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MAXIMAL FUNCTION IN BEURLING–ORLICZ AND CENTRAL MORREY–ORLICZ SPACES

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Abstract. We define Beurling–Orlicz spaces, weak Beurling–Orlicz spaces, Herz–Orlicz spaces, weak Herz–Orlicz spaces, central Morrey–Orlicz spaces and weak central Morrey–Orlicz spaces. Moreover, the strong-type and weak-type estimates of the Hardy–Littlewood maximal function on these spaces are investigated.

1. Introduction. Arne Beurling [B] introduced the spaces $B^p(\mathbb{R}^n)$, which we call the *Beurling spaces*, together with their preduals $A^p(\mathbb{R}^n)$, the *Beurling algebras* (they are, in fact, convolution algebras), and he proved the duality $(A^p(\mathbb{R}^n))^* = B^{p'}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$. Then Feichtinger [F] observed that the spaces $B^p(\mathbb{R}^n)$ can be described by the condition $\|f\|_{B^p} = \sup_{k\ge 0} 2^{-kn/p}\|f\chi_k\|_p < \infty$, where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$, $k = 1, 2, \ldots$, and $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^n)$. Note that this observation is a special case of earlier results of Gilbert [Gi]. In terms of duality the spaces $A^p(\mathbb{R}^n)$ can be described by the condition $\|f\|_{A^p} = \sum_{k=0}^{\infty} 2^{kn/p'}\|f\chi_k\|_p < \infty$.

Herz [H] further generalized $A^p(\mathbb{R}^n)$ and $B^p(\mathbb{R}^n)$ by defining the spaces $K^{\alpha}_{p,q}(\mathbb{R}^n)$ depending on $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$:

$$K^{\alpha}_{p,q}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha}_{p,q}} = \|\{2^{k\alpha}\|f\chi_k\|_p\}\|_{l^q(\mathbb{N}\cup\{0\})} < \infty\};$$

they are called *non-homogeneous Herz spaces*. In 1989 García-Cuerva [Ga, Proposition 1.2] showed that $f \in B^p(\mathbb{R}^n)$ if and only if

$$\sup_{r\ge 1}\left(\frac{1}{|B_r|}\int_{B_r} |f(x)|^p\,dx\right)^{1/p} < \infty, \tag{1}$$

where B_r is the ball with center at 0 and radius $r > 0$, and the quantity (1) is equivalent to the norm $\|f\|_{B^p}$ (see also [CL] for the one-dimensional case).

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The *homogeneous Herz* and *Beurling spaces* are defined as

$$\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{K}^{\alpha}_{p,q}} = \|\{2^{k\alpha}\|f\chi_k\|_p\}\|_{l^q(\mathbb{Z})} < \infty\}$$

and

$$\dot{B}^p(\mathbb{R}^n) = \bigg\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{B}^p} = \sup_{r>0}\bigg(\frac{1}{|B_r|}\int_{B_r}|f(x)|^p\,dx\bigg)^{1/p} < \infty\bigg\}.$$

The classical *Morrey spaces* $M^p_\lambda(\mathbb{R}^n)$ were introduced in 1938 by Morrey [Mo]. In today's language, these spaces consist of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M^p_\lambda} = \sup_{x_0\in\mathbb{R}^n,\, r>0}\bigg(\frac{1}{|B(x_0,r)|^\lambda}\int_{B(x_0,r)}|f(x)|^p\,dx\bigg)^{1/p} < \infty,$$

where $B(x_0,r)$ denotes the ball with center at $x_0 \in \mathbb{R}^n$ and radius $r > 0$. Note that if the supremum is taken over all sets of measure $\le t$, then we get the p-convexifications of the Marcinkiewicz spaces $M^{(p)}_{1-\lambda}$ on $(0,\infty)$ which consist of all Lebesgue measurable functions f such that

$$\|f\|_{M^{(p)}_{1-\lambda}} = \||f|^p\|^{1/p}_{M^1_{1-\lambda}} = \sup_{t>0}\bigg(\frac{1}{t^\lambda}\int_0^t f^*(s)^p\,ds\bigg)^{1/p} < \infty$$

(cf. [KPS, pp. 112–114], [M1, p. 164]). There are also the *local Morrey spaces* $LM^p_\lambda(\mathbb{R}^n, x_0)$ at any fixed $x_0 \in \mathbb{R}^n$ (cf. [BG]) and the *non-homogeneous central Morrey spaces* $B^{p,\lambda}(\mathbb{R}^n)$, which were first introduced in [ALG] as

$$(2)\quad B^{p,\lambda}(\mathbb{R}^n) = \bigg\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{B^{p,\lambda}} = \sup_{r\ge 1}\bigg(\frac{1}{|B_r|^\lambda}\int_{B_r}|f(x)|^p\,dx\bigg)^{1/p} < \infty\bigg\}.$$

Note that the *homogeneous central Morrey spaces* $\dot{B}^{p,\lambda}(\mathbb{R}^n)$, i.e., when the supremum in (2) is taken over all $r > 0$, are $LM^p_\lambda(\mathbb{R}^n, 0)$, the special cases of local Morrey spaces.

Chiarenza and Frasca [CF] proved that if $f \in M^p_\lambda(\mathbb{R}^n)$, then the maximal function Mf is finite almost everywhere, the strong-type estimate $\|Mf\|_{M^p_\lambda} \le \|f\|_{M^p_\lambda}$ holds if $p > 1$ and $0 \le \lambda < 1$, and for $p = 1$ a weak-type estimate is also valid. The results on boundedness of the Hardy–Littewood maximal operator in the local and global Morrey-type spaces $LM^{p,q}_w(\mathbb{R}^n)$ and $GM^{p,q}_w(\mathbb{R}^n)$, respectively, were investigated by Burenkov and Guliyev [BG].

We remark that the special case of the global Morrey–Orlicz spaces was investigated by Nakai [N1]–[N3] and Sawano, Sugano and Tanaka [SST].

In this paper we combine basic definitions from the theory of Orlicz spaces with the Beurling and local Morrey constructions and introduce Beurling–Orlicz spaces, central Morrey–Orlicz spaces, weak Beurling–Orlicz spaces, weak central Morrey–Orlicz spaces and their homogeneous versions. We also study some relations between them and their relations to some Herz–Orlicz spaces. Furthermore, the boundedness of the Hardy–Littlewood maximal operator between these spaces is proved as extension of the previous results. Related results between weak-type spaces are also investigated.

