COLLOQUIUM MATHEMATICUM

VOL. 106 2006 NO. 2

THE DUNFORD–PETTIS PROPERTY, THE GELFAND–PHILLIPS PROPERTY, AND L-SETS

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Abstract. The Dunford–Pettis property and the Gelfand–Phillips property are studied in the context of spaces of operators. The idea of L-sets is used to give a dual characterization of the Dunford–Pettis property.

1. Introduction. Numerous papers have investigated whether spaces of operators inherit the Dunford–Pettis property or the Gelfand–Phillips property when the co-domain and the dual of the domain have the respective property; e.g., see the introduction and Section 2 of [10], Theorem 3 through Corollary 11 of [15], and Sections 2 and 3 of [17]. In this paper weak precompactness and Schauder basis theory are used in spaces of operators to establish simple mapping results which extend and consolidate results in [10], [15], and [17]. The hereditary Dunford–Pettis property is also studied. Additionally, the Schur property is characterized in terms of Dunford–Pettis properties, and L-sets are used in a dual characterization of the Dunford–Pettis property.

2. Definitions and notation. Let each of X, Y, E, and F denote a real Banach space, let X^* denote the continuous linear dual of X, let $L(X,Y)$ denote the space of all continuous linear operators $T : X \to Y$, and let $K(X,Y)$ denote the compact linear maps. The w^*-w continuous operators will be denoted by $L_{w^*}(X^*,Y)$, and $K_{w^*}(X^*,Y)$ will denote the compact and w^*-w continuous operators.

DEFINITION 2.1. An operator $T : X \to Y$ is *completely continuous* if $(T(x_n))$ is norm convergent in Y whenever (x_n) is weakly convergent in X.

All compact operators are completely continuous. However, if weakly Cauchy sequences in X are norm convergent, then all operators with domain X are completely continuous. We say that X has the *Schur property* if every weakly Cauchy sequence in X is norm convergent.

2000 *Mathematics Subject Classification*: 46B20, 46B25, 46B28.

Key words and phrases: Dunford–Pettis property, Gelfand–Phillips property, L-sets, w^*-w continuity, completely continuous operator.

A combination of classical results of Dunford and Pettis [11] and Grothendieck [22] shows that if X is a $C(K)$-space or an L_1-space, then every weakly compact operator $T : X \to Y$ is completely continuous. (See the introduction to Section 4 of this paper for a quick proof.) Grothendieck suggested the following terminology.

DEFINITION 2.2. The Banach space X has the *Dunford–Pettis property*, DPP for short, if every weakly compact operator $T : X \to Y$ is completely continuous.

We note that some authors call completely continuous operators Dunford–Pettis operators. The survey article [7] by Diestel is an excellent source of information about classical contributions to the study of the DPP. Theorem 1 of [7] shows that X has the DPP iff $x_n^*(x_n) \to 0$ whenever (x_n^*) is weakly null in X^* and (x_n) is weakly null in X. Kevin Andrews localized this idea in [1].

DEFINITION 2.3. A bounded subset S of X is called a *Dunford–Pettis subset* of X if every weakly null sequence (x_n^*) in X^* tends to 0 uniformly on S, that is,

$$\lim_n \sup\{|x_n^*(x)| : x \in S\} = 0.$$

Every DP subset of X is *weakly precompact*, i.e., if S is a DP subset of X and (x_n) is a sequence from S, then (x_n) has a weakly Cauchy subsequence. See [1] and [26, p. 377] for proofs.

Diestel [7] modified Definition 2.1 and Emmanuele [15] modified Definition 2.3 to produce the next concepts.

DEFINITION 2.4.

(i) The Banach space X has the *hereditary* DPP if each closed linear subspace of X has the DPP.
(ii) The Banach space X has the *Dunford–Pettis relatively compact property*, DPrcP for short, if every Dunford–Pettis subset of X is relatively compact.

Note that ℓ_1 and c_0 have the hereditary DPP (cf. [7]) and every Schur space has the DPrcP.

DEFINITION 2.5. A bounded subset S of X is called a *limited* subset of X if each w^*-null sequence (x_n^*) in X^* tends to 0 uniformly on S, and X is said to have the *Gelfand–Phillips property* if every limited subset of X is relatively compact.

All separable Banach spaces have the Gelfand–Phillips property, but nonseparable spaces need not have this property. See Bourgain and Diestel [6], Drewnowski and Emmanuele [10], and especially Schlumprecht [28] for discussions of limited sets. Specifically, note that every limited subset of X is a

DP subset of X. If $\mathcal{P}$ is one of the properties we have defined, we sometimes indicate that X has property $\mathcal{P}$ by writing $X \in (\mathcal{P})$; e.g., the assertion that X has the Gelfand–Phillips property may appear as $X \in (\mathrm{GP})$.

DEFINITION 2.6. A bounded subset S of X^* is called an *L-subset* of X^* if every null sequence (x_n) in X tends to 0 uniformly on S.

