COLLOQUIUM MATHEMATICUM

VOL. 106 2006 NO. 2

CONSTRUCTING SPACES OF ANALYTIC FUNCTIONS THROUGH BINORMALIZING SEQUENCES

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Abstract. H. Jiang and C. Lin [Chinese Ann. Math. 23 (2002)] proved that there exist infinitely many Banach spaces, called *refined Besov spaces*, lying strictly between the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and $\bigcup_{t>s} B^t_{p,q}(\mathbb{R}^n)$. In this paper, we prove a similar result for the analytic Besov spaces on the unit disc $\mathbb{D}$. We base our construction of the intermediate spaces on operator theory, or, more specifically, the theory of symmetrically normed ideals, introduced by I. Gohberg and M. Krein. At the same time, we use these spaces as models to provide criteria for several types of operators on H^2, including Hankel and composition operators, to belong to certain symmetrically normed ideals generated by binormalizing sequences.

1. Introduction. Let $\mathcal{S}$ be the Schwartz space of rapidly decreasing C^∞ functions φ on $\mathbb{R}^n$:

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^N \sum_{|l|\le N} |D^l\varphi(x)| < \infty, \qquad N=1,2,\ldots,$$

where $D^l = \partial^{|l|}/\partial x_1^{l_1}\cdots\partial x_n^{l_n}$ with $|l| = \sum_{j=1}^n l_j$. Also, consider a system $\{\varphi_j\}_{j=0}^\infty$ of functions in $\mathcal{S}$ satisfying

(a) $\operatorname{supp}(\varphi_0) \subseteq \{y\in\mathbb{R}^n : |y|\le 2\}$;
(b) $\operatorname{supp}(\varphi_j) \subseteq \{y\in\mathbb{R}^n : 2^{j-1}\le |y|\le 2^{j+1}\}$, $j=1,2,\ldots$;
(c) for each $l=(l_1,\ldots,l_n)$, there exists $C_l>0$ such that $|D^l\varphi_j(x)| \le C_l 2^{-j|l|}$ for all $x\in\mathbb{R}^n$;
(d) $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x\in\mathbb{R}^n$.

Let $\mathcal{S}'$ be the dual space of $\mathcal{S}$, consisting of all tempered distributions on $\mathbb{R}^n$. For $-\infty<s<\infty$ and $0<p,q\le\infty$, the *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ is defined by

$$B^s_{p,q}(\mathbb{R}^n) = \{f\in\mathcal{S}' : \|f\|_{B^s_{p,q}} < \infty\},$$

2000 *Mathematics Subject Classification*: 30H05, 47B100.

Key words and phrases: analytic function spaces, operator ideals.

This article is part of the second author's doctoral thesis, written under the direction of Mark C. Ho at National Sun Yat-Sen University.

where

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j=0}^{\infty} \|2^{sj}\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|^q_{L^p}\Big)^{1/q},$$

and $\mathcal{F}$, $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms, respectively. These Besov spaces were introduced by J. Peetre in 1967 [14] and 1973 [15], as a generalization of the Lipschitz spaces [2] and the Zygmund spaces [22]. It can be shown that the definition above does not depend on any particular system of functions satisfying conditions (a)–(d), and that $B^s_{p,q} \subset B^t_{p,q}$ if $s > t$. For the details on these spaces we refer to [18].

In their paper [11], H. Jiang and C. Lin introduce the so-called *refined Besov spaces*: Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k) \in \mathbb{R}^{k+1}$, $0 < p, q \le \infty$, and $\{\varphi_j\}_{j=0}^{\infty}$ be a system of functions in $\mathcal{S}$ satisfying (a)–(d). Also for $r > 0$, let

$$r^{\alpha}_{\ln} = r^{\alpha_0}|\ln r|^{-\alpha_1}|\ln|\ln r||^{-\alpha_2}\cdots\underbrace{|\ln\cdots|\ln|\ln|\ln r|||\cdots|}_{k \text{ logarithms}}{}^{-\alpha_k}.$$

Then the *refined Besov space* $RB^{\alpha}_{p,q}(\mathbb{R}^n)$ is given by

$$RB^{\alpha}_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}' : \|f\|_{RB^{\alpha}_{p,q}} < \infty\},$$

where

$$\|f\|_{RB^{\alpha}_{p,q}} = \|\mathcal{F}^{-1}(\varphi_0 \mathcal{F} f)\|_{L^p} + \Big(\sum_{j=0}^{\infty} \|(2^j)^{\alpha}_{\ln}\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|^q_{L^p}\Big)^{1/q}$$

if $0 < q < \infty$ and, for $q = \infty$,

$$\|f\|_{RB^{\alpha}_{p,\infty}} = \sup_{j\ge 1}\{\|\mathcal{F}^{-1}(\varphi_0 \mathcal{F} f)\|_{L^p} + \|(2^j)^{\alpha}_{\ln}\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{L^p}\}.$$

It can be shown ([11, Theorem 3.1]) that $RB^{\alpha}_{p,q}$ is independent of the choice of the system $\{\varphi_j\}_{j=0}^{\infty}$, and, most importantly, one has ([11, Theorem 4.1])

$$B^s_{p,q} \supsetneq RB^{\alpha}_{p,q} \supsetneq \bigcup_{t>s} B^t_{p,q} \quad \text{ for } \alpha = (s, \alpha_1),\ -2/q < \alpha_1 \le -1/q.$$

In this article, we shall prove an “analytic” version of the above result, namely, we will show (Theorem 3.8) that there are infinitely many Banach spaces embedded strictly between the *analytic Besov spaces* $B_p(\mathbb{D})$ and $\bigcap_{q>p} B_q(\mathbb{D})$, where

$$B_p(\mathbb{D}) = \Big\{f \text{ analytic on } \mathbb{D} : \int_{\mathbb{D}} |f''(z)|^p (1-|z|^2)^{2p-2}\,dx\,dy < \infty\Big\}, \quad p \ge 1.$$

In particular, we will base our construction of the intermediate spaces on operator theory, or, more specifically, the theory of symmetrically normed ideals of operators on Hilbert space. Furthermore, we shall use these spaces as models to study criteria for several types of operators on the Hardy space $H^2 = H^2(\mathbb{D})$, including Hankel, composition and Toeplitz operators, to belong to so-called symmetrically normed ideals generated by binormalizing sequences, which also generalizes the results in [9].

2. Symmetrically normed ideals generated by binormalizing sequences. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}$ be the space of compact operators on $\mathcal{H}$. For $T \in \mathcal{K}$, the *singular values* of T is a decreasing sequence $\{\lambda_n(T)\}$ of positive numbers defined by

$$\lambda_n(T) = \inf\{\|T - S\| : \operatorname{rank}(S) < n\}.$$

Clearly $\lambda_n(T) \searrow 0$ since T is compact. There are other ways to describe the $\lambda_n(T)$'s. For instance, one can easily show that the $\lambda_n(T)$'s are in fact the eigenvalues of $|T| = (T^*T)^{1/2}$.

On the other hand, let c_0 be the space of real sequences which converge to 0, and set $\widehat{c} = \{(x_1, x_2, \dots) \in c_0 : x_k = 0$ for all but finitely many $k\}$. A function $\varPhi : \widehat{c} \to \mathbb{R}$ is called a *symmetric norming function* if

(a) $\varPhi(x) > 0$ for $x \in \widehat{c}$, $x \neq 0$;
(b) $\varPhi(\alpha x) = |\alpha|\varPhi(x)$ for any $\alpha \in \mathbb{R}$, $x \in \widehat{c}$;
(c) $\varPhi(x + y) \leq \varPhi(x) + \varPhi(y)$, $x, y \in \widehat{c}$;
(d) $\varPhi(1, 0, 0, \dots) = 1$;
(e) $\varPhi(x_1, \dots, x_n, 0, 0, \dots) = \varPhi(x_{\sigma(1)}, \dots, x_{\sigma(n)}, 0, 0, \dots)$ for any n and any permutation $\sigma(1), \dots, \sigma(n)$ of $1, \dots, n$.

