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A SPHERICAL TRANSFORM ON SCHWARTZ FUNCTIONS ON THE HEISENBERG GROUP ASSOCIATED TO THE ACTION OF $U(p,q)$

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Abstract. Let $\mathcal{S}(H_n)$ be the space of Schwartz functions on the Heisenberg group H_n. We define a spherical transform on $\mathcal{S}(H_n)$ associated to the action (by automorphisms) of $U(p,q)$ on H_n, $p+q=n$. We determine its kernel and image and obtain an inversion formula analogous to the Godement–Plancherel formula.

1. Introduction. Let $n \geq 2$ and let p,q be natural numbers such that $p+q=n$. Let H_n be the Heisenberg group defined by $H_n = \mathbb{C}^n \times \mathbb{R}$ with group law

$$(z,t)(z',t') = \big(z+z', t+t' - \tfrac{1}{2}\operatorname{Im} B(z,z')\big)$$

where

$$B(z,w) = \sum_{j=1}^{p} z_j \overline{w}_j - \sum_{j=p+1}^{n} z_j \overline{w}_j.$$

For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^p, x'' \in \mathbb{R}^q$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ via the map

$$\varphi(x',x'',y',y'') = (x'+iy', x''-iy''), \qquad x',\, y' \in \mathbb{R}^p,\, x'', y'' \in \mathbb{R}^q.$$

In this setting, the form $-\operatorname{Im} B(z,w)$ agrees with the standard symplectic form on $\mathbb{R}^{2(p+q)}$, and the vector fields

$$X_j = -\frac{1}{2} y_j \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}, \quad Y_j = \frac{1}{2} x_j \frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}, \quad j = 1,\ldots,n, \quad T = \frac{\partial}{\partial t}$$

form a standard basis for the Lie algebra h_n of H_n. Thus H_n can be viewed as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ via the map $(x,y,t) \mapsto (\varphi(x,y),t)$. From now on, we will use freely this identification.

Let $\mathcal{S}(H_n)$ be the Schwartz space on H_n and let $\mathcal{S}'(H_n)$ be the space of corresponding tempered distributions. Consider the action of $U(p,q)$ on H_n given by $g \cdot (z,t) = (gz,t)$ (note that since we have assumed that $p,q \geq 1$, $U(p,q)$ is noncompact). So $U(p,q)$ acts on $L^2(H_n)$, $\mathcal{S}(H_n)$ and $\mathcal{S}'(H_n)$ in

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the canonical way. The subalgebra $\mathcal{U}_{U(p,q)}(h_n)$ of left invariant differential operators which commute with this action is generated by L and T where

$$L = \sum_{j=1}^{p}(X_j^2 + Y_j^2) - \sum_{j=p+1}^{n}(X_j^2 + Y_j^2)$$

and T is as above (cf. [5]). We observe that it is commutative, since T belongs to the center of h_n.

Moreover, for $\lambda \in \mathbb{R} - \{0\}$ and $k \in \mathbb{Z}$, there exists a tempered $U(p,q)$-invariant distribution (on H_n) $S_{\lambda,k}$ satisfying

$$LS_{\lambda,k} = -|\lambda|(2k + p - q)S_{\lambda,k}, \qquad iTS_{\lambda,k} = \lambda S_{\lambda,k} \tag{1.1}$$

and such that, for all $f \in \mathcal{S}(H_n)$,

$$f = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f * S_{\lambda,k}|\lambda|^n \, d\lambda \tag{1.2}$$

(cf. [5]).

Let us recall some facts concerning the compact case $p = n$, $q = 0$, i.e., when $U(p,q) = U(n)$. In this case it is well known (see [6]) that $\mathcal{U}_{U(n)}(h_n)$ is a commutative algebra if and only if the convolution algebra $L^1_{U(n)}(H_n)$ of $U(n)$-invariant integrable functions is commutative, that is, $(H_n, U(n))$ is a Gelfand pair. Its spectrum, denoted by $\Delta(U(n), H_n)$, can be identified, via integration, with the set of bounded spherical functions of the pair $(U(n), H_n)$. These spherical functions can be classified (see [2]) as:

a) The *spherical functions of type I*, i.e., those that restricted to the center of H_n are nontrivial characters. These are given by

$$\Phi^{n-1}_{\lambda,k}(z,t) := e^{-i\lambda t}\mathcal{L}_k^{n-1}(|\lambda|\,|z|^2/2)e^{-|\lambda|\,|z|^2/4}, \qquad \lambda \neq 0,\ k \geq 0,$$

where $\mathcal{L}_k^{n-1}$ is the Laguerre polynomial of order $n-1$ and degree k normalized by $\mathcal{L}_k^{n-1}(0) = 1$.

b) The *spherical functions η_w of type II*, i.e., those that are constant on the center. They are given, for $w \in \mathbb{C}^n - \{0\}$, by

$$\eta_w(z,t) = \frac{2^{n-1}(n-1)!}{(|z|\,|w|)^{n-1}} J_{n-1}(|z|\,|w|)$$

where J_{n-1} is the Bessel function of order $n-1$ of the first kind, and by

$$\eta_0(z,t) = 1.$$

We set

$$\Delta_1(U(n), H_n) = \{\Psi \in \Delta(U(n), H_n) : \Psi \text{ is of type I}\},$$
$$\Delta_2(U(n), H_n) = \{\Psi \in \Delta(U(n), H_n) : \Psi \text{ is of type II}\}.$$

For $f \in L^1_{U(n)}(H_n)$, its *spherical transform* $\widehat{f} : \Delta(U(n), H_n) \to \mathbb{C}$ is defined by

$$\widehat{f}(\Psi) = \int_{H_n} f(z,t)\,\overline{\Psi(z,t)}\,dz\,dt$$

where $dzdt$ is the Haar measure (i.e., the Lebesgue measure) on H_n.

In this case ($p = n$, $q = 0$) the image of the radial Schwartz functions on H_n under the map $f \mapsto \widehat{f}$ is explicitly described in [3]. The notion of rapidly decreasing functions on $\Delta(U(n), H_n)$ is introduced and it is proved that the image of $\mathcal{S}(H_n)$ under the spherical transform is the space $\widehat{\mathcal{S}}(U(n), H_n)$ of rapidly decreasing functions F on $\Delta(U(n), H_n)$ such that certain "derivatives" of F are also rapidly decreasing (see Definitions 6.1 and 6.3 in [3]).

Also, in [4], a map $\mathcal{E} : \Delta(U(n), H_n) \to [0, \infty) \times \mathbb{R}$ is defined by $\mathcal{E}(\Psi) = (-\widehat{L}(\Psi), i\widehat{T}(\Psi))$, where $\widehat{L}(\Psi)$ and $\widehat{T}(\Psi)$ denote the eigenvalues of L and T respectively, associated to Ψ. The image of $\mathcal{E}$ is the so-called *Heisenberg fan* $\mathcal{A}(U(n), H_n)$ and it is the set

$$\{(|\lambda|(2k+n), \lambda) : \lambda \neq 0,\ k \in \mathbb{N} \cup \{0\}\} \cup \{[0, \infty) \times \{0\}\}.$$

It is proved that $\mathcal{E}$ is a homeomorphism from $\Delta(U(n), H_n)$ (equipped with the Gelfand topology) onto the Heisenberg fan (provided with the topology induced from $\mathbb{R}^2$).

From the above considerations it is natural to consider, for arbitrary $p, q \in \mathbb{N}$ with $p + q = n$ and for $f \in \mathcal{S}(H_n)$, the "spherical transform" $\mathcal{F}(f) : (\mathbb{R} - \{0\}) \times \mathbb{Z} \to \mathbb{C}$ defined by

$$\mathcal{F}(f)(\lambda, k) = \langle S_{\lambda,k}, f\rangle. \tag{1.3}$$

Our aim is to characterize $\mathcal{F}(\mathcal{S}(H_n))$ and $\mathrm{Ker}(\mathcal{F})$. In order to state our results, let us introduce some additional notations.

For $m : (\mathbb{R}-\{0\}) \times \mathbb{Z} \to \mathbb{C}$ and $(\lambda, k) \in (\mathbb{R}-\{0\}) \times \mathbb{Z}$ define

$$m^*(\lambda,k) = \begin{cases} m(\lambda,k) & \text{if } k \geq 0, \\ (-1)^{n-2} m(\lambda,k) & \text{if } k < 0. \end{cases}$$

$$m^{**}(\lambda,k) = \begin{cases} m(\lambda,k) & \text{if } k < 0, \\ (-1)^{n-2} m(\lambda,k) & \text{if } k \geq 0. \end{cases}$$

We also set

$$\begin{aligned} E(m)(\lambda,k) &= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k-l), \\ \widetilde{E}(m)(\lambda,k) &= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k+l). \end{aligned} \tag{1.4}$$

Our main result is the following

THEOREM 1.1. *Assume that* $p,q\geq 1$ *with* $p+q=n$. *Then* $\mathcal{F}(\mathcal{S}(H_n))$ *is the space of functions* $m:(\mathbb{R}-\{0\})\times\mathbb{Z}\to\mathbb{C}$ *such that*

(i) *we have the estimate*

$$|m(\lambda,k)|\leq c_N\left(|k|^{n-1}+\frac{1}{|\lambda|^{n-1}}\right)\frac{1}{|\lambda|^N(|k|+1)^N},\qquad N\in\mathbb{N}\cup\{0\},\tag{1.5}$$

(ii) *the functions defined on* $(\mathbb{R}-\{0\})\times(\mathbb{N}\cup\{0\})$ *by*

$$(\lambda,k)\mapsto E(m^*)(\lambda,k+q),\qquad (\lambda,k)\mapsto\widetilde{E}(m^{**})(\lambda,-k-p)$$

extend to two functions belonging to $\widehat{\mathcal{S}}(U(1),H_1)$.

We also obtain an inversion formula for $\mathcal{F}$ analogous to the Godement–Plancherel formula and we determine the kernel of $\mathcal{F}$.

