A SPHERICAL TRANSFORM ON SCHWARTZ FUNCTIONS
ON THE HEISENBERG GROUP ASSOCIATED
TO THE ACTION OF U(p, q)

BY

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Abstract. Let $S(H_n)$ be the space of Schwartz functions on the Heisenberg group $H_n$. We define a spherical transform on $S(H_n)$ associated to the action (by automorphisms) of $U(p, q)$ on $H_n$, $p + q = n$. We determine its kernel and image and obtain an inversion formula analogous to the Godement–Plancherel formula.

1. Introduction. Let $n \geq 2$ and let $p, q$ be natural numbers such that $p + q = n$. Let $H_n$ be the Heisenberg group defined by $H_n = \mathbb{C}^n \times \mathbb{R}$ with group law

$$(z, t)(z', t') = (z + z', t + t' - \frac{1}{2} \text{Im} B(z, z'))$$

where

$$B(z, w) = \sum_{j=1}^{p} z_j \overline{w}_j - \sum_{j=p+1}^{n} z_j \overline{w}_j.$$ 

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^p, x'' \in \mathbb{R}^q$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ via the map

$$\varphi(x', x'', y', y'') = (x' + iy', x'' - iy''), \quad x', y' \in \mathbb{R}^p, x'', y'' \in \mathbb{R}^q.$$ 

In this setting, the form $-\text{Im} B(z, w)$ agrees with the standard symplectic form on $\mathbb{R}^{2(p+q)}$, and the vector fields

$$X_j = -\frac{1}{2} y_j \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}, \quad Y_j = \frac{1}{2} x_j \frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, n, \quad T = \frac{\partial}{\partial t}$$

form a standard basis for the Lie algebra $h_n$ of $H_n$. Thus $H_n$ can be viewed as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ via the map $(x, y, t) \mapsto (\varphi(x, y), t)$. From now on, we will use freely this identification.

Let $S(H_n)$ be the Schwartz space on $H_n$ and let $S'(H_n)$ be the space of corresponding tempered distributions. Consider the action of $U(p, q)$ on $H_n$ given by $g \cdot (z, t) = (gz, t)$ (note that since we have assumed that $p, q \geq 1$, $U(p, q)$ is noncompact). So $U(p, q)$ acts on $L^2(H_n)$, $S(H_n)$ and $S'(H_n)$ in.

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the canonical way. The subalgebra \( \mathcal{U}_{U(p,q)}(h_n) \) of left invariant differential operators which commute with this action is generated by \( L \) and \( T \) where

\[
L = \sum_{j=1}^{p} (X_j^2 + Y_j^2) - \sum_{j=p+1}^{n} (X_j^2 + Y_j^2)
\]

and \( T \) is as above (cf. [5]). We observe that it is commutative, since \( T \) belongs to the center of \( h_n \).

Moreover, for \( \lambda \in \mathbb{R} - \{0\} \) and \( k \in \mathbb{Z} \), there exists a tempered \( U(p,q) \)-invariant distribution (on \( H_n \)) \( S_{\lambda,k} \) satisfying

\[
(1.1) \quad LS_{\lambda,k} = -|\lambda|(2k + p - q)S_{\lambda,k}, \quad iTS_{\lambda,k} = \lambda S_{\lambda,k}
\]

and such that, for all \( f \in S(H_n) \),

\[
(1.2) \quad f = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} f \ast S_{\lambda,k} |\lambda|^n d\lambda
\]

(cf. [5]).

Let us recall some facts concerning the compact case \( p = n, q = 0 \), i.e., when \( U(p,q) = U(n) \). In this case it is well known (see [6]) that \( \mathcal{U}_{U(n)}(h_n) \) is a commutative algebra if and only if the convolution algebra \( L^1_{U(n)}(H_n) \) of \( U(n) \)-invariant integrable functions is commutative, that is, \( (H_n, U(n)) \) is a Gelfand pair. Its spectrum, denoted by \( \Delta(U(n), H_n) \), can be identified, via integration, with the set of bounded spherical functions of the pair \( (U(n), H_n) \). These spherical functions can be classified (see [2]) as:

a) The spherical functions of type I, i.e., those that restricted to the center of \( H_n \) are nontrivial characters. These are given by

\[
\Phi_{\lambda,k}^{n-1}(z,t) := e^{-i\lambda t} L_k^{n-1}(|\lambda| |z|^2/2) e^{-|\lambda||z|^2/4}, \quad \lambda \neq 0, \ k \geq 0,
\]

where \( L_k^{n-1} \) is the Laguerre polynomial of order \( n - 1 \) and degree \( k \) normalized by \( L_k^{n-1}(0) = 1 \).

b) The spherical functions \( \eta_w \) of type II, i.e., those that are constant on the center. They are given, for \( w \in \mathbb{C}^n - \{0\} \), by

\[
\eta_w(z,t) = \frac{2^{n-1}(n-1)!}{(|z| |w|)^{n-1}} J_{n-1}(|z| |w|)
\]

where \( J_{n-1} \) is the Bessel function of order \( n - 1 \) of the first kind, and by

\[
\eta_0(z,t) = 1.
\]

We set

\[
\Delta_1(U(n), H_n) = \{ \Psi \in \Delta(U(n), H_n) : \Psi \text{ is of type I} \},
\]

\[
\Delta_2(U(n), H_n) = \{ \Psi \in \Delta(U(n), H_n) : \Psi \text{ is of type II} \}.
\]
For \( f \in L^1_{U(n)}(H_n) \), its spherical transform \( \hat{f} : \Delta(U(n), H_n) \to \mathbb{C} \) is defined by

\[
\hat{f}(\Psi) = \int_{H_n} f(z, t) \Psi(z, t) \, dz \, dt
\]

where \( dz \, dt \) is the Haar measure (i.e., the Lebesgue measure) on \( H_n \).

In this case \((p = n, q = 0)\) the image of the radial Schwartz functions on \( H_n \) under the map \( f \mapsto \hat{f} \) is explicitly described in [3]. The notion of rapidly decreasing functions on \( \Delta(U(n), H_n) \) is introduced and it is proved that the image of \( S(H_n) \) under the spherical transform is the space \( \hat{S}(U(n), H_n) \) of rapidly decreasing functions \( F \) on \( \Delta(U(n), H_n) \) such that certain “derivatives” of \( F \) are also rapidly decreasing (see Definitions 6.1 and 6.3 in [3]).

Also, in [4], a map \( E : \Delta(U(n), H_n) \to [0, \infty) \times \mathbb{R} \) is defined by

\[
E(\Psi) = (\hat{\Lambda}(\Psi), i\hat{T}(\Psi)),
\]

where \( \hat{\Lambda}(\Psi) \) and \( \hat{T}(\Psi) \) denote the eigenvalues of \( L \) and \( T \) respectively, associated to \( \Psi \). The image of \( E \) is the so-called Heisenberg fan \( A(U(n), H_n) \) and it is the set

\[
\{(|\lambda|2k + n), \lambda \neq 0, k \in \mathbb{N} \cup \{0\}\} \cup \{[0, \infty) \times \{0\}\}.
\]

It is proved that \( E \) is a homeomorphism from \( \Delta(U(n), H_n) \) (equipped with the Gelfand topology) onto the Heisenberg fan (provided with the topology induced from \( \mathbb{R}^2 \)).

From the above considerations it is natural to consider, for arbitrary \( p, q \in \mathbb{N} \) with \( p + q = n \) and for \( f \in S(H_n) \), the “spherical transform” \( F(f) : (\mathbb{R} - \{0\}) \times \mathbb{Z} \to \mathbb{C} \) defined by

\[
F(f)(\lambda, k) = \langle S_{\lambda,k}, f \rangle.
\]

Our aim is to characterize \( F(S(H_n)) \) and \( \text{Ker}(F) \). In order to state our results, let us introduce some additional notations.

For \( m : (\mathbb{R} - \{0\}) \times \mathbb{Z} \to \mathbb{C} \) and \( (\lambda, k) \in (\mathbb{R} - \{0\}) \times \mathbb{Z} \) define

\[
m^*(\lambda, k) = \begin{cases} 
m(\lambda, k) & \text{if } k \geq 0, \\
(-1)^{n-2}m(\lambda, k) & \text{if } k < 0. \end{cases}
\]

\[
m^{**}(\lambda, k) = \begin{cases} 
m(\lambda, k) & \text{if } k < 0, \\
(-1)^{n-2}m(\lambda, k) & \text{if } k \geq 0. \end{cases}
\]

We also set

\[
E(m)(\lambda, k) = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k - l),
\]

\[
\tilde{E}(m)(\lambda, k) = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k + l).
\]

Our main result is the following
Theorem 1.1. Assume that $p, q \geq 1$ with $p + q = n$. Then $\mathcal{F}(S(H_n))$ is the space of functions $m : (\mathbb{R} - \{0\}) \times \mathbb{Z} \to \mathbb{C}$ such that

(i) we have the estimate

$$|m(\lambda, k)| \leq c_N \left( |k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \right) \frac{1}{|\lambda|^N(|k| + 1)^N}, \quad N \in \mathbb{N} \cup \{0\},$$

(ii) the functions defined on $(\mathbb{R} - \{0\}) \times (\mathbb{N} \cup \{0\})$ by

$$(\lambda, k) \mapsto E(m^*)(\lambda, k + q), \quad (\lambda, k) \mapsto \tilde{E}(m^{**})(\lambda, -k - p)$$

extend to two functions belonging to $\tilde{S}(U(1), H_1)$.

We also obtain an inversion formula for $\mathcal{F}$ analogous to the Godement–Plancherel formula and we determine the kernel of $\mathcal{F}$.

