|T_{\Delta_{k_0},\varphi_k} f\|_{p_A,q_A} \leq c$$

with c independent of k. From Remark 2, we get $\|T_{\Delta,\varphi}\|_{p_A,q_A} \leq c$ and so, since $E_{\Delta,\varphi}$ is symmetric with respect to the non-principal diagonal $1/p + 1/q = 1$ (see Remark 1(i)), we have $\Sigma_m \subset E_{\Delta,\varphi}$ and so, since $V_\delta \subset \Delta$ for δ small enough, $\Sigma_m \subset E_{V_\delta,\varphi}$ and then, by Lemma 2, $E_{V_\delta,\varphi} = \Sigma_m$. ■

Our next step will be to prove Proposition 1(iii).

Let $\widetilde{R} = [1/4, 2] \times [-2^{-k_0+1}, 2^{-k_0+1}]$ and pick $\theta_0 \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq \theta_0 \leq 1$, $\theta_0 \equiv 1$ on R and $\operatorname{supp} \theta_0 \subset \widetilde{R}$. We observe that $\sum_{j\in\mathbb{Z}} \theta_0(2^j y) \leq 3$ for $y \in \bigcup_{j=0}^\infty 2^{-j} R$. For $j \in \mathbb{N} \cup \{0\}$, let $\mu^{(j)}$ be the measure defined as $\mu_{j,k}$ in (3.8) but with θ_0 instead of θ and φ instead of φ_k, and let $T^{(j)}$ be the associated convolution operator.

For $\xi \in \mathbb{R}^3$ we set

$$\Phi_\xi(y_1, y_2) := \xi_1 y_1 + \xi_2 y_2 + \varphi(y_1, y_2)\xi_3.$$

LEMMA 11. *Suppose $l = 0$. Then there exist positive constants c_1, c_2 and c_3 such that, if we set*

$$C_0 = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : c_1|\xi_3| < |(\xi_1, \xi_2)| < c_2|\xi_3|\},$$

then $|\nabla \Phi_\eta(y)| \geq c_3$ for all $y = (y_1, y_2) \in \widetilde{R}$ and $\eta = (\eta_1, \eta_2, \eta_3) \notin C_0$ such that $|\eta| = 1$.

Proof. We have $D_1\Phi_\eta(y) = \eta_1 + \eta_3 D_1\varphi(y)$. Since $D_1\varphi(1, 0) \neq 0$ and $\varphi \in C^\infty(\mathbb{R}^2)$, there exist $M_1, M_2, M_3 > 0$ such that $M_1 \leq |D_1\varphi(y)| \leq M_2$ and $|D_2\varphi(y)| \leq M_3$ for all $y \in \widetilde{R}$.

For $\eta \notin C_0$ we have either $c_1|\eta_3| \geq |(\eta_1, \eta_2)|$ or $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$. If the first inequality holds, then, for $y \in R$,

$$\begin{aligned}|D_1\Phi_\eta(y)| &\geq |D_1\varphi(y)|\,|\eta_3| - |(\eta_1, \eta_2)| \\ &\geq (M_1 - c_1)|\eta_3| \geq (M_1 - c_1)(1 + c_1^2)^{-1/2},\end{aligned}$$

the last inequality because $c_1|\eta_3| \geq |(\eta_1, \eta_2)|$ and $|\eta| = 1$.

If $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$ and $|\eta_1| \geq |\eta_2|$, a similar computation gives, for $y \in \widetilde{R}$, $|D_1\Phi_\eta(y)| \geq (1/2 - c_2^{-1}M_2)(1 + c_2^{-2})^{-1/2}$.

Suppose now $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$ and $|\eta_2| \geq |\eta_1|$. We have, in this case,

$$\begin{aligned}|D_2\Phi_\eta(y)| &\geq |\eta_2| - M_3|\eta_3| \geq (1/2 - M_3 c_2^{-1})|(\eta_1, \eta_2)| \\ &\geq (1/2 - M_3 c_2^{-1})(c_2^{-1} + 1)^{-1/2}\end{aligned}$$

for all $y \in \widetilde{R}$, the last inequality because $|\eta| = 1$ and $|(\eta_1, \eta_2)| \geq c_2|\eta_3|$. So the lemma holds if we choose $c_1 < \frac{1}{2}M_1$ and $c_2 > 4\max(M_2, M_3)$. ■

Using Lemma 11 instead of Lemma 6, the proof given in Lemma 7 applies to yield

LEMMA 12. *For all $N \in \mathbb{N}$ and for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,*

$$\sup_{\xi \notin C_0} |\xi|^N |D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}((\mu^{(0)})^\wedge)(\xi)| < \infty.$$