2. Beurling–Orlicz spaces $B^{\Phi}(\mathbb{R}^n)$ and $\dot{B}^{\Phi}(\mathbb{R}^n)$. We start by giving necessary definitions. For a measurable set $A \subset \mathbb{R}^n$ we denote its Lebesgue measure by $|A|$ and its characteristic function by χ_A. Recall that $B(x,r)$ denotes the open ball with center at $x \in \mathbb{R}^n$ and radius $r > 0$, that is, $\{y \in \mathbb{R}^n : |x-y| < r\}$, and let $B_r = B(0,r)$. Moreover, for $k \in \mathbb{Z}$, let $C_k = B_{2^k} \setminus B_{2^{k-1}}$, and for $k \in \mathbb{N}$, let $P_k = C_k$ and $P_0 = B_1$. For two Banach or quasi-Banach spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the embedding $X \subset Y$ is continuous with norm at most C, i.e., $\|f\|_Y \le C\|f\|_X$ for all $f \in X$. When $X \overset{C}{\hookrightarrow} Y$ holds with some (unknown) constant $C > 0$, we simply write $X \hookrightarrow Y$. Furthermore, $X = Y$ means that the spaces are the same and the norms are equivalent.

We also need the definition of Orlicz spaces on $\mathbb{R}^n$, of weak Orlicz spaces on $\mathbb{R}^n$ and some of their properties to be used later on (see [M1] for details).

A function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is an increasing, continuous and convex function with $\Phi(0) = 0$. Each such function Φ has an integral representation

$$\Phi(u) = \int_0^u p(s)\,ds, \tag{3}$$

where p is a non-decreasing right-continuous function. Here, $\Phi'(u) = p(u)$ a.e. on $(0,\infty)$.

If we want to include in the Orlicz spaces, for example, $L^\infty(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we need to consider the so-called Young functions. A *Young function* is a non-decreasing convex function $\Phi : [0,\infty) \to [0,\infty]$ with $\lim_{u\to+0} \Phi(u) = \Phi(0) = 0$, and not identically 0 or ∞ in $(0,\infty)$. It may have a jump to ∞ at some point $u > 0$, but then it should be left-continuous at u.

For any Young function Φ the *Orlicz space* $L^{\Phi}(\mathbb{R}^n)$ consists of all classes of Lebesgue measurable real functions on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \Phi(\varepsilon|f(x)|)\,dx < \infty$ for some $\varepsilon = \varepsilon(f) > 0$ with the *Luxemburg–Nakano norm*

$$\|f\|_{L^{\Phi}} = \|f\|_{L^{\Phi}(\mathbb{R}^n)} = \inf\Big\{\varepsilon > 0 : \int_{\mathbb{R}^n} \Phi(|f(x)|/\varepsilon)\,dx \le 1\Big\};$$

it is a Banach space (cf. [M2, pp. 125–127]).

The *fundamental function* of the Orlicz space $L^{\Phi}(\mathbb{R}^n)$ is

$$\varphi_{L^{\Phi}}(t) = \|\chi_A\|_{L^{\Phi}(\mathbb{R}^n)} = \|\chi_{[0,|A|]}\|_{L^{\Phi}([0,\infty))} = 1/\Phi^{-1}(1/t),$$

where $|A| = t$ and Φ^{-1} is the right-continuous inverse of Φ defined by $\Phi^{-1}(v) = \inf\{u \geq 0 : \Phi(u) > v\}$ with $\inf \emptyset = \infty$.

The *weak Orlicz space* $WL^{\Phi}(\mathbb{R}^n)$ generated by the Young function Φ is a space larger than $L^{\Phi}(\mathbb{R}^n)$, determined by the quasi-norm

$$\|f\|_{WL^{\Phi}} = \inf\Big\{\varepsilon > 0 : \sup_{u>0} \Phi(u/\varepsilon) d_f(u) \leq 1\Big\},$$

where $d_f(u) = |\{x \in \mathbb{R}^n : |f(x)| > u\}|$. In fact, if $f \in L^{\Phi}(\mathbb{R}^n)$, then for any $\varepsilon > \|f\|_{L^{\Phi}}$ and arbitrary $u > 0$, we have

$$1 \geq \int_{\mathbb{R}^n} \Phi(|f(x)|/\varepsilon)\,dx \geq \int_{\{x\in\mathbb{R}^n : |f(x)|>u\}} \Phi(|f(x)|/\varepsilon)\,dx \geq \Phi(u/\varepsilon) d_f(u),$$

and so $f \in WL^{\Phi}(\mathbb{R}^n)$ with $\|f\|_{WL^{\Phi}} \leq \varepsilon$. Hence, $L^{\Phi}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WL^{\Phi}(\mathbb{R}^n)$. Also we remark that

$$\|f\|_{WL^{\Phi}} = \sup_{t>0} t\varphi_{L^{\Phi}}(d_f(t)) = \sup_{t>0} \varphi_{L^{\Phi}}(t) f^*(t),$$

where f^* is the non-increasing rearrangement of f. Therefore, $WL^{\Phi}(\mathbb{R}^n)$ given by the last quasi-norm is also the Marcinkiewicz space $M^*_{\varphi_{L^{\Phi}}}(\mathbb{R}^n)$ (cf. [O, Section 9] and [M3, Part 4.1.2]).

To each Young function Φ one can associate another convex function Φ^*, i.e., the *complementary function* to Φ, which is defined by

$$\Phi^*(v) = \sup_{u>0} [uv - \Phi(u)] \quad \text{for } v \geq 0.$$

Then Φ^* is also a Young function and $\Phi^{**} = \Phi$. Note that

$$u \leq \Phi^{-1}(u) \Phi^{*-1}(u) \leq 2u \quad \text{for all } u > 0.$$

We say that a Young function Φ satisfies the Δ_2-*condition*, and we write $\Phi \in \Delta_2$, if $0 < \Phi(u) < \infty$ for $u > 0$ and there exists a constant $C \geq 1$ such that $\Phi(2u) \leq C\Phi(u)$ for all $u > 0$.

Sometimes in the investigations of Orlicz spaces or spaces based on Orlicz spaces, it is enough to consider only the case of Orlicz functions, because the first author proved that for any Young function Φ there is an Orlicz function Ψ such that one of the four cases holds: $L^{\Phi} = L^{\Psi}$, $L^{\Phi} = L^{\Psi} \cap L^{\infty}$, $L^{\Phi} = L^{\Psi} + L^{\infty}$ and $L^{\Phi} = L^{\infty}$ (see [M1, Theorem 12.4]).