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2. Notations and preliminaries. Let us introduce some notation and recall some known facts. Let $H$ denote the Heaviside function (i.e., $H(\tau) = \lambda(0, \infty)(\tau)$) and let $\mathcal{H}$ be the space of functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that

$$\varphi(\tau) = \varphi_1(\tau) + \tau^{n-1}\varphi_2(\tau)H(\tau), \quad \varphi_1, \varphi_2 \in S(\mathbb{R}).$$

It is proved in [9] that $\mathcal{H}$, provided with a suitable topology, is a Fréchet space. Moreover, $\mathcal{H}$ is the space of functions $\varphi \in C^\infty(\mathbb{R} - \{0\})$ that are rapidly decreasing at $\pm \infty$ in the usual sense, have the limits $\lim_{\tau \to 0^+} \partial^j \varphi/\partial \tau^j$ and $\lim_{\tau \to 0^-} \partial^j \varphi/\partial \tau^j$ for all $j \in \mathbb{N}$, and admit $n - 2$ continuous derivatives at the origin. For $p + q = n$, $p, q \geq 1$, in [9] there is also given a linear, continuous and surjective map $N : S(\mathbb{R}^n) \to \mathcal{H}$ whose adjoint $N' : \mathcal{H}' \to S'(\mathbb{R}^n)^{O(p, q)}$ is a linear homeomorphism onto the space of $O(p, q)$-invariant tempered distributions on $\mathbb{R}^n$. As pointed out in [5], this construction also works to describe the space $S'(\mathbb{C}^n)^{U(p, q)}$, i.e., there exists a linear, continuous and surjective map, still denoted by $N : S(\mathbb{C}^n) \to \mathcal{H}$, whose adjoint $N' : \mathcal{H}' \to S'(\mathbb{C}^n)^{U(p, q)}$ is a homeomorphism. For $f \in S(H_n)$, we will write $Nf(\tau, t)$ for $N(f(\cdot, t))(\tau)$. We have (cf. (2.11) in [5])

$$Nf(\tau, t) = \int_{\varrho > |\tau|} Mf(\cdot, t)(\varrho, \tau)(\varrho + \tau)^{p-1}(\varrho - \tau)^{q-1}d\varrho,$$

where for $\varrho \geq |\sigma|$,

$$Mf(\cdot, t)(\varrho, \sigma) := \int_{S^{2p-1} \times S^{2q-1}} f \left( \frac{\varrho + \sigma}{2} \right)^{1/2} w_{u}, \left( \frac{\varrho - \sigma}{2} \right)^{1/2} w_{v}, t \right) dw_{u} dw_{v}.$$
Let $\mathcal{H}^\#$ be the space of functions $\varphi$ on $\mathbb{R}^2$ of the form
\[
\varphi(\tau, t) = \varphi_1(\tau, t) + \tau^{n-1}H(\tau)\varphi_2(\tau, t), \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^2).
\]

Remark 2.1. A straightforward adaptation of the proofs of Lemmas 4.2 and 4.3 in [9] shows that $N : \mathcal{S}(H_n) \to \mathcal{H}^\#$ is surjective.

In order to give an explicit expression of the distributions $S_{\lambda,k}$ we recall the definition of the Laguerre polynomials. For nonnegative integers $m$ and $\lambda$, let $L_m^\alpha$ (see, e.g., [8, pp. 99–101]) be given by
\[
L_m^\alpha = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{\tau^j}{j!}, \quad L_{m-1}^\alpha = -\frac{d}{d\tau} L_m^\alpha(\tau).
\]

For $\lambda \in \mathbb{R}$, $k, s \in \mathbb{N} \cup \{0\}$ and $(\tau, t) \in [0, \infty) \times \mathbb{R}$ we set
\[
\psi_{\lambda,k}^s(\tau, t) := e^{-i\lambda t} L_k^s(|\lambda|\tau/2)e^{-|\lambda|\tau/4},
\]
\[
\varphi_{\lambda,k}^s(\tau, t) := e^{-i\lambda t} L_k^s(|\lambda|\tau/2)e^{-|\lambda|\tau/4},
\]

where $L_k^s$ denotes the Laguerre polynomial of degree $k$ and order $s$ normalized by $L_k^s(0) = 1$, i.e., given by $L_k^s(\tau) = L_k^s(\tau)/(k+s)$.

It is well known that the family $e^{-\tau/2}L_m^0(\tau)$, $m \geq 0$, is an orthonormal basis of $L^2(0, \infty)$. Thus (cf. [5, Theorem 4.1 and Remarks 4.2, 4.3])
\[
S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t},
\]
with $F_{\lambda,k} \in \mathcal{S}'(\mathbb{C}^n)$ defined by
\[
\langle F_{\lambda,k}, g \rangle = \langle (L_{k-q+n-1}^0)_{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau) \rangle
\]
for $k \geq 0$, $\lambda \neq 0$ and by
\[
\langle F_{\lambda,k}, g \rangle = \langle (L_{-k+p+n-1}^0)_{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(-2|\lambda|^{-1}\tau) \rangle
\]
for $k < 0$, $\lambda \neq 0$.

For $\varphi \in \mathcal{H}$ and $j \in \mathbb{N} \cup \{0\}$ a computation gives
\[
\langle (L_j^0)_{(n-1)}, \varphi \rangle
= \int_0^{\infty} (L_j^0)_{(n-1)}(\tau) \, d\tau + \sum_{0 \leq s \leq n-2} (L_j^0)_{(n-2-s)}(0) \langle \delta^s, \varphi \rangle.
\]

Lemma 2.2. For $r \in \mathbb{Z}$ such that $0 \leq r \leq n-2$ and for $\varphi \in \mathcal{H}$,
\[
\langle (L_r^0)_{(n-1)}, \tau \mapsto e^{-\tau/2}\varphi(\tau) \rangle = (-1)^{n-2} \langle (L_{n-2-r})_{(n-1)}, \tau \mapsto e^{-\tau/2}\varphi(-\tau) \rangle.
\]

Proof. A computation using (2.7) gives
\[
\langle (L_r^0)_{(n-1)}, \tau \mapsto e^{-\tau/2}\varphi(\tau) \rangle
= \sum_{0 \leq l \leq n-2} \sum_{\max(n-2-r,l) \leq j \leq n-2} \frac{1}{2^{j-l}} \binom{j}{l} (-1)^{n-j} \binom{r}{n-2-j} \langle \delta^l, \varphi \rangle
\]
and also

\[
\langle (L^0_{n-2-r}H)^{(n-1)} , \tau \mapsto e^{-\tau/2} \varphi(-\tau) \rangle = \sum_{0 \leq l \leq n-2} \sum_{\max(r,l) \leq j \leq n-2} \frac{1}{2^{j-l}} \binom{j}{l} (-1)^{n-j+l} \binom{n-2-r}{n-2-j} \langle \delta(l), \varphi \rangle.
\]

To show the lemma it is enough to see that for \(0 \leq r \leq n-2\) and \(0 \leq l \leq n-2\),

\[
\sum_{\max(n-2-r,l) \leq j \leq n-2} \frac{1}{2^j} \binom{j}{l} (-1)^{n-j} \binom{r}{n-2-j} = (-1)^{n-2} \sum_{\max(r,l) \leq j \leq n-2} \frac{1}{2^j} \binom{j}{l} (-1)^{n-j+l} \binom{n-2-r}{n-2-j},
\]
i.e., to show that for \(0 \leq r \leq n-2\), the following polynomial identity holds:

\[
(2.8) \quad \sum_{0 \leq l \leq n-2} \sum_{\max(n-2-r,l) \leq j \leq n-2} \frac{1}{2^j} \binom{j}{l} (-1)^{n-j} \binom{r}{n-2-j} = (-1)^{n-2} \sum_{0 \leq l \leq n-2} t^l \sum_{\max(r,l) \leq j \leq n-2} \frac{1}{2^j} \binom{j}{l} (-1)^{n-j+l} \binom{n-2-r}{n-2-j}.
\]

If we change the summation order, (2.8) becomes

\[
(2.9) \quad (-1)^n \sum_{n-2-r \leq j \leq n-2} (-1)^j \binom{r}{n-2-j} \frac{1}{2^j} \sum_{0 \leq l \leq j} \binom{j}{l} t^l = \sum_{r \leq j \leq n-2} \frac{1}{2^j} \binom{n-2-r}{n-2-j} (-1)^j \sum_{0 \leq l \leq j} \binom{j}{l} (-1)^l l^l,
\]
which, by the binomial formula, is equivalent to

\[
(2.10) \quad (-1)^n \sum_{n-2-r \leq j \leq n-2} \left( \frac{r}{n-2-j} \right) \left( -\frac{t+1}{2} \right)^j = \sum_{r \leq j \leq n-2} \left( \frac{n-2-r}{n-2-j} \right) \left( \frac{t-1}{2} \right)^j,
\]
i.e., to

\[
(2.11) \quad \left( -\frac{1+t}{2} \right)^{n-2} \sum_{n-2-r \leq j \leq n-2} \left( \frac{r}{n-2-j} \right) \left( -\frac{2}{1+t} \right)^{n-2-j} = (-1)^n \left( \frac{t-1}{2} \right)^{n-2} \sum_{r \leq j \leq n-2} \left( \frac{n-2-r}{n-2-j} \right) \left( \frac{2}{t-1} \right)^{n-2-j}.
\]
After changing $j$ to $n-2-j$ and recalling that $0 \leq r \leq n-2$, by the binomial formula (2.11) reduces to
\[
\left(-\frac{1+t}{2}\right)^{n-2}\left(1-\frac{2}{1+t}\right)^r = (-1)^n\left(t-\frac{1}{2}\right)^{n-2}\left(1+\frac{2}{t-1}\right)^{n-2-r},
\]
which clearly holds. ■

**Corollary 2.3.** Let $g \in S(\mathbb{C}^n)$. For $0 \leq k \leq q-1$, $\lambda \neq 0$ we have
\[
\langle F_{\lambda,k}, g \rangle = (-1)^{n-2}\langle (L_{-k-p+n-1}^0 H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(-2|\lambda|^{-1}\tau) \rangle,
\]
and
\[
\langle F_{\lambda,k}, g \rangle = (-1)^{n-2}\langle (L_{k-q+n-1}^0 H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau) \rangle
\]
for $-p+1 \leq k < 0$. ■

For a given set $X$ and for $f : X \times \mathbb{R} \to \mathbb{C}$, $\lambda \in \mathbb{R}$ we set $f(z, \hat{\lambda}) := (t \mapsto f(z, t))^\wedge(\lambda)$ where $(\cdot)^\wedge$ denotes the one-dimensional Fourier transform (provided that it exists).

