KEMPİSTY’S THEOREM FOR THE
INTEGRAL PRODUCT QUASICONTINUITY

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Abstract. A function \( f : \mathbb{R}^n \to \mathbb{R} \) satisfies the condition \( Q_i(x) \) (resp. \( Q_s(x), Q_o(x) \)) at a point \( x \) if for each real \( r > 0 \) and for each set \( U \ni x \) open in the Euclidean topology of \( \mathbb{R}^n \) (resp. strong density topology, ordinary density topology) there is an open set \( I \) such that \( I \cap U \neq \emptyset \) and \( \left| \frac{1}{\mu(U \cap I)} \int_{U \cap I} f(t) \, dt - f(x) \right| < r \). Kempisty’s theorem concerning the product quasicontinuity is investigated for the above notions.

For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and positive reals \( r_1, \ldots, r_n \) put

\[
I_i = (x_i - r_i, x_i + r_i) \quad \text{for} \ i = 1, \ldots, n,
\]

\[
P(x; r_1, \ldots, r_n) = I_1 \times \cdots \times I_n, \quad Q(x, r) = P(x; r, \ldots, r).
\]

Denote by \( \mu \) the Lebesgue measure and by \( \mu_e \) the outer Lebesgue measure in \( \mathbb{R}^n \). For \( A \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \) we define the upper (resp. lower) outer strong density \( D_u(A, x) \) (resp. \( D_l(A, x) \)) of \( A \) at \( x \) as

\[
\limsup_{h_1, \ldots, h_n \to 0^+} \frac{\mu_e(A \cap P(x; h_1, \ldots, h_n))}{\mu(P(x; h_1, \ldots, h_n))}
\]

and

\[
\liminf_{h_1, \ldots, h_n \to 0^+} \frac{\mu_e(A \cap P(x; h_1, \ldots, h_n))}{\mu(P(x; h_1, \ldots, h_n))}
\]

respectively. Similarly for \( A \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \) we define the upper (resp. lower) outer ordinary density \( d_u(A, x) \) (resp. \( d_l(A, x) \)) of \( A \) at \( x \) as

\[
\limsup_{h \to 0^+} \frac{\mu_e(A \cap Q(x, h))}{\mu(Q(x, h))} \quad \text{and} \quad \liminf_{h \to 0^+} \frac{\mu_e(A \cap Q(x, h))}{\mu(Q(x, h))}
\]

respectively. A point \( x \) is said to be an outer strong density point (resp. a strong density point) of \( A \) if \( D_l(A, x) = 1 \) (resp. if there is a Lebesgue measurable set \( B \subset A \) such that \( D_l(B, x) = 1 \)).

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Similarly we define the notions of an outer ordinary density point and of an ordinary density point.

The family $T_{s,d}$ (resp. $T_{o,d}$) of all sets all of whose points are strong (resp. ordinary) density points is a topology called the strong (resp. ordinary) density topology ([1, 2, 9, 7]). If $n = 1$ then $T_{s,d} = T_{o,d}$ is called the density topology.

If $T_e$ denotes the Euclidean topology in $\mathbb{R}^n$ then evidently $T_e \subset T_{s,d} \subset T_{o,d}$ and all sets in $T_{o,d}$ are Lebesgue measurable ([1, 2, 9]).

The continuity of mappings $f$ from $(\mathbb{R}^n, T_{s,d})$ (resp. from $(\mathbb{R}^n, T_{o,d})$) to $(\mathbb{R}, T_e)$ is called the strong (resp. ordinary) approximate continuity ([1, 2, 9]).

In [5, 6] the following notion is investigated.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasicontinuous at a point $x$ (written $f \in Q(x)$) if for each $r > 0$ and each $U \in T_e$ containing $x$ there is a nonempty set $I \in T_e$ such that $I \subset U$ and $|f(t) - f(x)| < r$ for all $t \in I$.

