CB-DEGENERATIONS AND RIGID DEGENERATIONS
OF ALGEBRAS

BY

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Abstract. The main aim of this note is to prove that if $k$ is an algebraically closed field and a $k$-algebra $A_0$ is a CB-degeneration of a finite-dimensional $k$-algebra $A_1$, then there exists a factor algebra $A$ of $A_0$ of the same dimension as $A_1$ such that $A$ is a CB-degeneration of $A_1$. As a consequence, $A$ is a rigid degeneration of $A_1$, provided $A_0$ is basic.

Introduction. There are at least three different concepts of geometric degenerations for $k$-algebras: degenerations in the classical sense referring to the geometry of orbits in a variety of algebras (the idea goes back to nineteenth century algebraists, see [8]), the so-called rigid degenerations using the notion of degeneration of (ordered) locally bounded categories (see [5]), and the CB-degenerations introduced by Crawley-Boevey in [2] (see [3] for the precise definitions). All these three concepts are useful for deciding in some specific situations whether a fixed algebra is tame. This method is based on three “degeneration theorems” (see [5, 6, 2]), each of which states that, if a finite-dimensional tame $k$-algebra $A_0$ is a degeneration of a fixed algebra $A_1$, then $A_1$ is also tame. For classical and rigid degenerations this was proved by Geiss, who uses ordered locally bounded categories, avoiding the so-called Gabriel lemma whose proof is rather involved and requires at least the use of projective geometry (see [7] and also [4, 10]). The result of Crawley-Boevey appeared a little later and is mainly applied in the study of biserial algebras.

In the last fifteen years the degeneration technique has found many interesting applications; in particular, it was successfully used in solving several important classification problems for tame algebras. Also certain natural theoretical questions concerning degenerations have been considered. In [3] some interrelations between the three notions of degeneration are studied. It is shown there that a basic algebra $A_0$ is a CB-degeneration of a (basic) algebra $A_1$ of the same dimension as $A_0$ over a field $k$ if and only if $A_0$ is a rigid degeneration of $A_1$ ([3, Theorem 5.1]). Moreover, a reduction of CB-
degeneration problems for nonbasic algebras to those for their basic representatives in Morita equivalence classes is discussed. Finally, it is proved that for every CB-degeneration of an algebra $A_1$ to $A_0$, obtained along an affine line, there exists a factor algebra $\overline{A}_0$ of $A_0$ such that $\dim_k \overline{A}_0 = \dim_k A_1$ and $\overline{A}_0$ is also a CB-degeneration of $A_1$. As a consequence, $\overline{A}_0$ is also a rigid degeneration of $A_1$, provided $A_0$ is basic ([3, Theorem 6.1]).

The aim of this note is to prove a generalization of this result to the case of all CB-degenerations, without any restriction on the variety involved (see Theorem of Section 2). Consequently, the theoretical scope of [2, Theorem B] is exactly the same as that of the original version of the Geiss theorem from [5].

1. Preliminaries. Throughout the paper, we use the well known definitions (see [2, 6]) and notation introduced in [3]. We now briefly recall the most important of them.

Throughout the paper $k$ denotes an algebraically closed field. By an algebra we mean a finite-dimensional $k$-algebra.

For any $m, n \in \mathbb{N}$, we denote by $M_{m \times n}(k)$ the set of all $m \times n$-matrices with coefficients in $k$, by $M_n(k)$ the algebra $M_{n \times n}(k)$ of square $n \times n$-matrices and by $\text{Gl}_n(k)$ the group of invertible matrices in $M_n(k)$. For a fixed dimension vector $d \in \mathbb{N}^n_+$, we set

$$H_d(k) = \prod_{i,j=1,\ldots,n} \text{Gl}_{d_{i,j}}(k).$$

Following [2], we introduce a useful definition (see [2, Theorem B]).

**DEFINITION.** Given two algebras $A_0$ and $A_1$, the algebra $A_0$ is a CB-degeneration of $A_1$ if there exists a finite-dimensional algebra $A$, an irreducible variety $X$ and regular maps $f_1, \ldots, f_r : X \to A$ such that $A_1 \cong A_x$ for all $x$ in some nonempty open subset $U$ of $X$, and $A_0 \cong A_{x_0}$ for some $x_0 \in X$, where $A_y = A/(f_1(y), \ldots, f_r(y))$ for any $y \in X$.

In the situation as above, the data sequence $D = (A, X, \mathcal{F}, U, x_0)$ is called a degenerating collection defining a CB-degeneration of $A_1$ to $A_0$ along $X$ by use of $A$, where $\mathcal{F} = \{f_1, \ldots, f_r\}$.