We remark that Bator [2] showed that $\ell_1 \not\hookrightarrow X$ iff X^* has the DPrcP, and Emmanuele [13] showed that $\ell_1 \not\hookrightarrow X$ iff every L-subset of X^* is relatively compact.

We refer the reader to [8] and [25] for undefined terminology and notation. In particular, (e_n) will denote the canonical unit vector basis of c_0, and (e_n^*) the canonical unit vector basis of ℓ_1.

3. Spaces of operators

THEOREM 3.1.

(i) *Suppose that H is a weakly precompact subset of $L(E,F)$. If H is not compact and $\|A_n^*(y^*) - B_n^*(y^*)\| \to 0$ whenever $y^* \in F^*$ and $(A_n - B_n)$ is a weakly null sequence in $H - H$, then there is a separable linear subspace X of F and an operator $A : X \to c_0$ which is not completely continuous.*

(ii) *Suppose that H is a weakly precompact subset of $L_{w^*}(E^*,F)$. If H is not compact and $\|A_n(x^*) - B_n(x^*)\| \to 0$ whenever $x^* \in E^*$ and $(A_n - B_n)$ is a weakly null sequence in $H - H$, then there is a separable subspace X of E and an operator $A : X \to c_0$ which is not completely continuous.*

Proof. (i) Suppose that H is not compact. Choose $\varepsilon > 0$ and sequences (A_n), (B_n) from H so that $A_n - B_n \xrightarrow{w} 0$ and $\|A_n - B_n\| > \varepsilon$ for each n. Choose a normalized sequence (x_n) from E so that $\|A_n(x_n) - B_n(x_n)\| > \varepsilon$ for each n. Since $\|A_n^*(y^*) - B_n^*(y^*)\| \to 0$ for all $y^* \in F^*$, we have $A_n(x_n) - B_n(x_n) \xrightarrow{w} 0$.

By the Bessaga–Pełczyński selection principle ([8], [5]), we may (and do) assume that $(y_k)_{k=1}^\infty := (A_k(x_k) - B_k(x_k))_{k=1}^\infty$ is a seminormalized weakly null basic sequence in F. Let $X = [\{y_k : k \in \mathbb{N}\}]$, let (y_k^*) be the sequence of coefficient functionals associated with (y_k), and define $A : X \to c_0$ by $A(x) = (y_k^*(x))$. Then A is a bounded linear operator defined on a separable space, and A is not completely continuous.

(ii) Suppose that $(A_n), (B_n)$, and ε are as in (i). Choose a normalized sequence (y_n^*) in F^* so that $\|A_n^*(y_n^*) - B_n^*(y_n^*)\| > \varepsilon$ for each n. Since $\|A_n(x^*) - B_n(x^*)\| \to 0$ for each $x^* \in E^*$, the w^*-w continuity of the operators ensures that $(A_n^*(y_n^*) - B_n^*(y_n^*)) =: (z_n)$ is a weakly null sequence in E.

Thus we may assume that (z_n) is a weakly null and seminormalized basic sequence in E. We finish the argument as in (i). ∎

Corollary 3.2 ([17, Theorem 2]). *If $E^* \in$ (GP) and F has the Schur property, then $L(E,F) \in$ (GP).*

Proof. Deny the conclusion. Apply part (i) of Theorem 3.1 to obtain a non-completely continuous operator defined on a closed linear subspace X of F. This is a clear contradiction since X also has the Schur property. ∎

Corollary 3.3. *Suppose that $F \in$ (DPrcP) and S is a closed linear subspace of $L_{w^*}(E^*,F)$. If $S \notin$ (DPrcP), then there is a separable subspace X of E and a non-completely continuous operator $T : X \to c_0$.*

Proof. Let H be a DP subset of S which is not relatively compact. Apply (ii) of 3.1. ∎

Corollary 3.3 significantly extends Theorem 7 of [15]: *Let E have the Schur property and F the* DPrcP. *Then the Banach space $K_{w^*}(E^*,F)$ has the* DPrcP.

Corollary 3.4. *If $E^* \in$ (DPrcP) and F has the Schur property, then*

$$L(E,F) \in \text{(DPrcP)}.$$

The next three corollaries follow from the proof of 3.1.

Corollary 3.5 ([10, Theorem 2.1]). *If E and F belong to* (GP), *then*

$$K_{w^*}(E^*,F) \in \text{(GP)}.$$

Proof. Suppose not and let $(z_n) = (A_n^*(y_n^*) - B_n^*(y_n^*))$ be as in (ii) above. Then (z_n) is a seminormalized and weakly null basic sequence in E. If (x_n^*) is w^*-null in E^*, $T \in K_{w^*}(E^*,F)$ and $x_n^* \otimes y_n^*(T)$ is defined to be $\langle T(x_n^*), y_n^*\rangle$, then $x_n^* \otimes y_n^*(T) \to 0$; that is, $(x_n^* \otimes y_n^*)$ is w^*-null as a sequence of continuous linear functionals defined on $K_{w^*}(E^*,F)$. Combine this observation with the fact that $(A_n - B_n)$ is a limited sequence to see that (z_n) is also a limited sequence. Thus, since $E \in$ (GP), $\|z_n\| \to 0$, and we have a contradiction. ∎

A Banach space X has the *Grothendieck property*, or X is a *Grothendieck space* [9], if w^*-null sequences (x_n^*) in X^* are weakly null. If X is a Grothendieck space, then the limited and DP subsets of X coincide.