Now, we are ready to define the non-homogeneous and the homogeneous Beurling–Orlicz and weak Beurling–Orlicz spaces. For any Young function Φ and a set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$, let

$$\|f\|_{\Phi,A} = \inf\Big\{\varepsilon > 0 : \frac{1}{|A|} \int_A \Phi(|f(x)|/\varepsilon)\,dx \leq 1\Big\},$$

$$\|f\|_{\Phi,A,\infty} = \inf\Big\{\varepsilon > 0 : \sup_{u>0} \Phi(u/\varepsilon) \frac{1}{|A|}\, d_{f\chi_A}(u) \leq 1\Big\}.$$

Then the *non-homogeneous Beurling–Orlicz space* $B^{\Phi}(\mathbb{R}^n)$ and the *non-homogeneous weak Beurling–Orlicz space* $WB^{\Phi}(\mathbb{R}^n)$ are defined by

$$B^{\Phi}(\mathbb{R}^n) = \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{B^{\Phi}} = \sup_{r\ge 1} \|f\|_{\Phi,B_r} < \infty \Big\}, \tag{4}$$

$$WB^{\Phi}(\mathbb{R}^n) = \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WB^{\Phi}} = \sup_{r\ge 1} \|f\|_{\Phi,B_r,\infty} < \infty \Big\}. \tag{5}$$

If in (4) and (5) the supremums are taken over all $r > 0$, then we have the definitions of the *homogeneous Beurling–Orlicz space* $\dot{B}^{\Phi}(\mathbb{R}^n)$ and the *homogeneous weak Beurling–Orlicz space* $W\dot{B}^{\Phi}(\mathbb{R}^n)$. In particular, for $\Phi(u) = u^p$, $1 \le p < \infty$, these spaces are the classical spaces $B^p(\mathbb{R}^n)$, $WB^p(\mathbb{R}^n)$, $\dot{B}^p(\mathbb{R}^n)$ and $W\dot{B}^p(\mathbb{R}^n)$ (cf. [CL], [F], [Ga] and [Ma]).

Note that since $L^{\Phi}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WL^{\Phi}(\mathbb{R}^n)$, we obviously have $B^{\Phi}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WB^{\Phi}(\mathbb{R}^n)$ and $\dot{B}^{\Phi}(\mathbb{R}^n) \overset{1}{\hookrightarrow} W\dot{B}^{\Phi}(\mathbb{R}^n)$. It is also easy to prove that

$$B^{\Phi}(\mathbb{R}^n) \overset{\Phi^{-1}(1)}{\hookrightarrow} B^1(\mathbb{R}^n) \quad \text{and} \quad \dot{B}^{\Phi}(\mathbb{R}^n) \overset{\Phi^{-1}(1)}{\hookrightarrow} \dot{B}^1(\mathbb{R}^n).$$

In fact, if $f \in B^{\Phi}(\mathbb{R}^n)$ and $\|f\|_{B^{\Phi}} \le 1$, then $\|f\|_{\Phi,B_r} \le 1$ for any $r \ge 1$. Therefore, $\frac{1}{|B_r|}\int_{B_r} \Phi\big(\frac{|f(x)|}{1+\varepsilon}\big)\,dx \le 1$ for any $r \ge 1$ and any $\varepsilon > 0$. Then, by the Jensen inequality, we obtain

$$\Phi\bigg(\frac{1}{|B_r|}\int_{B_r} \frac{|f(x)|}{1+\varepsilon}\,dx\bigg) \le \frac{1}{|B_r|}\int_{B_r} \Phi\bigg(\frac{|f(x)|}{1+\varepsilon}\bigg)\,dx \le 1,$$

and so $\frac{1}{|B_r|}\int_{B_r} |f(x)|\,dx \le (1+\varepsilon)\Phi^{-1}(1)$ for any $r \ge 1$ and any $\varepsilon > 0$, i.e., $f \in B^1(\mathbb{R}^n)$ and $\|f\|_{B^1} \le \Phi^{-1}(1)$. The proof of the embedding for $\dot{B}^{\Phi}(\mathbb{R}^n)$ is the same.

We can also describe the above spaces as some non-homogeneous and homogeneous Herz–Orlicz and weak Herz–Orlicz spaces in the way Feichtinger [F] did for $B^p(\mathbb{R}^n)$ and $\dot{B}^p(\mathbb{R}^n)$.

Proposition 1. *Let Φ be an Orlicz function. Then with equivalent norms the following hold:*

(i) $$B^{\Phi}(\mathbb{R}^n) = K_{\Phi,\infty}(\mathbb{R}^n) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_{\Phi,\infty}} = \sup_{k\in\mathbb{N}\cup\{0\}} \|f\|_{\Phi,P_k} < \infty \Big\},$$

(ii) $$WB^{\Phi}(\mathbb{R}^n) = WK_{\Phi,\infty}(\mathbb{R}^n) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WK_{\Phi,\infty}} = \sup_{k\in\mathbb{N}\cup\{0\}} \|f\|_{\Phi,P_k,\infty} < \infty \Big\},$$

(iii) $$\dot{B}^{\Phi}(\mathbb{R}^n) = \dot{K}_{\Phi,\infty}(\mathbb{R}^n) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{K}_{\Phi,\infty}} = \sup_{k\in\mathbb{Z}} \|f\|_{\Phi,C_k} < \infty \Big\},$$

(iv) $W\dot{B}^{\Phi}(\mathbb{R}^n) = W\dot{K}_{\Phi,\infty}(\mathbb{R}^n)$

$$:= \Big\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{W\dot{K}_{\Phi,\infty}} = \sup_{k\in\mathbb{Z}} \|f\|_{\Phi,C_k,\infty} < \infty \Big\}.$$

Proof. (i) Let $f \in K_{\Phi,\infty}(\mathbb{R}^n)$. Taking $r \geq 1$ we can find $k \in \mathbb{N}\cup\{0\}$ such that $2^{k-1} < r \leq 2^k$. Then

$$\int_{B_r} \Phi\Big(\frac{|f(x)|}{\|f\|_{K_{\Phi,\infty}}}\Big)\,dx \leq \sum_{j=0}^{k} \int_{P_j} \Phi\Big(\frac{|f(x)|}{\|f\|_{K_{\Phi,\infty}}}\Big)\,dx \leq \sum_{j=0}^{k} \int_{P_j} \Phi\Big(\frac{|f(x)|}{\|f\|_{\Phi,P_j}}\Big)\,dx$$

$$\leq \sum_{j=0}^{k} |P_j| = |B_{2^k}| \leq |B_{2r}| = 2^n |B_r|.$$

Therefore, by the convexity of Φ, we obtain

$$\frac{1}{|B_r|}\int_{B_r} \Phi\Big(\frac{|f(x)|}{2^n\|f\|_{K_{\Phi,\infty}}}\Big)\,dx \leq 1,$$

which implies that

$$\|f\|_{B^{\Phi}} = \sup_{r\geq 1} \|f\|_{\Phi,B_r} \leq 2^n \|f\|_{K_{\Phi,\infty}}.$$

On the other hand, if $f \in B^{\Phi}(\mathbb{R}^n)$, then for any $k \in \mathbb{N}\cup\{0\}$ we have

$$\int_{P_k} \Phi\Big(\frac{|f(x)|}{\|f\|_{B^{\Phi}}}\Big)\,dx \leq \int_{B_{2^k}} \Phi\Big(\frac{|f(x)|}{\|f\|_{\Phi,B_{2^k}}}\Big)\,dx \leq |B_{2^k}| = \frac{2^n}{2^n-1}|P_k|.$$

Thus, for $C = \frac{2^n}{2^n-1} > 1$, again by the convexity of Φ, we obtain

$$\frac{1}{|P_k|}\int_{P_k} \Phi\Big(\frac{|f(x)|}{C}\,\|f\|_{B^{\Phi}}\Big)\,dx \leq 1,$$

which gives $\|f\|_{K_{\Phi,\infty}} \leq C\|f\|_{B^{\Phi}} \leq 2\|f\|_{B^{\Phi}}$.