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2. Notations and preliminaries. Let us introduce some notation and recall some known facts. Let H denote the Heaviside function (i.e., $H(\tau)=\chi_{(0,\infty)}(\tau)$) and let $\mathcal{H}$ be the space of functions $\varphi:\mathbb{R}\to\mathbb{C}$ such that

$$\varphi(\tau)=\varphi_1(\tau)+\tau^{n-1}\varphi_2(\tau)H(\tau),\qquad \varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}).$$

It is proved in [9] that $\mathcal{H}$, provided with a suitable topology, is a Fréchet space. Moreover, $\mathcal{H}$ is the space of functions $\varphi\in C^\infty(\mathbb{R}-\{0\})$ that are rapidly decreasing at $\pm\infty$ in the usual sense, have the limits $\lim_{\tau\to 0^+}\partial^j\varphi/\partial\tau^j$ and $\lim_{\tau\to 0^-}\partial^j\varphi/\partial\tau^j$ for all $j\in\mathbb{N}$, and admit $n-2$ continuous derivatives at the origin. For $p+q=n$, $p,q\geq 1$, in [9] there is also given a linear, continuous and surjective map $N:\mathcal{S}(\mathbb{R}^n)\to\mathcal{H}$ whose adjoint $N':\mathcal{H}'\to\mathcal{S}'(\mathbb{R}^n)^{O(p,q)}$ is a linear homeomorphism onto the space of $O(p,q)$-invariant tempered distributions on $\mathbb{R}^n$. As pointed out in [5], this construction also works to describe the space $\mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$, i.e., there exists a linear, continuous and surjective map, still denoted by $N:\mathcal{S}(\mathbb{C}^n)\to\mathcal{H}$, whose adjoint $N':\mathcal{H}'\to\mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$ is a homeomorphism. For $f\in\mathcal{S}(H_n)$, we will write $Nf(\tau,t)$ for $N(f(\cdot,t))(\tau)$. We have (cf. (2.11) in [5])

$$Nf(\tau,t)=\int\limits_{\varrho>|\tau|}Mf(\cdot,t)(\varrho,\tau)(\varrho+\tau)^{p-1}(\varrho-\tau)^{q-1}\,d\varrho,$$

where for $\varrho\geq|\sigma|$,

$$Mf(\cdot,t)(\varrho,\sigma):=\int\limits_{S^{2p-1}\times S^{2q-1}}f\left(\left(\frac{\varrho+\sigma}{2}\right)^{1/2}w_u,\left(\frac{\varrho-\sigma}{2}\right)^{1/2}w_v,t\right)dw_u\,dw_v.$$

Let $\mathcal{H}^{\#}$ be the space of functions φ on $\mathbb{R}^2$ of the form

$$\varphi(\tau,t)=\varphi_1(\tau,t)+\tau^{n-1}H(\tau)\varphi_2(\tau,t),\qquad \varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R}^2).$$

REMARK 2.1. A straightforward adaptation of the proofs of Lemmas 4.2 and 4.3 in [9] shows that $N:\mathcal{S}(H_n)\to\mathcal{H}^{\#}$ is surjective.

In order to give an explicit expression of the distributions $S_{\lambda,k}$ we recall the definition of the Laguerre polynomials. For nonnegative integers m and α let $L_m^\alpha(\tau)$ (see, e.g., [8, pp. 99–101]) be given by

$$L_m^0(\tau)=\sum_{j=0}^{m}\binom{m}{j}(-1)^j\frac{\tau^j}{j!},\qquad L_{m-1}^{\alpha+1}(\tau)=-\frac{d}{d\tau}L_m^\alpha(\tau). \tag{2.1}$$

For $\lambda\in\mathbb{R}$, $k,s\in\mathbb{N}\cup\{0\}$ and $(\tau,t)\in[0,\infty)\times\mathbb{R}$ we set

$$\psi_{\lambda,k}^s(\tau,t):=e^{-i\lambda t}\mathcal{L}_k^s(|\lambda|\tau/2)e^{-|\lambda|\tau/4}, \tag{2.2}$$

$$\varphi_{\lambda,k}^s(\tau,t):=e^{-i\lambda t}L_k^s(|\lambda|\tau/2)e^{-|\lambda|\tau/4}, \tag{2.3}$$

where $\mathcal{L}_k^s$ denotes the Laguerre polynomial of degree k and order s normalized by $\mathcal{L}_k^s(0)=1$, i.e., given by $\mathcal{L}_k^s(\tau)=L_k^s(\tau)/\binom{k+s}{k}$.

It is well known that the family $e^{-\tau/2}L_m^0(\tau)$, $m\ge 0$, is an orthonormal basis of $L^2(0,\infty)$. Thus (cf. [5, Theorem 4.1 and Remarks 4.2, 4.3])

$$S_{\lambda,k}=F_{\lambda,k}\otimes e^{-i\lambda t}, \tag{2.4}$$

with $F_{\lambda,k}\in\mathcal{S}'(\mathbb{C}^n)$ defined by

$$\langle F_{\lambda,k},g\rangle=\langle (L_{k-q+n-1}^0H)^{(n-1)},\tau\mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau)\rangle \tag{2.5}$$

for $k\ge 0$, $\lambda\ne 0$ and by

$$\langle F_{\lambda,k},g\rangle=\langle (L_{-k-p+n-1}^0H)^{(n-1)},\tau\mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(-2|\lambda|^{-1}\tau)\rangle \tag{2.6}$$

for $k<0$, $\lambda\ne 0$.

For $\varphi\in\mathcal{H}$ and $j\in\mathbb{N}\cup\{0\}$ a computation gives

$$\begin{aligned}\langle (L_j^0H)^{(n-1)},\varphi\rangle\\ &=\int_0^\infty (L_j^0)^{(n-1)}\varphi(\tau)\,d\tau+\sum_{0\le s\le n-2}(L_j^0)^{(n-2-s)}(0)\langle\delta^{(s)},\varphi\rangle.\end{aligned} \tag{2.7}$$

LEMMA 2.2. *For $r\in\mathbb{Z}$ such that $0\le r\le n-2$ and for $\varphi\in\mathcal{H}$,*
$$\langle (L_r^0H)^{(n-1)},\tau\mapsto e^{-\tau/2}\varphi(\tau)\rangle=(-1)^{n-2}\langle (L_{n-2-r}^0H)^{(n-1)},\tau\mapsto e^{-\tau/2}\varphi(-\tau)\rangle.$$

Proof. A computation using (2.7) gives

$$\begin{aligned}&\langle (L_r^0H)^{(n-1)},\tau\mapsto e^{-\tau/2}\varphi(\tau)\rangle\\ &=\sum_{0\le l\le n-2}\ \sum_{\max(n-2-r,l)\le j\le n-2}\frac{1}{2^{j-l}}\binom{j}{l}(-1)^{n-j}\binom{r}{n-2-j}\langle\delta^{(l)},\varphi\rangle\end{aligned}$$

and also

$$\langle (L^0_{n-2-r}H)^{(n-1)}, \tau \mapsto e^{-\tau/2}\varphi(-\tau)\rangle = \sum_{0\le l\le n-2}\ \sum_{\max(r,l)\le j\le n-2} \frac{1}{2^{j-l}}\binom{j}{l}(-1)^{n-j+l}\binom{n-2-r}{n-2-j}\langle \delta^{(l)},\varphi\rangle.$$

To show the lemma it is enough to see that for $0 \le r \le n-2$ and $0 \le l \le n-2$,

$$\sum_{\max(n-2-r,l)\le j\le n-2} \frac{1}{2^{j}}\binom{j}{l}(-1)^{n-j}\binom{r}{n-2-j} = (-1)^{n-2}\sum_{\max(r,l)\le j\le n-2} \frac{1}{2^{j}}\binom{j}{l}(-1)^{n-j+l}\binom{n-2-r}{n-2-j},$$

i.e., to show that for $0 \le r \le n-2$, the following polynomial identity holds:

$$\sum_{0\le l\le n-2} t^l \sum_{\max(n-2-r,l)\le j\le n-2} \frac{1}{2^{j}}\binom{j}{l}(-1)^{n-j}\binom{r}{n-2-j} = (-1)^{n-2}\sum_{0\le l\le n-2} t^l \sum_{\max(r,l)\le j\le n-2} \frac{1}{2^{j}}\binom{j}{l}(-1)^{n-j+l}\binom{n-2-r}{n-2-j}. \tag{2.8}$$

If we change the summation order, (2.8) becomes

$$(-1)^n \sum_{n-2-r\le j\le n-2} (-1)^j\binom{r}{n-2-j}\frac{1}{2^j}\sum_{0\le l\le j}\binom{j}{l}t^l = \sum_{r\le j\le n-2}\frac{1}{2^j}\binom{n-2-r}{n-2-j}(-1)^j\sum_{0\le l\le j}\binom{j}{l}(-1)^l t^l, \tag{2.9}$$

which, by the binomial formula, is equivalent to

$$(-1)^n \sum_{n-2-r\le j\le n-2}\binom{r}{n-2-j}\left(-\frac{t+1}{2}\right)^j = \sum_{r\le j\le n-2}\binom{n-2-r}{n-2-j}\left(\frac{t-1}{2}\right)^j, \tag{2.10}$$

i.e., to

$$\left(-\frac{1+t}{2}\right)^{n-2}\sum_{n-2-r\le j\le n-2}\binom{r}{n-2-j}\left(-\frac{2}{1+t}\right)^{n-2-j} = (-1)^n\left(\frac{t-1}{2}\right)^{n-2}\sum_{r\le j\le n-2}\binom{n-2-r}{n-2-j}\left(\frac{2}{t-1}\right)^{n-2-j}. \tag{2.11}$$

After changing j to $n-2-j$ and recalling that $0 \le r \le n-2$, by the binomial formula (2.11) reduces to

$$\left(-\frac{1+t}{2}\right)^{n-2}\left(1-\frac{2}{1+t}\right)^{r} = (-1)^n\left(\frac{t-1}{2}\right)^{n-2}\left(1+\frac{2}{t-1}\right)^{n-2-r},$$

which clearly holds. ∎

Corollary 2.3. *Let $g \in \mathcal{S}(\mathbb{C}^n)$. For $0 \le k \le q-1$, $\lambda \ne 0$ we have*

$$\text{(2.12)}\quad \langle F_{\lambda,k}, g\rangle = (-1)^{n-2}\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(-2|\lambda|^{-1}\tau)\rangle,$$

and

$$\text{(2.13)}\quad \langle F_{\lambda,k}, g\rangle = (-1)^{n-2}\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau)\rangle$$

for $-p+1 \le k < 0$. ∎

For a given set X and for $f : X \times \mathbb{R} \to \mathbb{C}$, $\lambda \in \mathbb{R}$ we set $f(z,\widehat{\lambda}) := (t \mapsto f(z,t))^\wedge(\lambda)$ where $()^\wedge$ denotes the one-dimensional Fourier transform (provided that it exists).

Proposition 2.4. Ker$(\mathcal{F})$ = Ker(N).

Proof. If $f \in \mathcal{S}(H_n)$ and $Nf = 0$, then, by (2.5) and (2.6), $\mathcal{F}(f)(\lambda,k) = \langle S_{\lambda,k}, f\rangle = \langle F_{\lambda,k} \otimes e^{-i\lambda t}, f\rangle = 0$ and so $\mathcal{F}(f) = 0$.