Corollary 3.6. *If E and F have the* DPrcP *and $K_{w^*}(E^*,F)$ is a Grothendieck space, then $K_{w^*}(E^*,F)$ has the* DPrcP.

Proof. If $K_{w^*}(E^*,F)$ is a Grothendieck space, then E and F are Grothendieck spaces. Thus $E, F \in$ (GP). Apply 3.5. ∎

Corollary 3.7. *If $X^*, Y \in$ (GP), then $K(X,Y) \in$ (GP).*

Proof. Suppose not and let $(A_n - B_n)$ be a weakly null limited sequence in $K(X,Y)$ so that $\|A_n - B_n\| > \varepsilon > 0$ for all n. Let (x_n) be a normalized sequence in X so that $\|A_n(x_n) - B_n(x_n)\| > \varepsilon$ for all n. Arguing as in 3.5 above, one sees that $(A_n(x_n) - B_n(x_n))$ is weakly null and limited in Y. Thus $\|A_n(x_n) - B_n(x_n)\| \to 0$, and we have a contradiction. ∎

See [15] for results related to the next theorem.

THEOREM 3.8. *If X^* and Y have the* DPrcP *and $L(Y^*, X^*) = K(Y^*, X^*)$, then $L(X,Y)$ has the* DPrcP .

Proof. Suppose not and let (T_n) be a weakly null DP sequence in $L(X,Y)$ so that $\|T_n\| = 1$ for each n. Let (y_n^*) be a sequence in B_{Y^*} and (x_n) be a sequence in B_X so that $y_n^*(T_n(x_n)) > 1/2$ for each n. Note that $(T_n(x_n))$ is weakly null since $\|T_n^*(y^*)\| \to 0$ for $y^* \in Y^*$.

Suppose that $v_n^* \xrightarrow{w} 0$ in Y^*, and let $T \in L(X, Y^{**}) \cong (X \otimes_\gamma Y^*)^*$. Then $T^* \in L(Y^{***}, X^*)$ and $T^*_{|Y^*}$ is a compact operator. Therefore $|\langle x_n \otimes v_n^*, T\rangle| \le \|T^*(v_n^*)\|$ and $(T^*(v_n^*))$ is relatively compact and weakly null. Thus $(x_n \otimes v_n^*)$ is weakly null in $X \otimes_\gamma Y^*$.

Now $L(X,Y)$ embeds isometrically in $L(X, Y^{**})$ and (T_n) is a DP sequence in $L(X, Y^{**})$. Since a DP subset of a dual space is necessarily an L-subset of the dual space, $v_n^*(T_n(x_n)) \to 0$. Thus $(T_n(x_n))$ is a weakly null DP sequence in Y, $\|T_n(x_n)\| \to 0$, and we have a contradiction. ∎

The arguments in this section—particularly the proof of Theorem 3.1—also produce the next two results:

- (†) If $E^* \in$ (GP), B_{F^*} is w^*-sequentially compact, and all operators $T : F \to c_0$ are completely continuous, then $L(E,F) \in$ (GP).
- (††) If E and F have the DPrcP and all operators $T : E \to c_0$ are completely continuous, then $K_{w^*}(E^*, F)$ has the DPrcP.

We remark that if F is infinite-dimensional and all operators $T : F \to c_0$ are completely continuous, then $\ell_1 \hookrightarrow F$. To see this, begin by using the Josefson–Nissenzweig theorem to obtain a normalized and w^*-null sequence (x_n^*), and then choose any sequence (x_n) so that $\|x_n\| \le 1$ and $x_n^*(x_n) > 1/2$ for each n. Since the map $x \mapsto (x_n^*(x))_{n=1}^\infty$ is completely continuous by hypothesis, (x_n) cannot have a weakly Cauchy subsequence. Rosenthal's classical ℓ_1-theorem then puts a copy of ℓ_1 in F.

Moreover, if one assumes that all operators $T : X \to \ell_\infty$ are completely continuous, then it is easy to see that X has the Schur property. In fact, if S is a separable subspace of X, $A : S \to \ell_\infty$ is an isometrically isomorphic embedding of S into ℓ_∞, and $T : X \to \ell_\infty$ is a continuous linear extension of A, then the complete continuity of T immediately guarantees that every weakly null sequence in S is norm null. Clearly X has the Schur property iff every separable closed linear subspace of X has the Schur property.