Now consider $x = (x_1, x_2, \dots) \in c_0$. Write $x^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$ and define

$$c_\varPhi = \{x \in c_0 : \sup_n \varPhi(x^{(n)}) < \infty\}.$$

Given $T \in \mathcal{K}$, we say that T is in the *symmetrically normed ideal* $\mathfrak{S}_\varPhi$ if $(\lambda_1(T), \lambda_2(T), \dots) \in c_\varPhi$. We endow $\mathfrak{S}_\varPhi$ with the norm

$$\|T\|_\varPhi = \sup_n \varPhi((\lambda_1(T), \lambda_2(T), \dots)^{(n)}).$$

For example, the usual Schatten p-class $\mathfrak{S}_p$ $(p \geq 1)$ is the symmetrically normed ideal with norming function

$$\varPhi_p(x) = \Big(\sum_{n=1}^{n} |x_k|^p\Big)^{1/p}, \qquad x = (x_1, \dots, x_n, 0, \dots) \in \widehat{c}.$$

In this article, we shall focus on symmetrically normed ideals given by norming functions of the form

$$\varPhi(x) = \sup_n \frac{\sum_{k=1}^n x_k^*}{\sum_{k=1}^n \pi_k}, \qquad x = (x_1, \dots, x_n, 0, \dots) \in \widehat{c},$$

where $x_1^*, \dots, x_n^*$ is the rearrangement of $|x_1|, \dots, |x_n|$ in descending order, and $\pi_1 \geq \pi_2 \geq \cdots \geq 0$ with $\pi_1 = 1$. In fact, we will only consider π_k's such that $\pi_k \searrow 0$ and $\sum \pi_k = \infty$, called *binormalizing sequences*. The interested reader can find a detailed discussion on this subject in [7].

Let $\sum \pi_n$ be a divergent series with $\pi_n \searrow 0$ and $\pi_1 = 1$, and let $L(n) = \sum_{k=1}^n \pi_k$. Then the norming function defined above can be written as

$$\Phi_L(x) = \sup_n \frac{1}{L(n)} \sum_{k=1}^n x_k^*, \qquad x = (x_1, \dots, x_n, 0, \dots) \in \widehat{c}.$$

We are interested in the case when $\mathfrak{S}_1 \subsetneq \mathfrak{S}_{\Phi_L}$ and $\mathfrak{S}_{\Phi_L} \subsetneq \mathfrak{S}_p$ for all $p > 1$. Choose a nondecreasing function ≥ 1 on $[0, \infty)$ whose value at n is $L(n)$ for each positive integer n, and therefore we shall denote this function by $L(t)$. It is clear, from the definition of $L(t)$ and the fact that $\pi_k \searrow 0$, that there is, for each $x_0 > 0$ and $y_0 \geq 1$, a $\kappa > 0$ (which may depend on x_0 and y_0) such that $\kappa^{-1} L(t) \leq L(y_0 + x_0 t) \leq \kappa L(t)$ for all $t \geq 0$. On the other hand, let $a(t)$ be a strictly decreasing function on $[0, \infty)$ so that $a(t) \searrow 0$ as $t \to \infty$. Given the pair $\{L(t), a(t)\}$ above, we say that the pair is *regular* if $a(t_0) = 1$ for some $t_0 > 0$ and there is a $C > 0$ such that:

$\mathfrak{R}1$. for every $\lambda_n \searrow 0$ such that $\sum_{k=1}^n \lambda_k = O(L(n))$, $\limsup_{t\to\infty} \beta(t) < \infty$, where $\beta(t) = L(N_t)/L(t)$ and $N_t = \max\{n : \lambda_n > a(t)\}$;

$\mathfrak{R}2$. $\limsup_{t\to\infty} \alpha(t)^x L(t) = O(L(\beta(e^{-x^{-1}})))$ for all $x > 0$ and $\alpha(n)^C \leq \pi_n$ for all n;

$\mathfrak{R}3$. the function $\alpha(u,t)$, defined by $L(b(u^t)) = \alpha(u,t) L(b(u))$ for $u, t > 0$, satisfies

$$\limsup_{u\to 0} \int_0^\infty e^{-t} \alpha(u,t)\, dt \leq C.$$

Here $b(s)$ is the inverse of $s = a(t)$.

The following result shows that if there is a function $a(t)$ strictly decreasing to 0 so that the pair $\{L(t), a(t)\}$ is regular, then we can define $\mathfrak{S}_{\Phi_L}$ in terms of the zeta function of the singular values of the operators in $\mathfrak{S}_{\Phi_L}$. The proof will be omitted.

THEOREM 2.1. *Let $\lambda_n \searrow 0$. If there is a function $a(t)$ strictly decreasing to 0 as $t \to \infty$ such that $\{L(t), a(t)\}$ is regular then the following conditions are equivalent*:

(i) $\displaystyle \limsup_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^n \lambda_k < \infty$,

(ii) $\displaystyle \limsup_{s\to 1} \frac{1}{L(b(e^{-(s-1)^{-1}}))} \sum_{n=1}^\infty \lambda_n^s < \infty$.

Moreover, if $\lim_{t\to\infty} a(t)^x L(t) \to 0$ *for all* $x > 0$ *and* $\lim_{x\to 0} \int_0^\infty e^{-t} \alpha(x,t)\, dt = \nu$ *then convergence of the sequence in* (i) *implies convergence in* (ii), *and* *we have, in this case,*

$$\nu \lim_{n\to\infty} \frac{1}{L(n)} \sum_{k=1}^{n} \lambda_k = \lim_{s\to 1} \frac{1}{L(b(e^{-(s-1)^{-1}}))} \sum_{n=1}^{\infty} \lambda_n^s.$$

By simply replacing λ_k with λ_k^p in Theorem 2.1, we obtain

COROLLARY 2.2. *Let $\pi_k \searrow 0$ and $L(n) = \sum_{k=1}^n \pi_k$ and define the symmetric norming function*

$$\Phi_L(x) = \sup_n \left(\frac{1}{L(n)} \sum_{k=1}^{n} x_k^{*p} \right)^{1/p}, \qquad x = (x_1, \ldots, x_n, 0, \ldots) \in \widehat{c}.$$

Assume that $\{L(t), a(t)\}$ is regular. Then $T \in \mathfrak{S}_{\Phi_L}$ if and only if

$$\limsup_{s\searrow p} \frac{1}{L(b(e^{-(s-p)^{-1}}))} \sum_{n=1}^{\infty} \lambda_n(T)^s < \infty.$$

REMARK. The statement about convergence in Theorem 2.1 is a typical result of the kind that Cesàro summability implies Abel summability. On the other hand, there are more specific conditions on L so that convergence in (ii) implies convergence in (i) (i.e., Abel summability implies Cesàro summability, or, Tauberian theory). For instance, this is the case when $L(t) = 1 + (\log t)^\gamma F(t)$, where $F(t^x)/F(t) \to 1$ as $t \to \infty$, for every $x > 0$. The reader can find details on this subject in, for example, [8].

EXAMPLES. Let

$$L(t) = 1 + (\log t)^\gamma, \qquad \gamma > 0,\ t \geq 1.$$

Let $\lambda_n \searrow 0$ be such that $\sum_{k=1}^n \lambda_k = O(L(n))$, $n \geq 1$. Since it is possible to find $t_0 > 0$ so that $\gamma t^{-1}(\log t)^{\gamma-1}$ is strictly decreasing and bounded above by 1 if $t \geq t_0$, we may choose a sequence $\pi_k \searrow 0$ strictly such that $\pi_1 = 1$ and $\pi_k = \gamma k^{-1}(\log k)^{\gamma-1}$, $k \geq k_0$ for some $k_0 \in \mathbb{N}$. Obviously $\sum_{k=1}^n \pi_k = O(L(n))$.