**Proposition 2.4.** Ker($F$) = Ker($N$).

**Proof.** If $f \in S(H_n)$ and $Nf = 0$, then, by (2.5) and (2.6), $F(f)(\lambda,k) = \langle S_{\lambda,k}, f \rangle = \langle F_{\lambda,k} \otimes e^{-i\lambda t}, f \rangle = 0$ and so $F(f) = 0$.

If $F(f) = 0$, from the definition of $S_{\lambda,k}$, for $k \geq 0$ and $\lambda \neq 0$ we have
\[
\langle (L_{k-q+n-1}^0 H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle = 0
\]
and, by Lemma 2.2, for $-p+1 \leq k < 0$,
\[
\langle (L_{k-q+n-1}^0 H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle = (-1)^{n-2}\langle (L_{-k-p+n-1}^0 H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle = 0.
\]
Thus, for $j \geq 0$,
\[
2|\lambda|^{-1}\int_0^\infty e^{-\tau/2}L_j^0(\tau)e^{\tau/2} d_{n-1}^{-1}(e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda})) \, d\tau = 0.
\]
Thus
\[
d_{n-1}^{-1}(e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda})) = 0 \quad \text{for } \tau \geq 0, \lambda \neq 0.
\]
So for such $\tau$ and $\lambda$, $e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) = P_\lambda(\tau)$ where $P_\lambda(\tau)$ is a polynomial of degree at most $n-2$ with coefficients which (in principle) depend on $\lambda$. Thus $Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) = e^{\tau/2}P_\lambda(\tau)$. For each $\lambda \neq 0$, $\lim_{\tau \to \infty} Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) = 0$ and so $P_\lambda \equiv 0$. This implies $Nf(\tau, \hat{\lambda}) = 0$ for $\tau \geq 0$ and $\lambda \in \mathbb{R}$. 


A similar argument starting with the fact that, for $k < 0$,
\[(L_{-k}^0 - p + n - 1)H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau, \lambda)) = 0\]
shows that $Nf(\tau, \lambda) = 0$ for $\tau < 0, \lambda \in \mathbb{R}$. 

3. Necessary conditions. In this section we find necessary conditions for a function $m$ defined on $(\mathbb{R} - \{0\}) \times \mathbb{Z}$ to belong to the image of $F$. To do this, we recall the definition of the space $\hat{\mathcal{S}}(U(n), H_n)$. We say that $F : \Delta(U(n), H_n) \rightarrow \mathbb{C}$ is rapidly decreasing (cf. [3, Definition 6.1]) if

(i) $F$ is continuous,

(ii) for $w \in \mathbb{C}^n, w \mapsto F(\eta_w)$ belongs to $S_{U(n)}(\mathbb{C}^n)$ where $\eta_w$ is the spherical function of type II described in the introduction,

(iii) the map $\lambda \mapsto F(\lambda, k)$ is smooth on $\mathbb{R} - \{0\},$

(iv) for each $j, N \geq 0$ there exists a constant $c_{j,N}$ such that

\[\left| \frac{\partial^j}{\partial \lambda^j} F(\lambda, k) \right| \leq \frac{c_{j,N}}{|\lambda|^{j+N}(2k + n)^N}.\]

Also we set (see [3, Definition 6.2])

\[M^-F(\lambda, k) = \begin{cases} \frac{\partial F}{\partial \lambda}(\lambda, k) - \frac{k}{\lambda} [F(\lambda, k) - F(\lambda, k - 1)] & \text{for } \lambda > 0, \\ \frac{\partial F}{\partial \lambda}(\lambda, k) - \frac{k + n}{\lambda} [F(\lambda, k + 1) - F(\lambda, k)] & \text{for } \lambda < 0, \end{cases}\]

and

\[M^+F(\lambda, k) = \begin{cases} \frac{\partial F}{\partial \lambda}(\lambda, k) - \frac{k + n}{\lambda} [F(\lambda, k + 1) - F(\lambda, k)] & \text{for } \lambda > 0, \\ \frac{\partial F}{\partial \lambda}(\lambda, k) - \frac{k}{\lambda} [F(\lambda, k) - F(\lambda, k - 1)] & \text{for } \lambda < 0. \end{cases}\]

The space $\hat{\mathcal{S}}(U(n), H_n)$ is defined as the set of all functions $F : \Delta(U(n), H_n) \rightarrow \mathbb{C}$ for which $(M^+)^l(M^-)^mF$ is rapidly decreasing for all $l, m \geq 0$.

Our results in this section are as follows:

**Theorem 3.1.** For $f \in \mathcal{S}(H_n)$ and $k \in \mathbb{Z}$, $\partial^j(Ff(\lambda, k))/\partial \lambda^j$ exists for all $j \in \mathbb{N}$ and $\lambda \neq 0$. Moreover, for each $j, N \in \mathbb{N} \cup \{0\}$ there exists a positive constant $c$ independent of $\lambda$ and $k$ such that

\[|\frac{\partial^j(Ff(\lambda, k))}{\partial \lambda^j}| \leq c(|k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \frac{1}{|\lambda|^{N+j}(k+1)^N}}.\]

**Theorem 3.2.** Let $f \in \mathcal{S}(H_n)$ and let $m = Ff$. Then the function defined on $(\mathbb{R} - \{0\}) \times (\mathbb{N} \cup \{0\})$ by $(\lambda, k) \mapsto E(m^*)(\lambda, k + q)$ (with $E, m^*$ as in the introduction) can be extended to a function belonging to $\hat{\mathcal{S}}(U(1), H_1)$. 
Moreover, for \( k \geq 0 \) and \( \lambda \neq 0 \),

\[
E(m^*)(\lambda, k + q) = (-1)^{n-1} \int_0^\infty L_k^0(|\lambda| \tau/2) e^{-|\lambda| \tau/4} N f(\tau, \lambda) \, d\tau. 
\]  

**Theorem 3.3.** Let \( f \in \mathcal{S}(H_n) \) and let \( m \) as in Theorem 3.2. Then the function defined on \((\mathbb{R} - \{0\}) \times (\mathbb{N} \cup \{0\})\) by \((\lambda, k) \mapsto \tilde{E}(m^{**})(\lambda, -k - p)\) (\( \tilde{E} \) and \( m^{**} \) as in the introduction) extends to a function in \( \tilde{S}(U(1), H_1) \). Furthermore

\[
\tilde{E}(m^{**})(\lambda, -k - p) = (-1)^{n-1} \int_0^\infty L_k^0(|\lambda| \tau/2) e^{-|\lambda| \tau/4} N f(-\tau, \lambda) \, d\tau. 
\]

For \( j, s \in \mathbb{N} \cup \{0\} \), let \( \varphi_{\lambda,j}(\tau, t) \) be defined by (2.3). From (2.7) and the definition of \( S_{\lambda,k} \) we have

\[
\mathcal{F} f(\lambda, k) = I(\lambda, k) + II(\lambda, k)
\]

where

\[
I(\lambda, k) = \begin{cases} 
(-1)^{n-1} \int_{\mathbb{R}} \int_{\tau > 0} e^{-i\lambda \tau} \varphi_{\lambda,k-q}(\tau, t) e^{-|\lambda| \tau/4} N f(\tau, t) \, d\tau \, d\tau dt & \text{for } k \geq q, \\
(-1)^{n-1} \int_{\mathbb{R}} \int_{\tau > 0} \varphi_{\lambda,-k-p}(\tau, t) N f(-\tau, t) \, d\tau dt & \text{for } k \leq -p,
\end{cases}
\]

\[
II(\lambda, k) = \sum_{r=0}^{n-2} c_{r,k} |\lambda|^{-(l+1)} \langle \delta^{(r)}, N f(\cdot, \lambda) \rangle \quad \text{for } k \in \mathbb{Z},
\]

with

\[
c_{r,k} = \begin{cases} 
4^r \sum_{j=0}^{n-2} \frac{1}{2^j} \frac{r}{j} (L_{k-q+n-1}^0)^{(n-j-2)}(0) & \text{for } k \geq 0, \\
(-1)^r 4^r \sum_{j=0}^{n-2} \frac{1}{2^j} \frac{r}{j} (L_{-k-p+n-1}^0)^{(n-j-2)}(0) & \text{for } k < 0.
\end{cases}
\]

**Proof of Theorem 3.1.** Since \( N f \in \mathcal{H}^\# \) we have \( \frac{\partial}{\partial \tau}(\tau N f(\tau, t)) \in \mathcal{H}^\# \), so by Remark 2.1, there is \( g \in \mathcal{S}(H_n) \) such that \( N g(\tau, t) = \frac{\partial}{\partial \tau}(\tau N f(\tau, t)) \).

We claim that for \( \lambda \neq 0 \) and \( k \in \mathbb{Z} \), \( \partial \mathcal{F} f(\lambda, k)/\partial \lambda \) exists and

\[
\frac{\partial \mathcal{F} f(\lambda, k)}{\partial \lambda} = -i \mathcal{F}(tf)(\lambda, k) - \frac{1}{\lambda} \mathcal{F}g(\lambda, k). 
\]

Indeed, consider the case \( k \geq q \). Let \( I(\lambda, k) \) and \( II(\lambda, k) \) be given by (3.5) and (3.6) respectively. Since for \( j \geq 0 \) we have

\[
\frac{\partial}{\partial \lambda} \varphi_{\lambda,j}(\tau, t) = -it \varphi_{\lambda,j}(\tau, t) + \frac{\tau}{\lambda} \frac{\partial}{\partial \tau} \varphi_{\lambda,j}(\tau, t),
\]
after integration by parts we obtain

\[
\frac{\partial I}{\partial \lambda}(\lambda, k) = \frac{\partial}{\partial \lambda} \left( \int_{\mathbb{R}} \int_{\tau > 0} (-1)^{n-1} \varphi_{\lambda,k-q}(\tau, t) N f(\tau, t) \, d\tau \, dt \right)
\]

\[
= \int_{\mathbb{R}} \int_{\tau > 0} (-1)^{n-1} \varphi_{\lambda,k-q}(\tau, t) (-i t N f(\tau, t)) \, d\tau \, dt
\]

\[
- \frac{1}{\lambda} \int_{\mathbb{R}} \int_{\tau > 0} (-1)^{n-1} \varphi_{\lambda,k-q}(\tau, t) \frac{\partial}{\partial \tau} (\tau N f(\tau, t)) \, d\tau \, dt.
\]