If $\mathcal{F}(f) = 0$, from the definition of $S_{\lambda,k}$, for $k \ge 0$ and $\lambda \ne 0$ we have

$$\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda})\rangle = 0$$

and, by Lemma 2.2, for $-p+1 \le k < 0$,

$$\begin{aligned}&\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda})\rangle \\ &= (-1)^{n-2}\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau,\widehat{\lambda})\rangle = 0.\end{aligned}$$

Thus, for $j \ge 0$,

$$2|\lambda|^{-1}\int_0^\infty e^{-\tau/2}L^0_j(\tau)e^{\tau/2}\,\frac{d^{n-1}}{d\tau^{n-1}}(e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda}))\,d\tau = 0.$$

Thus

$$\frac{d^{n-1}}{d\tau^{n-1}}(e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda})) = 0 \quad \text{for } \tau \ge 0,\ \lambda \ne 0.$$

So for such τ and λ, $e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda}) = P_\lambda(\tau)$ where $P_\lambda(\tau)$ is a polynomial of degree at most $n-2$ with coefficients which (in principle) depend on λ. Thus $Nf(2|\lambda|^{-1}\tau,\widehat{\lambda}) = e^{\tau/2}P_\lambda(\tau)$. For each $\lambda \ne 0$, $\lim_{\tau\to\infty} Nf(2|\lambda|^{-1}\tau,\widehat{\lambda}) = 0$ and so $P_\lambda \equiv 0$. This implies $Nf(\tau,\widehat{\lambda}) = 0$ for $\tau \ge 0$ and $\lambda \in \mathbb{R}$.

A similar argument starting with the fact that, for $k<0$,

$$\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau,\widehat{\lambda})\rangle = 0$$

shows that $Nf(\tau,\widehat{\lambda})=0$ for $\tau<0$, $\lambda\in\mathbb{R}$. ∎

3. Necessary conditions. In this section we find necessary conditions for a function m defined on $(\mathbb{R}-\{0\})\times\mathbb{Z}$ to belong to the image of $\mathcal{F}$. To do this, we recall the definition of the space $\widehat{\mathcal{S}}(U(n),H_n)$. We say that $F:\Delta(U(n),H_n)\to\mathbb{C}$ is *rapidly decreasing* (cf. [3, Definition 6.1]) if

(i) F is continuous,
(ii) for $w\in\mathbb{C}^n$, $w\mapsto F(\eta_w)$ belongs to $\mathcal{S}_{U(n)}(\mathbb{C}^n)$ where η_w is the spherical function of type II described in the introduction,
(iii) the map $\lambda\mapsto F(\lambda,k)$ is smooth on $\mathbb{R}-\{0\}$,
(iv) for each $j,N\ge 0$ there exists a constant $c_{j,N}$ such that

$$\left|\frac{\partial^j}{\partial\lambda^j}F(\lambda,k)\right| \le \frac{c_{j,N}}{|\lambda|^{j+N}(2k+n)^N}.$$

Also we set (see [3, Definition 6.2])

$$M^-F(\lambda,k)=\begin{cases}\dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{k}{\lambda}\,[F(\lambda,k)-F(\lambda,k-1)] & \text{for } \lambda>0,\\ \dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{k+n}{\lambda}\,[F(\lambda,k+1)-F(\lambda,k)] & \text{for } \lambda<0,\end{cases}$$

and

$$M^+F(\lambda,k)=\begin{cases}\dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{k+n}{\lambda}\,[F(\lambda,k+1)-F(\lambda,k)] & \text{for } \lambda>0,\\ \dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{k}{\lambda}\,[F(\lambda,k)-F(\lambda,k-1)] & \text{for } \lambda<0.\end{cases}$$

The space $\widehat{\mathcal{S}}(U(n),H_n)$ is defined as the set of all functions $F:\Delta(U(n),H_n)\to\mathbb{C}$ for which $(M^+)^l(M^-)^mF$ is rapidly decreasing for all $l,m\ge 0$.

Our results in this section are as follows:

THEOREM 3.1. *For $f\in\mathcal{S}(H_n)$ and $k\in\mathbb{Z}$, $\partial^j(\mathcal{F}f(\lambda,k))/\partial\lambda^j$ exists for all $j\in\mathbb{N}$ and $\lambda\ne 0$. Moreover, for each $j,N\in\mathbb{N}\cup\{0\}$ there exists a positive constant c independent of λ and k such that*

$$\left|\frac{\partial^j(\mathcal{F}f(\lambda,k))}{\partial\lambda^j}\right| \le c\left(|k|^{n-1}+\frac{1}{|\lambda|^{n-1}}\right)\frac{1}{|\lambda|^{N+j}(|k|+1)^N}. \tag{3.1}$$

THEOREM 3.2. *Let $f\in\mathcal{S}(H_n)$ and let $m=\mathcal{F}f$. Then the function defined on $(\mathbb{R}-\{0\})\times(\mathbb{N}\cup\{0\})$ by $(\lambda,k)\mapsto E(m^*)(\lambda,k+q)$ (with E, m^* as in the introduction) can be extended to a function belonging to $\widehat{\mathcal{S}}(U(1),H_1)$.*

Moreover, for $k \geq 0$ and $\lambda \neq 0$,

$$E(m^*)(\lambda, k+q) = (-1)^{n-1} \int_0^\infty L_k^0(|\lambda|\tau/2) e^{-|\lambda|\tau/4} Nf(\tau, \widehat{\lambda})\, d\tau. \tag{3.2}$$

THEOREM 3.3. *Let $f \in \mathcal{S}(H_n)$ and let m as in Theorem 3.2. Then the function defined on $(\mathbb{R} - \{0\}) \times (\mathbb{N} \cup \{0\})$ by $(\lambda, k) \mapsto \widetilde{E}(m^{**})(\lambda, -k-p)$ ($\widetilde{E}$ and m^{**} as in the introduction) extends to a function in $\widehat{\mathcal{S}}(U(1), H_1)$. Furthermore*

$$\widetilde{E}(m^{**})(\lambda, -k-p) = (-1)^{n-1} \int_0^\infty L_k^0(|\lambda|\tau/2) e^{-|\lambda|\tau/4} Nf(-\tau, \widehat{\lambda})\, d\tau. \tag{3.3}$$

For $j, s \in \mathbb{N} \cup \{0\}$, let $\varphi_{\lambda,j}^s(\tau, t)$ be defined by (2.3). From (2.7) and the definition of $S_{\lambda,k}$ we have

$$\mathcal{F}f(\lambda, k) = I(\lambda, k) + II(\lambda, k) \tag{3.4}$$

where

$$I(\lambda, k) = \begin{cases} (-1)^{n-1} \int_{\mathbb{R}} \int_{\tau>0} e^{-i\lambda t} \varphi_{\lambda,k-q}(\tau, t) e^{-|\lambda|/4\tau} Nf(\tau, t)\, d\tau\, dt & \text{for } k \geq q, \\ (-1)^{n-1} \int_{\mathbb{R}} \int_{\tau>0} \varphi_{\lambda,-k-p}(\tau, t) Nf(-\tau, t)\, d\tau\, dt & \text{for } k \leq -p, \\ 0 \quad \text{for } -p+1 \leq k \leq q-1, \end{cases} \tag{3.5}$$

$$II(\lambda, k) = \sum_{r=0}^{n-2} c_{r,k} |\lambda|^{-(l+1)} \langle \delta^{(r)}, Nf(\cdot, \widehat{\lambda}) \rangle \quad \text{for } k \in \mathbb{Z}, \tag{3.6}$$

with

$$c_{r,k} = \begin{cases} 4^r \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (L_{k-q+n-1}^0)^{(n-j-2)}(0) & \text{for } k \geq 0, \\ (-1)^r 4^r \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (L_{-k-p+n-1}^0)^{(n-j-2)}(0) & \text{for } k < 0. \end{cases}$$

Proof of Theorem 3.1. Since $Nf \in \mathcal{H}^\#$ we have $\frac{\partial}{\partial\tau}(\tau Nf(\tau, t)) \in \mathcal{H}^\#$, so by Remark 2.1, there is $g \in \mathcal{S}(H_n)$ such that $Ng(\tau, t) = \frac{\partial}{\partial\tau}(\tau Nf(\tau, t))$.

We claim that for $\lambda \neq 0$ and $k \in \mathbb{Z}$, $\partial\mathcal{F}f(\lambda, k)/\partial\lambda$ exists and

$$\frac{\partial\mathcal{F}f(\lambda, k)}{\partial\lambda} = -i\mathcal{F}(tf)(\lambda, k) - \frac{1}{\lambda}\mathcal{F}g(\lambda, k). \tag{3.7}$$

Indeed, consider the case $k \geq q$. Let $I(\lambda, k)$ and $II(\lambda, k)$ be given by (3.5) and (3.6) respectively. Since for $j \geq 0$ we have

$$\frac{\partial}{\partial\lambda}\varphi_{\lambda,j}(\tau, t) = -it\varphi_{\lambda,j}(\tau, t) + \frac{\tau}{\lambda}\frac{\partial}{\partial\tau}\varphi_{\lambda,j}(\tau, t),$$

after integration by parts we obtain

$$\frac{\partial I}{\partial \lambda}(\lambda,k)=\frac{\partial}{\partial \lambda}\Big(\int_{\mathbb{R}}\int_{\tau>0}(-1)^{n-1}\varphi_{\lambda,k-q}(\tau,t)Nf(\tau,t)\,d\tau\,dt\Big) \tag{3.8}$$
$$=\int_{\mathbb{R}}\int_{\tau>0}(-1)^{n-1}\varphi_{\lambda,k-q}(\tau,t)(-itNf(\tau,t))\,d\tau\,dt$$
$$-\frac{1}{\lambda}\int_{\mathbb{R}}\int_{\tau>0}(-1)^{n-1}\varphi_{\lambda,k-q}(\tau,t)\,\frac{\partial}{\partial\tau}(\tau Nf(\tau,t))\,d\tau\,dt.$$