The concept of rigid degenerations is based on the notion of degeneration for finite locally bounded categories $R$ with a fixed linearly ordered set $(x_1, \ldots, x_n)$ of objects and with the dimension vector $d \in \mathbb{N}^n_+$, $n \geq 1$, where $d_{i,j} = \dim_k J_R(x_i, x_j)$ for all $i, j$ and $J_R$ is the Jacobson radical of $R$.

Given $d$ as above, we consider a group action

$$\cdot : H_d(k) \times \text{lbc}_d(k) \to \text{lbc}_d(k),$$

where $\text{lbc}_d(k)$ is the affine variety of constant structures for locally bounded $k$-categories with a fixed object set $\{1, \ldots, n\}$ and dimension vector $d$ (see [3]).

Suppose we are now given two basic $k$-algebras $A_0$ and $A_1$ of the same dimension. We say that $A_0$ is a rigid degeneration of $A_1$ if there exist complete sequences $e^{(0)} = (e_i^0, \ldots, e_n^0)$ and $e^{(1)} = (e_1^1, \ldots, e_n^1)$ of primitive pairwise orthogonal idempotents in $A_0$ and $A_1$, respectively, such that $$\dim_k(e_i^0 A^0 e_j^0) = \dim_k(e_i^1 A e_j^1)$$ for all $i, j = 1, \ldots, n$, and that for any constant structures $c^{(0)}, c^{(1)} \in \text{lbc}_d(k)$ of finite locally bounded $k$-categories $R_0 = R(A_0, e^{(0)})$ and $R_1 = R(A_1, e^{(1)})$ respectively, we have the inclusion $$H_d \cdot c^{(0)} \subset H_d \cdot c^{(1)}.$$ Here we treat $R_0$ and $R_1$ via the correspondence $e_i^0 \leftrightarrow i \leftrightarrow e_i^1$, as categories with the object set $\{1, \ldots, n\}$; see [3] for more details.

2. Main theorem. Now we are able to formulate the main result of the paper, generalizing [3, Theorem 6.1].

THEOREM. Let $A_0$ and $A_1$ be finite-dimensional $k$-algebras. Assume that $A_0$ is a CB-degeneration of $A_1$ (with respect to a finite-dimensional algebra $A$). Then $A_1$ admits a CB-degeneration (with respect to $A$) to some factor algebra $\overline{A}_0$ of $A_0$ such that $\dim_k \overline{A}_0 = \dim_k A_1$. In particular, if $A_0$ is basic, then $A_1$ admits a rigid degeneration to the same $\overline{A}_0$.

For the proof we need some auxiliary facts.

LEMMA. Let $X$ be an irreducible affine $k$-variety, $X' \subseteq X$ a nonempty open subset, and $x_0 \in X \setminus X'$. Then there exists an irreducible closed curve $\Gamma \subseteq X$ such that $x_0 \in \Gamma$ and $X' \cap \Gamma \neq \emptyset$.

Proof. We proceed by induction on $\dim X$. If $\dim X = 1$, then obviously $\Gamma = X$. Suppose that $\dim X > 1$ and the lemma is proved for all varieties of dimension less than $\dim X$. We can assume that $X \subseteq \overline{A}^n(k)$ is a closed set (in the Zariski topology). Let $X \setminus X' = X_1 \cup \cdots \cup X_s$ be a decomposition of $X \setminus X'$ into irreducible components, and $x_1 \in X_1, \ldots, x_s \in X_s$ a fixed selection of elements. Choose a polynomial $F \in k[T_1, \ldots, T_n]$ such that $F(x_0) = 0$ and $F(x_i) \neq 0$ for $i = 1, \ldots, s$. Then the set $V = X \cap V(F)$ contains no $X_i$ for $i = 1, \ldots, s$. Let $Z$ be an irreducible component of $V$ passing through $x_0$. Then $Z$ contains no $X_i$ since $Z \subseteq V$. By [9, Theorem 3.3] we have $\dim X_i \leq \dim (X \setminus X') \leq \dim X - 1 = \dim Z$. Thus $X_i$ contains $Z$, otherwise $\dim X_i = \dim Z$ and by [9, Proposition 3.2] we get $X_i = Z$, a contradiction. Therefore the open subset $Z' = Z \cap X'$ of $Z$ is nonempty, and by definition of $Z$ the point $x_0$ belongs to $Z$. By inductive assumption
(dim \(Z = \dim X - 1\)) there exists an irreducible affine curve \(\Gamma \subseteq Z\) such that \(x_0 \in \Gamma\) and \(\Gamma \cap Z' \neq \emptyset\). Notice that \(\Gamma \subseteq X\) is closed and \(\Gamma \cap X' \neq \emptyset\), hence \(\Gamma\) is the required curve. 

**Corollary.** Every CB-degeneration \(A_0\) of an algebra \(A_1\) can be obtained along a nonsingular irreducible affine curve.