Also,

\[
\frac{\partial II}{\partial \lambda}(\lambda, k) = \frac{\partial}{\partial \lambda} \left( \sum_{l=0}^{n-2} c_{l,k} |\lambda|^{-(l+1)} \langle \delta^{(l)}, N f(\cdot, \lambda) \rangle \right)
\]

\[
= - \sum_{l=0}^{n-2} (l+1) c_{l,k} |\lambda|^{-(l+2)} s(g) \langle \delta^{(l)}, N f(\cdot, \lambda) \rangle
\]

\[
+ \sum_{l=0}^{n-2} c_{l,k} |\lambda|^{-(l+1)} \langle \delta^{(l)}, -i(t N f(\cdot, t))^\wedge(\lambda) \rangle
\]

where \((\cdot)^\wedge\) denotes the Fourier transform in the variable \(t\). Thus the derivative \(\partial F f(\lambda, k)/\partial \lambda\) exists. On the other hand,

\[
-iF(t f(z, t))(\lambda, k) = \int_{\mathbb{R}} \int_{\tau > 0} (-1)^{n-1} \varphi_{\lambda,k-q}(\tau, t) (-i t N f(\tau, t)) \, d\tau \, dt
\]

\[
+ \sum_{l=0}^{n-2} c_{l,k} |\lambda|^{-(l+1)} \langle \delta^{(l)}, -i(t N f(\cdot, t))^\wedge(\lambda) \rangle.
\]

Since \(\langle \delta^{(l)}, \frac{\partial}{\partial \tau} (\tau N f(\tau, t)) \rangle = (l+1) \langle \delta^{(l)}, N f(\cdot, t) \rangle\) we have

\[
\frac{1}{\lambda} F g(\lambda, k) = - \sum_{l=0}^{n-2} (l+1) c_{l,k} |\lambda|^{-(l+2)} s(g) \langle \delta^{(l)}, N f(\cdot, \lambda) \rangle
\]

\[
- \frac{1}{\lambda} \int_{\mathbb{R}} \int_{\tau > 0} (-1)^{n-1} \varphi_{\lambda,k-q}(\tau, t) \frac{\partial}{\partial \tau} (\tau N f(\tau, t)) \, d\tau \, dt
\]

and now (3.8)–(3.11) give (3.7) for \(k \geq q\). The case \(k < q\) follows from a similar argument and using the corresponding expressions for \(I(\lambda, k)\) and \(II(\lambda, k)\).

Now, induction on \(j\) implies that \(\partial^j F f(\lambda, k)/\partial \lambda^j\) exists for \(\lambda \neq 0\), \(k \in \mathbb{Z}\) and all \(j\).

In the rest of the proof, \(c_1, c_2, \ldots, c', c''\), will denote positive constants independent of \(\lambda\) and \(k\). To prove (3.1) we first consider the case \(k \geq q\).
From (3.4), we have
\[
|\mathcal{F}f(\lambda, k)| \leq L_{k-q}^{n-1}(0)\|Nf\|_{L^1((0,\infty) \times \mathbb{R})} + c_1 \sum_{l=0}^{n-2} |c_l, k| |\lambda|^{-(l+1)}.
\]
Since \(L_{k-q}^{n-1}(0) = (k-q+n-1) < c_2 k^{n-1}\) and \(|c_l, k| \leq c_3 k^{n-l-2}\) we have
\[
|\mathcal{F}f(\lambda, k)| \leq c_4 \left( k^{n-1} + \sum_{l=0}^{n-2} k^{n-1-(l+1)}|\lambda|^{-(l+1)} \right) \\
\leq c_4 \left( k + \frac{1}{|\lambda|} \right)^{n-1} \leq c_5 \left( k^{n-1} + \frac{1}{|\lambda|^{n-1}} \right).
\]
Applying (3.12) to \(L^Nf\) instead of \(f\) and recalling (1.1) we get
\[
|2k + p - q|N|\lambda|^N|\mathcal{F}f(\lambda, k)| = |\mathcal{F}(L^Nf)(\lambda, k)| \leq c' \left( |k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \right)
\]
and since \(2k + p - q \neq 0\) because \(k \geq q\), this gives
\[
|\mathcal{F}f(\lambda, k)| \leq c'' \left( |k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \right) \frac{1}{2k + p - q|\lambda|^N}.
\]
A similar argument applies to the case \(k < q\), giving (3.13) except when \(q - p \in 2\mathbb{Z}\) and \(k = (q - p)/2\). In this case we take \((iT)^Nf\) instead of \(L^Nf\) above to get
\[
|\mathcal{F}f(\lambda, k)| \leq c \left( |k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \right) \frac{1}{|\lambda|^N}.
\]
for \(k = (q - p)/2\). From (3.13) and (3.14) we obtain (3.1) for \(j = 0\) and all \(k\) and \(N\).

Observe that for \(r \in \mathbb{N} \cup \{0\}\), (3.1) used with \(j = 0\) and \(N + r\) instead of \(N\) gives immediately that
\[
|\lambda|^r |\mathcal{F}f(\lambda, k)| \leq c \left( |k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \right) \frac{1}{|\lambda|^{N(|k| + 1)^N}}.
\]
An easy induction using (3.7) shows that for \(j \geq 1\),
\[
\lambda^j \frac{\partial^j \mathcal{F}f}{\partial \lambda^j} (\lambda, k) = \sum_{0 \leq r \leq j} \lambda^r \mathcal{F}f_r (\lambda, k)
\]
for some \(f_1, \ldots, f_j\) belonging to \(S(H_n)\) and independent of \(\lambda\) and \(k\). Now, (3.15) and (3.16) give (3.1) for all \(j\). ~

**Lemma 3.4.** Let \(f \in S(H_n)\). If either \(k \geq q\) or \(k \leq -p\), then
\[
\sum_{r=0}^{n-2} |\lambda|^{-(r+1)} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} c_{r,k-l} (\delta(r), Nf(\cdot, \hat{\lambda})) = 0.
\]
Proof. Assume \( k \geq q \). For \( r = 0, 1, \ldots, n - 2 \) we have
\[
\sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} c_{r,k-l} = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (-1)^{n-j} \binom{k-l-q+n-1}{n-j-2} = \sum_{j=r}^{n-1} \frac{1}{2^j} \binom{j}{r} (-1)^{n-j} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \binom{k-l-q+n-1}{n-j-2}.
\]
Let
\[
\beta := \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \binom{k-l-q+n-1}{n-j-2}.
\]
We claim that if \( 0 \leq r \leq j \leq n - 2 \) then \( \beta = 0 \). To see this we note that \( \beta \) is the coefficient of \( y^s \) in the polynomial \( \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (1+y)^{m-l} \) (where \( m = k - q + n - 1 \) and \( s = n - j - 2 \)), i.e. \( \beta \) is the coefficient of \( y^s \) in \( (1+y)^{m-(n-1)} \sum_{l=0}^{n} (-1)^l \binom{n-1}{l} (1+y)^{n-l} = (1+y)^{m-(n-1)} y^{n-1} \). So \( \beta = 0 \) since \( s = n - j - 2 < n - 1 \). The proof for the case \( k \leq -p \) is similar, replacing \( k-q \) by \( -k-p \).

We recall that (cf. [8, p. 101])
\[
(3.17) \quad L_j^n(x) = L_j^{n+1}(x) - L_{j-1}^{n+1}(x).
\]

Lemma 3.5. For \( j \geq 0, \)
\[
(3.18) \quad \sum_{l=0}^\infty (-1)^l \binom{n-1}{l} L_{j-l}^{n-1}(x) = L_j^0(x).
\]

Proof. We first give the proof for the case \( j \geq n - 1 \). We proceed by induction on \( n \). For \( n = 1 \) the lemma is clear. Suppose that it holds for \( n \) and \( j \geq n - 1 \). Then for \( j \geq n, \)
\[
\sum_{l=0}^{n} (-1)^l \binom{n}{l} L_{j-l}^n(x) = L_j^n(x) + (-1)^n L_{j-n}(x)
\]
\[
+ \sum_{l=1}^{n-1} (-1)^l \binom{n-1}{l} L_{j-l}^n(x) + \sum_{l=1}^{n-1} (-1)^l \binom{n-1}{l-1} L_{j-l}^n(x).
\]
An index change in the last sum gives
\[
\sum_{l=0}^{n} (-1)^l \binom{n}{l} L_{j-l}^n(x) = L_j^n(x) + (-1)^n L_{j-n}(x)
\]
\[
+ \sum_{l=1}^{n-1} (-1)^l \binom{n-1}{l} L_{j-l}^n(x) + \sum_{l=0}^{n-2} (-1)^{l-1} \binom{n-1}{l} L_{j-l-1}^n(x).
\]

Thus, for $k < n - 1$ we write

$$
\sum_{l=0}^{j} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) L_{j-l}^{n-1}(x) = \sum_{l=0}^{n-1} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) c_l L_{j-l}^{n-1}(x),
$$

where $c_l = 1$ for $0 \leq l \leq j$ and $c_l = 0$ for $j \leq l \leq n - 1$, and now we proceed as above. ■

Proof of Theorem 3.2. Let $m = \mathcal{F}f$. For $k \geq n - 1$,

$$
E(m^*)(\lambda, k + q) = \sum_{l=0}^{n-1} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) m(\lambda, k + q - l)
\begin{align*}
&= \sum_{l=0}^{n-1} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) (-1)^{n-1} \int_{0}^{\infty} L_{k-l}^{n-1}(|\lambda|\tau/2)e^{-|\lambda|\tau/4} Nf(\tau, \hat{\lambda}) d\tau \\
&\quad + \sum_{l=0}^{n-1} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) \sum_{r=0}^{n-2} c_{\tau, k+q-l}|\lambda|^{-r+1}\langle \delta^{(r)}, Nf(\tau, \hat{\lambda}) \rangle = I + II.
\end{align*}
\]

Now, by Lemma 3.4, $II = 0$ and Lemma 3.5 gives

$$
I = (-1)^{n-1} \int_{0}^{\infty} L_{k}^{0}(|\lambda|\tau/2)e^{-|\lambda|\tau/4} Nf(\tau, \hat{\lambda}) d\tau.
$$

Thus, for $k \geq n - 1$,

$$
E(m^*)(\lambda, k + q) = (-1)^{n-1} \int_{0}^{\infty} L_{k}^{0}(|\lambda|\tau/2)e^{-|\lambda|\tau/4} Nf(\tau, \hat{\lambda}) d\tau.
\]

On the other hand, if $0 \leq k < n - 1$,

$$
E(m^*)(\lambda, k + q) = \sum_{0 \leq l \leq \min(k+q, n-1)} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) m(\lambda, k + q - l) \\
+ \sum_{k+q < l \leq n-1} (-1)^l \left( \begin{array}{c} n - 1 \\ l \end{array} \right) (-1)^{n-2} m(\lambda, k + q - l)
\]
(with the convention that a sum on an empty set is zero). Since, for $0 \leq l \leq \min(k + q, n - 1),$
\[ m(\lambda, k + q - l) = \langle (L_{k-l+n-1}^0H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau) \rangle \]
and since for $k + q < l \leq n - 1$ Corollary 2.3 gives
\[ (-1)^{n-2}m(\lambda, k + q - l) = \langle (L_{k-l+n-1}^0H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau) \rangle, \]
we obtain $E(m^*)(\lambda, k + q) = I + II$ also for $0 \leq k < n - 1$ (with $I$ and $II$ as in (3.19)). Proceeding as in the case $k \geq n - 1$ we conclude that (3.20) holds for all $k$.

