ON A LINEAR HOMOGENEOUS CONGRUENCE

A. SCHINZEL (Warszawa) and M. ZAKARCZEMNY (Kraków)

Abstract. The number of solutions of the congruence \( a_1 x_1 + \cdots + a_k x_k \equiv 0 \pmod{n} \)
in the box \( 0 \leq x_i \leq b_i \) is estimated from below in the best possible way, provided for all \( i, j \) either \( (a_i, n) \mid (a_j, n) \) or \( (a_j, n) \mid (a_i, n) \) or \( n \mid [a_i, a_j] \).

1. Introduction. We shall consider the following conjecture proposed in [1]:

**CONJECTURE.** Let \( k, n \) and \( b_i \) \( (1 \leq i \leq k) \) be positive integers, and let \( a_i \) \( (1 \leq i \leq k) \) be any integers. The number \( N(n; a_1, b_1, \ldots, a_k, b_k) \) of solutions of the congruence

\[
\sum_{i=1}^{k} a_i x_i \equiv 0 \pmod{n}
\]
in the box \( 0 \leq x_i \leq b_i \)
satisfies the inequality

\[
N(n; a_1, b_1, \ldots, a_k, b_k) \geq 2^{1-n} \prod_{i=1}^{k} (b_i + 1).
\]

Since for \( k = n - 1 \),

\[
N(n; 1, 1, \ldots, 1) = 2^{1-n} \prod_{i=1}^{k} (1 + 1),
\]

if the above conjecture is true, then \( 2^{1-n} \) is the best possible coefficient independent of \( a_i, b_i \), and dependent only on \( n \), with which the inequality (2) holds. The first named author proved in [1] that (2) holds if \( (n, a_i) = 1 \) for all \( i \leq k \). The aim of this paper is to prove

**Theorem.** The inequality (2) holds if for all \( i, j \leq k \) we have either \( (n, a_i) \mid (n, a_j) \) or \( (n, a_j) \mid (n, a_i) \), or \( n \mid [a_i, a_j] \).

**Corollary 1.** The inequality (2) holds for \( n = p^\alpha \) and for \( n = pq \) \((p, q \text{ primes})\).

2000 Mathematics Subject Classification: Primary 11D79.

Key words and phrases: linear homogeneous congruence.
2. Lemmas. We shall use the following lemmas taken from [1]:

**Lemma A.** Inequality (2) holds for \( n = 4, a_1 \) and \( a_2 \) odd, \( b_1 = b_2 = 2 \).

**Lemma B.** Let \( B \) be a set of residues mod \( m \), and let \( a, b \in \mathbb{N} \) with \((a, m) = 1\). If \( x \) runs through the integers of the interval \([0, b]\) and \( y \) through the elements of \( B \), then \( ax + y \) gives at least \( \min\{m, |B| + b\} \) residues mod \( m \).

**Lemma C.** For positive integers \( a \) and \( x \leq a \) we have

\[
\left(1 + \frac{a}{x}\right)^{x+1} \leq 2^{a+1},
\]

except for \( a = 2 \) and \( x = 1 \).

From Lemma B we deduce

**Lemma 1.** Let \( A \) be a set of residues mod \( m \), and let \( a, b \in \mathbb{N} \) with \((a, m) = 1\) and \( b \geq m - |A| \). For every \( r \) the number of solutions of the congruence \( ax + y \equiv r \mod m \) such that \( 0 \leq x \leq b \), \( y \in A \) is at least

\[
s = \left\lceil \frac{b+1}{m+1-|A|} \right\rceil \geq \max\left\{1, \frac{b+1}{2(m+1-|A|)-1}\right\}.
\]

**Proof.** Put \( m - |A| = c \) and consider the intervals (reduced to a point for \( c = 0 \))

\[
I_i = [ci + i, (c(i+1) + i], \quad 0 \leq i \leq s - 1.
\]

Each interval \( I_i \) contains \( c + 1 \) consecutive integers, hence by Lemma B, \( ax + y \) with \( y \in A \) gives \( c + |A| = m \) residues mod \( m \), thus in particular \( r \). Since the \( s \) intervals \( I_i \) are disjoint we obtain the first part of the lemma. The second part (the inequality) follows from the inequality

\[
u \geq \frac{uv + v - 1}{2v - 1}
\]

valid for \( u, v \geq 1 \), in which we take \( u = \left\lceil \frac{b+1}{m+1-|A|} \right\rceil \), \( v = m + 1 - |A| \).