Proof of Proposition 1(iii). As in Lemma 5 we have $\|T_{\widetilde{R},\varphi}\|_{p_A,q_A} < \infty$ and so, from (3.10), we get $\sup_{j\ge 0} \|T_{2^{-j}\widetilde{R},\varphi}\|_{p_A,q_A} < \infty$. Let P_j, Q_j and $\widetilde{Q}_j$ be as in Lemmas 8–10 and observe that, by Lemma 12, these lemmas remain true if we replace $T_{j,k}$ by $T^{(j)}$. So this part of the proposition follows as in the proof of (i). ∎

Proof of Proposition 2. Let V be a closed connected cone with vertex at the origin and let $V_0 = \{y \in V : 1/4 \le |y| \le 2\}$. Since $\det \varphi''$ does not vanish on V_0, as in Lemma 5 we obtain $\|T_{V_0,\varphi}\|_{p_A,q_A} < \infty$. Now, the proof follows similar lines to the proof of Proposition 1(iii). ∎

REMARK 4. If $m/2 < l < m$, then $\alpha > m-2$, so $2 + (m+2)/q_B - (m+2)/p_B > 0$. Thus from (3.10) we will obtain $\|T_{\bigcup_{j\ge 0} 2^{-j}R,\varphi}\|_{p_B,q_B} \le c$ once we have proved that $\|T_{R,\varphi}\|_{p_B,q_B} \le c'$ for some positive constant c'.

Again the proof of this estimate will follow from an adaptation of Christ's argument, in this case concerning a Littlewood–Paley decomposition of the operator $T_{R,\varphi}$.

Let T_k be the convolution operator with the measure μ_k defined by

$$\mu_k(E) = \int \chi_E(y, \varphi(y))\theta(y_1, 2^{k-k_0}y_2)\,dy.$$

A computation shows that

$$(\mu_k)^\wedge(\xi) = 2^{k_0-k}(\mu_{0,k})^\wedge(2^{k_0-k} \bullet \xi) = 2^{k_0-k}\int e^{-i|\xi|(y_1\eta_1 + y_2 2^{k_0-k}\eta_2 + \varphi_k(y_1,y_2)2^{(k_0-k)l}\eta_3)}\theta(y_1,y_2)\,dy \tag{3.14}$$

where $\eta = \xi/|\xi|$.

Let C_0 be as in Lemma 6. Then $\{2^k \bullet C_0\}_{k\in\mathbb{Z}}$ has the finite overlapping property. Indeed, suppose that $\xi \in C_0 \cap 2^{-k} \bullet C_0$. Then $\xi_3 \ne 0$. If $k \ge 0$ then

$$c_1 2^{kl}|\xi_3| < |(\xi_1, 2^k\xi_2)| \le 2^k|(\xi_1,\xi_2)| < c_2 2^k|\xi_3|$$

and so $k \le (l-1)^{-1}(\ln 2)^{-1}\ln(c_2/c_1)$. If $k < 0$ then

$$c_1 2^k|\xi_3| < 2^k|(\xi_1,\xi_2)| \le |(\xi_1, 2^k\xi_2)| < c_2 2^{kl}|\xi_3|$$

and thus $k \ge (l-1)^{-1}(\ln 2)^{-1}\ln(c_1/c_2)$. Let $C_0^{\#}$ be the cone defined as C_0 but with $c_1/2$ and $2c_2$ instead of c_1 and c_2. Let $\widetilde{C}_0^{\#}$ be the similar cone with $c_1/4$ and $4c_2$ in place of c_1 and c_2. Let $m_0^{\#} \in C^\infty(\mathbb{R}^3 - \{0\})$ be a homogeneous function (with respect to the euclidean dilations on $\mathbb{R}^3$) of degree zero such that $\operatorname{supp}(m_0^{\#}) \subset C_0^{\#}$ and $m_0^{\#} \equiv 1$ on C_0, and let $\widetilde{m}_0^{\#} \in C^\infty(\mathbb{R}^3 - \{0\})$

be a homogeneous function (again with respect to the euclidean dilations on $\mathbb{R}^3$) of degree zero such that $\widetilde{m}_0^{\#} \equiv 1$ on $\operatorname{supp}(m_0^{\#})$ and $\operatorname{supp}(\widetilde{m}_0^{\#}) \subset \widetilde{C}_0^{\#}$. For $k \in \mathbb{Z}$, let $m_k^{\#}(\xi) = m_0^{\#}(2^{k_0-k} \bullet \xi)$ and $\widetilde{m}_k^{\#}(\xi) = \widetilde{m}_0^{\#}(2^{k_0-k} \bullet \xi)$. Let $Q_k^{\#}$ and $\widetilde{Q}_k^{\#}$ be the operators with multipliers $m_k^{\#}$ and $\widetilde{m}_k^{\#}$ respectively. From (3.14) and Lemma 7 we find that

$$(\mu_{0,k})^\wedge(1-h)(1-\widetilde{m}_0^{\#}) \in S(\mathbb{R}^3) \tag{3.15}$$

and each seminorm of these functions is bounded in k for $k \geq k_0$.