(ii) Let $f \in WK_{\Phi,\infty}(\mathbb{R}^n)$. For $r \geq 1$ there exists $k \in \mathbb{N}\cup\{0\}$ such that $2^{k-1} < r \leq 2^k$. Then

$$\Phi(u)\Big|\Big\{x \in B_r : \frac{|f(x)|}{\|f\|_{WK_{\Phi,\infty}}} > u\Big\}\Big| \leq \sum_{j=0}^{k} \Phi(u)\Big|\Big\{x \in P_j : \frac{|f(x)|}{\|f\|_{WK_{\Phi,\infty}}} > u\Big\}\Big|$$

$$\leq \sum_{j=0}^{k} \Phi(u)\Big|\Big\{x \in P_j : \frac{|f(x)|}{\|f\|_{\Phi,P_j,\infty}} > u\Big\}\Big| \leq \sum_{j=0}^{k} |P_j| = |B_{2^k}| \leq 2^n|B_r|.$$

Therefore, by the convexity of Φ, we obtain

$$\Phi(u)\frac{1}{|B_r|}\Big|\Big\{x \in B_r : \frac{|f(x)|}{2^n\|f\|_{WK_{\Phi,\infty}}} > u\Big\}\Big| \leq 1,$$

which implies that

$$\|f\|_{WB^{\Phi}} = \sup_{r\geq 1} \|f\|_{\Phi,B_{r,\infty}} \leq 2^n \|f\|_{WK_{\Phi,\infty}}.$$

On the other hand, if $f \in WB^{\Phi}(\mathbb{R}^n)$, then for any $k \in \mathbb{N} \cup \{0\}$,

$$\Phi(u)\left|\left\{x \in P_k : \frac{|f(x)|}{\|f\|_{WB^{\Phi}}} > u\right\}\right| \le \Phi(u)\left|\left\{x \in B_{2^k} : \frac{|f(x)|}{\|f\|_{\Phi,B_{2^k},\infty}} > u\right\}\right| \le |B_{2^k}| = \frac{2^n}{2^n-1}|P_k| = C|P_k|.$$

Thus,

$$\Phi(u)\frac{1}{|P_k|}\left|\left\{x \in P_k : \frac{|f(x)|}{C2^n\|f\|_{WB^{\Phi}}} > u\right\}\right| \le 1,$$

and so $\|f\|_{WK_{\Phi,\infty}} \le C\|f\|_{WB^{\Phi}} \le 2\|f\|_{WB^{\Phi}}$. The proofs of (iii) and (iv) are the same as those of (i) and (ii), respectively. ∎

Remark 2. We can prove in the same way as in Proposition 1 that

$$B^{\Phi}(\mathbb{R}^n) = K^*_{\Phi,\infty}(\mathbb{R}^n) := \Big\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^*_{\Phi,\infty}} = \sup_{k\in\mathbb{N}\cup\{0\}} \|f\|_{\Phi,P_k,2^{kn}} < \infty\Big\},$$

where

$$\|f\|_{\Phi,P_k,2^{kn}} = \inf\Big\{\varepsilon > 0 : 2^{-kn}\int_{P_k} \Phi(|f(x)|/\varepsilon)\,dx \le 1\Big\},$$

and $\|f\|_{K^*_{\Phi,\infty}} \le |B_1|\,\|f\|_{B^{\Phi}} \le \frac{4^n}{2^n-1}\|f\|_{K^*_{\Phi,\infty}}$. Similar results are true for the other three cases.

3. Central Morrey–Orlicz spaces $B^{\Phi,\lambda}(\mathbb{R}^n)$ and $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$. For an Orlicz function Φ, and numbers $\lambda \in \mathbb{R}$ and $r > 0$, let $\|f\|_{\Phi,\lambda,B_r}$ denote the *λ-central mean Luxemburg–Nakano norm* of f on B_r defined by

$$\|f\|_{\Phi,\lambda,B_r} = \inf\Big\{\varepsilon > 0 : \frac{1}{|B_r|^{\lambda}}\int_{B_r} \Phi(|f(x)|/\varepsilon)\,dx \le 1\Big\},$$

and the corresponding (smaller) *weak λ-central mean Luxemburg–Nakano norm* $\|f\|_{\Phi,\lambda,B_r,\infty}$ is defined by

$$\|f\|_{\Phi,\lambda,B_r,\infty} = \inf\Big\{\varepsilon > 0 : \sup_{u>0}\Phi(u)\frac{1}{|B_r|^{\lambda}}\,d(f\chi_{B_r},\varepsilon u) \le 1\Big\}.$$

Then using these notions we can define the *non-homogeneous central Morrey–Orlicz space* $B^{\Phi,\lambda}(\mathbb{R}^n)$ and the *non-homogeneous weak central Morrey–Orlicz space* $WB^{\Phi,\lambda}(\mathbb{R}^n)$:

$$B^{\Phi,\lambda}(\mathbb{R}^n) = \Big\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{B^{\Phi,\lambda}} = \sup_{r\ge1}\|f\|_{\Phi,\lambda,Br} < \infty\Big\}, \tag{6}$$

$$WB^{\Phi,\lambda}(\mathbb{R}^n) = \Big\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WB^{\Phi,\lambda}} = \sup_{r\ge1}\|f\|_{\Phi,\lambda,Br,\infty} < \infty\Big\}. \tag{7}$$

If in (6) and (7) the supremums are taken over all $r > 0$, then we have the definitions of the *homogeneous central Morrey–Orlicz* space $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$ and the *homogeneous weak central Morrey–Orlicz* space $W\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$.

REMARK 3. Clearly $B^{\Phi,0}(\mathbb{R}^n) = \dot{B}^{\Phi,0}(\mathbb{R}^n) = L^{\Phi}(\mathbb{R}^n)$, $WB^{\Phi,0}(\mathbb{R}^n) = W\dot{B}^{\Phi,0}(\mathbb{R}^n) = WL^{\Phi}(\mathbb{R}^n)$ and $B^{\Phi,1}(\mathbb{R}^n) = B^{\Phi}(\mathbb{R}^n)$, $WB^{\Phi,1}(\mathbb{R}^n) = WB^{\Phi}(\mathbb{R}^n)$. The last two equalities also hold for the homogeneous cases. In particular, if $\Phi(u) = u^p$, $1 \le p < \infty$, and $\lambda \in \mathbb{R}$, then $B^{\Phi,\lambda}(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n)$ and $WB^{\Phi,\lambda}(\mathbb{R}^n) = WB^{p,\lambda}(\mathbb{R}^n)$, where $WB^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WB^{p,\lambda}} < \infty\}$ with

$$\|f\|_{WB^{p,\lambda}} = \sup_{r\ge 1}\sup_{u>0} u\left(\frac{1}{|B_r|^{\lambda}}|\{x \in B_r : |f(x)| > u\}|\right)^{1/p}$$

is the *non-homogeneous weak central Morrey space*. The same properties hold for the homogeneous cases, using $W\dot{B}^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{W\dot{B}^{p,\lambda}} < \infty\}$ with

$$\|f\|_{W\dot{B}^{p,\lambda}} = \sup_{r>0}\sup_{u>0} u\left(\frac{1}{|B_r|^{\lambda}}|\{x \in B_r : |f(x)| > u\}|\right)^{1/p}$$

which is the *homogeneous weak central Morrey space*. For $WB^{p,\lambda}(\mathbb{R}^n)$ and $W\dot{B}^{p,\lambda}(\mathbb{R}^n)$, see [KMNS].