We have further

**Lemma 2.** If \( a > (\log 2)^{-1} \), then the function \( (a - x)2^x \) is unimodal in the interval \([0, a]\) with the maximum at \( x = a - (\log 2)^{-1} \).

**Proof.** By differentiation.

For the proof of further lemmas we need the following definitions and corollaries.

**Definition 1.** \( d_i = (n_i, a_i), n_i = n/d_i \ (1 \leq i \leq k) \).

**Corollary 2.** Under the assumption of the Theorem we have for all \( i, j \leq k \) either \( n_i | n_j \) or \( n_j | n_i \) or \( (n_i, n_j) = 1 \).

**Definition 2.** We write \( i \prec j \) if \( n_i | n_j \) and either \( n_i < n_j \) or \( i < j \).

**Corollary 3.** \( \prec \) is a partial ordering of the set \( \{1, \ldots, k\} \).
DEFINITION 3. $N_l$ is the number of residues mod $n$ given by the numbers
$
\sum_{i \leq l} a_i x_i,$
where $0 \leq x_i \leq b_i$.

DEFINITION 4. $c(i)$ is the number of $j \leq i$ such that $n_j = n_i$.

Now we can formulate

**Lemma 3.** If for $0 \leq x_i \leq b_i$ the sum $\sum_{i \leq l} a_i x_i$ gives at least $1 + \sum_{i \leq l} b_i$ residues mod $n$, then

\[ N_l \geq \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\}. \]

**Proof.** Since $(n, a_l) = d_l$ divides $a_i$ for $i \prec l$,

\[ \sum_{i \prec l} \frac{a_i}{d_l} x_i \text{ gives at least } 1 + \sum_{i \prec l} b_i \text{ residues mod } n_l. \]

We apply Lemma B with $B$ being the set of these residues and with $m = n_l$, $a = a_l/d_l$. The assumptions are satisfied, since

\[ \left( \frac{a_i}{d_l}, n_l \right) = \left( \frac{a_i}{d_l}, \frac{n}{d_l} \right) = 1. \]

Therefore, the number of residues mod $n_l$ of $\sum_{i \leq l} (a_i/d_l)x_i$ is at least

\[ \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\} \]

and hence

\[ \sum_{i \leq l} a_i x_i \text{ gives at least } \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\} \text{ residues mod } n. \]

**Lemma 4.** If for an $l \leq k$ and all $g \prec l$ we have

\[ \sum_{i \geq g} b_i \leq n_g - 1, \]

then

\[ N_l \geq \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\}. \]

**Proof.** Let $I_{h}$ be the set of $i \leq k$ for which there exists a sequence $i_1, \ldots, i_h$ such that $i_1 = i$, $n_{i_\nu} | n_{i_{\nu+1}}$, $n_{i_\nu} < n_{i_{\nu+1}}$ ($1 \leq \nu < h$) and there exists no longer sequence with this property. Clearly, for a certain $s$,

\[ \{1, \ldots, k\} = \bigcup_{h=1}^{s} I_h \]

and $I_g \cap I_h = \emptyset$ for $g \neq h$. Moreover, by Corollary 2,

\[ \text{if } i, j \in I_h \text{ and } n_i \neq n_j, \text{ then } (n_i, n_j) = 1. \]
If \( l \in I_h \) we shall write \( h(l) = h \). We shall prove the lemma by a double induction, with respect to \( s - h(l) \) and with respect to \( c(l) \). If \( s - h(l) = 0 \) and \( c(l) = 1 \), then \( i \leq l \) implies \( i = l \). We have two possibilities.

If \( b_l + 1 \geq n_l \), then \((a_l/d_l)x_l \) \( (0 \leq x_l \leq b_l) \) gives all residues mod \( n_l \), hence \( a_lx_l \) gives \( n_l \) residues mod \( n \), thus \( N_l = n_l \).