A function $f$ is quasicontinuous if $f \in Q(x)$ for every $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous at a point $x$ ($f \in Q_i(x)$, [4]) if for each $r > 0$ and each $U \in T_e$ containing $x$ there is a nonempty set $I \in T_e$ such that $I \subset U$ and

$$\left| \frac{\int_I f(t) \, dt}{\mu(I)} - f(x) \right| < r.$$

A function $f$ is integrally quasicontinuous ($f \in Q_i$) if $f \in Q_i(x)$ for all $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_s(x)$ (resp. $f \in Q_o(x)$, [4]) if for each $\eta > 0$ and each $U \in T_{s,d}$ (resp. $U \in T_{o,d}$) containing $x$ there is a nonempty set $I \in T_e$ such that $I \cap U \neq \emptyset$, $f$ is Lebesgue integrable on $I \cap U$ and

$$\left| \frac{1}{\mu(I \cap U)} \int_{I \cap U} f(t) \, dt - f(x) \right| < \eta.$$

If $f \in Q_s(x)$ (resp. $f \in Q_o(x)$) for all $x \in \mathbb{R}^n$ then we write $f \in Q_s$ (resp. $f \in Q_o$).

The inclusions $Q_o \subset Q_s \subset Q_i$ are true and each measurable quasicontinuous function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous ([4]). If $n = 1$ then $Q_o = Q_s$.

Now let $n_1, n_2$ be two positive integers with $n_1 + n_2 = n$ and let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For $x = (x_1, \ldots, x_{n_1}) \in \mathbb{R}^{n_1}$ and $y = (x_{n_1+1}, \ldots, x_n) \in \mathbb{R}^{n_2}$ we write $(x, y) = (x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_n) \in \mathbb{R}^n$.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ and for points $t \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ we define the sections $f_t : \mathbb{R}^{n_2} \to \mathbb{R}$ and $f_y : \mathbb{R}^{n_1} \to \mathbb{R}$ by

$$f_t(y) = f(t, y) \quad \text{and} \quad f^y(t) = f(t, y).$$
If \( n = n_1 + n_2 \) with \( n_1, n_2 > 0 \) then we refer to different types of quasi-continuity of functions \( f : \mathbb{R}^n \to \mathbb{R} \) as product quasicontinuities.

The following theorem of Kempisty is well known ([5, 6]).

**THEOREM 1.** If all sections \( f_t \) and \( f^y \) of a function \( f : \mathbb{R}^n \to \mathbb{R} \) are quasicontinuous then \( f \) is also quasicontinuous.

To prove that analogues of Kempisty’s theorem for integral quasicontinuities are not true, we start from the following lemma.

**LEMMA 1.** Let \( A, B \subset \mathbb{R} \) be disjoint countable nonempty sets. There are disjoint measurable sets \( E, G \) such that
\[
E \supset A, \quad G \supset B, \quad E \cup G = \mathbb{R},
\]
\[
D_u(E, x) > 0 \quad \text{for each } x \in E,
\]
\[
D_u(G, y) > 0 \quad \text{for each } y \in G.
\]

*Proof.* This is an immediate consequence of Lemma 3 from [3].

**REMARK 1.** Assume the Continuum Hypothesis CH. There is a function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that all sections \( f_t \) and \( f^y \), \( t, y \in \mathbb{R} \), belong to \( Q_s = Q_o \) and the restriction \( f \mid A \) is not measurable for any measurable set \( A \subset \mathbb{R}^2 \) with \( \mu(A) > 0 \).

*Proof.* Let
\[
a_0, a_1, \ldots, a_\alpha, \ldots, \quad \alpha < \omega_1,
\]
be a transfinite sequence of all reals such that \( a_\alpha \neq a_\beta \) for \( \alpha < \beta < \omega_1 \), where \( \omega_1 \) denotes the first uncountable ordinal.

Let \( S \subset \mathbb{R}^2 \) be such that the inner Lebesgue measures \( \mu_i(S) \) and \( \mu_i(\mathbb{R}^2 \setminus S) \) are 0 and \( \text{card}(p \cap S) \leq 2 \) for each straight line \( p \) ([8]).