Also,

$$\frac{\partial II}{\partial \lambda}(\lambda,k)=\frac{\partial}{\partial \lambda}\Big(\sum_{l=0}^{n-2}c_{l,k}|\lambda|^{-(l+1)}\langle\delta^{(l)},Nf(\cdot,\widehat{\lambda})\rangle\Big) \tag{3.9}$$
$$=-\sum_{l=0}^{n-2}(l+1)c_{l,k}|\lambda|^{-(l+2)}sg(\lambda)\langle\delta^{(l)},Nf(\cdot,\widehat{\lambda})\rangle$$
$$+\sum_{l=0}^{n-2}c_{l,k}|\lambda|^{-(l+1)}\langle\delta^{(l)},-i(tNf(\cdot,t))^{\wedge}(\lambda)\rangle$$

where $(\cdot)^{\wedge}$ denotes the Fourier transform in the variable t. Thus the derivative $\partial\mathcal{F}f(\lambda,k)/\partial\lambda$ exists. On the other hand,

$$-i\mathcal{F}(tf(z,t))(\lambda,k)=\int_{\mathbb{R}}\int_{\tau>0}(-1)^{n-1}\varphi_{\lambda,k-q}(\tau,t)(-itNf(\tau,t))\,d\tau\,dt \tag{3.10}$$
$$+\sum_{l=0}^{n-2}c_{l,k}|\lambda|^{-(l+1)}\langle\delta^{(l)},-i(tNf(\cdot,t))^{\wedge}(\lambda)\rangle.$$

Since $\big\langle\delta^{(l)},\frac{\partial}{\partial\tau}(\tau Nf(\tau,t))\big\rangle=(l+1)\langle\delta^{(l)},Nf(\cdot,t)\rangle)$ we have

$$-\frac{1}{\lambda}\mathcal{F}g(\lambda,k)=-\sum_{l=0}^{n-2}(l+1)c_{l,k}|\lambda|^{-(l+2)}sg(\lambda)\langle\delta^{(l)},Nf(\cdot,\widehat{\lambda})\rangle \tag{3.11}$$
$$-\frac{1}{\lambda}\int_{\mathbb{R}}\int_{\tau>0}(-1)^{n-1}\varphi_{\lambda,k-q}(\tau,t)\,\frac{\partial}{\partial\tau}(\tau Nf(\tau,t))\,d\tau\,dt$$

and now (3.8)–(3.11) give (3.7) for $k\geq q$. The case $k<q$ follows from a similar argument and using the corresponding expressions for $I(\lambda,k)$ and $II(\lambda,k)$.

Now, induction on j implies that $\partial^j\mathcal{F}f(\lambda,k)/\partial\lambda^j$ exists for $\lambda\neq 0$, $k\in\mathbb{Z}$ and all j.

In the rest of the proof, $c_1,c_2,\ldots,c',c''$, will denote positive constants independent of λ and k. To prove (3.1) we first consider the case $k\geq q$.

From (3.4), we have

$$|\mathcal{F}f(\lambda,k)| \le L_{k-q}^{n-1}(0)\|Nf\|_{L^1((0,\infty)\times\mathbb{R})} + c_1\sum_{l=0}^{n-2}|c_{l,k}|\,|\lambda|^{-(l+1)}.$$

Since $L_{k-q}^{n-1}(0) = \binom{k-q+n-1}{n-1} \le c_2k^{n-1}$ and $|c_{l,k}| \le c_3k^{n-l-2}$ we have

$$|\mathcal{F}f(\lambda,k)| \le c_4\Big(k^{n-1} + \sum_{l=0}^{n-2}k^{n-1-(l+1)}|\lambda|^{-(l+1)}\Big) \le c_4\Big(k+\frac{1}{|\lambda|}\Big)^{n-1} \le c_5\Big(k^{n-1}+\frac{1}{|\lambda|^{n-1}}\Big). \tag{3.12}$$

Applying (3.12) to L^Nf instead of f and recalling (1.1) we get

$$|2k+p-q|^N|\lambda|^N|\mathcal{F}f(\lambda,k)| = |\mathcal{F}(L^Nf)(\lambda,k)| \le c'\Big(|k|^{n-1}+\frac{1}{|\lambda|^{n-1}}\Big)$$

and since $2k+p-q \neq 0$ because $k \ge q$, this gives

$$|\mathcal{F}f(\lambda,k)| \le c''\Big(|k|^{n-1}+\frac{1}{|\lambda|^{n-1}}\Big)\frac{1}{|2k+p-q|^N|\lambda|^N}. \tag{3.13}$$

A similar argument applies to the case $k < q$, giving (3.13) except when $q-p \in 2\mathbb{Z}$ and $k = (q-p)/2$. In this case we take $(iT)^Nf$ instead of L^Nf above to get

$$|\mathcal{F}f(\lambda,k)| \le c\Big(|k|^{n-1}+\frac{1}{|\lambda|^{n-1}}\Big)\frac{1}{|\lambda|^N} \tag{3.14}$$

for $k = (q-p)/2$. From (3.13) and (3.14) we obtain (3.1) for $j = 0$ and all k and N.

Observe that for $r \in \mathbb{N}\cup\{0\}$, (3.1) used with $j = 0$ and $N+r$ instead of N gives immediately that

$$|\lambda|^r|\mathcal{F}f(\lambda,k)| \le c\Big(|k|^{n-1}+\frac{1}{|\lambda|^{n-1}}\Big)\frac{1}{|\lambda|^N(|k|+1)^N}. \tag{3.15}$$

An easy induction using (3.7) shows that for $j \ge 1$,

$$\lambda^j\,\frac{\partial^j\mathcal{F}f}{\partial\lambda^j}(\lambda,k) = \sum_{0\le r\le j}\lambda^r\mathcal{F}f_r(\lambda,k) \tag{3.16}$$

for some $f_1,\ldots,f_j$ belonging to $\mathcal{S}(H_n)$ and independent of λ and k. Now, (3.15) and (3.16) give (3.1) for all j. ∎

LEMMA 3.4. *Let $f \in \mathcal{S}(H_n)$. If either $k \ge q$ or $k \le -p$, then*

$$\sum_{r=0}^{n-2}|\lambda|^{-(r+1)}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}c_{r,k-l}\langle\delta^{(r)},Nf(\cdot,\widehat{\lambda})\rangle = 0.$$

Proof. Assume $k \geq q$. For $r = 0, 1, \ldots, n-2$ we have

$$\sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} c_{r,k-l}$$

$$= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (-1)^{n-j} \binom{k-l-q+n-1}{n-j-2}$$

$$= \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (-1)^{n-j} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \binom{k-l-q+n-1}{n-j-2}.$$

Let

$$\beta := \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \binom{k-l-q+n-1}{n-j-2}.$$

We claim that if $0 \leq r \leq j \leq n-2$ then $\beta = 0$. To see this we note that β is the coefficient of y^s in the polynomial $\sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (1+y)^{m-l}$ (where $m = k-q+n-1$ and $s = n-j-2$), i.e. β is the coefficient of y^s in $(1+y)^{m-(n-1)} \sum_{l=0}^{n} (-1)^l \binom{n-1}{l} (1+y)^{n-1-l} = (1+y)^{m-(n-1)} y^{n-1}$. So $\beta = 0$ since $s = n-j-2 < n-1$. The proof for the case $k \leq -p$ is similar, replacing $k-q$ by $-k-p$. ■

We recall that (cf. [8, p. 101])

$$L_j^n(x) = L_j^{n+1}(x) - L_{j-1}^{n+1}(x). \tag{3.17}$$

LEMMA 3.5. *For $j \geq 0$,*

$$\sum_{l=0}^{\min(j,n-1)} (-1)^l \binom{n-1}{l} L_{j-l}^{n-1}(x) = L_j^0(x). \tag{3.18}$$

Proof. We first give the proof for the case $j \geq n-1$. We proceed by induction on n. For $n = 1$ the lemma is clear. Suppose that it holds for n and $j \geq n-1$. Then for $j \geq n$,

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} L_{j-l}^n(x) = L_j^n(x) + (-1)^n L_{j-n}^n(x)$$
$$+ \sum_{l=1}^{n-1} (-1)^l \binom{n-1}{l} L_{j-l}^n(x) + \sum_{l=1}^{n-1} (-1)^l \binom{n-1}{l-1} L_{j-l}^n(x).$$

An index change in the last sum gives

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} L_{j-l}^n(x) = L_j^n(x) + (-1)^n L_{j-n}^n(x)$$
$$+ \sum_{l=1}^{n-1} (-1)^l \binom{n-1}{l} L_{j-l}^n(x) + \sum_{l=0}^{n-2} (-1)^{l-1} \binom{n-1}{l} L_{j-l-1}^n(x)$$

$$= L_j^n(x) + (-1)^n L_{j-n}^n(x) - L_{j-1}^n(x) + (-1)^{n-1} L_{j-(n-1)}^n(x)$$
$$+ \sum_{l=1}^{n-2} (-1)^l \binom{n-1}{l} (L_{j-l}^n - L_{j-l-1}^n)(x)$$
$$= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (L_{j-l}^n(x) - L_{j-l-1}^n(x))$$
$$= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} L_{j-l}^{n-1}(x) = L_j^0(x).$$

The last equality follows from (3.17) and the inductive hypothesis.

For the case $j < n-1$ we write

$$\sum_{l=0}^{j} (-1)^l \binom{n-1}{l} L_{j-l}^{n-1}(x) = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} c_l L_{j-l}^{n-1}(x),$$

where $c_l = 1$ for $0 \le l \le j$ and $c_l = 0$ for $j \le l \le n-1$, and now we proceed as above. ∎

Proof of Theorem 3.2. Let $m = \mathcal{F}f$. For $k \ge n-1$,

$$E(m^*)(\lambda, k+q) = \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} m(\lambda, k+q-l) \tag{3.19}$$
$$= \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (-1)^{n-1} \int_0^\infty L_{k-l}^{n-1}(|\lambda|\tau/2) e^{-|\lambda|\tau/4} Nf(\tau, \widehat{\lambda})\, d\tau$$
$$+ \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \sum_{r=0}^{n-2} c_{r,k+q-l} |\lambda|^{-(r+1)} \langle \delta^{(r)}, Nf(\cdot, \widehat{\lambda})\rangle = I + II.$$

Now, by Lemma 3.4, $II = 0$ and Lemma 3.5 gives

$$I = (-1)^{n-1} \int_0^\infty L_k^0(|\lambda|\tau/2) e^{-|\lambda|\tau/4} Nf(\tau, \widehat{\lambda})\, d\tau.$$

Thus, for $k \ge n-1$,

$$E(m^*)(\lambda, k+q) = (-1)^{n-1} \int_0^\infty L_k^0(|\lambda|\tau/2) e^{-|\lambda|\tau/4} Nf(\tau, \widehat{\lambda})\, d\tau. \tag{3.20}$$

On the other hand, if $0 \le k < n-1$,

$$E(m^*)(\lambda, k+q) = \sum_{0 \le l \le \min(k+q, n-1)} (-1)^l \binom{n-1}{l} m(\lambda, k+q-l)$$
$$+ \sum_{k+q < l \le n-1} (-1)^l \binom{n-1}{l} (-1)^{n-2} m(\lambda, k+q-l)$$

(with the convention that a sum on an empty set is zero). Since, for $0 \leq l \leq \min(k+q, n-1)$,

$$m(\lambda, k+q-l) = \langle (L^0_{k-l+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau)\rangle$$

and since for $k+q < l \leq n-1$ Corollary 2.3 gives

$$(-1)^{n-2}m(\lambda, k+q-l) \\ = \langle (L^0_{k-l+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Ng(2|\lambda|^{-1}\tau)\rangle,$$

we obtain $E(m^*)(\lambda, k+q) = I + II$ also for $0 \leq k < n-1$ (with I and II as in (3.19)). Proceeding as in the case $k \geq n-1$ we conclude that (3.20) holds for all k.