Let $h_k^{\#}(\xi) = h(2^{k_0-k} \bullet \xi)$ with h as in Section 3 and let $P_k^{\#}$ be the Fourier multiplier operator with symbol $h_k^{\#}$.

Remark 5. It can be checked that Lemma 8 still holds for $\widetilde{Q}_j^{\#}$ and $P_j^{\#}$ in place of $\widetilde{Q}_j$ and P_j.

Lemma 13. *There exists a constant* $C > 0$, *independent of* $J \in \mathbb{N}$, *such that*

$$\Big\| \sum_{k_0 \leq k \leq J} T_k P_k^{\#} \Big\|_{p_B, q_B} \leq C.$$

Proof. The kernel K_k of $T_k P_k^{\#}$ is given by

$$(K_k)^\wedge(\xi) = 2^{k_0-k}(\mu_{0,k})^\wedge h(2^{k_0-k} \bullet \xi).$$

Now,

$$\sum_{k_0 \leq k \leq J} |K_k(\xi)| \leq \sum_{k \in \mathbb{Z}} 2^{(k-k_0)l} |G_k(2^{k-k_0} \bullet \xi)|$$

with $G_k \in S(\mathbb{R}^3)$ defined by $(G_k)^\wedge = (\mu_{0,k})^\wedge h$ for $k \geq k_0$ and $G_k \equiv 0$ for $k < k_0$. We decompose

$$\sum_{k \in \mathbb{Z}} 2^{(k-k_0)l} |G_k(2^{k-k_0} \bullet \xi)| = \sum_{k \leq M_0} + \sum_{k > M_0}$$

with M_0 chosen such that $2^{M_0-1} < (|\xi_2|^{2l} + |\xi_3|^2)^{1/2l} \leq 2^{M_0}$ to obtain from (3.15),

$$\sum_{k \in \mathbb{Z}} 2^{(k-k_0)l} |G_k(2^{k-k_0} \bullet \xi)| \leq c(1+|\xi_1|)^{-1}(|\xi_2|^{2l} + |\xi_3|^2)^{-1/2}.$$

Since this last function belongs to weak $L^{(l+1)/l}$, by Young's weak inequality, the lemma follows. ∎

Lemma 14. *There exists a constant* $C > 0$, *independent of* $J \in \mathbb{N}$, *such that*

$$\Big\| \sum_{1 \leq k \leq J} T_k(I - P_k^{\#})(I - \widetilde{Q}_k^{\#}) \Big\|_{p_B, q_B} \leq C.$$

Proof. Since the kernel of $T_k(I - P_k^{\#})(I - \widetilde{Q}_k^{\#})$ is given by

$$\xi \mapsto 2^{k_0-k}(\mu_{0,k})^\wedge(1-h)(1-\widetilde{m}_0^{\#})(2^{k_0-k} \bullet \xi),$$

the assertion follows as in the previous lemma, by taking account of (3.15). ∎

Proof of Proposition 1(ii). As before, from (3.9), Lemma 4 and the last three lemmas we obtain $\|T_{R,\varphi}f\|_{p_B,q_B} \le c$, so from Remark 4, we get $\|T_{\bigcup_{j\ge 0} 2^{-j}R,\varphi}\|_{p_B,q_B} \le c$. Then, taking into account Remark 1(i) and Lemma 3 we conclude the proof. ∎

4. Proof of the main result

Proof of Theorem 1. If $\det \varphi''(y) \neq 0$ for $y \neq 0$, Proposition 2 gives (ii).

Suppose now that $\det \varphi''(y) \equiv 0$. Then Lemma 1 gives $\varphi(y_1, y_2) = (ay_1 + by_2)^m$ for some $a, b \in \mathbb{R}$. Thus there exists $S \in \mathrm{GL}(2, \mathbb{R})$ such that $(\varphi \circ S)(y) = y_2^m$. Then, by Remark 1(ii), E_μ coincides with the type set corresponding to the measure associated to the function $(y_1, y_2) \mapsto y_2^m$. So, in order to prove (i), we can assume that $\varphi(y_1, y_2) = y_2^m$. Let $\widetilde{\mu}$ be the Borel measure on $\mathbb{R}^2$ defined by $\widetilde{\mu}(E) = \int_{-1}^{1} \chi_E(t, t^m)\, dt$ and let $E_{\widetilde{\mu}}$ be its type set. Thus (see the case $n = 1$ of Theorem 3.12 in [3]) $E_{\widetilde{\mu}}$ is the closed polygonal region $\Sigma_m^{\#}$. We claim that $E_\mu = \Sigma_m^{\#}$. Indeed, let ν be the Borel measure on $\mathbb{R}^3$ given by $\nu(E) = \int_Q \chi_E(y, \varphi(y))\, dy$ where $Q = [-1, 1] \times [-1, 1]$. Since $D \subset Q \subset 2D$ a computation using the homogeneity of φ shows that $E_\mu = E_\nu$. For $f \in S(\mathbb{R}^2)$ and $g \in S(\mathbb{R})$ we have $\nu * (f \otimes g) = (\widetilde{\mu} * f) \otimes (\chi_I * g)$, where $I = [-1, 1]$. If $(1/p, 1/q) \in E_{\widetilde{\mu}}$ and $1/r = 1 + 1/q - 1/p$ then