For \( \alpha < \omega_1 \) we will define by transfinite induction two functions \( g_\alpha, h_\alpha : \mathbb{R} \to \{0, 1\} \).

If the vertical straight line \( p_0 \) defined by the equation \( t = a_0 \) is such that \( p_0 \cap S = \emptyset \) then we put \( h_0(y) = 0 \) for \( y \in \mathbb{R} \). Analogously if the horizontal straight line \( q_0 \) defined by \( y = a_0 \) is such that \( q_0 \cap S = \emptyset \) then we put \( g_0(t) = 0 \) for \( t \in \mathbb{R} \).

If \( p_0 \cap S \neq \emptyset \) then we put \( h_0(y) = 1 \) for \( y \in \mathbb{R} \); if \( q_0 \cap S \neq \emptyset \) then we put \( g_0(t) = 1 \) for \( t \in \mathbb{R} \).

Fix a countable ordinal number \( \alpha > 0 \) and assume that we have defined \( g_\beta, h_\beta : \mathbb{R} \to \{0, 1\} \) for \( \beta < \alpha \).

Let \( p_\alpha \) be defined by \( t = a_\alpha \) and let \( q_\alpha \) be defined by \( y = a_\alpha \). Set
\[
A_{1, \alpha} = \{a_\beta; \beta < \alpha \text{ and } h_\beta(a_\alpha) = 1\} \cup \{(t \in \mathbb{R}; (t, a_\alpha) \in q_\alpha \cap S\}.
\]
Moreover let $A_{2,\alpha} \subset \mathbb{R} \setminus A_{1,\alpha}$ be a countable dense set. By Lemma 1, there are disjoint measurable sets $E_{1,\alpha} \supset A_{1,\alpha}$ and $E_{2,\alpha} \supset A_{2,\alpha}$ such that $\mathbb{R} = E_{1,\alpha} \cup E_{2,\alpha}$, $D_u(E_{1,\alpha}, t) > 0$ for each $t \in E_{1,\alpha}$ and $D_u(E_{2,\alpha}, t) > 0$ for each $t \in E_{2,\alpha}$. Put

$$g_\alpha(t) = \begin{cases} 1 & \text{for } t \in E_{1,\alpha}, \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Set

$$B_{1,\alpha} = \{a_\beta; \beta \leq \alpha \text{ and } g_\beta(a_\alpha) = 1\} \cup \{y \in \mathbb{R}; (a_\alpha, y) \in p_\alpha \cap S\}$$

and let $B_{2,\alpha} \subset \mathbb{R} \setminus B_{1,\alpha}$ be a countable dense set. By Lemma 1, there are disjoint measurable sets $G_{1,\alpha} \supset B_{1,\alpha}$, and $G_{2,\alpha} \supset B_{2,\alpha}$ such that $\mathbb{R} = G_{1,\alpha} \cup G_{2,\alpha}$, $D_u(G_{1,\alpha}, t) > 0$ for each $t \in G_{1,\alpha}$ and $D_u(G_{2,\alpha}, t) > 0$ for each $t \in G_{2,\alpha}$. Let

$$h_\alpha(y) = \begin{cases} 1 \text{ for } t \in G_{1,\alpha} \\ 0 \text{ otherwise on } \mathbb{R.} \end{cases}$$

Now for $x \in \mathbb{R}$ we find an ordinal $\alpha$ such that $x = a_\alpha$ and put

$$f(x, v) = h_\alpha(v) \quad \text{for } v \in \mathbb{R},$$

$$f(u, x) = g_\alpha(u) \quad \text{for } u \in \mathbb{R.}$$

Let

$$\text{Pr}(S) = \{t; \exists y(t, y) \in S\}.$$ 

Since $f(t, y) = 1$ for $(t, y) \in S$ and $\mu_i(f_t^{-1}(0)) > 0$ for $t \in \text{Pr}(S)$, the restriction $f|A$ is not measurable for any measurable $A \subset \mathbb{R}^2$ with $\mu(A) > 0$. If $t = a_\alpha$ then $f^t = g_\alpha \in Q_s$ and $f_t = h_\alpha \in Q_s$ (see [4, Th. 2]). This finishes the proof.