**Proof.** Let \(A_0, A_1\) be fixed finite-dimensional algebras and \(\mathcal{D} = (A, X, \mathcal{F}, U, x_0)\) a collection defining a CB-degeneration from \(A_1\) to \(A_0\), where \(\mathcal{F} = \{f_1, \ldots, f_s\}\) are regular maps from \(X\) to \(A\). Changing \(X\) to a suitable principal open set containing \(x_0\), we can assume that \(X\) is an irreducible affine variety. By the Lemma there exists an irreducible curve \(\Gamma \subseteq X\) such that \(x_0 \in \Gamma\) and \(\Gamma \cap U \neq \emptyset\). Then replacing \(\mathcal{F}\) by \(\mathcal{F}_{|\Gamma} = \{f_1|\Gamma, \ldots, f_s|\Gamma\}\) and \(U\) by \(U|\Gamma = U \cap \Gamma\) we can assume that \(X\) is an irreducible affine curve.

Let \(p : \tilde{X} \rightarrow X\) be a normalization of \(X\) (see [11]). It is known that \(\tilde{X}\) is a nonsingular curve, since \(\dim Y - \dim \text{Sing} Y \geq 2\) for any normal variety \(Y\), where \(\text{Sing} Y\) denotes the set of singular points of \(Y\). We now define a collection \(\mathcal{D} = (A, \tilde{X}, \tilde{\mathcal{F}}, \tilde{U}, \tilde{x}_0)\), where \(\tilde{\mathcal{F}} = \{f_1 \circ p, \ldots, f_s \circ p\}\), \(\tilde{U} = p^{-1}(U)\), \(\tilde{x}_0\) is a fixed point in \(p^{-1}(x_0)\). It is easily seen that \(\mathcal{D}\) defines a CB-degeneration from \(A_1\) to \(A_0\).

Now we can prove the main result of this note.

**Proof of Theorem.** We carry out the proof by induction on \(n = \dim_k A_0 - \dim_k A_1\). If \(n = 0\) then we simply get \(A_0 = A_0\). Assume that \(n > 0\) and let \(\mathcal{D} = (A, X, \mathcal{F}, U, x_0)\) be a collection defining a CB-degeneration from \(A_1\) to \(A_0\), where as usual \(\mathcal{F} = \{f_1, \ldots, f_s\}\). Denote by \(v_1, \ldots, v_m \in A\) a basis of \(A\), where \(m = \dim_k A\). By the Corollary we can assume that \(X\) is an irreducible nonsingular curve. Without loss of generality we can also assume that \(I_x = \langle f_1(x), \ldots, f_s(x) \rangle\) for all \(x \in X\). For any \(i = 1, \ldots, s\) we denote by \(\{f_i^j\}_{j=1,\ldots,m}\) the family of regular functions on \(X\) such that \(f_i(x) = \sum_{j=1}^m f_i^j(x)v_j\) for \(x \in X\). We set

\[f(x) = [f_i^j(x)]\]

for any \(x \in X\) ([\(f_i^j(x) \in M_{s \times m}(k)\]) and \(r = r(f(x_0))\). Note that \(r < r(f(x))\) for all \(x \in U\), since \(r = \dim_k I_{x_0}\) and \(r(f(x)) = \dim_k I_x\). By the definition of \(r\) there exists a nonzero \(r \times r\)-minor of the matrix \(f(x_0)\). We can assume that it is the determinant of the upper-left \(r \times r\)-submatrix of \(f(x_0)\). Let \(h : X \rightarrow k\) be the regular function defined by \(x \mapsto \det(f(x))_r\) for \(x \in X\), where \(f(x)_r = [f_i^j(x)]_{i,j=1,\ldots,r} \in M_r(k)\). Clearly \(h(x_0) \neq 0\). Now we use the identification

\[M_{s \times m}(k) = \begin{bmatrix} M_{r \times r}(k) & M_{r \times (m-r)}(k) \\ M_{(s-r) \times r}(k) & M_{(s-r) \times (m-r)}(k) \end{bmatrix} \]

\((m, s > r\), since \(r(f(x)) > r\) for any \(x \in U\)). By applying two-step Gaussian-
row elimination, we transform \( f(x) \) to a matrix \( \tilde{f}(x) = [\tilde{f}^i_j(x)] \in M_{s \times m}(k) \), for \( x \in X \) such that \( h(x) \neq 0 \), as follows:

\[
\begin{pmatrix} \text{id}_r & \ast & \ast \\ \ast & \ast & \ast \\ 0 & \ast & \ast \end{pmatrix} \quad \begin{pmatrix} \ast & \ast & \ast \\ 0 & \text{id}_{s-r} & \ast \\ \ast & \ast & \ast \end{pmatrix} = \begin{pmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix} = \tilde{f}(x),
\]