Note that since $L^{\Phi}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WL^{\Phi}(\mathbb{R}^n)$, for any Orlicz function Φ and $\lambda \in \mathbb{R}$ we have $B^{\Phi,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WB^{\Phi,\lambda}(\mathbb{R}^n)$ and $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} W\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$.

4. Boundedness of the Hardy–Littlewood maximal function on $B^{\Phi,\lambda}(\mathbb{R}^n)$ and $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$. The *Hardy–Littlewood maximal function* Mf of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ at $x \in \mathbb{R}^n$ is defined by

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\,dy,$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^n$ containing x. A sublinear operator M sending f to Mf is also called the *Hardy–Littlewood maximal operator*.

The following modular strong-type and weak-type inequalities concerning the Hardy–Littlewood maximal operator M hold on the Orlicz space $L^{\Phi}(\mathbb{R}^n)$.

THEOREM 4. *Let M be the Hardy–Littlewood maximal operator and Φ be an Orlicz function.*

(i) *$\Phi^* \in \Delta_2$ if and only if there exists a constant $C_1 \ge 1$ such that*

$$\int_{\mathbb{R}^n} \Phi(Mf(x)/C_1)\,dx \le \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx \tag{8}$$

provided that the right side of inequality (8) *is finite.*

(ii) *There exists a constant* $C_2 > 1$ *such that*

$$\sup_{u>0} \Phi(u)|\{x \in \mathbb{R}^n : Mf(x)/C_2 > u\}| \leq \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx \tag{9}$$

provided that the right side of inequality (9) *is finite.*

In order to prove (ii) of Theorem 4 we need the following lemma.

LEMMA 5. *If* Φ *is an Orlicz function and* $Mf(x) < \infty$ *for* $x \in \mathbb{R}^n$, *then*

$$\Phi(Mf(x)) \leq M\Phi(|f|)(x) \quad \text{for } x \in \mathbb{R}^n.$$

Proof. Let $x \in \mathbb{R}^n$ and suppose $Mf(x) < \infty$. Then, for any $0 < \epsilon < 1$, there exists a ball $B_0 \subset \mathbb{R}^n$ such that $B_0 \ni x$ and

$$Mf(x) < \frac{1}{|B_0|} \int_{B_0} |f(y)|\,dy + \epsilon.$$

Further, for an Orlicz function Φ, by the representation (3),

$$\Phi(u+\varepsilon) = \int_0^{u+\varepsilon} p(s)\,ds = \int_0^u p(s)\,ds + \int_u^{u+\varepsilon} p(s)\,ds \leq \Phi(u) + p(u+\varepsilon)\,\varepsilon.$$

Consequently, by the Jensen inequality we obtain

$$\begin{aligned}
\Phi(Mf(x)) &\leq \Phi\Big(\frac{1}{|B_0|} \int_{B_0} |f(y)|\,dy + \varepsilon\Big) \\
&\leq \Phi\Big(\frac{1}{|B_0|} \int_{B_0} |f(y)|\,dy\Big) + p\Big(\frac{1}{|B_0|} \int_{B_0} |f(y)|\,dy + \varepsilon\Big)\varepsilon \\
&\leq \frac{1}{|B_0|} \int_{B_0} \Phi(|f(y)|)\,dy + p(Mf(x)+1)\varepsilon \\
&\leq M\Phi(|f|)(x) + p(Mf(x)+1)\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows $\Phi(Mf(x)) \leq M\Phi(|f|)(x)$. ∎

Proof of Theorem 4. (i) The strong-type estimate for the maximal function on $[0,1]$ was proved already by Lorentz [L, Theorem 4] and also by Shimogaki [S, Theorem 3], who has the result even for rearrangement invariant spaces on $[0,1]$ with the Fatou property (cf. also [KPS, Theorem 6.6, p. 138]). The modular estimate for the maximal function on $\mathbb{R}^n$ with the restriction on Φ to be the so-called N-function was found by Gallardo [G, Theorem 2.1]. The modular estimate for an Orlicz function Φ was presented by Krbec and Kokilashvili [KK, Theorem 1.2.1] with some constant $C \geq 1$ inside and outside of the integral on the right side of (8).

Now, we give a direct proof of (8). First of all, if $\int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx < \infty$, then $\|f\|_{L^\Phi} \leq 1$ and so we get $Mf(x) < \infty$ for almost all $x \in \mathbb{R}^n$, because $L^{\Phi}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n) + L^{\infty}(\mathbb{R}^n)$ (cf. [M1, Theorem 12.1c]) and $L^1(\mathbb{R}^n) + L^{\infty}(\mathbb{R}^n) \subset \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : Mf < \infty \text{ a.e. in } \mathbb{R}^n\}$ (cf. [FK, Theorem 2.2]) hold.

Second, it is well-known that the maximal operator M is of weak-type $(1,1)$ (this was proved in 1939 independently by Wiener [W] and Marcinkiewicz–Zygmund—see [M3, Theorem 15, p. 196]), that is, there exists a constant $C_3 > 1$ such that

$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \le C_3 \int_{\mathbb{R}^n} |f(x)|\, dx \tag{10}$$

for any $f \in L^1(\mathbb{R}^n)$. Also Grafakos [Gr, Theorem 2.1.6] proved (10) with the constant C_3 being at most 3^n. Further, Wiener [W] observed the validity of a stronger inequality,

$$\lambda |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \le 2\,C_3 \int_{\{x\in\mathbb{R}^n :\, |f(x)|>\lambda/2\}} |f(x)|\, dx \tag{11}$$

for any $f \in L^1(\mathbb{R}^n)$, which is called the *Wiener inequality* (cf. [AKMP, pp. 109 and 118]). In fact, $|f| = g + h$, where $g = |f|\chi_{\{|f|\le\lambda/2\}}$ and $h = |f|\chi_{\{|f|>\lambda/2\}}$. Then $Mf \le Mg + Mh \le \lambda/2 + Mh$ and

$$\{Mf > \lambda\} \subset \{Mg > \lambda/2\} \cup \{Mh > \lambda/2\}.$$

Thus, by (10), we have

$$\begin{aligned} \lambda |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| &\le \lambda |\{x \in \mathbb{R}^n : Mh(x) > \lambda/2\}| \\ &\le 2C_3 \int_{\mathbb{R}^n} |h(x)|\, dx = 2C_3 \int_{\{x\in\mathbb{R}^n :\, |f(x)|>\lambda/2\}} |f(x)|\, dx. \end{aligned}$$