**Corollary 1.** The function $f$ constructed in the proof of Remark 1 is not in $Q_i$, so analogues of Kempisty’s theorem for the integral quasicontinuities are not true.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly (resp. ordinarily) approximately quasicontinuous at a point $x \in \mathbb{R}^n$ if for each $\eta > 0$ and each $U \in T_{s,d}$ (resp. $U \in T_{o,d}$) containing $x$ there is a nonempty set $V \subset U$ belonging to $T_{s,d}$ (resp. to $T_{o,d}$) for which $f(V) \subset (f(x) - \eta, f(x) + \eta)$ (cf. [3]). If $n = 1$ then the notions of strong and ordinary approximate quasicontinuity are equivalent and in this case we say that $f$ is approximately quasicontinuous.

Observe that all sections $f_t$ and $f^y$, $t, y \in \mathbb{R}$, of the function $f : \mathbb{R}^2 \to \mathbb{R}$ constructed in the proof of Remark 1 are approximately quasicontinuous at each point.

By the Lebesgue density theorem, functions strongly (and ordinarily) approximately quasicontinuous at all points are measurable (cf. [3]).
For $A \subset \mathbb{R}^n$, $t \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ define the sections

$$A_t = \{ v \in \mathbb{R}^{n_2}; (t, v) \in A \} \quad \text{and} \quad A^y = \{ u \in \mathbb{R}^{n_1}; (u, y) \in A \}.$$ 

Let

$$T_{s,d}^+ = \{ A \subset \mathbb{R}^n; A \text{ is measurable and} \quad A_u, A^v \in T_{s,d} \text{ for all } u \in \mathbb{R}^{n_1} \text{ and } v \in \mathbb{R}^{n_2} \},$$

$$T_{o,d}^+ = \{ A \subset \mathbb{R}^n; A \text{ is measurable and} \quad A_u, A^v \in T_{o,d} \text{ for all } u \in \mathbb{R}^{n_1} \text{ and } v \in \mathbb{R}^{n_2} \}.$$ 

In connection with Remark 1 we have the following.

**Theorem 2.** If all sections $f_u$ and $f^v$, $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ are strongly (resp. ordinarily) approximately quasicontinuous at all points then for each $(t, y) \in \mathbb{R}^n$, each $\eta > 0$ and each $A \in T_{s,d}^+$ (resp. $A \in T_{o,d}^+$) containing $(t, y)$ there is a measurable subset $B \subset A$ such that $\mu(B) > 0$ and $f(B) \subset (f(t, y) - \eta, f(t, y) + \eta)$.

**Proof.** Fix $(t, y) \in \mathbb{R}^n$, $A \in T_{s,d}^+$ containing $(t, y)$, and $\eta > 0$. Since $f^y$ is strongly approximately quasicontinuous at $t$, there is a measurable set $U \subset A^y$ such that $\mu(U) > 0$ and $f^y(U) \subset (f(t, y) - \eta/3, f(t, y) + \eta/3)$. Since all sections $f_u$, $u \in U$, are strongly approximately quasicontinuous at $y$, for each $u \in U$ there is a measurable set $V(u) \subset A_u$ of positive measure such that $f_u(V(u)) \subset (f(u, y) - \eta/3, f(u, y) + \eta/3)$. Let

$$E = \{ (u, v); u \in U \text{ and } v \in V(u) \}$$

and let $H \subset A$ be a measurable cover of $E$, i.e. $H \supset E$ is a measurable set and each measurable subset of $B \setminus E$ is of measure zero. Evidently the set

$$B = H \cap \{ (u, v) \in \mathbb{R}^n; |f(u, v) - f(t, y)| < \eta \}$$

is as required. The proof for the case of ordinary approximate quasicontinuity is the same.