Let $\mathcal{F}_1$ be the $U(1)$-spherical transform on $\mathcal{S}(H_1)$ defined in [3] and let f_1 be the radial function in $\mathcal{S}(H_1)$ given by $f_1(z,t) = Nf(|z|^2, t)$. Then, by definition,

$$\mathcal{F}_1(f_1)(\lambda, k) = \int_{\mathbb{C}} L^0_k(|\lambda|\,|z|^2/2)e^{-|\lambda|\,|z|^2/4}Nf(|z|^2, \widehat{\lambda})\,dz.$$

We use polar coordinates $z = re^{i\theta}$ and then we perform the change of variable $s = r^2$ to get

$$\mathcal{F}_1(f_1)(\lambda, k) = \pi \int_0^\infty L^0_k(|\lambda|s/2)e^{-|\lambda|s/4}Nf(s, \widehat{\lambda})\,ds,$$

i.e. $(-1)^{n-1}E(\mathcal{F}f)(\lambda, k+q) = \mathcal{F}_1(f_1)(\lambda, k)$ for $k \geq 0$. ∎

Proof of Theorem 3.3. As before, it is enough to find $g_1 \in \mathcal{S}(H_1)$ such that for $k \geq 0$, $\mathcal{F}_1 g_1(\lambda, k) = (-1)^{n-1}\widetilde{E}(m^{**})(\lambda, -k-p)$. Set $g_1(z,t) = Nf(-|z|^2, t)$. Following the lines of the proof of Theorem 3.2 we obtain

$$\mathcal{F}_1(g_1)(\lambda, k) = \pi \int_0^\infty L^0_k(|\lambda|s/2)e^{-|\lambda|s/4}Nf(-s, \widehat{\lambda})\,ds \\ = (-1)^{n-1}\widetilde{E}(m^{**})(\lambda, -k-p)$$

for $k \geq 0$. ∎

4. The image of the spherical transform

LEMMA 4.1. *For $k \geq 0$,*

$$\frac{d^{n-1}}{d\tau^{n-1}}\left(\frac{1}{(n-1)!}\,\tau^{n-1}\mathcal{L}^{n-1}_k(\tau)e^{-\tau}\right) = L^0_{k+n-1}(\tau)e^{-\tau}. \tag{4.1}$$

Proof. We have

$$\frac{1}{(n-1)!}\,\frac{d^{n-1}}{d\tau^{n-1}}(\tau^{n-1}\mathcal{L}_k^{n-1}(\tau)e^{-\tau})$$
$$= \frac{1}{(n-1)!}\,\frac{(n-1)!k!}{(k+n-1)!}\,\frac{d^{n-1}}{d\tau^{n-1}}(\tau^{n-1}L_k^{n-1}(\tau)e^{-\tau})$$
$$= \frac{1}{(k+n-1)!}\left(\frac{d}{d\tau}\right)^{n-1+k}(\tau^{n-1+k}e^{-\tau}) = L^0_{k+n-1}(\tau)e^{-\tau}$$

where we have used (twice) the fact that

$$L_j^{\alpha}(\tau)\tau^{\alpha}e^{-\tau} = \frac{1}{j!}\,\frac{d^j}{d\tau^j}(\tau^{\alpha+j}e^{-\tau}) \quad \text{for } j \geq 0 \text{ (Rodrigues formula).} \blacksquare$$

Let D be the linear operator defined on the space of polynomial functions by $DL_k^0 = L_k^0 - L_{k-1}^0$ for $k \geq 1$ and $D1 = 1$.

LEMMA 4.2. *For all* $m \geq 0$,

$$\left(\frac{d}{d\tau}\right)^m (e^{-\tau}D^m(P(\tau))) = (-1)^m e^{-\tau}P(\tau). \tag{4.2}$$

Proof. We proceed by induction on m. For $m = 0$ there is nothing to prove. Assume that (4.2) holds. Then, for $k \geq 0$,

$$\left(\frac{d}{d\tau}\right)^{m+1}(e^{-\tau}D^{m+1}(L_k^0(\tau))) = \frac{d}{d\tau}\left(\frac{d}{d\tau}\right)^m (e^{-\tau}D^m(DL_k^0(\tau)))$$
$$= (-1)^m\,\frac{d}{d\tau}(e^{-\tau}DL_k^0(\tau)) = (-1)^{m+1}e^{-\tau}L_k^0(\tau).$$

In fact, the last equality follows from a direct computation for $k = 0, 1$, and for $k \geq 2$ observe that, taking into account (2.1) and (3.17),

$$(-1)^m\,\frac{d}{d\tau}(e^{-\tau}DL_k^0(\tau)) = (-1)^m\,\frac{d}{d\tau}(e^{-\tau}(L_k^0(\tau) - L_{k-1}^0(\tau)))$$
$$= (-1)^m(-e^{-\tau}L_k^0(\tau) + e^{-\tau}L_{k-1}^0(\tau) - e^{-\tau}L_{k-1}^1(\tau) + e^{-\tau}L_{k-2}^1(\tau))$$
$$= (-1)^m(-e^{-\tau}L_k^0(\tau) + e^{-\tau}L_{k-1}^0(\tau) - e^{-\tau}L_{k-1}^0(\tau)) = (-1)^{m+1}e^{-\tau}L_k^0(\tau). \blacksquare$$

LEMMA 4.3. (a) *For* $k \geq 0$ *and* $m \geq 0$,

$$D^m(L_k^0) = \sum_{l=0}^{\min(m,k)} (-1)^l \binom{m}{l} L_{k-l}^0. \tag{4.3}$$

(b) *If* $k > m$ *then* $D^m(L_k^0)(0) = 0$.

Proof. The proof proceeds along similar lines to the proof of Lemma 3.5. ∎

LEMMA 4.4. *For $r \geq n-1$,*

$$D^{n-1}(L_r^0)(\tau) = (-1)^{n-1}\,\frac{1}{(n-1)!}\,\tau^{n-1}\mathcal{L}_{r-(n-1)}^{n-1}(\tau) \tag{4.4}$$

Proof. From Lemma 4.2 we have

$$\left(\frac{d}{d\tau}\right)^{n-1}(e^{-\tau}D^{n-1}(L_r^0(\tau))) = (-1)^{n-1}e^{-\tau}L_r^0(\tau),$$

thus $e^{-\tau}D^{n-1}(L_r^0(\tau))$ is an $(n-1)$-primitive of $(-1)^{n-1}e^{-\tau}L_r^0(\tau)$ and then, by Lemma 4.1,

$$e^{-\tau}D^{n-1}(L_r^0(\tau)) = (-1)^{n-1}\,\frac{1}{(n-1)!}\,\tau^{n-1}\mathcal{L}_{r-n-1}^{n-1}(\tau)e^{-\tau} + Q(\tau)$$

for some polynomial Q of degree at most $n-2$. But this is impossible if Q does not vanish identically. ∎

THEOREM 4.5. *Let $a : (\mathbb{R}-\{0\}) \times (\mathbb{N}\cup\{0\}) \to \mathbb{C}$ be such that for each $N \in \mathbb{N}\cup\{0\}$ there exists a positive constant c independent of λ and k such that*

$$|a(\lambda,k)| \leq c_N\left(|k|^{n-1} + \frac{1}{|\lambda|^{n-1}}\right)\frac{1}{|\lambda|^N(|k|+1)^N}. \tag{4.5}$$

Then for each $s \in \mathbb{N}\cup\{0\}$ the function $\Psi : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ defined by

$$\Psi(\tau,t) := \sum_{k\geq 0}\int_{-\infty}^{\infty} a(\lambda,k)\mathcal{L}_k^s(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t}|\lambda|^n\,d\lambda \tag{4.6}$$

is well defined and belongs to $C^\infty([0,\infty)\times\mathbb{R})$. Moreover, the series in (4.6) converges absolutely and uniformly on $[0,\infty)\times\mathbb{R}$.

Proof. For $\lambda \neq 0$ and $k,s \in \mathbb{N}\cup\{0\}$ let $\psi_{\lambda,k}^s$ be defined by (2.2). Since $|\psi_{\lambda,k}^s| \leq 1$ (cf. [3]), in order to prove the absolute and uniform convergence of the series in (4.6) it is enough to show that

$$\sum_{k\geq q}\int_{-\infty}^{\infty}|a(\lambda,k)|\,|\lambda|^n\,d\lambda < \infty. \tag{4.7}$$

From (4.5) used with $N=0$ and since $k^{n-1}+1/|\lambda|^{n-1} \leq 2/|\lambda|^{n-1}$ if $|\lambda(k+1)| \leq 1$, we get

$$\sum_{k\geq 0}\int_{|\lambda(k+1)|\leq 1}|a(\lambda,k)|\,|\lambda|^n\,d\lambda \leq c\sum_{k\geq 0}\int_{|\lambda(k+1)|\leq 1}\frac{|\lambda|}{2}\,d\lambda \leq c''\sum_{k\geq 0}\frac{1}{(k+1)^2} < \infty.$$

Also, from (4.5) used with $N = n+2$ and since $k^{n-1}+1/|\lambda|^{n-1} \le 2(k+1)^{n-1}$ if $|\lambda(k+1)| > 1$, we get

$$\sum_{k\ge 0} \int_{|\lambda(k+1)|>1} |a(\lambda,k)|\,|\lambda|^n\,d\lambda \le c\sum_{k\ge 0}\int_{|\lambda(k+1)|>1} \frac{(k+1)^{n-1}|\lambda|^n}{(k+1)^{n+2}|\lambda|^{n+2}}\,d\lambda$$
$$= c\sum_{k>0}\frac{1}{(k+1)^2} < \infty.$$

Thus we have (4.7) and so the series in (4.6) converges absolutely and uniformly.