The next result extends the observations in these two paragraphs.

THEOREM 3.9. *If X is a Banach space, then the following are equivalent:*

(i) *X is a Schur space.*
(ii) *All operators $T : X \to \ell_\infty$ are completely continuous.*
(iii) *Every weakly null sequence in X is limited in its closed linear span.*
(iv) *$X \in$ (DPrcP) and all operators $T : X \to c_0$ are completely continuous.*
(v) *$X \in$ (GP) and all operators $T : X \to c_0$ are completely continuous.*

Proof. That (ii) implies (i) was noted above. Certainly (i) implies (ii). Also, since a DP subset is weakly precompact, (i) (or (ii)) implies (iv), and (iv) clearly implies (v).

Now suppose that (ii) holds, and let (x_n) be a weakly null sequence in X. Suppose that $x_n^* \xrightarrow{w^*} 0$ in $[\{x_n : n \in \mathbb{N}\}]^*$, and define $A : [\{x_n\}] \to c_0$ by $A(x) = (x_n^*(x))$. Let $T : X \to \ell_\infty$ be a continuous linear extension of A. Since T is completely continuous, $x_n^*(x_n) \to 0$, and it follows that (x_n) is limited. Thus (ii) implies (iii).

Suppose that (iii) holds, $x_n \xrightarrow{w} 0$ in X, and $\|x_n\| = 1$ for each n. Without loss of generality, one may assume that (x_n) is basic. Let (x_n^*) be the coefficient functionals, and observe that $x_n^* \xrightarrow{w^*} 0$ in $[\{x_n\}]^*$. Since $x_n^*(x_n) = 1$ for each n, (x_n) cannot be a limited sequence. This contradiction shows that (iii) implies (i).

Now suppose that (every) $T : X \to c_0$ is completely continuous and $X \in$ (GP). Recall that the operators from X to c_0 correspond to the w^*-null sequences in X^*. Let (x_n^*) be w^*-null in X^* so that $T(x) = (x_n^*(x))$. If $x_n \xrightarrow{w} 0$ in X, then $\|T(x_n)\| \to 0$. Consequently, (x_n) is a limited sequence in X. Thus $\{x_n : n \in \mathbb{N}\}$ is relatively compact. Since (x_n) is weakly null, $\|x_n\| \to 0$, and (v) implies (i). ∎

This argument and the separable injectivity of c_0 immediately yield the next result.

COROLLARY 3.10. *If X is separable, then the following are equivalent:*

(i) *X is Schur.*
(ii) *Every operator $T : X \to c_0$ is completely continuous.*
(iii) *Every weakly null sequence in X is limited in X.*

As a consequence of Theorem 3.9, it is clear that (††) is subsumed by Corollary 3.3 above.

The fact that the continuous linear image of a Dunford–Pettis (resp. limited) set is Dunford–Pettis (resp. limited) can be coupled with the Bator–

Emmanuele characterization of the DPrcP for dual spaces [2], [13] to easily produce results for quotient spaces. See [7, p. 42] and [10] for discussions of the subtleties and complexity of the general problem.

THEOREM 3.11. *If* $X^* \in$ (DPrcP) (*respectively*, $X^* \in$ (GP)) *and* Z *is a quotient of* X, *then* $Z^* \in$ (DPrcP) (*respectively*, $Z^* \in$ (GP)).

Proof. Let $Q : X \to Z$ be a quotient map. Then $Q^* : Z^* \to X^*$ is an isomorphism. If K is a DP (resp. limited) subset of Z^*, then $Q^*(K)$ is a DP (resp. limited) subset of X^*. Thus $Q^*(K)$ and K must be relatively compact. ■

COROLLARY 3.12. *The following are equivalent*:

(i) $\ell_1 \not\hookrightarrow X$.

(ii) *If* Y *is a closed linear subspace of* X, *then* $\ell_1 \not\hookrightarrow Y$ *and* $\ell_1 \not\hookrightarrow X/Y$.

Proof. Bator [2] and Emmanuele [15] showed that $X^* \in$ (DPrcP) iff $\ell_1 \not\hookrightarrow X$. This characterization and 3.11 immediately yield the corollary. ■

In the next theorem, $\mathrm{CC}(X, c_0)$ denotes the space of completely continuous operators from X to c_0.

THEOREM 3.13. *If* X *has the* DPP *and* $L(X, c_0) \neq \mathrm{CC}(X, c_0)$, *then* $\ell_1 \hookrightarrow X^*$. *If* X *has the hereditary* DPP *and* $L(X, c_0) \neq \mathrm{CC}(X, c_0)$, *then* ℓ_1 *embeds complementably in* X^* *and* $c_0 \hookrightarrow X$.