Theorem 2 implies the following:

**Theorem 3.** If all sections $f_u$ and $f^v$, $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, of a bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$ are strongly (resp. ordinarily) approximately quasicontinuous at all points then for each $(t, y) \in \mathbb{R}^n$, each $\eta > 0$ and each $A \in T_{s,d}^+$ (resp. $A \in T_{o,d}^+$) containing $(t, y)$ there is a bounded set $E \in T_e$ such that $E \cap A \neq \emptyset$ and

$$\left| \frac{\int_{A \cap E} f}{\mu(A \cap E)} - f(t, y) \right| < \eta.$$ 

**Proof.** Fix $(t, y) \in \mathbb{R}^n$, $A \in T_{s,d}^+$ containing $(t, y)$, and $\eta > 0$. By Theorem 2 there is a measurable set $B \subset A$ such that $\mu(B) > 0$ and
\[ f(B) \subset (f(t, y) - \eta/2, f(t, y) + \eta/2). \] So,

\[
\left| \frac{\int_B f}{\mu(B)} - f(t, y) \right| \leq \frac{\eta}{2}.
\]

From the absolute continuity of the Lebesgue integral it follows that there is a nonempty set \( E \subset \mathbb{R}^n \) belonging to \( T_e \) such that \( E \supset B \) and

\[
\left| \frac{\int_{A \cap E} f}{\mu(A \cap E)} - f(t, y) \right| < \eta.
\]

This completes the proof.

Let \( Z \) be a nonempty set of indices. We will say that functions \( h_\alpha : \mathbb{R}^{n_2} \to \mathbb{R}, \alpha \in Z, \) are strongly (resp. ordinarily) integrally equiquasicontinuous at a point \( v \in \mathbb{R}^{n_2} \) if for each set \( V \subset \mathbb{R}^{n_2} \) containing \( v \) and belonging to \( T_{s,d} \) (resp. to \( T_{o,d} \)) and for each \( \eta > 0 \) there is a set \( G \subset \mathbb{R}^{n_2} \) belonging to \( T_e \) and such that \( \emptyset \neq V \cap G \) and

\[
\left| \frac{\int_{V \cap G} f_\alpha}{\mu(G \cap V)} - f_\alpha(v) \right| < \eta \quad \text{for } \alpha \in Z.
\]

**Theorem 4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a locally bounded measurable function such that

(i) for each \((u, v) \in \mathbb{R}^n\) there is a set \( A(u, v) \subset \mathbb{R}^{n_1}\) belonging to \( T_{s,d}\) and containing \( u \) for which the sections \( f_t, t \in A(u, v), \) are strongly integrally equiquasicontinuous at \( v. \)

If \( f^y \in Q_s \) for all \( y \in \mathbb{R}^{n_2}, \) then \( f \) satisfies the following condition:

(a) for each \((t, y) \in \mathbb{R}^n, \) each \( \eta > 0 \) and all \( U \in T_{s,d} \) with \( t \in U \) and \( V \in T_{s,d} \) with \( y \in V \) there are \( Z \subset \mathbb{R}^{n_1} \) and \( Y \subset \mathbb{R}^{n_2} \) belonging to \( T_e \) and such that \( \emptyset \neq U \cap Z, \emptyset \neq V \cap Y \) and

\[
\left| \frac{\int_{(U \times V) \cap (Z \times Y)} f}{\mu((U \times V) \cap (Z \times Y))} - f(t, y) \right| < \eta.
\]

**Proof.** Let \((t, y) \in \mathbb{R}^n, \) \( \eta > 0 \) and \( U, V \in T_{s,d} \) such that \((t, y) \in U \times V. \) Since \( f^y \in Q_s \) and \( t \in U \cap A(t, y) \in T_{s,d}, \) there is a bounded set \( W \in T_e \) such that \( K = W \cap U \cap A(t, y) \neq \emptyset \) and

\[
\left| \frac{\int_K f^y}{\mu(K)} - f(t, y) \right| < \frac{\eta}{2}.
\]