To prove the remaining assertion of the lemma we observe that for $k \ge 1$,

$$\frac{d}{d\tau}\mathcal{L}_k^s(|\lambda|\tau/2) = -\frac{|\lambda|}{2}\,\frac{k}{s+1}\mathcal{L}_{k-1}^{s+1}(|\lambda|\tau/2)$$

and so, for $k \ge 1$,

$$\frac{\partial}{\partial\tau}(a(\lambda,k)\mathcal{L}_k^s(|\lambda|\tau/2)e^{-|\lambda|\tau/4})$$
$$= \left(-\frac{1}{2(s+1)}\,a_1(\lambda,k)\mathcal{L}_{k-1}^{s+1}(|\lambda|\tau/2) - \frac{1}{4}\,a_2(\lambda,k)\mathcal{L}_k^s(|\lambda|\tau/2)\right)e^{-|\lambda|\tau/4}$$

where $a_1(\lambda,k) := |\lambda|ka(\lambda,k)$ and $a_2(\lambda,k) := |\lambda|a(\lambda,k)$. A similar identity holds for $k=0$ with the term involving $\mathcal{L}_{k-1}^{s+1}$ deleted. Since a_1 and a_2 satisfy the same estimates assumed for a, it follows that the series defining Ψ can be differentiated term by term and that $\partial\Psi/\partial\tau$ is a series of the form (4.6) with $a(\lambda,k)$ replaced by a new $\widetilde{a}(\lambda,k)$ satisfying the estimates (4.5). Similarly, we can show that the same conclusion holds for $\partial\Psi/\partial t$. Now the lemma follows by induction. ■

Remark 4.6. Let $a = a(\lambda,k)$ satisfy the conditions of Theorem 4.5. Then for $\tau \ge 0$ and $\lambda \ne 0$, the series

$$\Psi(\tau,\lambda) = \frac{|\lambda|}{2}\sum_{k\ge 0} a(\lambda,k)L_k^0(|\lambda|\tau/2)e^{-|\lambda|\tau/4}$$

converges absolutely (so it can be rearranged) and $\Psi(\tau,\cdot) \in L^1(\mathbb{R})$. Indeed, this follows from the assumption on $a(\lambda,k)$ and the fact that $|\varphi_{\lambda,k}^0| \le 1$. Moreover, for each $l \ge 0$,

$$\frac{\partial^l\Psi(2|\lambda|^{-1}\tau,\lambda)}{\partial\tau^l} = \frac{|\lambda|}{2}\sum_{k\ge 0} a(\lambda,k)\,\frac{\partial^l}{\partial\tau^l}(L_k^0(\tau)e^{-\tau/2}).$$

Theorem 4.7. *Let $f \in \mathcal{S}(H_n)$. Then, for $(\tau,t) \in [0,\infty)\times\mathbb{R}$,*

$$Nf(\tau,t) = (-1)^{n-1}\int_{\mathbb{R}}\frac{|\lambda|}{2}\sum_{k\ge 0} E(m^*)(\lambda,k+q)L_k^0(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t}\,d\lambda \tag{4.8}$$

and for $(\tau, t) \in (-\infty, 0] \times \mathbb{R}$,

(4.9) $$Nf(\tau,t) = (-1)^{n-1} \int_{\mathbb{R}} \frac{|\lambda|}{2} \sum_{k\geq 0} \widetilde{E}(m^{**})(\lambda, -k-p) L_k^0(-|\lambda|\tau/2) e^{|\lambda|\tau/4} e^{-i\lambda t}\, d\lambda.$$

Proof. Let $f \in \mathcal{S}(H_n)$ and $m = \mathcal{F}f$. Since $\{L_k^0(\tau)e^{-\tau/2}\}_{k\geq 0}$ is an orthonormal basis of $L^2(0,\infty)$, Theorems 3.2 and 3.3 imply that for all $\lambda \neq 0$,

(4.10) $$Nf(\tau, \widehat{\lambda}) = (-1)^{n-1} \frac{|\lambda|}{2} \sum_{k\geq 0} E(m^*)(\lambda, k+q) L_k^0(|\lambda|\tau/2) e^{-|\lambda|\tau/4}$$

for a.e. $\tau > 0$, and

(4.11) $$Nf(\tau, \widehat{\lambda}) = (-1)^{n-1} \frac{|\lambda|}{2} \sum_{k\geq 0} \widetilde{E}(m^{**})(\lambda, -k-p) L_k^0(-|\lambda|\tau/2) e^{|\lambda|\tau/4}$$

for a.e. $\tau < 0$. We multiply these equalities by $e^{-i\lambda t}$ and then integrate with respect to λ. Since, by Lemma 4.5, the above series can be integrated term by term, (4.8) and (4.9) follow (because they hold for a.e. $\tau > 0$ and a.e. $\tau < 0$ respectively and have both sides continuous in τ). ∎

REMARK 4.8. Theorem 4.7 also follows from formula (1.1) in [3] since the restrictions to $(\mathbb{R}-\{0\}) \times (\mathbb{N} \cup \{0\})$ can be extended to $\widehat{\mathcal{S}}(U(1), H_1)$.

In order to obtain, for a given $m(\lambda, k)$ satisfying the hypothesis of Theorem 1.1, a function $f \in \mathcal{S}(H_n)$ such that $\mathcal{F}f = m$, Theorem 4.7 suggests considering the functions $\varphi_1 : [0,\infty) \times \mathbb{R} \to \mathbb{C}$ and $\varphi_2 : (-\infty, 0] \times \mathbb{R} \to \mathbb{C}$ defined by the right sides of (4.8) and (4.9) respectively. After checking that they agree for $\tau = 0$, we will prove that the function $\varphi : \mathbb{R}^2 \to \mathbb{C}$ given by φ_1 and φ_2 belongs to $\mathcal{H}^{\#}$, and then we will choose f such that $\mathcal{N}f = \varphi$. We fix such φ_1 and φ_2 from now on.

$\mathcal{S}([0,\infty) \times \mathbb{R})$ will denote the space of functions $h : [0,\infty) \times \mathbb{R} \to \mathbb{C}$ which are C^∞ and rapidly decreasing at infinity (with the derivatives at $\tau = 0$ understood as lateral derivatives).

LEMMA 4.9. *Assume that m satisfies the conditions of Theorem* 1.1. *Then $\varphi_1 \in \mathcal{S}([0,\infty) \times \mathbb{R})$ and $\varphi_2 \in \mathcal{S}((-\infty, 0] \times \mathbb{R})$.*

Proof. From our assumptions on m, Theorem 6.1 in [3] gives functions $f_1 = f_1(z,t)$ and $f_2 = f_2(z,t)$ which are radial in z, belong to $\mathcal{S}(H_1)$ and

$$f_1(z,t) = (-1)^{n-1} \int_{\mathbb{R}} \frac{|\lambda|}{2} \sum_{k\geq 0} E(m^*)(\lambda, k+q) L_k^0(|\lambda|\, |z|^2/2) e^{-|\lambda|\, |z|^2/4} e^{-i\lambda t}\, d\lambda$$

and

$$f_2(z,t) = (-1)^{n-1}\int_{\mathbb{R}} \frac{|\lambda|}{2} \sum_{k\geq 0} E(m^{**})(\lambda,-k-p) L_k^0(|\lambda|\,|z|^2/2) e^{-|\lambda|\,|z|^2/4} e^{-i\lambda t}\, d\lambda.$$

So $\varphi_1(\tau,t) = f_1(\tau^{1/2},t)$ for $(\tau,t)\in[0,\infty)\times\mathbb{R}$ and $\varphi_2(\tau,t) = f_2(|\tau|^{1/2},t)$ for $(\tau,t)\in(-\infty,0]\times\mathbb{R}$, and the lemma follows by proceeding as in the proof of Theorem 6.1 in [3, pp. 410–412]. ∎

From the definition of φ_1 we have

$$\varphi_1(\tau,t) = (-1)^{n-1}\sum_{k\geq 0}\sum_{0\leq l\leq n-1}\int_{\mathbb{R}} (-1)^l \binom{n-1}{l} \frac{|\lambda|}{2}\, m^*(\lambda,k+q-l)\varphi^0_{\lambda,k}(\tau,t)\, d\lambda.$$

Note that this series can be rearranged by Theorem 4.5. We first change the summation order, then we change the index in the sum on k setting $j = k-q-l$, and finally we change l to $n-1-l$ to obtain

$$\varphi_1(\tau,t) = \sum_{0\leq l\leq n-1}\sum_{j\geq -p+1+l}\int_{\mathbb{R}} (-1)^l \binom{n-1}{n-1-l} \frac{|\lambda|}{2}\, m^*(\lambda,j)\varphi^0_{\lambda,j-q+n-1-l}(\tau,t)\, d\lambda.$$

Now we change the summation order again to get

$$\varphi_1(\tau,t) = \sum_{j\geq -p+1}\int_{\mathbb{R}} \frac{|\lambda|}{2}\, m^*(\lambda,j) \sum_{0\leq l\leq \min(j+p-1,n-1)} (-1)^l \binom{n-1}{l} \varphi^0_{\lambda,j-q+n-1-l}(\tau,t)\, d\lambda$$

and so by Lemma 4.3,

$$\begin{aligned}\varphi_1(\tau,t) &= \sum_{j\geq q}\int_{\mathbb{R}} \frac{|\lambda|}{2}\, m^*(\lambda,j)(D^{n-1}L^0_{j-q+n-1})(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t}\, d\lambda \\ &+ \sum_{-p+1\leq j\leq q-1}\int_{\mathbb{R}} \frac{|\lambda|}{2}\, m^*(\lambda,j)(D^{n-1}L^0_{j-q+n-1})(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t}\, d\lambda.\end{aligned} \tag{4.12}$$

Then, by Lemma 4.4,

$$\begin{aligned}\varphi_1(\tau,t) &= \sum_{j\geq q}\int_{\mathbb{R}} \frac{|\lambda|}{2}\, m(\lambda,j)(-1)^{n-1}\frac{1}{(n-1)!}(|\lambda|\tau/2)^{n-1}\psi^{n-1}_{\lambda,j-q}(\tau,t)\, d\lambda \\ &+ \sum_{-p+1\leq j\leq q-1}\int_{\mathbb{R}} \frac{|\lambda|}{2}\, m^*(\lambda,j)(D^{n-1}L^0_{j-q+n-1})(|\lambda|\tau/2)e^{-|\lambda|\tau/4}e^{-i\lambda t}\, d\lambda\end{aligned}$$