Proof. Choose a non-completely continuous $T \in L(X, c_0)$. Since $(T^*(e_i^*))$ is w^*-null in X^* and T is not completely continuous, there is a weakly null sequence (x_n) in X which is not limited. By a result of Schlumprecht ([28], [16, p. 126]) we may choose a w^*-null sequence (x_n^*) in X^* so that $x_m^*(x_n) = \delta_{mn}$. Now suppose that (x_n^*) has a weakly Cauchy subsequence. In fact, suppose that $x_n^* - x_{n+1}^* \xrightarrow{w} 0$. Since X has the DPP, (x_n) is a DP sequence, and $1 = \langle x_n^* - x_{n+1}^*, x_n \rangle \to 0$. This contradiction and Rosenthal's ℓ_1-theorem finishes the first assertion.

Now suppose that X has the hereditary DPP. As in the previous paragraph, we may assume that (x_n) is weakly null and not limited in X. Thus we may (and do) assume that (x_n) is basic and normalized. Suppose that no subsequence of (x_n) is equivalent to (e_n). By a fundamental result of J. Elton [7, pp. 27–30], we obtain a subsequence (y_n) of (x_n) so that if (w_n) is any subsequence of (y_n) and (t_n) is a non-null sequence of real numbers, then $\sup_k \|\sum_{n=1}^k t_n w_n\| = \infty$. Arguing precisely as on p. 28 of [7], one sees that the coefficient functionals (w_n^*) are weakly null. However, since (w_n) is weakly null and $W = [(w_n)]$ has the DPP, (w_n) is a DP sequence in W, $1 = w_n^*(w_n) \to 0$, and we have an obvious contradiction. Thus some subsequence of (x_n) is equivalent to the unit vector basis of c_0. The main result of [24] ensures that ℓ_1 is complemented in X^*. ■

COROLLARY 3.14. *Suppose that X is an infinite-dimensional Banach space with the hereditary* DPP*. Then either*

(i) *$c_0 \hookrightarrow Y$ and Y^* contains a complemented copy of ℓ_1 whenever Y is a separable and infinite-dimensional closed linear subspace of X, or*
(ii) *$\ell_1 \hookrightarrow X$.*

Proof. Suppose that X is infinite-dimensional and has the hereditary DPP. Either $L(Y, c_0) = \mathrm{CC}(Y, c_0)$ for some separable and infinite-dimensional subspace Y of X, or the equality holds for no separable and infinite-dimensional subspace of X. Apply 3.10 and 3.13. ∎

Theorem 1 of [7] and another application of Rosenthal's ℓ_1-theorem easily produce the following dichotomy for spaces with the DPP.

THEOREM 3.15. *If the Banach space X has the* DPP, *then either X is a Schur space or $\ell_1 \hookrightarrow X^*$.*

Proof. Suppose that X is not a Schur space, and let (x_n) be a normalized and weakly null sequence in X. Choose (x_n^*) in B_{X^*} so that $x_n^*(x_n) = 1$ for all n. By part (f) of Theorem 1 of [7], (x_n^*) has no weakly Cauchy subsequence. Rosenthal's ℓ_1-theorem guarantees that $\ell_1 \hookrightarrow X^*$. ∎

Since $\ell_1 \hookrightarrow X^*$ whenever $\ell_1 \hookrightarrow X$ ([8, p. 211]), it follows directly from 3.15 that if X is an infinite-dimensional space with the DPP, then $\ell_1 \hookrightarrow X^*$.

The next corollary provides a counterpoint to Corollary 3.14 above and to the comment immediately following Theorem 7 on p. 28 of [7]. Rosenthal's ℓ_1-theorem shows that every infinite-dimensional Schur space contains ℓ_1.

COROLLARY 3.16. *If X is infinite-dimensional and $\ell_1 \not\hookrightarrow X^*$, then every infinite-dimensional closed linear subspace of X fails to have the* DPP*.*

COROLLARY 3.17. *Suppose that X is a separable Banach space which has the* DPP*. If $c_0 \hookrightarrow Y$, then the space $W(X,Y)$ of weakly compact operators is not complemented in $L(X,Y)$.*

Proof. Choose (x_n^*) in X^* so that $(x_n^*) \sim (e_n^*)$. Using the separability of X, one may assume that $x_n^* \xrightarrow{w^*} x^*$. Thus X^* contains a weak*-null sequence which is not weakly null. Theorem 4 of [3] ensures that $W(X,Y)$ is not complemented in $L(X,Y)$. ∎

Schlumprecht's result [16, p. 126] also leads to a non-complementation result when $X \in (\mathrm{GP})$ but $X \notin (\mathrm{DPrcP})$.

THEOREM 3.18. *Suppose that X fails to have the* DPrcP *but $X \in (\mathrm{GP})$. If $c_0 \hookrightarrow Y$, then $W(X,Y)$ is not complemented in $L(X,Y)$.*

Proof. Suppose that K is a DP subset which is not relatively compact. Then there is a weakly null sequence (x_n) in $K - K$ and a $\delta > 0$ so that

$\|x_n\| > \delta$ for each n. Therefore (x_n) is not a limited sequence. Then we can find (x_n^*) in X^* so that $x_n^* \xrightarrow{w^*} 0$ and $x_n^*(x_m) = \delta_{nm}$. Thus (x_n^*) is w^*-null and not w-null. Again by Theorem 4 of [3], $W(X,Y)$ is not complemented in $L(X,Y)$. ∎

4. L-sets. It is well known that X must have the DPP if X^* has the DPP and that the reverse implication is false (see e.g. [7, pp. 19–23]). In this section we identify a natural property involving L-subsets of X^* which is in complete duality with the DPP.