Now consider $a(t) \searrow 0$ strictly so that $a(k) = \pi_k$ for all k and $a(t) = \gamma t^{-1}(\log t)^{\gamma-1}$ if $t \geq k_0$. Then $N_t = \max\{k : \mu_k > a(t)\} \geq [t] - 1$ for $t \geq 1$, where $\mu_k = \lambda_k + \pi_k$ and $[t]$ is the greatest integer $\leq t$. Set $\beta(t) = (L(N_t) - 1)/(L(t) - 1)$. Then for large t (say, $t \geq k_0$), we have

$$\frac{1}{L(N_t)} \sum_{n=0}^{N_t} \mu_n = \frac{1}{L(N_t)} \Big(\sum_{n=0}^{[t]-1} \mu_n + \sum_{n=[t]}^{N_t} \mu_n \Big) \geq a(t) \frac{N_t - [t] + 1}{L(N_t)}$$
$$\geq \gamma \frac{t^{\beta(t)^{\gamma^{-1}}-1} - t^{-1}[t] + t^{-1}}{\log t^{\beta(t)}}.$$

This implies that $\limsup_{t\to\infty} \beta(t) \leq 1$ since $\sum_{k=1}^n \mu_k = O(L(n))$. Also, given $x > 0$, $a(t)^x L(t) \to 0$ as $t \to \infty$.

Next, fix $t > 0$. Given $0 < \varepsilon < 1$ and $\sigma > 1$, since $y^{-\sigma} \le a(y) \le y^{-\varepsilon}$ for sufficiently large y, we have $s^{-\sigma^{-1}} \le b(s) \le s^{-\varepsilon^{-1}}$ for sufficiently small s. Therefore

$$\frac{t^\gamma \varepsilon^\gamma}{\sigma^\gamma} \le \frac{L(b(u^t))}{L(b(u))} \le \frac{t^\gamma \sigma^\gamma}{\varepsilon^\gamma}$$

for $0 < u < 1$ sufficiently small. This leads to

$$\lim_{u\to 0} \int_0^\infty e^{-t}\alpha(u,t)\,dt = \int_0^\infty e^{-t} t^\gamma\,dt = \Gamma(\gamma+1),$$

and therefore $\{L(t), a(t)\}$ is regular. In fact, given $0 < \varepsilon < 1$ and $\sigma > 1$, since $e^{\sigma^{-1}t} \le b(e^{-t}) \le e^{\varepsilon^{-1}t}$ for sufficiently large t,

$$\sigma^{-\gamma}(\log t)^\gamma \le (\log(b(e^{-t})))^\gamma \le \varepsilon^{-\gamma}(\log t)^\gamma \quad \text{for large } t.$$

This means that $(s-1)^\gamma / L(b(e^{-(s-1)^{-1}})) \to 1$ since ε and σ are arbitrary. Therefore $T \in \mathfrak{S}_L$ if and only if

$$\limsup_{s\searrow 1} (s-1)^\gamma \sum_{n=1}^\infty \lambda_n(T)^s < \infty.$$

Here we refer the reader to Theorem 108 of [8].

3. Analytic functions on $\mathbb{D}$ related to symmetrically normed ideals. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $z, \omega \in \mathbb{D}$. Let $\varrho(z,\omega)$ be the hyperbolic distance between z and ω, i.e.,

$$\varrho(z,\omega) = \frac{1}{2}\log\frac{1 + \left|\frac{z-\omega}{1-\bar{z}\omega}\right|}{1 - \left|\frac{z-\omega}{1-\bar{z}\omega}\right|}.$$

Also, for $r > 0$, we denote by $D(z,r)$ the hyperbolic ball with center at z and radius r, i.e., $D(z,r) = \{\omega : \varrho(z,\omega) < r\}$ and, for a set $\mathcal{X} \subseteq \mathbb{D}$, let $E_r(\mathcal{X}) = \bigcup_{z\in\mathcal{X}} D(z,r)$. On the other hand, let $K(z,\omega) = (1 - z\overline{\omega})^{-2}$ be the Bergman kernel on $\mathbb{D}$ and define $\eta(\mathcal{X}) = 1 + \int_{\mathcal{X}} K(z,z)\,dv(z)$, where $dv(z) = (1/\pi)dxdy$ is the normalized Lebesgue area measure on $\mathbb{D}$. The following are some useful facts concerning the hyperbolic metric.

Given $r, s > 0$, there is a $C > 0$ depending only on r and s so that:

- **F1.** $C^{-1}(1-|a|^2)^2 \le |D(z,r)| \le C(1-|a|^2)^2$, where $|D(z,r)|$ is the area of $D(z,r)$, for all $z \in D(a,r)$ and $a \in \mathbb{D}$.
- **F2.** $C^{-1}|D(z,r)| \le |D(\omega,s)| \le C|D(z,r)|$ if $\beta(\omega,z) < r$.
- **F3.** $C^{-1}K(a,\omega) \le K(z,\omega) \le CK(a,\omega)$ for all $\omega \in \mathbb{D}$ if $z \in D(a,r)$.
- **F4.** (Subnormality)

$$h(a) \le \frac{C}{|D(a,r)|}\int_{D(a,r)} h(z)\,dv(z)$$

for any nonnegative subharmonic function h on $\mathbb{D}$.

F5. There is a sequence $\{\omega_n\}$ in $\mathbb{D}$ and measurable sets $D_n \subseteq \mathbb{D}$ so that (1) $|\omega_n| \to 1$ and $\bigcup_{n=1}^{\infty} D_n = \mathbb{D}$, (2) $D(\omega_n, r/4) \subseteq D_n \subseteq D(\omega_n, r)$ for $n \geq 1$, (3) $D_n \cap D_m = \emptyset$ if $n \neq m$ and (4) there is an $N \in \mathbb{N}$ depending only on r such that any z in $\mathbb{D}$ belongs to at most N of the sets $\{D(\omega_n, 2r)\}$.

The reader can find details about these properties in, for instance, [21].

Now let $\pi_1 = 1$, $\pi_k \searrow 0$ and $L(t) > 0$ be nondecreasing continuous for $t \geq 1$ so that $L(n) = \sum_{k=1}^{n} \pi_k$ for each n. Fix $r > 0$, $p \geq 1$ and consider the space

$$\left\{ f \text{ analytic on } \mathbb{D} : \sup_{\mathcal{X} \subset\subset D} \frac{1}{L(\eta(E_r(\mathcal{X})))} \int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p}\, dv(z) < \infty \right\},$$

where $\mathcal{X} \subset\subset \mathbb{D}$ means $\mathcal{X}$ is a compact subset of $\mathbb{D}$. We denote this space by $\widetilde{B}_{L,p}(\mathbb{D})$.

Proposition 3.1. *Given $r > 0$, the space $\widetilde{B}_{L,p}(\mathbb{D})$ is a Banach space with norm defined by*

$$\|f\| := \sup_{\mathcal{X} \subset\subset \mathbb{D}} \left(\frac{1}{L(\eta(E_r(\mathcal{X})))} \int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p}\, dv(z) \right)^{1/p}.$$

Furthermore, $f(z)K(z,z)^{-1} \to 0$ as $|z| \to 1$ if $f \in \widetilde{B}_{L,p}(\mathbb{D})$.

Proof. Let $z \in \mathbb{D}$. Then there is a $C > 0$, depending only on r, so that

(a) $C^{-1}K(z,z) \leq K(\zeta,\zeta) \leq CK(z,z)$ and $C^{-1}(1-|z|^2)^2 \leq |D(\zeta,s)| \leq C(1-|z|^2)^2$ if $\zeta \in D(z,2r)$, $s = r$ or $2r$,

(b) $h(z) \leq (C/|D(z,r)|) \int_{D(z,r)} h\, dv$ for any subharmonic function $h \geq 0$.