Without loss of generality we can assume that an Orlicz function Φ is differentiable on $(0,\infty)$. Otherwise we consider the equivalent Orlicz function $\Phi_1(u) = \int_0^u \frac{\Phi(t)}{t}\, dt$ with this property, for which

$$\Phi(u/2) \le \int_{u/2}^{u} \frac{\Phi(t)}{t}\, dt \le \int_0^u \frac{\Phi(t)}{t}\, dt = \Phi_1(u) \le \Phi(u) \quad \text{for all } u > 0.$$

Using twice the Fubini theorem and the Wiener inequality (11) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(Mf(x))\, dx &= \int_0^\infty \Phi'(\lambda) |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}|\, d\lambda \\ &\le 2C_3 \int_0^\infty \frac{\Phi'(\lambda)}{\lambda} \Big(\int_{\{x\in\mathbb{R}^n :\, |f(x)|>\lambda/2\}} |f(x)|\, dx \Big)\, d\lambda \\ &= 2C_3 \int_{\mathbb{R}^n} |f(x)| \Big(\int_0^{2|f(x)|} \frac{\Phi'(\lambda)}{\lambda}\, d\lambda \Big)\, dx. \end{aligned}$$

The complementary function Φ^* satisfies the Δ_2-condition if and only if there exists a constant $p > 1$ such that $p\Phi(u) < u\Phi'(u)$ for all $u > 0$ (cf. [KR, Theorem 4.3]) and the latter is equivalent to $u^{-p}\Phi(u)$ being increasing on $(0,\infty)$ because $\left[\frac{\Phi(u)}{u^p}\right]' = \frac{u\Phi'(u)-p\Phi(u)}{u^{p+1}}$. Thus, using integration by parts and the above fact, we obtain

$$\int_0^u \frac{\Phi'(\lambda)}{\lambda}\,d\lambda \le \frac{\Phi(u)}{u} + \int_0^u \frac{\Phi(\lambda)}{\lambda^2}\,d\lambda = \frac{\Phi(u)}{u} + \int_0^u \frac{\Phi(\lambda)}{\lambda^p}\lambda^{p-2}\,d\lambda$$
$$\le \frac{\Phi(u)}{u} + \frac{\Phi(u)}{u^p}\,\frac{u^{p-1}}{p-1} = \frac{p}{p-1}\,\frac{\Phi(u)}{u}.$$

Therefore,

$$\int_{\mathbb{R}^n} \Phi(Mf(x))\,dx \le 2C_3 \int_{\mathbb{R}^n} |f(x)|\frac{p}{p-1}\,\frac{\Phi(2|f(x)|)}{2|f(x)|}\,dx$$
$$= C_3\,\frac{p}{p-1} \int_{\mathbb{R}^n} \Phi(2|f(x)|)\,dx$$

and for $C \ge 2C_3\frac{p}{p-1}$ by the convexity of Φ we obtain

$$\int_{\mathbb{R}^n} \Phi(Mf(x)/C)\,dx \le \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx,$$

which means that estimate (8) holds with $C_1 \ge 2C_3\,\frac{p}{p-1}$, where $p = p(\Phi)$.

If (8) holds and $0 \ne f \in L^\Phi$, then since $\int_{\mathbb{R}^n} \Phi(|f(x)|/\|f\|_{L^\Phi})\,dx \le 1$, estimate (8) means that

$$\|Mf\|_{L^\Phi} \le C_1\|f\|_{L^\Phi} \quad \text{for any } f \in L^\Phi.$$

In particular,

$$(8') \qquad \|M\chi_A\|_{L^\Phi} \le C_1\|\chi_A\|_{L^\Phi} \quad \text{for any } 0 < |A| < \infty.$$

Taking in (8′) $A = B_r$ with $r = (a_1 uv)^{-1/n}$, where $a_r = |B_r|$, $u > 0$ and $v > 1$, we get

$$\|\chi_{B_r}\|_{L^\Phi} = \frac{1}{\Phi^{-1}(1/|B_r|)} = \frac{1}{\Phi^{-1}(1/(r^n|B_1|))} = \frac{1}{\Phi^{-1}(uv)} \le \frac{1}{uv}\Phi^{*-1}(uv).$$

On the other hand, if $x \notin B_r$ then $B_r \subset B(x,2|x|)$ since for $y \in B_r$ we have

$$|x-y| \le |x| + |y| \le |x| + r \le 2|x|$$

and

$$M\chi_{B_r}(x) \ge \frac{1}{|B(x,2|x|)|} \int_{B(x,2|x|)} \chi_{B_r}(y)\,dy = \frac{|B(x,2|x|)\cap B_r|}{|B(x,2|x|)|} = \left(\frac{r}{2|x|}\right)^n.$$

For $g = \Phi^{*-1}(u)\chi_{B_s}$ with $s = (a_1 u)^{-1/n}$ we obtain

$$\int_{\mathbb{R}^n} \Phi^*(|g(x)|)\,dx = u|B_s| = us^n|B_1| = 1.$$

Since the Luxemburg–Nakano norm is equivalent to the *Orlicz norm*

$$\|f\|^0_{L^\Phi} = \sup\Big\{ \int_{\mathbb{R}^n} |f(x)g(x)|\,dx : \int_{\mathbb{R}^n} \Phi^*(|g(x)|)\,dx \le 1 \Big\}$$

(more precisely, $\|f\|_{L^\Phi} \le \|f\|^0_{L^\Phi} \le 2\|f\|_{L^\Phi}$—cf. [KR] or [M1]), it follows that

$$\begin{aligned}
\|M\chi_{B_r}\|^0_{L^\Phi} &= \sup\Big\{ \int_{\mathbb{R}^n} |M\chi_{B_r}(x)g(x)|\,dx : \int_{\mathbb{R}^n} \Phi^*(|g(x)|)\,dx \le 1 \Big\} \\
&\ge \Phi^{*-1}(u) \int_{B_s} M\chi_{B_r}(x)\,dx \ge \Phi^{*-1}(u) \int_{B_s\setminus B_r} \Big(\frac{r}{2|x|}\Big)^n dx \\
&= \frac{\Phi^{*-1}(u)}{2^n a_1 uv} \int_{r<|x|<s} \frac{1}{|x|^n}\,dx \qquad \text{(using spherical coordinates)} \\
&= \frac{\Phi^{*-1}(u)}{2^n a_1 uv} n a_1 \ln\frac{s}{r} = \frac{\Phi^{*-1}(u)}{2^n uv} \ln v.
\end{aligned}$$

Hence, (8′) implies that

$$\frac{{\Phi^*}^{-1}(u)}{2^n uv} \ln v \le 2C_1 \frac{1}{uv} \Phi^{*-1}(uv) \qquad \text{for } u > 0 \text{ and } v > 1.$$

Thus, taking $v = \exp(C_1 \cdot 2^{n+2})$ we obtain $2\Phi^{*-1}(u) \le \Phi^{*-1}(u\exp(C_1 \cdot 2^{n+2}))$ for $u > 0$ or $\Phi^*(2t) \le \exp(C_1 \cdot 2^{n+2})\,\Phi^*(t)$ for every $t > 0$, and so Φ^* satisfies the Δ_2-condition.