If X is a Banach space, then we say that X^* has the *L-property* (or $X^* \in$ (LP)) if every operator $T \in L_{w^*}(X^*, c_0)$ is completely continuous. See Theorem 3.1 of [4] for related ideas. Since the operators $T \in L_{w^*}(X^*, c_0)$ correspond to the weakly null sequences in X, the statement that $X^* \in$ (LP) is equivalent to the assertion that every weakly null sequence in X is a DP sequence in X. A direct application of Theorem 2.6 of [20] then shows that X has the DPP if and only if $X^* \in$ (LP).

This simple characterization provides a particularly easy way to show that $C(K)$ (and $L_1(\mu)$) enjoy the DPP. Suppose that $T : C(K)^* \to c_0$ is a w^*-w continuous operator and let (f_n) be a w-null (and therefore bounded) sequence in $C(K)$ so that $T(\mu) = (\int f_n d\mu)_{n=1}^{\infty}$. If (λ_n) is a weakly null sequence of regular Borel measures in $C(K)^*$, choose a non-negative regular measure λ so that $\lambda_n \ll \lambda$ uniformly in n. Now $f_n \to 0$ uniformly except on sets of arbitrarily small λ-measure. Consequently, $\|T(\lambda_n)\|_{c_0} \to 0$. See also pp. 113–114 of [8].

One can check that X has the DPP if and only if each of its weakly compact sets is a DP subset of X. Further, it is well known that a subset S of X is a DP subset of X iff $L(S)$ is relatively compact whenever $L : X \to Y$ is a weakly compact operator [1]. The next two lemmas and theorems continue to emphasize the duality that exists between L-subsets of X^* and DP subsets of X.

LEMMA 4.1. *If A is an L-subset of X^*, B_{Y^*} is w^*-sequentially compact, and $T \in L_{w^*}(X^*, Y)$, then $T(A)$ is relatively compact.*

Proof. Suppose that $T \in L_{w^*}(X^*, Y)$ and $T(A)$ is not relatively compact. Since any element in $L_{w^*}(X^*, Y)$ sends L-sets to DP sets, we choose sequences (u_k^*) and (v_k^*) in A and $\varepsilon > 0$ so that $\|T(u_k^*) - T(v_k^*)\| > \varepsilon$ for all k and $T(u_k^*) - T(v_k^*) \xrightarrow{w} 0$. Let (y_k^*) be a sequence in B_{Y^*} so that $y_k^*(T(u_k^*) - T(v_k^*)) > \varepsilon$, and, without loss of generality, suppose that $y_k^* \xrightarrow{w^*} y^*$. Consequently, $T^*(y_k^*) \xrightarrow{w} T^*(y^*)$ in X, and $\langle T^*(y_k^*) - T^*(y^*), x^* \rangle \to 0$ uniformly for $x^* \in A$. Since $\langle T^*(y^*), u_k^* - v_k^* \rangle = y^*(T(u_k^*) - T(v_k^*)) \to 0$, it follows that $\langle T^*(y_k^*), u_k^* - v_k^* \rangle \to 0$, and we have a contradiction. ∎

Lemma 4.2. *If $T(A)$ is relatively compact for each $T \in L_{w^*}(X^*, c_0)$, then A is an L-subset of X^*.*

Proof. Suppose that $x_n \xrightarrow{w} 0$ in X, and define $T : X^* \to c_0$ by $T(x^*) = (x^*(x_n))_{n=1}^{\infty}$. If $\lambda = (\lambda_n) \in \ell_1$, then $T^*(\lambda) = \sum \lambda_n x_n \in X$, and T is w^*-w continuous. Thus $T(A)$ is relatively compact, and $\lim_n \sup_{x^* \in A} x^*(x_n) = 0$. ∎

Remark. A combination of 4.1 and 4.2 directly shows that a subset A of X^* is an L-subset of X^* iff $T(A)$ is relatively compact for each $T \in L_{w^*}(X^*, c_0)$. These two lemmas also facilitate two additional characterizations of the L-property.

Theorem 4.3. *Every weakly compact subset of X^* is an L-subset of X^* iff $X^* \in$ (LP).*

Proof. If $X^* \in$ (LP) and A is a weakly compact subset of X^*, then, by the Eberlein–Shmul'yan theorem, $T(A)$ is relatively compact whenever $T \in L_{w^*}(X^*, c_0)$. Thus A is an L-subset of X^*.