If \( b_l + 1 < n_l \), then \((a_l/d_l)x_l \) \( (0 \leq x_l \leq b_l) \) gives \( b_l + 1 \) residues mod \( n_l \), hence \( a_lx_l \) gives \( b_l + 1 \) residues mod \( n \), thus \( N_l \geq b_l + 1 \). Assume now that the assertion is true for \( s - h(l') = 0 \) and \( c(l') = c - 1 \) (\( c \geq 2 \)), and that \( s - h(l) = 0 \), \( c(l) = c \). Then \( i < l \) if and only if \( i \leq l' \), where \( n_{l'} = n_l \) and \( c(l') = c - 1 \). Clearly \( s - h(l') = 0 \) and by the inductive assumption and by (3) with \( g = l' \),

\[
N_{l'} \geq \min \left\{ n_{l'}, 1 + \sum_{i \leq l'} b_i \right\} \geq 1 + \sum_{i \leq l'} b_i.
\]

Hence, by Lemma 3,

\[
N_l \geq \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\}.
\]

Assume now that the assumption is true for \( s - h(l') = s - h - 1 \) and that \( h(l) = h, c(l) = 1 \). Put

\[
A_l = \{ i < l : i \in I_{h+1} \land i = \max\{ q : n_q = n_i \} \} = \{ i_1, \ldots, i_l \}.
\]

If \( t = 0 \), then \( i \leq l \) implies \( i = l \) and the proof proceeds as above for \( s - h(l) = 0 \), \( c(l) = 1 \). Therefore we assume that \( t > 0 \) and infer from (4) that

\[
(n_{i_\mu}, n_{i_\nu}) = 1 \quad \text{for } \mu \neq \nu.
\]

Since \( c(l) = 1 \), \( i < l \) implies \( i \leq i_u \) for some \( u \leq t \). By the inductive assumption the assertion is true for every \( l' = i_u \in A_l \subset I_{h+1} \), hence by (3) for all \( u \leq t \),

\[
N_{i_u} \geq 1 + \sum_{i \leq i_u} b_i.
\]

For \( i \leq i_u \) we have

\[
n_i \mid n_{i_u}, \quad d_i \mid a_i,
\]

hence for each \( u \leq t \),

\[
\sum_{i \leq i_u} a_i d_i^{-1} x_i \quad (0 \leq x_i \leq b_i) \quad \text{gives} \quad N_{i_u} \geq 1 + \sum_{i \leq i_u} b_i \text{ residues mod } n_{i_u}.
\]

Now, by (5) for all integers \( z_1, \ldots, z_t, r_1, \ldots, r_t \) we have

\[
\sum_{u=1}^t \frac{n}{n_{i_u}} z_u \equiv \sum_{u=1}^t \frac{n}{n_{i_u}} r_u \pmod{n}
\]
if and only if \( z_u \equiv r_u \pmod{n_{i_u}} \) for all \( u \leq t \). It follows that the number of residues mod \( n \) given by

\[
\sum_{u=1}^{t} \frac{n}{n_{i_u}} \sum_{i \leq i_u} \frac{a_i}{n_{i_u}} x_i = \sum_{u=1}^{t} \frac{d_{i_u}}{d_{i_u}} \sum_{i \leq i_u} \frac{a_i}{d_{i_u}} x_i = \sum_{u=1}^{t} \sum_{i \leq i_u} a_i x_i = \sum_{i \leq l} a_i x_i
\]

for \( 0 \leq x_i \leq b_i \) is equal to \( \prod_{u=1}^{t} N_{i_u} \), hence by (6) it is at least

\[
\prod_{u=1}^{t} \left( 1 + \sum_{i \leq i_u} b_i \right) \geq 1 + \sum_{u=1}^{t} \sum_{i \leq i_u} b_i = 1 + \sum_{i \leq l} b_i.
\]

Using Lemma 3 we obtain

\[
N_l \geq \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\},
\]

which proves the assertion for \( s - h(l) = s - h, c(l) = 1 \).