(ii) By applying Lemma 5 and estimate (10), it follows for $u > 0$, $C_3 > 1$ and $\Phi(|f|) \in L^1(\mathbb{R}^n)$ that

$$\begin{aligned}
&\Phi(u)|\{x \in \mathbb{R}^n : Mf(x) > C_3 u\}| \\
&\qquad = \Phi(u)|\{x \in \mathbb{R}^n : \Phi(Mf(x)) > \Phi(C_3 u)\}| \\
&\qquad = \Phi(u)|\{x \in \mathbb{R}^n : M\Phi(|f|)(x) > \Phi(C_3 u)\}| \\
&\qquad \le \frac{\Phi(u)C_3}{\Phi(C_3\,u)} \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx \le \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx,
\end{aligned}$$

and so (9) is proved with $C_2 = C_3$. ∎

Using Theorem 4 we can show the following strong-type and weak-type estimates for the Hardy–Littlewood maximal operator M on the spaces $B^{\Phi,\lambda}(\mathbb{R}^n)$ and $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$, which include the estimates on $B^{\Phi}(\mathbb{R}^n)$ and $\dot{B}^{\Phi}(\mathbb{R}^n)$, respectively, i.e., the cases of $\lambda = 1$.

Theorem 6. *Let M be the Hardy–Littlewood maximal operator, Φ be an Orlicz function and* $0 \le \lambda \le 1$.

(i) *If* $\Phi^* \in \Delta_2$, *then* M *is bounded on* $B^{\Phi,\lambda}(\mathbb{R}^n)$, *that is,* $\|Mf\|_{B^{\Phi,\lambda}} \leq C_4\|f\|_{B^{\Phi,\lambda}}$ *for all* $f \in B^{\Phi,\lambda}(\mathbb{R}^n)$ *with* $C_4 \leq 2\max(C_1 2^{n\lambda}, 4^n)$.

(ii) M *is bounded from* $B^{\Phi,\lambda}(\mathbb{R}^n)$ *to* $WB^{\Phi,\lambda}(\mathbb{R}^n)$, *that is,* $\|Mf\|_{WB^{\Phi,\lambda}} \leq C_5\|f\|_{B^{\Phi,\lambda}}$ *for all* $f \in B^{\Phi,\lambda}(\mathbb{R}^n)$ *with* $C_5 \leq 4 \cdot 6^n$.

The same conclusions hold for homogeneous spaces $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$.

Proof. (i) Let $r \geq 1$ and $x \in B_r$. Then

$$\begin{aligned} Mf(x) &\leq \sup_{x\in B\subset B_{2r}} \frac{1}{|B|}\int_B |f(y)|\,dy + \sup_{x\in B\setminus B_{2r}\neq\emptyset} \frac{1}{|B|}\int_B |f(y)|\,dy \\ &=: M^{(1)}f(x) + M^{(2)}f(x). \end{aligned}$$

Now, since $\Phi^* \in \Delta_2$, there exists a constant $C_1 > 1$ such that the strong-type modular inequality (8) holds. Moreover, for the ball B with radius $r_0 > 0$ satisfying $B \cap B_r \neq \emptyset$ and $B \setminus B_{2r} \neq \emptyset$ denote by B_0' the smallest ball centered at 0 and containing B. Then $B_0' \subset B_{r+2r_0}$, $r \leq 2r_0$, and so

$$|B_0'| \leq |B_{r+2r_0}| = (r+2r_0)^n|B_1| \leq (4r_0)^n|B_1| = 4^n r_0^n|B_1| = 4^n|B|.$$

Therefore, for $C = \max(C_1\, 2^{n\lambda}, 4^n)$ and $0 \neq f \in B^{\Phi,\lambda}(\mathbb{R}^n)$ we have

$$\begin{aligned} 2\int_{B_r} &\Phi\left(\frac{Mf(x)}{2C\|f\|_{B^{\Phi,\lambda}}}\right)dx \\ &\leq 2\int_{B_r}\Phi\left(\frac{M^{(1)}f(x)+M^{(2)}f(x)}{2C\|f\|_{B^{\Phi,\lambda}}}\right)dx \\ &\leq \int_{B_r}\Phi\left(\frac{M^{(1)}f(x)}{C\|f\|_{B^{\Phi,\lambda}}}\right)dx + \int_{B_r}\Phi\left(\frac{M^{(2)}f(x)}{C\|f\|_{B^{\Phi,\lambda}}}\right)dx \\ &\leq \int_{B_r}\Phi\left(\frac{M^{(1)}f(x)}{C_3 2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}}\right)dx + \int_{B_r}\Phi\left(\frac{M^{(2)}f(x)}{4^n\|f\|_{B^{\Phi,\lambda}}}\right)dx \\ &=: I_1 + I_2. \end{aligned}$$

First, we estimate I_1. Since $M^{(1)}f(x) \leq M(f\chi_{B_{2r}})(x)$ for $x \in B_r$, it follows from the strong-type modular inequality (8), definition of $B^{\Phi,\lambda}(\mathbb{R}^n)$ and $0 \leq \lambda \leq 1$ that

$$\begin{aligned} I_1 &\leq \int_{B_r}\Phi\left(\frac{M(f\chi_{B_{2r}})(x)}{C_1 2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}}\right)dx \leq \int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|\chi_{B_{2r}}(x)}{2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}}\right)dx \\ &= \int_{B_{2r}}\Phi\left(\frac{|f(x)|}{2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}}\right)dx \leq \frac{1}{2^{n\lambda}}\int_{B_{2r}}\Phi\left(\frac{|f(x)|}{\|f\|_{\Phi,\lambda,B_{2r}}}\right)dx \\ &\leq (|B_{2r}|/2^n)^\lambda = |B_r|^\lambda. \end{aligned}$$