Let $\mathcal{F}_1$ be the $U(1)$-spherical transform on $S(H_1)$ defined in [3] and let $f_1$ be the radial function in $S(H_1)$ given by $f_1(z, t) = Nf(|z|^2, t)$. Then, by definition,
\[ \mathcal{F}_1(f_1)(\lambda, k) = \int_\mathbb{C} L_{0}^0(|\lambda||z|^2/2)e^{-|\lambda||z|^2/4}Nf(|z|^2, \widehat{\lambda})\,dz. \]

We use polar coordinates $z = re^{i\theta}$ and then we perform the change of variable $s = r^2$ to get
\[ \mathcal{F}_1(f_1)(\lambda, k) = \pi \int_0^\infty L_{k}^0(|\lambda|s/2)e^{-|\lambda|s/4}Nf(s, \widehat{\lambda})\,ds, \]
i.e. $(-1)^{n-1}E(\mathcal{F}f)(\lambda, k + q) = \mathcal{F}_1(f_1)(\lambda, k)$ for $k \geq 0$. ■

**Proof of Theorem 3.3.** As before, it is enough to find $g_1 \in S(H_1)$ such that for $k \geq 0,$ $\mathcal{F}_1g_1(\lambda, k) = (-1)^{n-1}\mathcal{E}(m^{**})(\lambda, -k - p).$ Set $g_1(z, t) = Nf(-|z|^2, t).$ Following the lines of the proof of Theorem 3.2 we obtain
\[ \mathcal{F}_1(g_1)(\lambda, k) = \pi \int_0^\infty L_{k}^0(|\lambda|s/2)e^{-|\lambda|s/4}Nf(-s, \widehat{\lambda})\,ds \]
\[ = (-1)^{n-1}\mathcal{E}(m^{**})(\lambda, -k - p) \]
for $k \geq 0$. ■

**4. The image of the spherical transform**

**Lemma 4.1.** For $k \geq 0,$
\[ \frac{d^{n-1}}{d\tau^{n-1}} \left( \frac{1}{(n-1)!} \tau^{n-1}L_{k}^{n-1}(\tau)e^{-\tau} \right) = L_{k+n-1}^0(\tau)e^{-\tau}. \]
Proof. We have
\[
\frac{1}{(n-1)!} \frac{d^{n-1}}{d\tau^{n-1}} (\tau^{n-1} L_k^{n-1}(\tau) e^{-\tau})
\]
\[
= \frac{1}{(n-1)! (k+n-1)!} \frac{d^{n-1}}{d\tau^{n-1}} (\tau^{n-1} L_k^{n-1}(\tau) e^{-\tau})
\]
\[
= \frac{1}{(k+n-1)!} \left( \frac{d}{d\tau} \right)^{n-1+k} (\tau^{n-1+k} e^{-\tau}) = L_k^{0}(\tau) e^{-\tau}
\]
where we have used (twice) the fact that
\[
L_\alpha^{j}(\tau) L_\alpha^{\tau} e^{-\tau} = \frac{1}{j!} \frac{d^j}{d\tau^j} (\tau^\alpha e^{-\tau}) \quad \text{for } j \geq 0 \quad \text{(Rodrigues formula)}. \]

Let \( D \) be the linear operator defined on the space of polynomial functions by
\[
DL_k^{0} = L_k^{0} - L_{k-1}^{0} \quad \text{for } k \geq 1 \quad \text{and } D1 = 1.
\]

**Lemma 4.2.** For all \( m \geq 0 \),
\[
(4.2) \quad \left( \frac{d}{d\tau} \right)^m (e^{-\tau} D^m (P(\tau))) = (-1)^m e^{-\tau} P(\tau).
\]

Proof. We proceed by induction on \( m \). For \( m = 0 \) there is nothing to prove. Assume that (4.2) holds. Then, for \( k \geq 0 \),
\[
\left( \frac{d}{d\tau} \right)^{m+1} (e^{-\tau} D^{m+1}(L_k^{0}(\tau))) = \frac{d}{d\tau} \left( \frac{d}{d\tau} \right)^m (e^{-\tau} D^m(DL_k^{0}(\tau)))
\]
\[
= (-1)^m \frac{d}{d\tau} (e^{-\tau} DL_k^{0}(\tau)) = (-1)^{m+1} e^{-\tau} L_k^{0}(\tau).
\]

In fact, the last equality follows from a direct computation for \( k = 0, 1 \), and for \( k \geq 2 \) observe that, taking into account (2.1) and (3.17),
\[
(-1)^m \frac{d}{d\tau} (e^{-\tau} DL_k^{0}(\tau)) = (-1)^m \frac{d}{d\tau} (e^{-\tau} (L_k^{0}(\tau) - L_{k-1}^{0}(\tau)))
\]
\[
= (-1)^m (-e^{-\tau} L_k^{0}(\tau) + e^{-\tau} L_{k-1}^{0}(\tau) - e^{-\tau} L_{k-1}^{1}(\tau) + e^{-\tau} L_{k-2}^{1}(\tau))
\]
\[
= (-1)^m (-e^{-\tau} L_k^{0}(\tau) + e^{-\tau} L_{k-1}^{0}(\tau) - e^{-\tau} L_{k-1}^{0}(\tau)) = (-1)^{m+1} e^{-\tau} L_k^{0}(\tau). \]

**Lemma 4.3.** (a) For \( k \geq 0 \) and \( m \geq 0 \),
\[
(4.3) \quad D^m(L_k^{0}) = \sum_{l=0}^{\min(m,k)} (-1)^l \binom{m}{l} L_{k-l}^{0}.
\]

(b) If \( k > m \) then \( D^m(L_k^{0})(0) = 0 \).
**Proof.** The proof proceeds along similar lines to the proof of Lemma 3.5.

**Lemma 4.4.** For \( r \geq n - 1 \),

\[
D^{n-1}(L_r^0(\tau)) = (-1)^{n-1} \frac{1}{(n-1)!} \tau^{n-1} \mathcal{L}^{n-1}_{r-(n-1)}(\tau)
\]

**Proof.** From Lemma 4.2 we have

\[
\left( \frac{d}{d\tau} \right)^{n-1} (e^{-\tau} D^{n-1}(L_r^0(\tau))) = (-1)^{n-1} e^{-\tau} L_r^0(\tau),
\]

thus \( e^{-\tau} D^{n-1}(L_r^0(\tau)) \) is an \((n-1)\)-primitive of \((-1)^{n-1} e^{-\tau} L_r^0(\tau)\) and then, by Lemma 4.1,

\[
e^{-\tau} D^{n-1}(L_r^0(\tau)) = (-1)^{n-1} \frac{1}{(n-1)!} \tau^{n-1} \mathcal{L}^{n-1}_{r-n}(\tau)e^{-\tau} + Q(\tau)
\]

for some polynomial \( Q \) of degree at most \( n - 2 \). But this is impossible if \( Q \) does not vanish identically.

**Theorem 4.5.** Let \( a : (\mathbb{R} - \{0\}) \times (\mathbb{N} \cup \{0\}) \to \mathbb{C} \) be such that for each \( N \in \mathbb{N} \cup \{0\} \) there exists a positive constant \( c \) independent of \( \lambda \) and \( k \) such that

\[
|a(\lambda, k)| \leq c_N \left( |k|^{n-1} + \frac{1}{|\lambda|^{n-1}} \right) \frac{1}{|\lambda|^N(|k| + 1)^N}.
\]

Then for each \( s \in \mathbb{N} \cup \{0\} \) the function \( \Psi : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) defined by

\[
\Psi(\tau, t) := \sum_{k \geq 0} \int_{-\infty}^{\infty} a(\lambda, k) \mathcal{L}^s_k(\lambda \tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t}|\lambda|^n d\lambda
\]

is well defined and belongs to \( C^\infty([0, \infty) \times \mathbb{R}) \). Moreover, the series in (4.6) converges absolutely and uniformly on \([0, \infty) \times \mathbb{R}\).

**Proof.** For \( \lambda \neq 0 \) and \( k, s \in \mathbb{N} \cup \{0\} \) let \( \psi_{\lambda,k}^s \) be defined by (2.2). Since \( |\psi_{\lambda,k}^s| \leq 1 \) (cf. [3]), in order to prove the absolute and uniform convergence of the series in (4.6) it is enough to show that

\[
\sum_{k \geq q} \int_{-\infty}^{\infty} |a(\lambda, k)| |\lambda|^n d\lambda \leq c \sum_{k \geq 0} \int_{|\lambda(k+1)| \leq 1} |\lambda|^{n-1} d\lambda \leq c'' \sum_{k \geq 0} \frac{1}{(k+1)^2} < \infty.
\]

From (4.5) used with \( N = 0 \) and since \( k^{n-1} + 1/|\lambda|^{n-1} \leq 2/|\lambda|^{n-1} \) if \( |\lambda(k+1)| \leq 1 \), we get

\[
\sum_{k \geq q} \int_{|\lambda(k+1)| \leq 1} |a(\lambda, k)| |\lambda|^n d\lambda \leq c \sum_{k \geq 0} \int_{|\lambda(k+1)| \leq 1} |\lambda|^{n-1} d\lambda \leq c'' \sum_{k \geq 0} \frac{1}{(k+1)^2} < \infty.
\]
Moreover, for each $l$ if this follows from the assumption on $a$ formally. Thus we have (4.7) and so the series in (4.6) converges absolutely and uniformly.