By our hypothesis (i) there is a set \( Y \subset \mathbb{R}^{n_2} \) belonging to \( T_e \) and such that \( V \cap Y \neq \emptyset \) and

\[
\left| \frac{\int_{V \cap Y} f_u}{\mu(Y \cap V)} - f(u, y) \right| < \frac{\eta}{2} \quad \text{for } u \in K.
\]
Let $H = K \times (Y \cap V)$. Observe that

\[
\left| \frac{\int_{H} f(u,v) \, du \, dv}{\mu(H)} - f(t,y) \right| 
\leq \left| \frac{\int_{K} \left( \int_{Y \cap V} f(u,v) \, dv \right) \, du}{\mu(H)} - \frac{\int_{K} f(u,y) \mu(V \cap Y) \, du}{\mu(H)} \right| 
+ \left| \frac{\int_{K} f(u,y) \mu(V \cap Y) \, du}{\mu(K)} - f(t,y) \right|
\]

\[
\leq \frac{\int_{K} \left( \int_{Y \cap V} f(u,v) \, dv \right) \, du}{\mu(K)} - f(t,y) \left| + \frac{\int_{K} f(u,y) \, du}{\mu(K)} - f(t,y) \right|
\]

\[
< \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\]

Since $f$ is locally bounded and measurable, from the absolute continuity of the Lebesgue integral it follows that there is a bounded set $X \subset \mathbb{R}^{n_1}$ containing $K$, belonging to $T_e$ and such that for $M = X \times (Y \cap V)$ we have

\[
\left| \frac{\int_{M} f(u,y) \, du \, dv}{\mu(M)} - f(t,y) \right| < \eta.
\]

So the proof is finished.

In the same way we can prove the following theorem.

**Theorem 5.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally bounded measurable function such that

(ii) for each $(u,v) \in \mathbb{R}^n$ there is a set $A(u,v) \subset \mathbb{R}^{n_1}$ belonging to $T_{o,d}$ and containing $u$ for which the sections $f_t, t \in A(u,v)$, are ordinarily equiquasicontinuous at $v$.

If $f^y \in Q_o$ for all $y \in \mathbb{R}^{n_2}$, then $f$ satisfies the following condition:

(b) for each $(t,y) \in \mathbb{R}^n$, each $\eta > 0$ and all $U \subset T_{o,d}$ with $t \in U$ and $V \subset T_{o,d}$ with $y \in V$ there are $Z \subset \mathbb{R}^{n_1}$ and $Y \subset \mathbb{R}^{n_2}$ belonging to $T_e$ and such that $\emptyset \neq U \cap Z, \emptyset \neq V \cap Y$ and

\[
\left| \frac{\int_{(U \times V) \cap (Z \times Y)} f}{\mu((U \times V) \cap (Z \times Y))} - f(t,y) \right| < \eta.
\]

**Problem.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally bounded function satisfying condition (i) of Theorem 4 (resp. condition (ii) of Theorem 5) and having measurable sections $f^y$ for all $y \in \mathbb{R}^{n_2}$. Is $f$ measurable?

Let $Z$ be a nonempty set of indices. We will say that functions $h_\alpha : \mathbb{R}^{n_2} \to \mathbb{R}, \alpha \in Z$, are integrally equiquasicontinuous at a point $y \in \mathbb{R}^{n_2}$ if for each set $U \subset \mathbb{R}^{n_2}$ containing $y$ and belonging to $T_e$ and for each $\eta > 0$
there is a nonempty set $V \subset U$ belonging to $T_e$ such that
\[
\left| \frac{\int_{V} f_\alpha}{\mu(V)} - f_\alpha(y) \right| < \eta \quad \text{for } \alpha \in \mathbb{Z}.
\]

**Theorem 6.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function such that $f^v \in Q_i$ for all $v \in \mathbb{R}^{n^2}$, and for each $(t, y) \in \mathbb{R}^n$ there is a set $A(t, y) \subset \mathbb{R}^{n^1}$ containing $t$ and belonging to $T_e$ such that the sections $f_u$, $u \in A(t, y)$, are integrally equiquasicontinuous at $y$. Then $f \in Q_i$.

The proof of Theorem 6 is completely similar to the proof of Theorem 4.

**REFERENCES**


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