Conversely, suppose that every w-compact subset of X^* is an L-subset of X^*, and let $T \in L_{w^*}(X^*, c_0)$. If $x_n^* \xrightarrow{w} x_0^*$, then $U = \{x_n^* : n \geq 0\}$ is w-compact. Thus $T(U)$ is relatively compact, and $\|T(x_n^*) - T(x_0^*)\| \to 0$. ∎

Theorem 4.4. *A bounded subset S of X^* is an L-subset of X^* if and only if $T^*(S)$ is relatively compact whenever Y is a Banach space and $T : Y \to X$ is weakly compact.*

Proof. Suppose that $T : Y \to X$ is a weakly compact operator and let R be a reflexive space and $A : Y \to R$ and $B : R \to X$ be operators so that $T = BA$ ([8, p. 237]). Suppose that S is an L-subset of X^* and $T^*(S)$ is not relatively compact. Then $B^*(S)$ is an L-subset of R^*, and $B^*(S)$ is not relatively compact. Consequently, we may assume that Y itself is reflexive.

Now choose a sequence (x_n^*) in S, $\delta > 0$, and $y^* \in Y^*$ so that $T^*(x_n^*) \xrightarrow{w} y^*$ and $\|T^*(x_n^*) - y^*\| > \delta$ for each n. Choose $y_n \in B_Y$ so that

$$y_n(T^*(x_n^*) - y^*) > \delta, \qquad n \in \mathbb{N}.$$

Without loss of generality, suppose that $y_n \xrightarrow{w} y \in B_Y$ (Y is reflexive). Therefore $\langle y_n - y, T^*(x_n^*) \rangle \to 0$ since $T^*(S)$ is an L-subset of Y^*. Since $\langle y, T^*(x_n^*) - y^* \rangle \xrightarrow{n} 0$ and $y^*(y_n - y) \xrightarrow{n} 0$, it follows that $y_n(T^*(x_n^*) - y^*) \xrightarrow{n} 0$, and we have a clear contradiction.

Conversely, suppose that if $T : Y \to X$ is weakly compact, then $T^*(S)$ is relatively compact. Let (x_n) be weakly null in X, and let (x_n^*) be a sequence in S. Define $L : X^* \to c_0$ by $L(x^*) = (x^*(x_n))$. If $\lambda = (\lambda_n) \in \ell_1$, then $L^*(\lambda) = \sum \lambda_n x_n$, and $L^*(B_{\ell_1})$ is contained in the closed and absolutely convex hull of $\{x_n : n \in \mathbb{N}\}$. Thus L^* and L are weakly compact. Moreover, it is clear that L itself is an adjoint. Therefore $L(S)$ is relatively compact in c_0, $\lim_n x_n^*(x_n) = 0$, and S is an L-subset of X^*. ∎

COROLLARY 4.5. *The bounded subset S of X^* is an L-subset of X^* if and only if $T^*(S)$ is relatively compact in R^* whenever R is reflexive and $T : R \to X$ is an operator.*

Our next result gives an extension of Theorem 3 of [15]. An operator $T : X \to Y$ is called *limited* if $T(B_X)$ is limited in Y, and the set of all limited operators from X to Y is denoted by $\mathrm{ltd}(X, Y)$. Certainly every compact operator is limited. If $T : X \to Y$ is a limited operator and $y_n^* \xrightarrow{w^*} y^*$, note that

$$\lim_n \sup\{\langle y_n^* - y^*, T(x)\rangle : \|x\| \le 1\} \mapsto 0.$$

That is, $\|T^*(y_n^*) - T^*(y^*)\| \to 0$.

THEOREM 4.6. *Suppose that every operator $T : X \to Y^*$ is limited. If (x_n) is bounded and (y_n) is weakly null in Y, then $(x_n \otimes y_n)$ is weakly null in $X \otimes_\gamma Y$. Consequently, if (T_n) is a* DP *sequence in $L(X, Y^*)$, then $\{T_n(x_n) : n \in \mathbb{N}\}$ is an L-subset of Y^*.*

Proof. Recall that $(X \otimes_\gamma Y)^* \cong L(X, Y^*)$ ([9, p. 229]), and let $T \in L(X, Y^*)$. Since $L(X, Y^*) = \mathrm{ltd}(X, Y^*)$, $\|T^*(u_n^{**})\| \to 0$ if $u_n^{**} \xrightarrow{w^*} 0$ in Y^{**}. Therefore $|\langle T, x_n \otimes y_n\rangle| = |\langle T(x_n), y_n\rangle| = |\langle x_n, T^*(y_n)\rangle| \to 0$. Consequently, if (T_n) is a DP sequence in $L(X, Y^*)$, then $|\langle x_n \otimes y_n, T_n\rangle| \to 0$. ■

In Section 3 of this paper, compactness properties of Dunford–Pettis sets and limited sets were repeatedly used. Compactness questions involving L-sets naturally arise in this context. As noted in Section 2 above, Emmanuele [13] showed that L-subsets of X^* are relatively compact iff $\ell_1 \not\hookrightarrow X$. In fact, if $\ell_1 \hookrightarrow X$, then L-subsets of X^* may well fail to be even weakly precompact. Specifically, if X is any infinite-dimensional Schur space, then all bounded subsets of X^* are L-subsets, and thus there are L-subsets of X^* which fail to be weakly precompact. The next theorem presents a simple operator-theoretic characterization of weak precompactness, relative weak compactness, and relative norm compactness for L-sets. An operator $T : X \to Y$ is said to be *almost weakly compact* [7, pp. 17–18] if $T(B_X)$ is weakly precompact in Y.