To prove the remaining assertion of the lemma we observe that for $k \geq 1$,

$$\frac{d}{d\tau} L^s_k(|\tau|/2) = -\frac{|\lambda|}{2} \frac{k}{s+1} L^{s+1}_{k-1}(|\tau|/2)$$

and so, for $k \geq 1$,

$$\frac{\partial}{\partial \tau} (a(\lambda, k) L^s_k(|\tau|/2)e^{-|\lambda|\tau/4})$$

where $a_1(\lambda, k) := |\lambda|ka(\lambda, k)$ and $a_2(\lambda, k) := |\lambda|a(\lambda, k)$. A similar identity holds for $k = 0$ with the term involving $L^{s+1}_{k-1}$ deleted. Since $a_1$ and $a_2$ satisfy the same estimates assumed for $a$, it follows that the series defining $\Psi$ can be differentiated term by term and that $\partial \Psi / \partial \tau$ is a series of the form (4.6) with $a(\lambda, k)$ replaced by a new $\tilde{a}(\lambda, k)$ satisfying the estimates (4.5). Similarly, we can show that the same conclusion holds for $\partial \Psi / \partial t$. Now the lemma follows by induction. 

**Remark 4.6.** Let $a = a(\lambda, k)$ satisfy the conditions of Theorem 4.5. Then for $\tau \geq 0$ and $\lambda \neq 0$, the series

$$\Psi(\tau, \lambda) = \frac{|\lambda|}{2} \sum_{k \geq 0} a(\lambda, k) L_k^0(|\lambda|\tau/2)e^{-|\lambda|\tau/4}$$

converges absolutely (so it can be rearranged) and $\Psi(\tau, \cdot) \in L^1(\mathbb{R})$. Indeed, this follows from the assumption on $a(\lambda, k)$ and the fact that $|\varphi^{0}_{\lambda, k}| \leq 1$. Moreover, for each $l \geq 0$,

$$\frac{\partial^l \Psi(2|\lambda|^{-1}, \lambda)}{\partial \tau^l} = \frac{|\lambda|}{2} \sum_{k \geq 0} a(\lambda, k) \frac{\partial^l}{\partial \tau^l}(L_k^0(\tau)e^{-\tau/2}).$$

**Theorem 4.7.** Let $f \in S(H_n)$. Then, for $(\tau, t) \in [0, \infty) \times \mathbb{R}$,

$$Nf(\tau, t) = (-1)^{n-1} \int \frac{|\lambda|}{2} \sum_{k \geq 0} E(m^*)(\lambda, k + q)L_k^0(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t} d\lambda$$
and for \((\tau, t) \in (-\infty, 0] \times \mathbb{R}\),

\(\text{(4.9)}\)

\[
Nf(\tau, t) = (-1)^{n-1} \frac{|\lambda|}{2} \sum_{k \geq 0} \tilde{E}(m^{**})(\lambda, -k - p) L_k^0(-|\lambda|\tau/2)e^{\lambda|\tau|/4}e^{-i\lambda t} d\lambda.
\]

**Proof.** Let \(f \in \mathcal{S}(H_n)\) and \(m = \mathcal{F}f\). Since \(\{L_k^0(\tau)e^{-\tau/2}\}_{k \geq 0}\) is an orthonormal basis of \(L^2(0, \infty)\), Theorems 3.2 and 3.3 imply that for all \(\lambda \neq 0\),

\(\text{(4.10)}\)

\[
Nf(\tilde{\lambda}) = (-1)^{n-1} \frac{|\lambda|}{2} \sum_{k \geq 0} E(m^*)(\lambda, k + q)L_k^0(|\lambda|\tau/2)e^{-|\lambda|\tau/4}
\]

for a.e. \(\tau > 0\), and

\(\text{(4.11)}\)

\[
Nf(\tilde{\lambda}) = (-1)^{n-1} \frac{|\lambda|}{2} \sum_{k \geq 0} \tilde{E}(m^{**})(\lambda, -k - p)L_k^0(-|\lambda|\tau/2)e^{\lambda|\tau|/4}
\]

for a.e. \(\tau < 0\). We multiply these equalities by \(e^{-i\lambda t}\) and then integrate with respect to \(\lambda\). Since, by Lemma 4.5, the above series can be integrated term by term, (4.8) and (4.9) follow (because they hold for a.e. \(\tau > 0\) and a.e. \(\tau < 0\) respectively and have both sides continuous in \(\tau\)). \(\blacksquare\)

**Remark 4.8.** Theorem 4.7 also follows from formula (1.1) in [3] since the restrictions to \((\mathbb{R} - \{0\}) \times (\mathbb{N} \cup \{0\})\) can be extended to \(\hat{\mathcal{S}}(U(1), H_1)\).

In order to obtain, for a given \(m(\lambda, k)\) satisfying the hypothesis of Theorem 1.1, a function \(f \in \mathcal{S}(H_n)\) such that \(\mathcal{F}f = m\), Theorem 4.7 suggests considering the functions \(\varphi_1 : [0, \infty) \times \mathbb{R} \to \mathbb{C}\) and \(\varphi_2 : (-\infty, 0] \times \mathbb{R} \to \mathbb{C}\) defined by the right sides of (4.8) and (4.9) respectively. After checking that they agree for \(\tau = 0\), we will prove that the function \(\varphi : \mathbb{R}^2 \to \mathbb{C}\) given by \(\varphi_1\) and \(\varphi_2\) belongs to \(\mathcal{H}^\#\), and then we will choose \(f\) such that \(Nf = \varphi\). We fix such \(\varphi_1\) and \(\varphi_2\) from now on.

\(\mathcal{S}([0, \infty) \times \mathbb{R})\) will denote the space of functions \(h : [0, \infty) \times \mathbb{R} \to \mathbb{C}\) which are \(C^\infty\) and rapidly decreasing at infinity (with the derivatives at \(\tau = 0\) understood as lateral derivatives).

**Lemma 4.9.** Assume that \(m\) satisfies the conditions of Theorem 1.1. Then \(\varphi_1 \in \mathcal{S}([0, \infty) \times \mathbb{R})\) and \(\varphi_2 \in \mathcal{S}((-\infty, 0] \times \mathbb{R})\).

**Proof.** From our assumptions on \(m\), Theorem 6.1 in [3] gives functions \(f_1 = f_1(z, t)\) and \(f_2 = f_2(z, t)\) which are radial in \(z\), belong to \(\mathcal{S}(H_1)\) and

\[
f_1(z, t) = (-1)^{n-1} \frac{|\lambda|}{2} \sum_{k \geq 0} E(m^*)(\lambda, k + q)L_k^0(|\lambda||z|^2/2)e^{-|\lambda||z|^2/4}e^{-i\lambda t} d\lambda
\]
and
\[ f_2(z, t) = (-1)^{n-1} \int_R \frac{|\lambda|}{2} \sum_{k \geq 0} E(m^{**})(\lambda, -k - p) L_k^0(\|\lambda\|z^2/2)e^{-\|\lambda\|z^2/4}e^{-i\lambda t} d\lambda. \]

So \( \varphi_1(\tau, t) = f_1(\tau^{1/2}, t) \) for \((\tau, t) \in [0, \infty) \times \mathbb{R} \) and \( \varphi_2(\tau, t) = f_2(|\tau|^{1/2}, t) \) for \((\tau, t) \in (-\infty, 0] \times \mathbb{R} \), and the lemma follows by proceeding as in the proof of Theorem 6.1 in [3, pp. 410–412].

From the definition of \( \varphi_1 \) we have
\[
\varphi_1(\tau, t) = (-1)^{n-1} \sum_{k \geq 0} \sum_{0 \leq l \leq n-1} (-1)^l \left( \begin{array}{c} n-1 \\ l \end{array} \right) \frac{|\lambda|}{2} m^*(\lambda, k + q - l) \varphi_{\lambda, k}^0(\tau, t) d\lambda.
\]

Note that this series can be rearranged by Theorem 4.5. We first change the summation order, then we change the index in the sum on \( k \) setting \( j = k - q - l \), and finally we change \( l \) to \( n - 1 - l \) to obtain
\[
\varphi_1(\tau, t) = \sum_{0 \leq l \leq n-1} \sum_{j \geq -p+1+l} \sum_{0 \leq l \leq \min(j+p-1, n-1)} (-1)^l \left( \begin{array}{c} n-1 \\ l \end{array} \right) \frac{|\lambda|}{2} m^*(\lambda, j) \varphi_{\lambda, j-q+n-1-l}^0(\tau, t) d\lambda.
\]

Now we change the summation order again to get
\[
\varphi_1(\tau, t) = \sum_{j \geq -p+1} \int_{R} \frac{|\lambda|}{2} m^*(\lambda, j) \sum_{0 \leq l \leq \min(j+p-1, n-1)} (-1)^l \left( \begin{array}{c} n-1 \\ l \end{array} \right) \varphi_{\lambda, j-q+n-1-l}^0(\tau, t) d\lambda
\]
and so by Lemma 4.3,
\[
(4.12) \quad \varphi_1(\tau, t) = \sum_{j \geq q} \int_{R} \frac{|\lambda|}{2} m^*(\lambda, j) (D^{n-1} L_{j-q+n-1}^0)(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t} d\lambda
\]
\[
+ \sum_{-p+1 \leq j \leq q-1} \int_{R} \frac{|\lambda|}{2} m^*(\lambda, j) (D^{n-1} L_{j-q+n-1}^0)(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t} d\lambda.
\]