Thus, $\varphi_1(\tau,t)=\xi_1(\tau,t)+\eta_1(\tau,t)$ where

$$\xi_1(\tau,t)=\sum_{j\geq q}\int_{\mathbb{R}}\frac{|\lambda|}{2}\,m(\lambda,j)(-1)^{n-1}\,\frac{1}{(n-1)!}(|\lambda|\tau/2)^{n-1}\psi^{n-1}_{\lambda,j-q}(\tau,t)\,d\lambda,$$

$$\eta_1(\tau,t) = \sum_{-p+1\leq j\leq q-1}\int_{\mathbb{R}}\frac{|\lambda|}{2}\,m^*(\lambda,j)\sum_{0\leq l\leq j+p-1}(-1)^l\binom{n-1}{l}\psi^0_{\lambda,j-q+n-1-l}(\tau,t)\,d\lambda.$$

Similarly, $\varphi_2(\tau,t)=\xi_2(\tau,t)+\eta_2(\tau,t)$ where

$$\xi_2(\tau,t)=\sum_{j\leq -p}\int_{\mathbb{R}}\frac{|\lambda|}{2}\,m(\lambda,j)(-1)^{n-1}\frac{1}{(n-1)!}\,(|\lambda|\tau/2)^{n-1}\psi^{n-1}_{\lambda,-j-p}(-\tau,t)\,d\lambda,$$

$$\eta_2(\tau,t) = \sum_{-p<j\leq q-1}\int_{\mathbb{R}}\frac{|\lambda|}{2}\,m^{**}(\lambda,j)\sum_{0\leq l\leq q-1-j}(-1)^l\binom{n-1}{l}\psi^0_{\lambda,-j-p+n-1-l}(-\tau,t)\,d\lambda.$$

Observe that by Theorem 4.5, $\xi_1(\tau,t)=\tau^{n-1}\widetilde{\xi}_1(\tau,t)$ with $\widetilde{\xi}_1\in C^\infty([0,\infty)\times\mathbb{R})$, and so $\frac{\partial^l\xi_1}{\partial\tau^l}(0,t)=0$ for $0\leq l\leq n-2$ and all $t\in\mathbb{R}$. Analogously, $\frac{\partial^l\xi_2}{\partial\tau^l}(0,t)=0$ for $0\leq l\leq n-2$, $t\in\mathbb{R}$.

Our next step is to prove that

$$\frac{\partial^s\varphi_1}{\partial\tau^s}(0,t)=\frac{\partial^s\varphi_2}{\partial\tau^s}(0,t),\qquad 0\leq s\leq n-2,\ t\in\mathbb{R}, \tag{4.13}$$

i.e., for each t,

$$\frac{\partial^s\eta_1}{\partial\tau^s}(0,t)=\frac{\partial^s\eta_2}{\partial\tau^s}(0,t),\qquad 0\leq s\leq n-2. \tag{4.14}$$

Observe that from Theorem 4.5 we have, for $t\in\mathbb{R}$ and $0\leq l\leq n-2$,

$$\frac{\partial^l\eta_1}{\partial\tau^l}(0,t)=\int_{\mathbb{R}}\frac{|\lambda|}{2}\Big(\sum_{j=-p+1}^{-1}(-1)^{n-2}m(\lambda,j)G_{j,l}(\lambda)+\sum_{j=0}^{q-1}m(\lambda,j)G_{j,l}(\lambda)\Big)e^{-i\lambda t}\,d\lambda$$

and

$$\frac{\partial^l\eta_2}{\partial\tau^l}(0,t)=\int_{\mathbb{R}}\frac{|\lambda|}{2}\Big(\sum_{j=-p+1}^{-1}m(\lambda,j)H_{j,l}(\lambda)+\sum_{j=0}^{q-1}(-1)^{n-2}m(\lambda,j)H_{j,l}(\lambda)\Big)e^{-i\lambda t}\,d\lambda$$

with $G_{j,l}$, $H_{j,l}$ independent of m and the integrals being absolutely convergent. But (4.14) holds if and only if

$$\sum_{j=-p+1}^{-1}(-1)^{n-2}m(\lambda,j)G_{j,l}(\lambda)+\sum_{j=0}^{q-1}m(\lambda,j)G_{j,l}(\lambda) = \sum_{j=-p+1}^{-1}m(\lambda,j)H_{j,l}(\lambda)+\sum_{j=0}^{q-1}(-1)^{n-2}m(\lambda,j)H_{j,l}(\lambda) \tag{4.15}$$

for all $\lambda\neq 0$.

From Theorem 4.7, this clearly holds if $m = \mathcal{F}f$ for some $f \in \mathcal{S}(H_n)$, because the first $n-2$ derivatives of $Nf(\cdot, t)$ are continuous at the origin.

Moreover, for each λ and j such that $\lambda \neq 0$ and $-p+1 \leq j \leq q-1$, Proposition 4.10 below gives an $f \in \mathcal{S}(H_n)$ (depending of λ and j) such that for $-p+1 \leq k \leq q-1$, $\mathcal{F}f(\lambda, k) = 1$ if $k = j$, and $\mathcal{F}f(\lambda, k) = 0$ if $k \neq j$. So for such λ and j, $G_{j,l}(\lambda) = (-1)^{n-2} H_{j,l}(\lambda)$, $0 \leq l \leq n-2$.

Proposition 4.10. *Given $\lambda \neq 0$ and $n-1$ complex numbers $\{a_j\}_{j=p+1}^{q-1}$, there exists $f \in \mathcal{S}(H_n)$ such that*

$$\mathcal{F}f(\lambda, j) = a_j, \qquad -p+1 \leq j \leq q-1. \tag{4.16}$$

Proof. We take f such that $Nf(\tau, \widehat{\lambda}) := \omega(|\lambda|\tau/2) e^{|\lambda|\tau/4} \widehat{\psi}(\lambda)$ where $\omega, \psi \in \mathcal{S}(\mathbb{R})$, $\omega \in C_{\mathrm{c}}(\mathbb{R})$ and $\widehat{\psi}(\lambda) = 1$. We recall that from the definition of $\mathcal{F}f$,

$$\mathcal{F}f(\lambda, k) = \langle (L^0_{k-q+n-1} H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(2|\lambda|^{-1}\tau, \widehat{\lambda}) \rangle, \qquad 0 \leq k \leq q-1,$$

and

$$\mathcal{F}f(\lambda, k) = \langle (L^0_{-k-p+n-1} H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(-2|\lambda|^{-1}\tau, \widehat{\lambda}) \rangle, \qquad -p+1 \leq k < 0,$$

and from Corollary 2.3,

$$\mathcal{F}f(\lambda, k) = (-1)^{n-2} \langle (L^0_{k-q+n-1} H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1} e^{-\tau/2} Nf(2|\lambda|^{-1}\tau, \widehat{\lambda}) \rangle$$

for $-p+1 \leq k < 0$. So, from our choice of f and since $\widehat{\psi}(\lambda) = 1$, (4.16) reads $a_k = 2|\lambda_0|^{-1} \langle (L^0_{k-q+n-1} H)^{(n-1)}, \omega \rangle$ for $0 \leq k \leq q-1$, and $a_k = (-1)^{n-2} 2|\lambda_0|^{-1} \langle (L^0_{k-q+n-1} H)^{(n-1)}, \omega \rangle$ for $-p+1 \leq k < 0$. But for $-p+1 \leq k \leq q-1$ we have $0 \leq k-q+n-1 \leq n-2$. So, by (2.7),

$$(L^0_{k-q+n-1} H)^{(n-1)} = \sum_{s=0}^{n-2} (L^0_{k-q+n-1})^{(n-2-s)}(0) \delta^{(s)}.$$

To obtain (4.16) it is enough to find $\beta_0, \ldots, \beta_{n-2}$ solving

$$\sum_{s=0}^{n-2} (L^0_{k-q+n-1})^{(n-2-s)}(0)(-1)^{(s)} \beta_s = |\lambda| a_k/2, \qquad 0 \leq k \leq q-1,$$

$$\sum_{s=0}^{n-2} (L^0_{k-q+n-1})^{(n-2-s)}(0)(-1)^{(s)} \beta_s = (-1)^{n-2} |\lambda| a_k/2, \qquad -p+1 \leq k < 0,$$

and then to find $\omega \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\omega^{(s)}(0) = \beta_s$ for $s = 0, 1, \ldots, n-2$. This is a linear system in $\{\omega^{(s)}(0)\}_{s=0}^{n-2}$. Since $(L^0_k)^{(s)}(0) = (-1)^s \binom{k}{s}$, the associated $(n-1)\times(n-1)$ matrix A is lower triangular with ± 1 on the diagonal. So A is nonsingular and the existence of $\beta_0, \ldots, \beta_{n-2}$ follows. Now, we take

$\omega = P(\tau)\widetilde{\omega}(\tau)$, where P is a polynomial of degree $n-1$ with $P^{(s)}(0) = \beta_s$ for $s = 0,\dots,n-2$ and where $\widetilde{\omega} \in C_c^\infty(\mathbb{R})$, $\mathrm{supp}(\widetilde{\omega}) \subset (-2,2)$ and $\widetilde{\omega}(\tau) = 1$ for $\tau \in (-1,1)$. ■

A classical result due to Borel states that given a sequence $\{a_j\}_{j=1}^\infty$ of complex numbers, there exists a $C^\infty(\mathbb{R})$ function ψ such that $\psi^{(j)}(0) = a_j$ for all j. Moreover ψ can be taken in $C_c^\infty(\mathbb{R})$. A similar result holds in two variables. Since we have not been able to find it in the literature we give a proof for completeness.