It follows that for any $f \in \widetilde{B}_{L,p}$ we have

$$\begin{aligned}
|f(z)|^p &\leq \frac{C}{|D(z,r)|} \int_{D(z,r)} |f(\zeta)|^p\, dv(\zeta) \\
&\leq C^{3p-2}(1-|a|^2)^{2p-4} \int_{D(z,r)} |f(\zeta)|^p K(\zeta,\zeta)^{1-p}\, dv(\zeta) \\
&\leq \frac{C^{3p-2}(1-|a|^2)^{2p-4} L(1+C^2)}{L(\eta(D(z,2r)))} \int_{D(z,r)} |f(\zeta)|^p K(\zeta,\zeta)^{1-p}\, dv(\zeta) \\
&= \frac{C^{3p-2}(1-|a|^2)^{2p-4} L(1+C^2)}{L(\eta(E_r(D(z,r))))} \int_{D(z,r)} |f(\zeta)|^p K(\zeta,\zeta)^{1-p}\, dv(\zeta) \\
&\leq C^{3p-2}(1-|a|^2)^{2p-4} L(1+C^2) \|f\|^p
\end{aligned}$$

for all $z \in D(a,r)$. This means that for any Cauchy sequence $\{f_n\}$ in $\widetilde{B}_{L,p}$, $\{f_n\}$ is uniformly Cauchy on any compact subset of $\mathbb{D}$. So there is an f

analytic on $\mathbb{D}$ such that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. As a consequence, we see that

$$\sup_{\mathcal{X}\subset\subset\mathbb{D}} \frac{1}{L(\eta(E_r(\mathcal{X})))} \int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p} dv(z) \le M,$$

where $\|f_n\|^p \le M$ for all n. Therefore $f \in \widetilde{B}_{L,p}$. This proves the completeness of $\widetilde{B}_{L,p}$.

Now let $f \in \widetilde{B}_{L,p}$. Suppose that $f(z)K(z,z)^{-1} \not\to 0$ as $|z| \to 1$. Then there exist $\varepsilon > 0$ and a sequence $\{\omega_n\}$ in $\mathbb{D}$ such that $\beta(\omega_n, \omega_m) > 2r$ if $m \ne n$ and $|f(\omega_n)|K(\omega_n, \omega_n)^{-1} \ge \varepsilon$ for all n. Since the sets $\{D(\omega_n, 2r)\}$ are pairwise disjoint,

$$\|f\| \ge \limsup_{n\to\infty} \frac{n\varepsilon^p}{CL(1+nC)}.$$

However, since L is nondecreasing, and $L(n) = \sum_{k=1}^n \pi_k$ with $\pi_n \searrow 0$, the sequence $n\varepsilon^p/CL(1+nC)$ cannot be bounded, a contradiction. Hence we must have $f(z)K(z,z)^{-1} \to 0$ as $|z| \to 1$ if $f \in \widetilde{B}_{L,p}$. ∎

We will now seek alternative characterizations for functions in $\widetilde{B}_{L,p}$, in particular when $\{L(t), a(t)\}$ is regular for some $a(t)$. First, given $r > 0$, let us recall the decomposition $\{D_n\}$ of $\mathbb{D}$ (with respect to r) mentioned earlier in **F5** and let $\mathfrak{D}$ denote the collection of all possible finite unions of D_n's.

PROPOSITION 3.2. *Let $\pi_k \searrow 0$, $\pi_1 = 1$ and $L(t)$ be a nondecreasing function on $[1,\infty)$ with $L(n) = \sum_{k=1}^n \pi_k$ for all n. Then given $r \ge 4s > 0$, $p \ge 1$, and f analytic on $\mathbb{D}$, the following are equivalent*:

(i) $\displaystyle\limsup_{\mathcal{X}\subset\subset D} \frac{1}{L(\eta(E_s(\mathcal{X})))} \int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p}\, dv(z) < \infty.$

(ii) $\displaystyle\limsup_{\mathcal{D}\in\mathfrak{D}} \frac{1}{L(\eta(\mathcal{D}))} \int_{\mathcal{D}} |f(z)|^p K(z,z)^{1-p}\, dv(z) < \infty.$

Proof. As usual, we may choose a $C > 0$, depending only on r and s, so that

$$C^{-1}K(z,z) \le K(\zeta,\zeta) \le CK(z,z)$$

and

$$C^{-1}(1-|z|^2)^2 \le |D(\zeta,t)| \le C(1-|z|^2)^2 \quad \text{if } \zeta \in D(z,t),$$

for $t = r/4,\ r,\ 2r$ and s. Therefore, if $\mathcal{D} = \bigcup_{k=1}^n D_{n_k} \in \mathfrak{D}$, then $1 + C^{-1}n \le \eta(\mathcal{D}) \le 1 + Cn$.

Now let $\mathcal{X}$ be a compact subset of $\mathbb{D}$ and choose $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{X} \subseteq \mathcal{D}$ and $\mathcal{X} \not\subseteq \mathcal{D}'$ for any $\mathcal{D}' \subsetneq \mathcal{D}$. Let $\mathcal{D} = \bigcup_{k=1}^n D_{n_k}$. By the choice of $\mathcal{D}$, $\mathcal{X} \cap D_{n_k} \ne \emptyset$ for each k. Pick $z_k \in \mathcal{X} \cap D_{n_k}$ for $k = 1, \ldots, n$. Let N be the positive integer such that every z in $\mathbb{D}$ is covered by at most N of the sets

$\{D(\omega_n, 2r)\}$ (see **F5**). Then, since $D(z_k, s) \subseteq D(\omega_{n_k}, 2r)$ for each k, one has

$$(1) \qquad \int\limits_{\bigcup_{k=1}^n D(z_k,s)} g\,dv \le \sum_{k=1}^n \int\limits_{D(z_k,s)} g\,dv \le N \int\limits_{\bigcup_{k=1}^n D(z_k,s)} g\,dv$$

for any nonnegative measurable function g on $\mathbb{D}$. With $g(z) = K(z,z)$, this gives

$$\eta\Big(\bigcup_{k=1}^n D(z_k, s)\Big) \le 1 + \sum_{k=1}^n \int\limits_{D(z_k,s)} K(z,z)\,dv(z) \le N\eta\Big(\bigcup_{k=1}^n D(z_k, s)\Big),$$

which implies

$$(2) \quad C^{-2}\eta\Big(\bigcup_{k=1}^n D(z_k, s)\Big) \le \eta(\mathcal{D}) \le C^2 N\eta\Big(\bigcup_{k=1}^n D(z_k, s)\Big) \le C^2 N\eta(E_s(\mathcal{X})).$$

Hence there is a $\kappa > 0$, depending only on r and s, so that

$$\frac{1}{L(\eta(E_s(\mathcal{X})))} \int\limits_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p}\,dv(z) \le \frac{\kappa}{L(\eta(\mathcal{D}))} \int\limits_{\mathcal{D}} |f(z)|^p K(z,z)^{1-p}\,dv(z)$$

since $L(1 + x_0 t)/L(t)$ is bounded for $t \ge 1$ and every $x_0 > 0$. This means (ii) implies (i).