Then, by Lemma 4.4,
\[
\varphi_1(\tau, t) = \sum_{j \geq q} \int_{R} \frac{|\lambda|}{2} m(\lambda, j)(-1)^{n-1} \frac{1}{(n-1)!}(|\lambda|\tau/2)^{n-1} \psi_{\lambda, j-q}^{n-1}(\tau, t) d\lambda
\]
\[
+ \sum_{-p+1 \leq j \leq q-1} \int_{R} \frac{|\lambda|}{2} m^*(\lambda, j) (D^{n-1} L_{j-q+n-1}^0)(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t} d\lambda
\]
Thus, \( \varphi_1(\tau, t) = \xi_1(\tau, t) + \eta_1(\tau, t) \) where

\[
\xi_1(\tau, t) = \sum_{j \geq q} \int_{\mathbb{R}} \frac{|\lambda|}{2} m(\lambda, j)(-1)^{n-1} \frac{1}{(n-1)!} (|\lambda|\tau/2)^{n-1} \psi_{\lambda,j-q}(\tau, t) d\lambda,
\]

\[
\eta_1(\tau, t) = \sum_{-p+1 \leq j \leq q-1} \int_{\mathbb{R}} \frac{|\lambda|}{2} m^*(\lambda, j) \sum_{0 \leq l \leq j+p-1} (-1)^l \left( \frac{n-1}{l} \right) \psi_{\lambda,j+q+n-1-l}(\tau, t) d\lambda.
\]

Similarly, \( \varphi_2(\tau, t) = \xi_2(\tau, t) + \eta_2(\tau, t) \) where

\[
\xi_2(\tau, t) = \sum_{j \leq -p} \int_{\mathbb{R}} \frac{|\lambda|}{2} m(\lambda, j)(-1)^{n-1} \frac{1}{(n-1)!} (|\lambda|\tau/2)^{n-1} \psi_{\lambda,-j-p}(\tau, t) d\lambda,
\]

\[
\eta_2(\tau, t) = \sum_{-p < j \leq q-1} \int_{\mathbb{R}} \frac{|\lambda|}{2} m^*(\lambda, j) \sum_{0 \leq q_1 \leq j} (-1)^l \left( \frac{n-1}{l} \right) \psi_{\lambda,-j+p+n-1-l}(\tau, t) d\lambda.
\]

Observe that by Theorem 4.5, \( \xi_1(\tau, t) = \tau^{n-1} \tilde{\xi}_1(\tau, t) \) with \( \tilde{\xi}_1 \in C^\infty([0, \infty) \times \mathbb{R}) \), and so \( \frac{\partial \xi_1}{\partial \tau^s}(0, t) = 0 \) for \( 0 \leq l \leq n-2 \) and all \( t \in \mathbb{R} \). Analogously, \( \frac{\partial \xi_2}{\partial \tau^s}(0, t) = 0 \) for \( 0 \leq l \leq n-2, t \in \mathbb{R} \).

Our next step is to prove that

\[
\frac{\partial^s \varphi_1}{\partial \tau^s}(0, t) = \frac{\partial^s \varphi_2}{\partial \tau^s}(0, t), \quad 0 \leq s \leq n-2, \quad t \in \mathbb{R},
\]

i.e., for each \( t \),

\[
\frac{\partial^s \eta_1}{\partial \tau^s}(0, t) = \frac{\partial^s \eta_2}{\partial \tau^s}(0, t), \quad 0 \leq s \leq n-2.
\]

Observe that from Theorem 4.5 we have, for \( t \in \mathbb{R} \) and \( 0 \leq l \leq n-2 \),

\[
\frac{\partial^l \eta_1}{\partial \tau^l}(0, t) = \int_{\mathbb{R}} \frac{|\lambda|}{2} \left( \sum_{j=-p+1}^{-1} (-1)^{n-2} m(\lambda, j) G_{j,l}(\lambda) + \sum_{j=0}^{q-1} m(\lambda, j) \psi_{\lambda,l}(\lambda) \right) e^{-i\lambda t} d\lambda
\]

and

\[
\frac{\partial^l \eta_2}{\partial \tau^l}(0, t) = \int_{\mathbb{R}} \frac{|\lambda|}{2} \left( \sum_{j=-p+1}^{-1} m(\lambda, j) H_{j,l}(\lambda) + \sum_{j=0}^{q-1} (-1)^{n-2} m(\lambda, j) \psi_{\lambda,l}(\lambda) \right) e^{-i\lambda t} d\lambda
\]

with \( G_{j,l}, H_{j,l} \) independent of \( m \) and the integrals being absolutely convergent. But (4.14) holds if and only if

\[
\sum_{j=-p+1}^{-1} (-1)^{n-2} m(\lambda, j) G_{j,l}(\lambda) + \sum_{j=0}^{q-1} m(\lambda, j) \psi_{\lambda,l}(\lambda) = \sum_{j=-p+1}^{-1} m(\lambda, j) H_{j,l}(\lambda) + \sum_{j=0}^{q-1} (-1)^{n-2} m(\lambda, j) \psi_{\lambda,l}(\lambda)
\]

for all \( \lambda \neq 0 \).
From Theorem 4.7, this clearly holds if \( m = \mathcal{F}f \) for some \( f \in \mathcal{S}(H_n) \), because the first \( n - 2 \) derivatives of \( Nf(\cdot, t) \) are continuous at the origin.

Moreover, for each \( \lambda \) and \( j \) such that \( \lambda \neq 0 \) and \( -p + 1 \leq j \leq q - 1 \), Proposition 4.10 below gives an \( f \in \mathcal{S}(H_n) \) (depending of \( \lambda \) and \( j \)) such that for \( -p + 1 \leq k \leq q - 1 \), \( \mathcal{F}f(\lambda, k) = 1 \) if \( k = j \), and \( \mathcal{F}f(\lambda, k) = 0 \) if \( k \neq j \). So for such \( \lambda \) and \( j \), \( G_{j,l}(\lambda) = (-1)^{n-2}H_{j,l}(\lambda), \ 0 \leq l \leq n - 2 \).

**Proposition 4.10.** Given \( \lambda \neq 0 \) and \( n - 1 \) complex numbers \( \{a_j\}_{j=p+1}^{q-1} \), there exists \( f \in \mathcal{S}(H_n) \) such that

\[
(4.16) \quad \mathcal{F}f(\lambda, j) = a_j, \quad -p + 1 \leq j \leq q - 1.
\]

**Proof.** We take \( f \) such that \( Nf(\tau, \hat{\lambda}) := \omega(\|\lambda\|\tau/2)e^{\|\lambda\|\tau/4}\hat{\psi}(\lambda) \) where \( \omega, \hat{\psi} \in \mathcal{S}(\mathbb{R}) \), \( \omega \in C_c(\mathbb{R}) \) and \( \hat{\psi}(\lambda) = 1 \). We recall that from the definition of \( \mathcal{F}f \),

\[
\mathcal{F}f(\lambda, k) = \langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle, \quad 0 \leq k \leq q - 1,
\]

and

\[
\mathcal{F}f(\lambda, k) = \langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle, \quad -p + 1 \leq k < 0,
\]

and from Corollary 2.3,

\[
\mathcal{F}f(\lambda, k) = (-1)^{n-2}\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle
\]

for \( -p + 1 \leq k < 0 \). So, from our choice of \( f \) and since \( \hat{\psi}(\lambda) = 1 \), (4.16) reads \( a_k = 2|\lambda_0|^{-1}\langle (L^0_{k-q+n-1}H)^{(n-1)}, \omega \rangle \) for \( 0 \leq k \leq q - 1 \), and \( a_k = (-1)^{n-2}|\lambda_0|^{-1}\langle (L^0_{k-q+n-1}H)^{(n-1)}, \omega \rangle \) for \( -p + 1 \leq k < 0 \). But for \( -p + 1 \leq k \leq q - 1 \) we have \( 0 \leq k - q + n - 1 \leq n - 2 \). So, by (2.7),

\[
(L^0_{k-q+n-1}H)^{(n-1)} = \sum_{s=0}^{n-2}(L^0_{k-q+n-1})^{(n-2-s)}(0)\delta(s).
\]

To obtain (4.16) it is enough to find \( \beta_0, \ldots, \beta_{n-2} \) solving

\[
\sum_{s=0}^{n-2}(L^0_{k-q+n-1})^{(n-2-s)}(0)(-1)^{(s)}\beta_s = |\lambda|a_k/2, \quad 0 \leq k \leq q - 1,
\]

\[
\sum_{s=0}^{n-2}(L^0_{k-q+n-1})^{(n-2-s)}(0)(-1)^{(s)}\beta_s = (-1)^{n-2}|\lambda|a_k/2, \quad -p + 1 \leq k < 0,
\]

and then to find \( \omega \in C_c^\infty(\mathbb{R}) \) such that \( \omega^{(s)}(0) = \beta_s \) for \( s = 0, 1, \ldots, n - 2 \). This is a linear system in \( \{\omega^{(s)}(0)\}_{s=0}^{n-2} \). Since \( (L^0_k)^{(s)}(0) = (-1)^{s}\binom{k}{s} \), the associated \( (n-1) \times (n-1) \) matrix \( A \) is lower triangular with \( \pm 1 \) on the diagonal. So \( A \) is nonsingular and the existence of \( \beta_0, \ldots, \beta_{n-2} \) follows. Now, we take
$\omega = P(\tau)\tilde{\omega}(\tau)$, where $P$ is a polynomial of degree $n - 1$ with $P(s)(0) = \beta_s$ for $s = 0, \ldots, n - 2$ and where $\tilde{\omega} \in C_c^\infty(\mathbb{R})$, $\text{supp}(\tilde{\omega}) \subset (-2, 2)$ and $\tilde{\omega}(\tau) = 1$ for $\tau \in (-1, 1)$.

A classical result due to Borel states that given a sequence $\{a_j\}_{j=1}^\infty$ of complex numbers, there exists a $C^\infty(\mathbb{R})$ function $\psi$ such that $\psi^{(j)}(0) = a_j$ for all $j$. Moreover $\psi$ can be taken in $C_c^\infty(\mathbb{R})$. A similar result holds in two variables. Since we have not been able to find it in the literature we give a proof for completeness.

**Lemma 4.11.** Let $\{a_j(t)\}_{j=1}^\infty$ be a sequence of functions in $\mathcal{S}(\mathbb{R})$. Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^2)$ such that $\frac{\partial^j \psi}{\partial \tau^j}(0, t) = a_j(t)$.