where the first transformation corresponds to multiplication of \( f(x) \) from the left by the block diagonal matrix \( \begin{pmatrix} (f(x)_r)^{-1} & 0 & 0 \\ 0 & \text{id}_{s-r} & 0 \end{pmatrix} \). In this way all functions \( \tilde{f}^i_j(x) \) are rational and belong to the local ring \( O_{x_0}(X) \), since \( h(x_0) \neq 0 \) and therefore \( 1/h \in O_{x_0}(X) \). We regard here \( O_{x_0}(X) \) as a subring of \( k(X) \) (\( X \) is irreducible). Moreover, observe that \( \tilde{f}^i_j(x) = 0 \) for all \( r < i \leq s, 1 \leq j \leq r \), and \( \tilde{f}^i_j(x_0) = 0 \) for all \( r < i \leq s, r < j \leq m \).

Now, since \( \dim_k I_x > \dim_k I_{x_0} \) for \( x \in U \), and \( I_x = (\tilde{f}^1_1(x), \ldots, \tilde{f}^s_s(x)) \) for \( x \in X \) such that \( h(x) \neq 0 \), we infer that, for all \( x \in U \) such that \( h(x) \neq 0 \), there exists a pair \((i,j)\) with \( r < i \leq s, r < j \leq m \) such that \( \tilde{f}^i_j(x) \neq 0 \). Consequently, all functions \( \tilde{f}^i_j \), \( r < i \leq s, r < j \leq m \), belong to the maximal ideal \( \mathfrak{m}_{x_0}(X) \subseteq O_{x_0}(X) \) and not all of them are zero. By the Auslander–Buchsbaum theorem (see [1]), \( O_{x_0}(X) \) is a unique factorization domain, hence \( \mathfrak{m}_{x_0}(X) \) is a principal ideal generated by some \( g \in \mathfrak{m}_{x_0}(X) \), since Krull \( \dim O_{x_0}(X) = 1 \).

Note that \( p = \max\{k \in \mathbb{N} \mid g^k \mid \tilde{f}^i_j, r < i \leq s, r < j \leq m \} \) is finite. We get now equations \( \tilde{f}^i_j = g^p \cdot \tilde{f}^i_j \), \( r < i \leq s, r < j \leq m \), in \( O_{x_0}(X) \), for some rational functions \( \tilde{f}^i_j \in O_{x_0}(X) \). By definition of \( p \), not all \( \tilde{f}^i_j \) belong to \( \mathfrak{m}_{x_0}(X) \).

We can assume there exists \( r < j \leq m \) such that \( \tilde{f}^i_{r+1} \notin \mathfrak{m}_{x_0}(X) \). We now define a regular map \( \tilde{f}_{s+1} : X' \to A \) by \( \tilde{f}_{s+1}(x) = \sum_{i=r+1}^m \tilde{f}^i_{r+1}(x)v_j \) for \( x \in X' \), where \( X' \) is an open set (a neighbourhood of \( x_0 \)) obtained as the intersection of the domains of all rational functions \( \tilde{f}^i_{r+1}, r < j \leq m \). Observe that \( \tilde{f}_{s+1} \) is a regular function on \( X' \) and that \( \tilde{f}_{s+1}(x_0) \notin I_{x_0} \), since \( \tilde{f}^i_{r+1} \notin \mathfrak{m}_{x_0}(X) \) for some \( r < j \leq m \), and \( \tilde{f}_{s+1}(x) \in \text{Span}\{v_{r+1}, \ldots, v_m\} \) for \( x \in X' \).

We set \( \tilde{f}_i = f_i|_{X'} \) for \( i = 1, \ldots, s \), thus obtaining a collection \( D' = (A, X', \mathcal{F}', U', x_0) \) defining a CB-degeneration from \( A_1 \) to some factor algebra \( \overline{A}_0 \) of \( A_0 \) such that \( \dim_k \overline{A}_0 < \dim_k A_0 \), where \( \mathcal{F}' = \{\tilde{f}_1, \ldots, \tilde{f}_s, \tilde{f}_{s+1}\} \) and \( U' = U \cap X' \). By the inductive assumption \( \dim_k \overline{A}_0 - \dim_k A_1 < n \), \( A_1 \) admits a CB-degeneration to a factor algebra \( \overline{A}_0 \) of \( A_0 \) such that \( \dim_k \overline{A}_0 = \dim_k A_1 \). But \( \overline{A}_0 \) is also a factor algebra of \( A_0 \). This completes the proof of the first assertion.

The second assertion follows immediately from [3, Theorem 5.1], since \( \overline{A}_0 \) is a basic algebra, and consequently so is \( A_1 \) (see [3, Corollary 4.1]), provided \( A_0 \) is basic.
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