Conversely, since the hyperbolic metric β is invariant under Möbius transformations, there exists an integer $M > 0$ such that in any hyperbolic disc with radius $2r$ there can be at most M points which are at least $r/2$ apart (in the hyperbolic metric). This means that in every disc $D(z, 2r)$, there are at most M ω_n's (see **F5**). Consequently, for every $\mathcal{D} \in \mathfrak{D}$, there exists $\mathcal{D}_0 \in \mathfrak{D}$ such that (a) $\mathcal{D} \subseteq \mathcal{D}_0$; (b) $E_s(\overline{\mathcal{D}}) \subseteq \overline{\mathcal{D}}_0$ if $s \le r/4$, where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$; (c) if $\mathcal{D}$ is the union of k of the D_n's, then $\mathcal{D}_0$ is the union of at most kM of the D_n's. Hence, for $0 < s \le r/4$, we have

$$1 + C^{-1}k \le \eta(\mathcal{D}) \le \eta(E_s(\overline{\mathcal{D}})) \le \eta(\mathcal{D}_0) \le 1 + kCM,$$

and therefore

$$\frac{1}{L(\eta(\mathcal{D}))} \int\limits_{\mathcal{D}} |f(z)|^p K(z,z)^{1-p}\,dv(z) \le \frac{\kappa'}{L(\eta(E_s(\overline{\mathcal{D}})))} \int\limits_{\overline{\mathcal{D}}} |f(z)|^p K(z,z)^{1-p}\,dv(z)$$

for some $\kappa' > 0$. This means that (i) implies (ii). ∎

An immediate consequence of Proposition 3.2 is

Corollary 3.3. *The definition of $\widetilde{B}_{L,p}$ does not depend on r or the decomposition $\{D_n\}$.*

Let $\varepsilon > 0$. The *capacity function* $\mathbf{C}(\varepsilon, \mathcal{X})$ of $\mathcal{X} \subseteq \mathbb{D}$ with respect to the hyperbolic metric is the maximum number of points $z_1, z_2, \ldots$ in $\mathcal{X}$ such that the distance of any two distinct z_j is at least ε:

$$\beta(z_i, z_j) \ge \varepsilon, \qquad i \ne j.$$

It is clear that if $\mathcal{X}$ is bounded away from $\partial\mathbb{D}$, or bounded in terms of the hyperbolic metric, then $\mathbf{C}(\varepsilon,\mathcal{X})<\infty$. Given such an $\mathcal{X}$ and an $r>0$, we can always find a sequence $\{\omega_n\}$ in $\mathbb{D}$ and pairwise disjoint measurable sets $\{D_n\}$ satisfying the condition in **F5**, such that the number of ω_n's in $\mathcal{X}$ equals $\mathbf{C}(r/2,\mathcal{X})$. Therefore, as a consequence of Proposition 3.2, we obtain

PROPOSITION 3.4. *Let $\pi_k \searrow 0$, $\pi_1=1$ and $L(t)$ be a nondecreasing function on $[1,\infty)$ with $L(n)=\sum_{k=1}^n \pi_k$ for all n. Then given $r>0$ and $p\geq 1$, f belongs to $\widetilde{B}_{L,p}$ if and only if*

$$\sup_{\emptyset\neq\mathcal{X}\subset\subset\mathbb{D}} \frac{1}{L(\mathbf{C}(r/2,\mathcal{X}))}\int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p}\,dv(z)<\infty.$$

In the theory of symmetrically normed ideals, it is known ([7, p. 150]) that if $\sum_{k=1}^\infty \pi_k^p<\infty$ for some $1\leq p<\infty$, then $T\in\mathfrak{S}_{\Phi_L}$ implies $T\in\mathfrak{S}_p$, and we have

$$\|T\|_{\mathfrak{S}_p}\leq\Big(\sum_{k=1}^\infty \pi_k^p\Big)^{1/p}\|T\|_{\Phi_L}.$$

In the next result, we show that similar properties hold for $\widetilde{B}_{L,p}$:

PROPOSITION 3.5. *Let $\{\pi_k\}$ be a binormalizing sequence and let $L(t)$ be a nondecreasing function on $[1,\infty)$ such that $L(n)=\sum_{k=1}^n\pi_k$ for all n. Suppose that $\sum_{k=1}^\infty \pi_k^p<\infty$ for some $1\leq p<\infty$. Then $\widetilde{B}_{L,1}\subseteq\widetilde{B}_p$, where*

$$\widetilde{B}_p=\Big\{f \text{ analytic on } \mathbb{D}: \int_{\mathbb{D}}|f(z)|^pK(z,z)^{1-p}\,dv(z)<\infty\Big\}.$$

Proof. We will make use of the following ([7, Lemma 15.2]):

Let $\widehat{k}_n=\{(\xi_1,\dots,\xi_n):\xi_1\geq\cdots\geq\xi_n\geq 0\}$, and $\{\pi_k\}$ be a sequence of positive numbers. Then

$$\sum_{k=1}^n \eta_k\xi_k\leq\sum_{k=1}^n\pi_k\xi_k\cdot\sup_m\frac{\sum_{k=1}^m\eta_k}{\sum_{k=1}^m\pi_k}$$

for all $(\eta_1,\dots,\eta_n),(\xi_1,\dots,\xi_n)\in\widehat{k}_n$.

Now fix an $r>0$. There exists, as before, a $C>0$, depending on r, such that $C^{-1}K(z,z)\leq K(\zeta,\zeta)\leq CK(z,z)$ and $C^{-1}(1-|z|^2)^2\leq|D(\zeta,s)|\leq C(1-|z|^2)^2$ if $\zeta\in D(z,2r)$, $s=r$ or $2r$. Also, choose a decomposition $\{D_k\}$ of $\mathbb{D}$ as described in **F5**. By the mean value theorem, we can find $z_k\in\overline{D}_k$ for each k so that

$$\int_{D_k}|f(z)|^pK(z,z)^{-p}dv(z)=|f(z_k)|^pK(z_k,z_k)^{-p}|D_k|.$$

Therefore

$$\int_{D_k} |f(z)|^p K(z,z)^{1-p}\, dv(z) \le C^2 |f(z_k)|^p K(z_k,z_k)^{-p}.$$

Since $f(z)K(z,z)^{-1} \to 0$ (Proposition 3.1), we may assume, without loss of generality, that $\eta_k = |f(z_k)|K(z_k,z_k)^{-1} \searrow 0$. It is not difficult to see that for each n, there is a $(\xi_1,\dots,\xi_n) \in \widehat{k}_n$ such that

$$\sum_{k=1}^n \xi_k^q = 1, \qquad \sum_{k=1}^n \eta_k\xi_k = \Big(\sum_{k=1}^n \eta_k^p\Big)^{1/p}, \qquad \frac{1}{p}+\frac{1}{q} = 1.$$

Therefore

$$\int_{\mathbb{D}} |f(z)|^p K(z,z)^{1-p}\, dv(z) \le \kappa C^2 \Big(\sum_{k=1}^\infty \pi_k^p\Big) \|f\|^p_{\widetilde{B}_{L,1}}$$

for some $\kappa > 0$. This completes the proof. ∎

The above result establishes, at least in theory, the existence of Banach spaces between $\widetilde{B}_p$ and $\bigcap_{q>p} \widetilde{B}_q$. However, to show that strict embedding can occur, we need some tools to enable us to compute upper bounds for

$$\frac{1}{L(\eta(E_r(\mathcal{X})))} \int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p}\, dv(z)$$

more effectively. We shall do this when $\{L(t), a(t)\}$ is regular; in this case, we can also characterize $\widetilde{B}_{L,p}$ without the presence of r:

PROPOSITION 3.6. *Let $\pi_k \searrow 0$, $\pi_1 = 1$ and $L(t)$ be a nondecreasing function on $[1,\infty)$ with $L(n) = \sum_{k=1}^n \pi_k$ for all n. Assume that there exists $a(t) \searrow 0$ strictly as $t \to \infty$ such that $\{L(t), a(t)\}$ is regular. Let f be analytic on $\mathbb{D}$. Then the following are equivalent*:

(i) $\displaystyle \limsup_{\mathcal{X} \subset\subset \mathbb{D}} \frac{1}{L(\eta(E_r(\mathcal{X})))} \int_{\mathcal{X}} |f(z)|^p K(z,z)^{1-p} dv(z) < \infty.$

(ii) $\displaystyle \limsup_{q \searrow p} \frac{1}{L(b(e^{-(q-p)^{-1}}))} \int_{\mathbb{D}} |f(z)|^q K(z,z)^{1-q} dv(z) < \infty.$

Proof. Fix $r > 0$. Let $\{D_k\}$ be a decomposition of $\mathbb{D}$ with respect to r, as described in **F5**, and let ω_k be the corresponding sequence in $\mathbb{D}$. Recall from the proof of Proposition 3.2 that there exists an integer $M > 0$, depending on r only, so that for every disc $D(z,2r)$, there are at most M ω_n's belonging to $D(z,2r)$. Also, choose again a $C > 0$, depending only on r, so that

(a) $C^{-1}K(z,z) \le K(\zeta,\zeta) \le CK(z,z)$ and $C^{-1}(1-|z|^2)^2 \le |D(\zeta,s)| \le C(1-|z|^2)^2$ if $\zeta \in D(z,s)$, $s = r/4$, r, $2r$,

(b) $h(z) \le (C/|D(z,s)|)\int_{D(z,s)} h\, dv$ for any subharmonic function $h \ge 0$, if $s = r/4, r, 2r$.