Assume now that the assertion is true for \( s - h(l') = s - h \) and \( c(l') = c - 1 \) \( (c \geq 2) \) and that \( s - h(l) = s - h, c(l) = c \). Then \( i < l' \) if and only if \( i \leq l' \), where \( n_{l'} = n_l \) and \( c(l') = c - 1 \). Clearly \( s - h(l') = s - h \), thus by the inductive assumption and by (3),

\[
N_{l'} \geq \min \left\{ n_{l'}, 1 + \sum_{i \leq l'} b_i \right\} \geq 1 + \sum_{i \leq l'} b_i.
\]

Hence, by Lemma 3,

\[
N_l \geq \min \left\{ n_l, 1 + \sum_{i \leq l} b_i \right\}.
\]

**Definition 5.** \( M = \bigcup_{i=1}^{k} \{ n_i \} \).

**Lemma 5.** Let us order \( a_i \) in such a way that \( i \leq j \) implies \( n_i \leq n_j \). Under the assumption of the Theorem, for every \( l \leq k \) either

\( \sum_{n_i|n_l, i \leq l} a_i x_i \) (0 \( \leq x_i \leq b_i \)) gives at least

\[
\min \left\{ n_l, 1 + \sum_{n_i|n_l, i \leq l} b_i \right\} \text{ residues mod } n.
\]

**Proof.** We apply Lemma 4. If there exists \( g \) not satisfying (3) such that \( n_g \mid n_l, n_g < n_l \) then (7) holds with \( m' = n_g \). If there exist \( g \) not satisfying (3)
with \( n_g \mid n_l \), but for all of them \( n_g = n_l \), then taking the least such \( g \), by Lemma 4 we obtain
\[ N_l \geq N_g \geq \min \left\{ n_g, \sum_{n_i \mid n_g, i \leq g} b_i \right\} = n_g = \min \left\{ n_l, \sum_{n_i \mid n_l, i \leq l} b_i \right\}, \]
thus (8) holds.

Finally, if (3) is satisfied by all \( g \) with \( n_g \mid n_l, g < l \), then (8) holds by Lemma 4.

**Lemma 6.** Let \( t, x_1, \ldots, x_t \) be integers greater than 1. Then
\[ \sum_{u=1}^{t} (x_u - 2) \leq \frac{1}{2} \prod_{u=1}^{t} x_u - 2. \]

**Proof.** Since \( x + y \leq xy \) for \( x, y \geq 2 \), we have
\[ \sum_{u=1}^{t} (x_u - 2) = \sum_{u=1}^{t} x_u - 2t \leq \prod_{u=1}^{t} x_u + x_t - 2t. \]
Since \( x + y - 2 \leq xy/2 \) for \( x, y \geq 2 \), we have
\[ \prod_{u=1}^{t} x_u + x_t - 2t \leq \frac{1}{2} \prod_{u=1}^{t} x_u - 2(t - 1) \leq \frac{1}{2} \prod_{u=1}^{t} x_u - 2. \]
Combining both inequalities we obtain the lemma.

3. **Proof of the Theorem.** We may assume without loss of generality that if \( i \leq j \) then either \( n_i < n_j \), or \( n_i = n_j \) and \( b_i \geq b_j \). By Corollary 2, for all \( i, j \leq k \) we have
\[ \begin{cases} n_i \mid n_j & \text{or} \quad n_j \mid n_i & \text{or} \quad (n_i, n_j) = 1. \end{cases} \]
We proceed by induction on \( k \). For \( k = 1 \), (2) is trivially true. Assume it is true for all \( k' < k \). If \( n_1 = 1 \) then
\[ N(n; a_1, b_1, \ldots, a_k, b_k) = (b_1 + 1)N(n; a_2, b_2, \ldots, a_k, b_k), \]
hence (2) follows from the inductive assumption.
Therefore assume that \( n_i \geq 2 \), and \( n \geq 4 \) by the result of [1]. Suppose that there exist \( \overline{m} \in M \) such that \( \sum_{n_i \mid \overline{m}} b_i \geq \overline{m} - 1 \) and let \( m \) be the least number with this property. Hence for all \( m' < m, m' \in M \) we have
\[ \sum_{n_i \mid m'} b_i \leq m' - 2. \]
Let \( m_u \) (\( 1 \leq u \leq t \)) be all maximal elements with respect to divisibility in the set \( \{ \mu \in M \setminus \{m\} : \mu \mid m \} \). We have \( m_u \in M \setminus \{m\} \), \( m_u \mid m \), and the \( m_u \) are not divisible by one another, hence by (9), \( (m_u, m_v) = 1 \) for \( u \neq v \). It follows that

\[
\prod_{u=1}^{t} m_u \mid m.
\]

We take the least \( j \) such that

\[
\sum_{n_i \mid m, i \leq j} b_i \geq m - 1.
\]