LEMMA 4.11. *Let $\{a_j(t)\}_{j=1}^\infty$ be a sequence of functions in $\mathcal{S}(\mathbb{R})$. Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^2)$ such that $\frac{\partial^j \psi}{\partial \tau^j}(0,t) = a_j(t)$.*

Proof. Let $\widetilde{\omega}$ be as in the proof of Proposition 4.10. For a given sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers we set

$$g_n(\tau,t) := \frac{a_n(t)}{n!}\,\tau^n\widetilde{\omega}(\tau), \qquad f_n(\tau,t) := \frac{1}{\lambda_n^{2n}}\,g_n(\lambda_n\tau,t) = \frac{1}{\lambda_n^n}\,\frac{a_n(t)}{n!}\tau^n\widetilde{\omega}(\lambda_n\tau).$$

Let

$$f(\tau,t) := \sum_{n=1} f_n(\tau,t).$$

Clearly, the lemma will follow if we can prove (for a suitable sequence $\{\lambda_n\}$) that

$$\left\| t^s\,\frac{\partial^l}{\partial t^l}\frac{\partial^k}{\partial \tau^k} f_n \right\|_\infty \le \frac{1}{2^n} \qquad \text{for all } 0 \le k,l,s \le n-1, \tag{4.17}$$

We take $\lambda_n \ge 1$ for all n. Taking into account that $k \le n-1$ and $a_j(t) \in \mathcal{S}(\mathbb{R})$, we can apply the Leibniz rule to get a positive constant c_n such that

$$\left| t^s\,\frac{\partial^l}{\partial t^l}\frac{\partial^k}{\partial \tau^k} f_n \right| \le t^s\,\frac{c_n}{\lambda_n n!}\left|\frac{\partial^l a_n}{\partial t^l}\right| \le \frac{c_n}{\lambda_n n!}\sum_{s,l=0}^{n-1}\left\| t^s\,\frac{\partial^l a_n}{\partial^l t}\right\|_\infty.$$

Now (4.17) follows by choosing λ_n such that, in addition,

$$\frac{1}{\lambda_n} \le \frac{c_n}{2^n n!}\sum_{s,l=0}^{n-1}\left\| t^s\,\frac{\partial^l a_n}{\partial^l t}\right\|_\infty. \ ■$$

DEFINITION 4.12. Let $m = m(\lambda,k)$ be a function satisfying the conditions of the statement of Theorem 1.1. We define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(\tau,t) = \begin{cases} \varphi_1(\tau,t) & \text{for } \tau > 0,\ t \in \mathbb{R}, \\ \varphi(\tau,t) = \varphi_2(\tau,t) & \text{for } \tau \le 0,\ t \in \mathbb{R}. \end{cases} \tag{4.18}$$

Proof of Theorem 1.1. Let m and φ be as in Definition 4.12. By Theorems 3.2 and 3.3, it remains to see that φ belongs to $\mathcal{H}^\#$ and that if we take $f \in \mathcal{S}(H_n)$ such that $Nf = \varphi$ then $\mathcal{F}f = m$. To see that $\varphi \in \mathcal{H}^\#$ we must find ψ_1 and ψ_2 in $\mathcal{S}(\mathbb{R}^2)$ such that

$$\varphi(\tau,t) = \psi_2(\tau,t) + \tau^{n-1}\psi_1(\tau,t)H(\tau)$$

(where H is the Heaviside function), i.e.,

$$\psi_2(\tau,t) = \begin{cases} \varphi_2(\tau,t) & \text{for } \tau \le 0, \\ \varphi_1(\tau,t) - \tau^{n-1}\psi_1(\tau,t) & \text{for } \tau > 0. \end{cases} \tag{4.19}$$

For a given $\psi_1 \in \mathcal{S}(\mathbb{R}^2)$, we define ψ_2 by (4.19). In view of Lemma 4.9 and (4.13), $\psi_2 \in \mathcal{S}(\mathbb{R}^2)$ if and only if for a suitable $\psi_1 \in \mathcal{S}(\mathbb{R}^2)$,

$$\frac{\partial^j \varphi_2}{\partial \tau^j}(0,t) = \frac{\partial^j \varphi_1}{\partial \tau^j}(0,t) - \binom{j}{n-1}(n-1)!\,\frac{\partial^{j-(n-1)}\psi_1}{\partial \tau^{j-(n-1)}}(0,t). \tag{4.20}$$

for all $j \ge n-1$. But Lemma 4.11 gives a function $\psi_1 \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\frac{\partial^k \psi_1}{\partial \tau^k}(0,t) = \frac{1}{\binom{k+n-1}{n-1}(n-1)!}\,\frac{\partial^{k+n-1}(\varphi_2 - \varphi_1)}{\partial \tau^{k+n-1}}(0,t)$$

for $k \in \mathbb{N} \cup \{0\}$, i.e., (4.20) holds. Thus $\varphi \in \mathcal{H}^{\#}$.

Let $f \in \mathcal{S}(H_n)$ be such that $Nf = \varphi$. To see that $\mathcal{F}f = m$ we proceed as follows. For $k \ge 0$ and $\lambda \neq 0$ we have

$$\begin{aligned} \mathcal{F}f(\lambda,k) &= \langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda})\rangle \\ &= (-1)^{n-1}\int_0^\infty L^0_{k-q+n-1}(\tau)\frac{\partial^{n-1}}{\partial \tau^{n-1}}(2|\lambda|^{-1}e^{-\tau/2}\varphi_1(2|\lambda|^{-1}\tau,\widehat{\lambda}))\,d\tau. \end{aligned} \tag{4.21}$$

From the definition of φ_1 and Remark 4.6,

$$2|\lambda|^{-1}e^{-\tau/2}\varphi_1(2|\lambda|^{-1}\tau,\widehat{\lambda}) = \sum_{j\ge 0} E(m^*)(\lambda, j+q)L^0_j(\tau)e^{-\tau}.$$

Now, from similar computations to those that give (4.12) (allowed again by Remark 4.6) we get

$$\begin{aligned} 2|\lambda|^{-1}e^{-\tau/2}\varphi_1(2|\lambda|^{-1}\tau,\widehat{\lambda}) &= \frac{(-1)^{n-1}}{(n-1)!}\,\frac{|\lambda|}{2}\sum_{j\ge q} m(\lambda,j)D^{n-1}(L^0_{j-q+n-1})(\tau)e^{-\tau} \\ &\quad + \frac{|\lambda|}{2}\sum_{-p+1\le j\le q-1} m^*(\lambda,j)D^{n-1}(L^0_{j-q+n-1})(\tau)e^{-\tau}. \end{aligned}$$

Then, by Lemma 4.2,

$$\begin{aligned} &2|\lambda|^{-1}(-1)^{n-1}\left(\frac{d}{d\tau}\right)^{n-1} e^{-\tau/2}Nf(2|\lambda|^{-1}\tau,\widehat{\lambda}) \\ &\quad = \sum_{j\ge q} m(\lambda,j)L^0_{j-q+n-1}e^{-\tau} + \sum_{-p+1\le j\le q-1} m^*(\lambda,j)L^0_{j-q+n-1}(\tau)e^{-\tau}. \end{aligned}$$

Our assumptions on m imply that $\sum_{j\geq q} m(\lambda, j)L^0_{j-q+n-1}e^{-\tau/2}$ belongs to $L^2((0,\infty), d\tau)$. Also $\int_0^\infty L^0_{k-q+n-1}(\tau)L^0_{j-q+n-1}e^{-\tau} = \delta_{jk}$; then from (4.21) it follows that $Ff(\lambda, k) = m(\lambda, k)$ for $k \geq q$ and $Ff(\lambda, k) = m^*(\lambda, k)$ for $0 \leq k \leq q-1$. Since $m(\lambda, k) = m^*(\lambda, k)$ for $k \geq 0$ we have proved that $Ff(\lambda, k) = m(\lambda, k)$ for $k \geq 0$.

A completely similar argument starting with the facts that

$$\mathcal{F}f(\lambda, k) = \langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau, \widehat{\lambda})\rangle$$
$$= (-1)^{n-1}\int_0^\infty L^0_{-k-p+n-1}(\tau)\frac{\partial^{n-1}}{\partial\tau^{n-1}}(2|\lambda|^{-1}e^{-\tau/2}\varphi_2(-2|\lambda|^{-1}\tau, \widehat{\lambda}))\, d\tau$$

and that for $\tau < 0$,

$$2|\lambda|^{-1}e^{\tau/2}\varphi_2(2|\lambda|^{-1}\tau, \widehat{\lambda}) = \sum_{j\geq 0} E(m^{**})(\lambda, j+q)L^0_j(-\tau)e^{\tau}$$

can be used in the case $k < 0$ to complete the proof of the theorem. ∎

REMARK 4.13. Recall that for $h \in \mathcal{S}(H_1)$ and $H(\lambda, k) = \mathcal{F}_1 h(\lambda, k)$ we have $M^+H = \mathcal{F}_1((|z|^2/4 + it)h)$ and $M^-H = \mathcal{F}_1((|z|^2/4 - it)h)$ (cf. [3, p. 407]).

For $f \in \mathcal{S}(H_n)$ let $f_1 \in \mathcal{S}(H_1)$ be the function given by

$$f_1(z, t) = Nf(|z|^2, t).$$

We have seen that

$$\mathcal{F}_1(f_1)(\lambda, k) = E(\mathcal{F}f)(\lambda, k+q).$$

Consider the map $\varXi : \mathcal{S}(H_n) \to \mathcal{S}(H_1)$ defined by $\varXi(f) = f_1$, let $B(z, w)$ be the quadratic form given in the introduction and set $B(z) = B(z, z)$. It is immediate to see that $N(B(z)f) = \tau Nf$ and this says that $\varXi((B(z)/4 \pm it)f) = (|z|^2/4 \pm it)f_1$. Then we can conclude that

$$M^{\pm}(\mathcal{F}_1 f_1) = E(\mathcal{F}(B(z)/4 \pm it)f).$$

A similar expression can be obtained for $M^{\pm}(\mathcal{F}_1 g_1)(\lambda, k)$ (where $g_1(z, t) = Nf(-|z|^2, t)$) that involves $E(\mathcal{F}(B(z)/4 \pm it)f)(\lambda, -k-p)$ for $k \geq n-1$ and $\widetilde{E}(\mathcal{F}(B(z)/4 \pm it)f)(\lambda, k)$ for $0 \leq k \leq n-2$.

REFERENCES

[1] C. Benson, J. Jenkins and G. Ratcliff, *On Gelfand pairs associated with solvable Lie groups*, Trans. Amer. Math. Soc. 321 (1990), 85–116.

[2] —, —, —, *Bounded K-spherical functions on Heisenberg groups*, J. Funct. Anal. 105 (1992), 409–443.

[3] —, —, —, *The spherical transform of a Schwartz function on a Heisenberg group*, ibid. 154 (1998), 379–423.

[4] C. Benson, J. Jenkins, G. Ratcliff and T. Worku, *Spectra for Gelfand pairs associated with the Heisenberg group*, Colloq. Math. 71 (1996), 305–328.

[5] T. Godoy and L. Saal, L^2 *spectral decomposition on the Heisenberg group associated to the action of* $U(p,q)$, Pacific J. Math. 193 (2000), 327–353.

[6] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.

[7] E. Stein and G. Weiss, *Introduction to Fourier Analysis in Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

[8] G. Szegő, *Orthogonal Polynomials*, Colloq. Publ. 23, Amer. Math. Soc., 1939.

[9] A. Tengstrand, *Distributions invariant under an orthogonal group of arbitrary signature*, Math. Scand. 8 (1960), 201–218.

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