Now, for each q such that $p \le q < 2p$, choose $z_{k,q} \in \overline{D}_{\omega_k}$ for each k so that

$$\int\limits_{D_k} |f(z)|^q K(z,z)^{-q}\, dv(z) = |f(z_{k,q})|^q K(z_{k,q}, z_{k,q})^{-q} |D_k|$$

(mean value theorem). Hence

$$\int\limits_{D_k} |f(z)|^q K(z,z)^{1-q} dv(z) \le C^2 |f(z_{k,q})|^q K(z_{k,q}, z_{k,q})^{-q}$$

for each k. On the other hand, set $\mu_k = \sup_{p \le q < 2p}\{|f(z_{k,q})| K(z_{k,q}, z_{k,q})^{-1}\}$. Since f is continuous on $\mathbb{D}$ and $\overline{D}_k$ is compact, there is a z_k in $\overline{D}_k$ so that $\mu_k = |f(z_k)| K(z_k, z_k)^{-1}$ for each k. Therefore,

$$\mu_k^q \le C^{3-q} \int\limits_{D(z_k, r/4)} |f(z)|^q K(z,z)^{1-q}\, dv(z)$$

for all $k \ge 1$ and $q \ge p$ since $|f|^q$ is subharmonic.

Now assume that (i) holds. Then by Propositions 3.1 and 3.2, $\mu_k \to 0$ as $k \to \infty$. Let $\{\zeta_j\}$ be a rearrangement of $\{z_k\}$ so that $\lambda_j = |f(\zeta_j)| K(\zeta_j, \zeta_j)^{-1} \searrow 0$, and let $\{D_{k_j}\}$, $j = 1, 2, \ldots$, be the corresponding rearrangement of $\{D_k\}$. So

$$\begin{aligned}
\frac{1}{L(2+n)} \sum_{j=1}^{n} \lambda_j^p &\le \frac{C^{3-p}}{L(2+n)} \sum_{j=1}^{n} \int\limits_{D(\zeta_j, r/4)} |f(z)|^p K(z,z)^{1-p}\, dv(z) \\
&\le \frac{C^{3-p} N}{L(2+n)} \int\limits_{\bigcup_{j=1}^{n} D(\zeta_j, r/4)} |f(z)|^p K(z,z)^{1-p}\, dv(z) \\
&\le \frac{C^{3-p} N}{L(1 + M^{-1} C^{-1} \eta(\mathcal{D}_0))} \int\limits_{\mathcal{D}_0} |f(z)|^p K(z,z)^{1-p}\, dv(z) \\
&\le \frac{C^{3-p} N \kappa}{L(\eta(\mathcal{D}_0))} \int\limits_{\mathcal{D}_0} |f(z)|^p K(z,z)^{1-p}\, dv(z),
\end{aligned}$$

where $\mathcal{D}_0 = \bigcup_{j=1}^{n} D_{k_j} \cup \{D_i : D_i \text{ is adjacent to some } D_{k_j}\}$ and $\kappa > 0$, since $1 + C^{-1} n \le \eta(\mathcal{D}) \le 1 + Cn$ if $\mathcal{D} = \bigcup_{j=1}^{n} D_{k_j}$ (by (1), (2) in the proof of Proposition 3.2), and $L(t)/L(1 + x_0 t)$ is bounded for $t \ge 1$ and any $x_0 \ge 0$. Therefore, by (the proof of) Theorem 2.1, and the fact that

$$\int\limits_{\mathbb{D}} |f(z)|^q K(z,z)^{1-q}\, dv(z) = \sum_{k=1}^{\infty} \int\limits_{D_k} |f(z)|^q K(z,z)^{1-q}\, dv(z) \le C^2 \sum_{j=1}^{\infty} \lambda_j^q,$$

a constant multiple of the lim sup in (i) dominates that in (ii).

Suppose now, conversely, that (ii) holds. Then

$$\int\limits_{\mathbb{D}} |f(z)|^q K(z,z)^{1-q}\, dv(z) < \infty$$

for all $q > p$. Therefore, $|f(z)|K(z,z)^{-1} \to 0$ as $|z| \to 1$. Now choose $z_k \in \overline{D}_k$ for each k so that

$$\int_{D_k} |f(z)|^p K(z,z)^{-p}\, dv(z) = |f(z_k)|^p K(z_k, z_k)^{-p} |D_k|$$

and consider $\mu_k = |f(z_k)|K(z_k,z_k)^{-1}$. Then $\mu_k \to 0$ as $k \to \infty$ and we have

$$\int_{D_k} |f(z)|^p K(z,z)^{1-p}\, dv(z) = \int_{D_k} |f(z)|^p K(z,z)^{-p} K(z,z)\, dv(z) \le C^2 \mu_k^p.$$

Also, since $|f|^q$ is subharmonic for $q \ge 1$,

$$\mu_k^q \le C^{3-q} \int_{D(z_k, r/2)} |f(z)|^q K(z,z)^{1-q}\, dv(z).$$

Now let $\{\zeta_j\}$ be a rearrangement of $\{z_k\}$ so that $\lambda_j = |f(\zeta_j)|K(\zeta_j,\zeta_j)^{-1} \searrow 0$. Then by the definition of the λ_j's, and Theorem 2.1,

$$\begin{aligned}
&\limsup_{\mathcal{D} \in \mathfrak{D}} \frac{1}{L(\eta(\mathcal{D}))} \int_{\mathcal{D}} |f(z)|^p K(z,z)^{1-p}\, dv(z) \\
&\quad \le C^2 \limsup_{n \to \infty} \frac{1}{L(1 + C^{-1} n)} \sum_{j=1}^{n} \lambda_j^p \le \kappa' C^2 \limsup_{q \searrow p} \frac{1}{L(b(e^{-(q-p)^{-1}}))} \sum_{j=1}^{\infty} \lambda_j^q \\
&\quad \le \kappa' C^{5-q} \limsup_{q \searrow p} \frac{1}{L(b(e^{-(q-p)^{-1}}))} \sum_{j=1}^{\infty} \int_{D(\zeta_j, r/4)} |f(z)|^q K(z,z)^{1-q}\, dv(z) \\
&\quad \le \kappa' C^{5-q} N \limsup_{q \searrow p} \frac{1}{L(b(e^{-(q-p)^{-1}}))} \int_{\bigcup_{j=1}^{\infty} D(\zeta_j, r/4)} |f(z)|^q K(z,z)^{1-q}\, dv(z) \\
&\quad \le \kappa' C^{5-q} N \limsup_{q \searrow p} \frac{1}{L(b(e^{-(q-p)^{-1}}))} \int_{\mathbb{D}} |f(z)|^q K(z,z)^{1-q}\, dv(z)
\end{aligned}$$

for some $\kappa' > 0$ and $N > 0$. This completes the proof. ∎

Remark. The main idea of the proof of Proposition 3.6 is due to S. Y. Li.