By (10) we have

\[
\sum_{n_i \mid m_u} b_i \leq m_u - 2,
\]

hence by Lemma 6,

\[
\sum_{u=1}^{t} \sum_{n_i \mid m_u} b_i \leq \sum_{u=1}^{t} (m_u - 2) \leq \frac{1}{2} \prod_{u=1}^{t} m_u - 2 \leq \frac{1}{2} m - 2,
\]

unless \( t \leq 1 \). However, for \( t \leq 1 \) the inequality

\[
\sum_{u=1}^{t} (m_u - 2) \leq \frac{1}{2} m - 2
\]

is also true, thus (12) and (13) imply \( n_j \nmid m_u \) (\( 1 \leq u \leq t \)), hence \( n_j = m \).

Also, by Lemma 5, inequality (10) and the choice of \( j \) the number of residues mod \( n \) given by \( \sum_{n_i \mid m, i < j} a_i x_i \) (\( 0 \leq x_i \leq b_i \)) is at least \( 1 + \sum_{n_i \mid m, i < j} b_i \).

For every choice of \( x_i \) (\( n_i \nmid m \) or \( i > j \)) such that

\[
\sum_{n_i \nmid m \text{ or } i > j} a_i x_i \equiv 0 \pmod{\frac{n}{m}}
\]

there exist, by Lemma 1, at least

\[
\max \left\{ 1, \frac{b_j + 1}{2(m - \sum_{n_i \mid m, i < j} b_i)} - 1 \right\}
\]

solutions of the congruence

\[
\frac{m}{n} \sum_{n_i \mid m, i < j} a_i x_i + \frac{ma_j}{n} x_j + \frac{m}{n} \sum_{n_i \nmid m \text{ or } i > j} a_i x_i \equiv 0 \pmod{m},
\]

satisfying \( 0 \leq x_i \leq b_i \) (\( n_i \mid m \) and \( i \leq j \)). However, the number of summands in (14) is less than \( k \), hence, by the inductive assumption, the number of solutions of (14) with \( 0 \leq x_i \leq b_i \) is at least

\[
2^{1 - n/m} \prod_{n_i \nmid m \text{ or } i > j} (b_i + 1).
\]
Thus we obtain

\[ N(n; a_1, b_1, \ldots, a_k, b_k) \geq 2^{1-n/m} \prod_{n_i \mid m \text{ or } i > j} (b_i + 1) \max \left\{ 1, \frac{b_j + 1}{2(m - \sum_{n_i \mid m, i < j} b_i) - 1} \right\}. \]

We consider three cases:

\begin{align*}
(16) & \quad m < n, \\
(17) & \quad m = n \quad \text{and either } j = 1 \text{ or } n_{j-1} < n, \\
(18) & \quad m = n, \quad j \geq 2 \quad \text{and} \quad n_{j-1} = n.
\end{align*}

In the case (16) we have, by (15) and Bernoulli’s inequality,

\[ N(n; a_1, b_1, \ldots, a_k, b_k)^{-1} \prod_{i=1}^{k} (b_i + 1) \leq 2^{n/m} \left( m - \sum_{n_i \mid m, i < j} b_i - \frac{1}{2} \right) \prod_{n_i \mid m, i < j} (b_i + 1) \leq 2^{n/m} \left( m - b - \frac{1}{2} \right) 2^b, \]

where \( b = \sum_{n_i \mid m, i < j} b_i \). By the choice of \( j \) we have

\[ b \leq m - 2 < m - \frac{1}{2} - \frac{1}{\log 2}, \]

hence by Lemma 2,

\[ \left( m - b - \frac{1}{2} \right) 2^b \leq 3 \cdot 2^{m-3} < 2^{m-1}, \]

and by (19),

\[ N(n; a_1, b_1, \ldots, a_k, b_k)^{-1} \prod_{i=1}^{k} (b_i + 1) < 2^{n/m + m - 1} \leq 2^{n-1}, \]

because \( n - n/m - m = (n/m - 1)(m - 1) - 1 \geq 0. \)