Next, we estimate I_2. From Lemma 5, the Jensen inequality, the definition of $B^{\Phi,\lambda}(\mathbb{R}^n)$ and $0 \leq \lambda \leq 1$ it follows that

$$\begin{aligned} I_2 &\leq \int_{B_r} \Phi\bigg(\frac{1}{\|f\|_{B^{\Phi,\lambda}}} \sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} |f(y)|\, dy \bigg)\, dx \\ &\leq \int_{B_r} \sup_{B_0' \ni x} \Phi\bigg(\frac{1}{|B_0'|} \int_{B_0'} \frac{|f(y)|}{\|f\|_{B^{\Phi,\lambda}}}\, dy \bigg)\, dx \\ &\leq \int_{B_r} \sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} \Phi\bigg(\frac{|f(y)|}{\|f\|_{B_{\Phi,\lambda}}} \bigg)\, dy\, dx \\ &\leq \int_{B_r} \sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} \Phi\bigg(\frac{|f(y)|}{\|f\|_{\Phi,\lambda,B_0'}} \bigg)\, dy\, dx \\ &\leq |B_0'|^{\lambda-1} |B_r| \leq |B_r|^{\lambda-1}|B_r| = |B_r|^{\lambda}. \end{aligned}$$

Putting together the above estimates we obtain

$$\int_{B_r} \Phi\bigg(\frac{Mf(x)}{2C\|f\|_{B^{\Phi,\lambda}}} \bigg)\, dx \leq |B_r|^{\lambda},$$

and so

$$\|Mf\|_{B^{\Phi,\lambda}} \leq 2C\|f\|_{B^{\Phi,\lambda}},$$

where $C = \max(C_1 \cdot 2^{n\lambda}, 4^n)$.

(ii) If Φ is an Orlicz function, then there exists a constant $C_2 > 1$ such that the weak-type modular inequality (9) holds. Moreover, for any $r \geq 1$, $C = \max(C_2 2^{n\lambda}, 4^n)$ and $0 \neq f \in B^{\Phi,\lambda}(\mathbb{R}^n)$ we have

$$\begin{aligned} &2\Phi(u) \bigg| \bigg\{ x \in B_r : \frac{Mf(x)}{4C\|f\|_{B^{\Phi,\lambda}}} > u \bigg\} \bigg| \\ &\qquad \leq \Phi(2u) \bigg| \bigg\{ x \in B_r : \frac{Mf(x)}{4C\|f\|_{B^{\Phi,\lambda}}} > u \bigg\} \bigg| \\ &\qquad \leq \Phi(2u) \bigg| \bigg\{ x \in B_r : \frac{M^{(1)}f(x)}{4C\|f\|_{B^{\Phi,\lambda}}} > \frac{u}{2} \bigg\} \bigg| \\ &\qquad\quad + \Phi(2u) \bigg| \bigg\{ x \in B_r : \frac{M^{(2)}f(x)}{4C\|f\|_{B^{\Phi,\lambda}}} > \frac{u}{2} \bigg\} \bigg| \\ &\qquad \leq \Phi(2u) \bigg| \bigg\{ x \in B_r : \frac{M^{(1)}f(x)}{4C_2 2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}} > \frac{u}{2} \bigg\} \bigg| \\ &\qquad\quad + \Phi(2u) \bigg| \bigg\{ x \in B_r : \frac{M^{(2)}f(x)}{4 \cdot 4^n\|f\|_{B^{\Phi,\lambda}}} > \frac{u}{2} \bigg\} \bigg| \\ &\qquad =: I_3 + I_4. \end{aligned}$$

To estimate I_3 and I_4 we will apply the same argument as in (i). First, from the weak-type modular inequality (9) it follows that

$$\begin{aligned} I_3 &\le \Phi(2u)\left|\left\{x \in B_r : \frac{M(f\chi_{B_{2r}})(x)}{4C_2 2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}} > \frac{u}{2}\right\}\right| \\ &\le \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|\chi_{B_{2r}}(x)}{2^{n\lambda}\|f\|_{B^{\Phi,\lambda}}}\right) dx \le |B_r|^\lambda. \end{aligned}$$

Second, from Lemma 5, the Jensen inequality and $0 \le \lambda \le 1$ we obtain

$$\begin{aligned} I_4 &\le \Phi(2u)\left|\left\{x \in B_r : \frac{1}{4\|f\|_{B^{\Phi,\lambda}}} \sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} |f(y)|\, dy > \frac{u}{2}\right\}\right| \\ &\le \Phi\left(\frac{1}{\|f\|_{B^{\Phi,\lambda}}} \sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} |f(y)|\, dy\right) \cdot |B_r| \\ &\le |B_r| \Phi\left(\sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} \frac{|f(y)|}{\|f\|_{B^{\Phi,\lambda}}}\, dy\right) \\ &\le |B_r| \sup_{B_0' \ni x} \frac{1}{|B_0'|} \int_{B_0'} \Phi\left(\frac{|f(y)|}{\|f\|_{B^{\Phi,\lambda}}}\right) dy \le |B_r|\, |B_0'|^{\lambda-1} \le |B_r|^\lambda. \end{aligned}$$

Putting the above estimates together we get

$$\Phi(u)\left|\left\{x \in B_r : \frac{Mf(x)}{4C\|f\|_{B^{\Phi,\lambda}}} > u\right\}\right| \le |B_r|^\lambda$$

for all $u > 0$. Therefore, $\|Mf\|_{\Phi,\lambda,B_r,\infty} \le 4C\|f\|_{B^{\Phi,\lambda}}$ and

$$\|Mf\|_{WB^{\Phi,\lambda}} \le 4C\|f\|_{B^{\Phi,\lambda}},$$

where $C = \max(C_2 2^{n\lambda}, 4^n) \le \max(3^n 2^{n\lambda}, 4^n) \le 6^n$.

The proofs of the boundedness estimates in $\dot{B}^{\Phi,\lambda}(\mathbb{R}^n)$ are the same as above. ∎

Theorem 6, when $\lambda = 1$, gives the following strong-type and weak-type estimates on $B^\Phi(\mathbb{R}^n)$ and $\dot{B}^\Phi(\mathbb{R}^n)$.

Corollary 7. *Let M be the Hardy–Littlewood maximal operator and Φ be an Orlicz function.*

(i) *If $\Phi^* \in \Delta_2$, then M is bounded on $B^\Phi(\mathbb{R}^n)$, that is, $\|Mf\|_{B^\Phi} \le C_6\|f\|_{B^\Phi}$ for all $f \in B^\Phi(\mathbb{R}^n)$ with $C_6 \le 2\max(C_1 2^n, 4^n) = 2^{n+1}\max(C_1, 2^n)$.*

(ii) *M is bounded from $B^\Phi(\mathbb{R}^n)$ to $WB^\Phi(\mathbb{R}^n)$, that is, $\|Mf\|_{WB^\Phi} \le C_5\|f\|_{B^\Phi}$ for all $f \in B^\Phi(\mathbb{R}^n)$ with $C_5 \le 4 \cdot 6^n$.*

The same conclusions hold for homogeneous spaces $\dot{B}^\Phi(\mathbb{R}^n)$.

We think that the condition $\Phi^* \in \Delta_2$ in (i) of Theorem 6 is necessary, but we do not have the proof.

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