In the theory of symmetrically normed ideals, it is well known that the symmetrically normed ideal $\mathfrak{S}_{\Phi_L}$ is separable if and only if L is bounded, i.e., $\sum_{n=1}^{\infty} \pi_n < \infty$. Here we present a function space analog of this result, but in a weaker form:

Proposition 3.7. *Let $\{\omega_n\}$ be an interpolating sequence in $\mathbb{D}$. Let $\alpha_n \searrow 0$ be a sequence so that*

$$\liminf_{n \to \infty} \frac{\sum_{k=1}^{n} \alpha_k^p}{L(n)} > 0.$$

Assume that there is an $f \in \widetilde{B}_{L,p}$ *such that* $M^{-1}\alpha_n \le |f(\omega_n)|K(\omega_n,\omega_n)^{-1} \le M\alpha_n$ *for all* n *and for some* $M > 0$. *Then* $\widetilde{B}_{L,p}$ *is separable if and only if* L *is bounded.*

Proof. If L is bounded (i.e. $\sum \pi_k < \infty$), then $\widetilde{B}_{L,p}$ is simply the space

$$\Big\{f \text{ analytic on } \mathbb{D} : \int_{\mathbb{D}} |f(z)|^p K(z,z)^{1-p}\, dv(z) < \infty\Big\},$$

which is obviously separable. On the other hand, suppose that L is not bounded. Then, by the assumption, there is a $\delta > 0$ and $n_1 < n_2 < \cdots$ such that

$$\frac{1}{L(n_{k+1}-n_k)} \sum_{i=n_k}^{n_{k+1}-1} \alpha_i^p > \delta$$

for all k. Let $\varepsilon = \{\varepsilon_k\}$ be a sequence of 0's and 1's and let $g_\varepsilon \in H^\infty$ be such that $g_\varepsilon(\omega_i) = \varepsilon_k$ if $n_k \le i < n_{k+1}$. The cardinality of the set $\mathcal{F} = \{fg_\varepsilon : \varepsilon = \{\varepsilon_k\}$ is a 0-1 sequence$\}$ is apparently the cardinality of $\mathbb{R}$, and hence an uncountable subset of $\widetilde{B}_{L,p}$.

Now suppose that $\varepsilon \ne \varepsilon'$. Then $\varepsilon_k \ne \varepsilon'_k$ for some k, which means that $|g_\varepsilon(\omega_i) - g_{\varepsilon'}(\omega_i)| = 1$ if $n_k \le i < n_{k+1}$. On the other hand, since $\{\omega_n\}$ is an interpolating sequence, there is an $r > 0$ such that $\beta(\omega_m,\omega_n) > 2r$ if $m \ne n$. Again, choose $C > 0$, depending only on r, so that

(a) $C^{-1}K(z,z) \le K(\zeta,\zeta) \le CK(z,z)$ and $C^{-1}(1-|z|^2)^2 \le |D(\zeta,s)| \le C(1-|z|^2)^2$ if $\zeta \in D(z,2r)$, $s = r$ or $2r$,

(b) $h(z) \le (C/|D(z,r)|)\int_{D(z,r)} h\, dv$ for any subharmonic function $h \ge 0$.

Then

$$\begin{aligned}
&\|fg_\varepsilon - fg_{\varepsilon'}\| \\
&\ge \frac{1}{L(1+C(n_{k+1}-n_k))} \sum_{i=n_k}^{n_{k+1}-1} \int_{D(\omega_i,r)} |f(z)|\,|g_\varepsilon(z) - g_{\varepsilon'}(z)|^p K(z,z)^{1-p}\, dv(z) \\
&\ge \frac{C^{-p}M^{-1}}{L(1+C(n_{k+1}-n_k))} \sum_{i=n_k}^{n_{k+1}-1} \alpha_i^p \ge \frac{\kappa}{L(n_{k+1}-n_k)} \sum_{i=n_k}^{n_{k+1}-1} \alpha_i^p \ge \kappa\delta
\end{aligned}$$

for some $\kappa > 0$ since $L(1+Ct)/L(t)$ is bounded. Therefore $\{h \in \widetilde{B}_{L,p} : \|h - fg_\varepsilon\| < \kappa\delta\}$, $fg_\varepsilon \in \mathcal{F}$, is an uncountable collection of pairwise disjoint balls with the same radius in $\widetilde{B}_{L,p}$. This completes the proof. ∎

Example. Let $\log z$ be a branch of the logarithm defined on the slit plane

$$\mathbb{C} \setminus \{z : \mathrm{Re}(z) \le 0,\ \mathrm{Im}(z) = 0\}.$$

For $\gamma > 0$, consider the analytic function f on $\mathbb{D}$ defined by

$$f(z) = (\log(\log 6 - \log(1-z)))^{\gamma-1}(\log 6 - \log(1-z))^{-1}(1-z)^{-2}.$$

Here the $\log 6$ only serves to eliminate unnecessary singularities. We will show that $f \in \widetilde{B}_{1+(\log t)^\gamma,1}$. Since we have already seen that $\{1+(\log t)^\gamma, a(t)\}$ is regular for some $a(t)$ (example after Theorem 2.1), it suffices to show, by Proposition 3.6 and the definition of f, that

$$\sup_{1<p<2} \frac{1}{1+(\log(b(e^{-(p-1)^{-1}})))^\gamma} \int_{\Delta} |f(z)|^p K(z,z)^{1-p}\, dv(z) < \infty,$$

where $\Delta = \mathbb{D} \cap (\mathbb{D}+1)$, since f is bounded on $\mathbb{D} \setminus \Delta$. Moreover, since

$$\log 6 \le \log 6 - \log|\zeta| \le |\log\zeta - \log 6| \le ((\log 6 - \log|\zeta|)^2 + \pi^2)^{1/2}$$

and

$$|\arg(\log\zeta - \log 6)| \le \arctan\frac{\pi}{2\log 6}$$

for $\zeta \in \Delta$, we can find $c_\gamma > 0$, depending on γ, such that

$$|f(z)|^p K(z,z)^{1-p} \le c_\gamma |f(x)|^p K(x,x)^{1-p}$$

for all $z \in \mathbb{D}$ with $|1-z| = 1-x$, where $0 \le x < 1$. Therefore, we only need to show that

$$\sup_{1<p<2} \frac{1}{1+(\log(b(e^{-(p-1)^{-1}})))^\gamma} \int_0^1 |f(x)|^p K(x,x)^{1-p}(1-x)\, dx < \infty.$$

However, since there is a $t_0 > 0$ such that $(\log t)^{\gamma-1}/t$ strictly decreases for $t \ge t_0$, we have

$$\int_0^1 |f(x)|^p K(x,x)^{1-p}(1-x)\, dx \\ \le 4^p \int_0^\infty \frac{(\log(t+\log 6))^{(\gamma-1)p}}{(t+\log 6)^p}\, dt \le \kappa \sum_{n=2}^\infty \frac{(\log n)^{(\gamma-1)p}}{n^p}$$

for some $\kappa > 0$ which does not depend on p. Therefore, $f \in \widetilde{B}_{1+(\log t)^\gamma,1}$ by Theorem 2.1. Furthermore, since $x_n = 1 - 1/2^n$ is an interpolating sequence in $\mathbb{D}$ and $|f(x_n)|K(x_n,x_n)$ is obviously comparable to $(\log n)^{\gamma-1}/n$ for $n \ge 2$, we see, by Proposition 3.7, that $\widetilde{B}_{1+(\log t)^\gamma,1}$ is nonseparable for all $\gamma > 0$, and that f belongs precisely to $\widetilde{B}_{1+(\log t)^\gamma,1}$, which means that $\widetilde{B}_{1+(\log t)^{\gamma_1},1} \subsetneq \widetilde{B}_{1+(\log t)^{\gamma_2},1}$ if $0 < \gamma_1 < \gamma_2$.

With a similar argument, we obtain the following result, which may not be obvious from the function theory point of view:

THEOREM 3.8. *There are infinitely many nonseparable Banach spaces between B_p and all B_q with $q > p \geq 1$, where B_p is the analytic Besov space*

$$\Big\{f \text{ analytic on } \mathbb{D} : \int_{\mathbb{D}} |f''(z)|^p K(z,z)^{1-p} dv(z) < \infty\Big\}.$$

4. Operators on $H^2(\mathbb{D})$ which belong to a symmetrically normed ideal. Consider $H^2 = H^2(\mathbb{D})$, the Hardy space on $\mathbb{D}$. In this section, we discuss criteria for several types of operators on H^2 to belong to the ideal $\mathfrak{S}_{\Phi_L}$ generated by the sequence $\{\pi_k\}$ and the norming function

$$\Phi_L(x) = \sup_n \left(\frac{1}{L(n)} \sum_{k=1}^{n} x_k^{*p}\right)^{1/p}, \qquad x = (x_1, \ldots, x_n, 0, \ldots) \in \widehat{c},$$

where $L(t)$ is nondecreasing on $[0,\infty)$ with $L(n) = \sum_{k=1}^n \pi_k$ and $1 \leq p < \infty$.