In the case (17) we have again (19), but now, by (13),

\[ b \leq \frac{1}{2} n - 2 \leq n - \frac{1}{2} - \frac{1}{\log 2}, \]

hence by Lemma 2 and Bernoulli’s inequality

\[ \left( m - b - \frac{1}{2} \right) 2^b \leq \left( \frac{n}{2} + \frac{3}{2} \right) 2^{n/2 - 2} \leq 2^{n-3/2}. \]
In the case (18) we have, by (15),
\[
N(n; a_1, b_1, \ldots, a_k, b_k)^{-1} \prod_{i=1}^{k} (b_i + 1) \leq \prod_{n_i|m, i \leq j} (b_i + 1) \\
\leq \prod_{n_i|m, n_i < m} (b_i + 1) \cdot \prod_{n_i=m, i \leq j} (b_i + 1) \\
\leq 2^{\sum_{n_i|m, n_i < m} b_i} \prod_{n_i=m, i \leq j} (b_i + 1).
\]
Now, by the choice of \(j\),
\[
\sum_{n_i|m, i < j} b_i \leq n - 2, \quad \sum_{n_i=m, i < j} b_i \leq n - 2 - \sum_{n_i|m, n_i < m} b_i,
\]
thus \(b_j \leq b_{j-1} \leq a/x\), where
\[
a = n - 2 - \sum_{n_i=m, n_i < m} b_i, \quad x = \sum_{n_i=m, i < j} 1.
\]
By the inequality for the arithmetic and geometric mean and by Lemma C,
\[
\prod_{n_i=m, i \leq j} (b_i + 1) \leq \left(1 + \frac{a}{x}\right)^{x+1} \leq 2^{a+1},
\]
unless \(a = 2, x = 1\). Leaving this case for a further consideration we obtain from (20),
\[
N(n; a_1, b_1, \ldots, a_k, b_k)^{-1} \prod_{i=1}^{k} (b_i + 1) \\
\leq 2^{\sum_{n_i|m, n_i < m} b_i} \cdot 2^{n-1-\sum_{n_i|m, n_i < m} b_i} = 2^{n-1}.
\]
If \(a = 2, x = 1\) we obtain, because of (13),
\[
n - 4 = \sum_{n_i|m, n_i < m} b_i \leq \frac{1}{2} n - 2,
\]
hence \(n \leq 4\), that is, \(n = 4\); moreover, \(j = 2, n_1 = n_2 = 4, b_2 \leq b_1 \leq 2\) and
\[
N(n; a_1, b_1, \ldots, a_k, b_k)^{-1} \prod_{i=1}^{k} (b_i + 1) \leq (b_1 + 1)(b_2 + 1) \leq 2^{n-1}
\]
unless \(b_1 = b_2 = 2\). However, the last case is covered by Lemma A.

Assume now that for every \(m \in M\) we have
\[
(21) \sum_{n_i|m} b_i \leq m - 2.
\]
If \( n \in M \) it follows that
\[
\sum_{i=1}^{k} b_i \leq n - 2. \tag{22}
\]
If \( n \notin M \), then for every \( n_i \) there exists the greatest \( m \in M \) such that \( n_i | m \). Put \( m = f(n_i) \) and \( M_0 = f(M) \). It follows from (9) that the elements of \( M_0 \) are coprime. Hence
\[
\prod_{m \in M_0} m | n
\]
and, by Lemma 6,
\[
\sum_{n \in M_0} (m - 2) \leq \frac{1}{2} \prod_{m \in M_0} m \leq \frac{1}{2} n - 2
\]
unless \( M_0 \) has just one element \( m_0 \).

However, \( m_0 \leq \frac{1}{2} n \), thus in each case, by (21),
\[
\sum_{i=1}^{k} b_i \leq \sum_{m \in M_0} \sum_{n_i | m} b_i \leq \sum_{m \in M_0} (m - 2) \leq \frac{1}{2} n - 2
\]
and (22) holds generally. It follows by Bernoulli’s inequality that
\[
N(n; a_1, b_1, \ldots, a_k, b_k)^{-1} \prod_{i=1}^{k} (b_i + 1) \leq \frac{1}{2} \sum_{i=1}^{k} b_i \leq n - 2.
\]

**Added in proof.** As proved in [2], the inequality (2) holds if \( n = \prod_{j=1}^{l} q_j^{\alpha_j} \), where \( q_j \) are primes and \( \sum_{j=1}^{l} 1/q_j < 1 \).

---

**REFERENCES**