THEOREM 4.7. *Suppose that X is a Banach space.*

(I) *The following are equivalent*:

- I(i) *If $T : Y \to X^*$ is an operator and $T^*_{|X}$ is completely continuous, then T is almost weakly compact.*
- I(ii) *If $T : \ell_1 \to X^*$ is an operator and $T^*_{|X}$ is completely continuous, then T is almost weakly compact.*
- I(iii) *Any L-subset of X^* is weakly precompact.*

(II) *The following are equivalent*:

II(i) *If* $T : Y \to X^*$ *is an operator and* $T^*_{|X}$ *is completely continuous*, *then* T *is weakly compact.*

II(ii) *If* $T : \ell_1 \to X^*$ *is an operator and* $T^*_{|X}$ *is completely continuous*, *then* T *is weakly compact.*

II(iii) *Any* L*-subset of* X^* *is relatively weakly compact.*

(III) *The following are equivalent*:

III(i) *If* $T : Y \to X^*$ *is an operator and* $T^*_{|X} : X \to Y^*$ *is completely continuous*, *then* T *is compact.*

III(ii) *If* $T : \ell_1 \to X^*$ *is an operator and* $T^*_{|X} : X \to \ell_\infty$ *is completely continuous*, *then* T *is compact.*

III(iii) *Every* L*-subset of* X^* *is relatively compact.*

Proof. Since the proofs of (I), (II), and (III) are essentially the same, we present the argument for (III) only. Suppose that (iii) holds and $T_1 = T^*_{|X}$ is completely continuous. Let (x_n) be a w-null sequence in X. If (y_n) is a sequence in B_Y, then $|x_n(T(y_n))| = |T_1(x_n)(y_n)| \le \|T_1(x_n)\| \to 0$, and $T(B_Y)$ is an L-subset of X^*. Therefore T is compact and (iii) implies (i).

Certainly (i) implies (ii). Now suppose (ii) holds, and let (x^*_n) be a sequence from the L-subset A of X^*. Define $T : \ell_1 \to X^*$ by $T(\lambda) = \sum_{i=1}^\infty \lambda_i x^*_i$. Now suppose that (x_n) is weakly null in X. Since A is an L-subset of X,

$$\lim_n \sup_i |x^*_i(x_n)| = 0,$$

and (ii) ensures that T is compact. Since $T(e^*_i) = x^*_i$ for each i, the set $\{x^*_n : n \in \mathbb{N}\}$ is relatively compact. ∎

The Banach space X has the *reciprocal Dunford–Pettis property* (RDPP) ([14], [4]) provided that every completely continuous operator $T : X \to Y$ is weakly compact.

Corollary 4.8 ([14, Theorem 1]; [23]). *The Banach space* X *has the* RDPP *iff every* L*-subset of* X^* *is relatively weakly compact.*

Corollary 4.9. *The Banach space* X *has the* RDPP *iff every completely continuous operator* $T : X \to \ell_\infty$ *is weakly compact.*

Proof. Every L-subset of X^* is relatively weakly compact iff every completely continuous operator $T : X \to \ell_\infty$ is weakly compact. ∎

Corollary 4.10. *If* X *is a Banach space*, *then the following are equivalent*:

(i) *Every* L*-subset of* X^* *is relatively compact.*
(ii) *Every completely continuous operator with domain* X *is compact.*

Proof. The operator $T : X \to Y$ is completely continuous iff $T^*(B_{Y^*})$ is an L-subset of X^*. Therefore (i) certainly yields (ii).

Now suppose that $T : \ell_1 \to X^*$ is an operator and $T^*_{|X}$ is completely continuous. By (ii) this restriction is compact and thus T itself is compact. The preceding theorem then applies, and (i) follows. ∎

COROLLARY 4.11 ([7, Theorem 3]). *If X has the* DPP *and* $\ell_1 \not\hookrightarrow X$, *then X^* has the Schur property.*

Proof. If $x_n^* \xrightarrow{w} x^*$ in X^* and X has the DPP, then $A = \{x_n^* : n \in \mathbb{N}\}$ is an L-subset of X^*. Thus A is relatively compact by 4.10, and $\|x_n^* - x^*\| \to 0$. ∎

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Received 18 May 2005;
revised 28 February 2006 (4604)