4.1. *Hankel operators.* Let P denote the orthogonal projection from $L^2(\partial\mathbb{D})$ onto $H^2(\mathbb{D})$ (called the *Szegő projection*). Given f analytic on $\mathbb{D}$ so that $f = Pg$ for some g in $L^\infty(\partial\mathbb{D})$ (or $f \in$ BMOA, the space of functions in H^2 whose boundary values are of bounded mean oscillation), the (*small*) *Hankel operator* with symbol f is a bounded operator on H^2 defined as

$$h_f := PM_f R,$$

where $R : H^2 \to (H^2)^\perp$ is defined by $(Rg)(z) = g(\bar{z})$. The problem of characterizing the analytic functions f on $\mathbb{D}$ so that $h_f \in \mathfrak{S}_p$ for some $p > 0$ has attracted attention of many mathematicians including R. Coifman, S. Janson, J. Peetre, V. Peller, R. Rochberg, and K. Zhu. For example, Peller [16] shows that for $1 \leq p < \infty$, $h_f \in \mathfrak{S}_p$ if and only if $f \in B_p$, and there is a $C > 0$ such that

$$C^{-1}\|f\|_{B_p} \leq \|h_f\|_{\mathfrak{S}_p} \leq C\|f\|_{B_p}.$$

A similar result was obtained by Coifman and Rochberg [3] on the upper half-plane in $\mathbb{C}$ for $p = 1$ and by Rochberg [17] for $p > 1$. Results for weighted Bergman spaces on $\mathbb{D}$ (for definitions see, for example, [5, Chapter 2]) were proved in [10] for $1 \leq p < \infty$. In addition, several authors including M. Feldman and R. Rochberg [6], G. Zhang [19], and K. Zhu [20] have also studied criteria for Hankel operators to be in the Schatten classes on weighted Bergman spaces in higher dimensions. In the 80's and 90's, Hankel operators in symmetrically normed ideals generated by binormalizing sequences have found application in noncommutative geometry and quantum physics. For instance, the Hankel operators in the ideal $\mathcal{L}^{1,\infty}$ generated by the harmonic sequence played a central role in the construction of an important mathematical tool later known as *quantized calculus* (see [4]), while J. Bellisard and coworkers have connected Hankel operators in $\mathcal{L}^{1,\infty}$ to their study on the quantum Hall effect [1].

Now, by complex interpolation, there exists $C > 0$, depending on p only, so that

$$C^{-1}\|f\|_{B_q} \le \|h_f\|_{\mathfrak{S}_q} \le C\|f\|_{B_q}$$

for all $p \le q \le 2p$. So, according to Theorem 2.1, Corollary 2.2 and Proposition 3.6, one has

THEOREM 4.1. *Let f be analytic on $\mathbb{D}$. Let $\{\pi_k\}$ be a binormalizing sequence, and $L(t)$ be nondecreasing on $[0,\infty)$ such that $L(n) = \sum_{k=1}^{n} \pi_k$ for all $n \in \mathbb{N}$. Suppose that $\{L(t), a(t)\}$ is regular for some $a(t) \searrow 0$ strictly for $t > 0$. Then $h_f \in \mathfrak{S}_{\Phi_L}$ if and only if $f'' \in \widetilde{B}_{L,p}$.*

For the remaining types of operators to be discussed in this section, the following result proves to be useful:

PROPOSITION 4.2. *Let T be a positive definite compact operator on H^2. Assuming further that $\{L(t), a(t)\}$ is regular for some $a(t) \searrow 0$ strictly for $t > 0$. Then, given $r > 0$, the following are equivalent:*

(i) $\displaystyle\limsup_{X \subset\subset D} \frac{1}{L(\eta(E_r(\mathcal{X})))} \int_{\mathcal{X}} \langle TK_z, K_z\rangle dv(z) < \infty.$

(ii) $\displaystyle\limsup_{q \searrow p} \frac{1}{L(b(e^{-(q-p)^{-1}}))} \int_{\mathbb{D}} \langle TK_z, K_z\rangle^q K(z,z)^{1-q} dv(z) < \infty.$

The function K_z is the reproducing kernel defined by $K_z(\omega) = \overline{K(z,\omega)}$.

Proof. For any orthonormal basis $\{e_n(z)\}$ in H^2, one has

$$K_z(\omega) = \sum_{n=1}^{\infty} \overline{e_n(z)} e_n(\omega)$$

(see, for example, [21]). Now suppose that $\{e_n(z)\}$ is an orthonormal basis of H^2 such that

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n\rangle e_n$$

for all $f \in H^2$, where λ_n's are the singular values for T. Therefore

$$\langle TK_z, K_z\rangle = \sum_{n=1}^{\infty} \lambda_n |e_n(z)|^2.$$

So $\langle TK_z, K_z\rangle$ is, in particular, subharmonic on $\mathbb{D}$. Now the proof of the proposition is obtained by replacing $|f|$ with $\langle TK_z, K_z\rangle$ in the proof of Proposition 3.6. ∎

In the remaining part of this article, we still assume that $\{L(t), a(t)\}$ is regular for some $a(t) \searrow 0$ strictly for $t > 0$.

4.2. *Toeplitz operators defined by Borel measures.* Let μ be a positive Borel measure on $\mathbb{D}$. We define the *Toeplitz operator* T_μ (see [13] for details) by

$$T_\mu(f)(z) := \int_{\mathbb{D}} f(\omega)K(z,\omega)\,d\mu(\omega).$$

By Lemmas 2.1 and 4.5 in [12], for $1 \le p < \infty$, $T_\mu \in \mathfrak{S}_p$ if and only if

$$\int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |K(z,\omega)|^2\,d\mu(\omega)\Big)^p K(z,z)\,dv(z) < \infty,$$

and there is a $C > 0$ such that

$$C^{-1}\|T_\mu\|^q_{\mathfrak{S}_q} \le \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |K(z,\omega)|^2\,d\mu(\omega)\Big)^q K(z,z)\,dv(z) \le C\|T_\mu\|^q_{\mathfrak{S}_q}$$

for, say, all $p \le q \le 2p$. But on the other hand, it is evident that

$$\langle T_\mu K_z, K_z\rangle = \int_{\mathbb{D}} |K(z,\omega)|^2\,d\mu(\omega).$$

Therefore, by Theorem 2.2, Proposition 3.6 and Proposition 4.2 we have

THEOREM 4.3. *Let μ be a positive Borel measure on D. Then $T_\mu \in \mathfrak{S}_{\Phi_L}$ if and only if $\tau_\mu(z) = \int_{\mathbb{D}} |K(z,\omega)|^2 d\mu(\omega) \in \widetilde{B}_{L,p}$.*

4.3. *Composition operators.* Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and consider the operator C_φ defined by

$$C_\varphi(f)(z) := f(\varphi(z)),$$

called the composition operator with symbol φ (for a complete description and the history of the development of the theory of composition operators see, for example, [5]). The following result is a consequence of Theorem 2.1, Proposition 3.6, Proposition 4.2 (applied to the operator $T = |C_\varphi|$) and Theorem 1.1 in [12]:

THEOREM 4.4. *Consider the Berezin transform $\mathbf{B}_\varphi$ of φ defined as*

$$\mathbf{B}_\varphi(z) := \Big(K(z,z)^{-1} \int_{\mathbb{D}} |K(z,\varphi(\omega))|^2\,dv(\omega)\Big)^{1/2}.$$

Then $C_\varphi \in \mathfrak{S}_{\Phi_L}$ if and only if $\mathbf{B}_\varphi \in \widetilde{B}_{L,p}$.

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Received 6 December 2004 (4539)