**Proof.** Let $\tilde{\omega}$ be as in the proof of Proposition 4.10. For a given sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers we set

$$g_n(\tau, t) := \frac{a_n(t)}{n!} \tau^n \tilde{\omega}(\tau), \quad f_n(\tau, t) := \frac{1}{\lambda_n^2} g_n(\lambda_n \tau, t) = \frac{1}{\lambda_n^2} \frac{a_n(t)}{n!} \tau^n \tilde{\omega}(\lambda_n \tau).$$

Let

$$f(\tau, t) := \sum_{n=1}^\infty f_n(\tau, t).$$

Clearly, the lemma will follow if we can prove (for a suitable sequence $\{\lambda_n\}$) that

$$\left\| t^s \frac{\partial^l}{\partial t^l} \frac{\partial^k}{\partial \tau^k} f_n \right\|_{\infty} \leq \frac{1}{2^n} \quad \text{for all } 0 \leq k, l, s \leq n - 1, \quad (4.17)$$

We take $\lambda_n \geq 1$ for all $n$. Taking into account that $k \leq n - 1$ and $a_j(t) \in \mathcal{S}(\mathbb{R})$, we can apply the Leibniz rule to get a positive constant $c_n$ such that

$$\left| t^s \frac{\partial^l}{\partial t^l} \frac{\partial^k}{\partial \tau^k} f_n \right| \leq t^s \frac{c_n}{\lambda_n^{n!}} \left| \frac{\partial^l a_n}{\partial t^l} \right| \leq \frac{c_n}{\lambda_n^{n!}} \sum_{s, l=0}^{n-1} \left| t^s \frac{\partial^l a_n}{\partial t^l} \right|_{\infty}.$$ 

Now (4.17) follows by choosing $\lambda_n$ such that, in addition,

$$\frac{1}{\lambda_n} \leq \frac{c_n}{2^{n!}} \sum_{s, l=0}^{n-1} \left| t^s \frac{\partial^l a_n}{\partial t^l} \right|_{\infty}. \quad \blacksquare$$

**Definition 4.12.** Let $m = m(\lambda, k)$ be a function satisfying the conditions of the statement of Theorem 1.1. We define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(\tau, t) = \begin{cases} \varphi_1(\tau, t) & \text{for } \tau > 0, t \in \mathbb{R}, \\ \varphi(\tau, t) = \varphi_2(\tau, t) & \text{for } \tau \leq 0, t \in \mathbb{R}. \end{cases} \quad (4.18)$$

**Proof of Theorem 1.1.** Let $m$ and $\varphi$ be as in Definition 4.12. By Theorems 3.2 and 3.3, it remains to see that $\varphi$ belongs to $\mathcal{H}^\#$ and that if we take $f \in \mathcal{S}(H_n)$ such that $N f = \varphi$ then $\mathcal{F} f = m$. To see that $\varphi \in \mathcal{H}^\#$ we must
find \( \psi_1 \) and \( \psi_2 \) in \( S(\mathbb{R}^2) \) such that
\[
\varphi(\tau, t) = \psi_2(\tau, t) + \tau^{n-1}\psi_1(\tau, t)H(\tau)
\]
(where \( H \) is the Heaviside function), i.e.,
\[
\psi_2(\tau, t) = \begin{cases} 
\varphi_2(\tau, t) & \text{for } \tau \leq 0, \\
\varphi_1(\tau, t) - \tau^{n-1}\psi_1(\tau, t) & \text{for } \tau > 0.
\end{cases}
\]

(4.19)

For a given \( \psi_1 \in S(\mathbb{R}^2) \), we define \( \psi_2 \) by (4.19). In view of Lemma 4.9 and (4.13), \( \psi_2 \in S(\mathbb{R}^2) \) if and only if for a suitable \( \psi_1 \in S(\mathbb{R}^2) \),
\[
\frac{\partial^j \varphi_2}{\partial \tau^j}(0, t) = \frac{\partial^j \varphi_1}{\partial \tau^j}(0, t) - \left( \begin{array}{c} j \\ n - 1 \end{array} \right)(n - 1)! \frac{\partial^{j-(n-1)} \psi_1}{\partial \tau^{j-(n-1)}}(0, t).
\]

(4.20)

for all \( j \geq n - 1 \). But Lemma 4.11 gives a function \( \psi_1 \in S(\mathbb{R}^2) \) such that
\[
\frac{\partial^k \psi_1}{\partial \tau^k}(0, t) = \frac{1}{(k+n-1)(n-1)!} \frac{\partial^{k+n-1}(\varphi_2 - \varphi_1)}{\partial \tau^{k+n-1}}(0, t)
\]
for \( k \in \mathbb{N} \cup \{0\} \), i.e., (4.20) holds. Thus \( \varphi \in H^\# \).

Let \( f \in S(H^n) \) be such that \( Nf = \varphi \). To see that \( \mathcal{F}f = m \) we proceed as follows. For \( k \geq 0 \) and \( \lambda \neq 0 \) we have
\[
\mathcal{F}f(\lambda, k) = \langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \widehat{\lambda}) \rangle
\]
\[
= (-1)^{n-1} \int_0^\infty L^0_{k-q+n-1}(\tau) \frac{\partial^{n-1}}{\partial \tau^{n-1}}(2|\lambda|^{-1}e^{-\tau/2}\varphi_1(2|\lambda|^{-1}\tau, \widehat{\lambda})) d\tau.
\]

From the definition of \( \varphi_1 \) and Remark 4.6,
\[
2|\lambda|^{-1}e^{-\tau/2}\varphi_1(2|\lambda|^{-1}\tau, \widehat{\lambda}) = \sum_{j \geq 0} E(m^*)(\lambda, j + q) L^0_j(\tau)e^{-\tau}.
\]

Now, from similar computations to those that give (4.12) (allowed again by Remark 4.6) we get
\[
2|\lambda|^{-1}e^{-\tau/2}\varphi_1(2|\lambda|^{-1}\tau, \widehat{\lambda}) = \frac{(-1)^{n-1}}{(n-1)!} \frac{|\lambda|}{2} \sum_{j \geq q} m(\lambda, j) D^{n-1}(L^0_{j-q+n-1})(\tau)e^{-\tau}
\]
\[
+ \frac{|\lambda|}{2} \sum_{-p+1 \leq j \leq q-1} m^*(\lambda, j) D^{n-1}(L^0_{j-q+n-1})(\tau)e^{-\tau}.
\]

Then, by Lemma 4.2,
\[
2|\lambda|^{-1}(-1)^{n-1} \left( \frac{d}{d\tau} \right)^{n-1} e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \widehat{\lambda})
\]
\[
= \sum_{j \geq q} m(\lambda, j) L^0_{j-q+n-1}e^{-\tau} + \sum_{-p+1 \leq j \leq q-1} m^*(\lambda, j) L^0_{j-q+n-1}(\tau)e^{-\tau}.
\]
Our assumptions on \( m \) imply that \( \sum_{j \geq q} m(\lambda, j)L^0_{j-q+n-1}e^{-\tau/2} \) belongs to \( L^2((0, \infty), d\tau) \). Also \( \sum_{0}^{\infty} L^0_{k-q+n-1}(\tau)L^0_{j-q+n-1}e^{-\tau} = \delta_{jk} \); then from (4.21) it follows that \( Ff(\lambda, k) = m(\lambda, k) \) for \( k \geq q \) and \( Ff(\lambda, k) = m^*(\lambda, k) \) for \( 0 \leq k \leq q-1 \). Since \( m(\lambda, k) = m^*(\lambda, k) \) for \( k \geq 0 \) we have proved that \( Ff(\lambda, k) = m(\lambda, k) \) for \( k \geq 0 \).

A completely similar argument starting with the facts that

\[
\mathcal{F}f(\lambda, k) = \langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ff(-2|\lambda|^{-1}\tau, \hat{\lambda}) \rangle
\]

\[
= (-1)^{n-1} \int_{0}^{\infty} L^0_{-k-p+n-1}(\tau) \frac{\partial^{n-1}}{\partial \tau^{n-1}}(2|\lambda|^{-1}e^{-\tau/2}\varphi_2(-2|\lambda|^{-1}\tau, \hat{\lambda})) d\tau
\]

and that for \( \tau < 0 \),

\[
2|\lambda|^{-1/2}e^\tau \varphi_2(2|\lambda|^{-1}\tau, \hat{\lambda}) = \sum_{j \geq 0} E(m^{**}(\lambda, j+q)L^0_j(-\tau)e^{\tau}
\]

can be used in the case \( k < 0 \) to complete the proof of the theorem. □

**Remark 4.13.** Recall that for \( h \in S(H_1) \) and \( H(\lambda, k) = \mathcal{F}_1 h(\lambda, k) \) we have \( M^+H = \mathcal{F}_1 (|z|^2/4 + it)h \) and \( M^-H = \mathcal{F}_1 (|z|^2/4 - it)h \) (cf. [3, p. 407]).

For \( f \in S(H_n) \) let \( f_1 \in S(H_1) \) be the function given by

\[
f_1(z, t) = Nf(|z|^2, t).
\]

We have seen that

\[
\mathcal{F}_1(f_1)(\lambda, k) = E(\mathcal{F}f)(\lambda, k + q).
\]

Consider the map \( \Xi : S(H_n) \to S(H_1) \) defined by \( \Xi(f) = f_1 \), let \( B(z, w) \) be the quadratic form given in the introduction and set \( B(z) = B(z, z) \). It is immediate to see that \( N(B(z)f) = \tau Nf \) and this says that \( \Xi((B(z)/4 \pm it)f) = (|z|^2/4 \pm it)\hat{f}_1 \). Then we can conclude that

\[
M^\pm(\mathcal{F}_1f_1) = E(\mathcal{F}(B(z)/4 \pm it)f).
\]

A similar expression can be obtained for \( M^\pm(\mathcal{F}_1g_1)(\lambda, k) \) (where \( g_1(z, t) = Nf(-|z|^2, t) \)) that involves \( E(\mathcal{F}(B(z)/4 \pm it)f)(\lambda, -k-p) \) for \( k \geq n-1 \) and \( \tilde{E}(\mathcal{F}(B(z)/4 \pm it)f)(\lambda, k) \) for \( 0 \leq k \leq n-2 \).

**References**


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