$$\begin{aligned} \|\nu * (f \otimes g)\|_q &= \|\widetilde{\mu} * f\|_q \|\chi_I * g\|_q \\ &\le c\|\chi_I\|_r \|f\|_p \|g\|_p = c' \|f \otimes g\|_p. \end{aligned} \tag{4.1}$$

This implies $(1/p, 1/q) \in E_\nu$. Thus $E_{\widetilde{\mu}} \subset E_\mu$. Moreover, from the first line in (4.1) it is easy to show (taking there a suitable fixed g) that $E_\mu \subset E_{\widetilde{\mu}}$. So (i) holds.

Suppose now that $\det \varphi''(y) = 0$ somewhere in $\mathbb{R}^2 - \{0\}$ but $\det \varphi''(y)$ is not identically zero. Let $L_1, \ldots, L_k$ be the lines as in the introduction and for $\delta > 0$, let V_δ^j, $j = 1, \ldots, k$, be the sets defined by

$$V_\delta^j = D \cap \{y \in \mathbb{R}^2 : \mathrm{dist}(y, L_j) \le \delta |\pi_{L_j}(y)|\}$$

where π_{L_j} is the orthogonal projection from $\mathbb{R}^2$ onto L_j. We choose δ small enough such that no V_δ^j intersects L_s for $s \neq j$. Then

$$D = \bigcup_{j=1}^{k} (D \cap W_j) \cup \bigcup_{j=1}^{k} V_\delta^j$$

where each W_j is a closed and connected cone in $\mathbb{R}^2$ with the property that $\det \varphi''(y)$ does not vanish for $y \in W_j - \{0\}$. Now,

$$E_\mu = E_{D,\varphi} = \bigcap_{j=1}^{k} E_{V_\delta^j,\varphi} \cap \bigcap_{j=1}^{k} E_{D\cap W_j,\varphi}. \tag{4.2}$$

From Proposition 2 we have

$$E_{D\cap W_j} = \Sigma_m \quad \text{for } j = 1, \ldots, k. \tag{4.3}$$

Let $S_j \in \mathrm{GL}(2,\mathbb{R})$ be such that $S_j(L_j)$ is the y_1-axis. Remark 1(ii) says that $E_{V_\delta^j,\varphi} = E_{V_\delta,\varphi\circ S_j}$. Our aim now is to show that

$$E_{V_\delta,\varphi\circ S_j} = \Sigma_{\max(m,\alpha_j+2)} \quad \text{for } j = 1, \ldots, k. \tag{4.4}$$

For each $j = 1, \ldots, k$, let l, r and α be as in Lemma 1, with $\varphi \circ S_j$ in place of φ. Then $\alpha = \alpha_j$. Also (since $\det \varphi''$ is not identically zero), $l < m$. If $l = 0$ and $r = m - 1$ then $\alpha = m - 2$ and so $\max(m, \alpha + 2) = m$ in this case. Moreover, $\varphi(y_1, y_2) = (ay_1 + by_2)^m + dy_2^m$ for some $a, b, d \in \mathbb{R}$ with $a \neq 0$ and $d \neq 0$. Then there exists $S \in \mathrm{GL}(2,\mathbb{R})$ such that $(\varphi \circ S)(y) = y_1^m \pm y_2^m$ and so E_μ coincides with the type set corresponding to the measure associated to the function $y \mapsto y_1^m \pm y_2^m$. Then Theorem 3.12 in [3] gives $E_\mu = \Sigma_m$. Since $E_\mu \subset E_{V_\delta,\varphi\circ S_j}$ and also, by Lemma 2, $E_{V_\delta,\varphi\circ S_j} \subset \Sigma_m$, we obtain (4.4) in this case.

If $l = 0$ and $r \leq m - 2$ then $\alpha < m - 2$ and so Proposition 1 gives (4.4) in this case.

If $1 \leq l < m$ then $\alpha = 2l - 2 < m - 2$. Also, our hypothesis on φ implies that $l \neq m/2$ and so Proposition 1 gives (4.4) in this case.

Now, Theorem 1 follows from (4.2)–(4.4). ∎

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