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ON s-SETS IN SPACES OF HOMOGENEOUS TYPE

BY

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Abstract. Let (X, d, μ) be a space of homogeneous type. We study the relationship between two types of s-sets: relative to a distance and relative to a measure. We find a condition on a closed subset F of X under which F is an s-set relative to the measure μ if and only if F is an s-set relative to δ. Here δ denotes the quasi-distance defined by Macías and Segovia such that (X, δ, μ) is a normal space. In order to prove this result, we prove a covering type lemma and a type of Hausdorff measure based criterion for a given set to be an s-set relative to μ.

1. Introduction, notation and definitions. A *quasi-metric* on a set X is a non-negative function d defined on $X \times X$ satisfying the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) there exists a constant $K \geq 1$ such that $d(x, y) \leq K(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

We will refer to K as the *triangle constant* for d. A quasi-distance d on X induces a topology through the neighborhood system given by the family of all subsets of X containing a d-ball $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$ (see [4]). In a quasi-metric space (X, d) the *diameter* of a subset E is defined as

$$\operatorname{diam}(E) = \sup\{d(x, y) : x, y \in E\}.$$

Throughout this paper (X, d) will be a quasi-metric space such that all d-balls are open sets. Also we shall assume that (X, d) has *finite metric dimension.* This means that there exists a constant $N \in \mathbb{N}$ such that any d-ball $B(x, 2r)$ contains at most N points of any r-disperse subset of X. A set U is said to be *r-disperse* if $d(x, y) \geq r$ for any $x, y \in U$, $x \neq y$. If a quasi-metric space (X, d) has finite metric dimension, then every r-disperse subset of X has at most N^m points in each d-ball of radius $2^m r$ for all $m \in \mathbb{N}$ and every $r > 0$ (see [4] and [3]). Also it is well known that every bounded subset F of X is totally bounded, so that for every $r > 0$ there exists a finite

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maximal r-disperse subset of F, whose cardinality depends on $\operatorname{diam}(F)$ and on r.

We shall say that a closed subset F of X is an *s-set in* (X, d) with associated measure ν if ν is a Borel measure supported on F such that

$$c^{-1} r^s \leq \nu(B(x, r)) \leq c r^s \tag{1.1}$$

for all $x \in F$ and $0 < r < \operatorname{diam}(F)$, for some constant $c \geq 1$. When the above conditions hold for every $0 < r < r_0$, where r_0 is a positive number less than $\operatorname{diam}(F)$, we say that F is *locally an s-set in* (X, d). In some references related to problems of harmonic analysis and partial differential equations (see for example [1]), such sets are called *(locally) s-Ahlfors*. In geometric measure theory (see e.g. [7]), an s-set F is one for which $0 < \mathscr{H}^s(F) < \infty$ where $\mathscr{H}^s$ is the Hausdorff measure of dimension s. However, following [11] we shall use the term s-set for a set that supports a measure ν for which $\nu(B(x, r))$ behaves as r^s for r small.

In [1] it is proved that the concepts of s-set and locally s-set coincide when the set F is bounded and (X, d) has finite metric dimension.

We shall now recall the definitions of Hausdorff measure and Hausdorff dimension of a set in a quasi-metric space (X, d). The basic background related to these concepts can be found in [7]. For $\rho > 0$, we say that a sequence $\{B_i = B(x_i, r_i)\}$ of subsets of X is a *ρ-cover* by d-balls of a set F if $F \subseteq \bigcup B_i$ and $r_i \leq \rho$ for every i. Let $F \subseteq X$ and $s \geq 0$ be fixed. We define

$$\mathscr{H}^s_\rho(F) = \inf \Big\{ \sum_{i=1}^{\infty} r_i^s : \{B_i\} \text{ is a } \rho\text{-cover of } F \text{ by } d\text{-balls} \Big\}.$$

Clearly $\mathscr{H}^s_\rho(F)$ increases when ρ decreases, so that its limit when ρ tends to 0 exists (although it may be infinite). We define

$$\mathscr{H}^s(F) = \lim_{\rho \to 0} \mathscr{H}^s_\rho(F) = \sup_{\rho > 0} \mathscr{H}^s_\rho(F).$$

We shall refer to $\mathscr{H}^s(F)$ as the *Hausdorff measure* of F. The corresponding *Hausdorff dimension* of F is defined as $\dim_{\mathscr{H}}(F) = \inf\{s > 0 : \mathscr{H}^s(F) = 0\}$. It is easy to see that every s-set F in (X, d) has $\dim_{\mathscr{H}}(F) = s$ (see [11]).

We point out that if (F, d) is (locally) an s-set, then there exists essentially only one Borel measure ν satisfying the condition required in the definition. This fact is known in the Euclidean setting (see for instance [12]), and was proved for general quasi-metric spaces in [1]. More precisely, it is proved that if (X, d) has finite metric dimension and F is (locally) an s-set in (X, d) with measure ν, then F is (locally) an s-set in (X, d) with the restriction of $\mathscr{H}^s$ to F.

A sufficient condition for a quasi-metric space (X, d) to have finite metric dimension is that X supports a doubling measure (see [4]). A Borel measure

μ defined on d-balls is said to be *doubling* if for some constant $A \geq 1$,

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty$$

for all $x \in X$ and $r > 0$. We say that a point x in (X, d, μ) is an *atom* if $\mu(\{x\}) > 0$. When $\mu(\{x\}) = 0$ for every $x \in X$, we say that μ is *non-atomic.* Macías and Segovia [9] proved that a point is an atom if and only if it is topologically isolated, and that the set of such points is at most countable. Throughout this paper we shall say that (X, d, μ) is a *space of homogeneous type* if μ is a non-atomic doubling measure on the quasi-metric space (X, d).

Given a space of homogeneous type (X, d, μ), the Hausdorff measure and the Hausdorff dimension *relative to* μ are considered in [11]. Precisely, the *Hausdorff measure relative to* μ is defined as $H^s(F) := \lim_{\rho\to 0} H^s_\rho(F)$, where

$$H^s_\rho(F) = \inf\Big\{\sum_{i=1}^{\infty} \mu^s(B_i) : F \subseteq \bigcup_i B_i \text{ and } \mu(B_i) \leq \rho\Big\},$$

where the B_i are d-balls on X. The *Hausdorff dimension relative to* μ is defined by

$$\dim_H(F) = \inf\{s > 0 : H^s(F) = 0\}.$$

These concepts lead to a definition of an s-set relative to the measure μ, compatible with H^s. Given a space of homogeneous type (X, d, μ), we shall say that a closed subset F of X is an *s-set in* (X, d, μ) with associated measure m if m is a Borel measure supported on F such that

$$c^{-1}\mu(B(x, r))^s \leq m(B(x, r)) \leq c\mu(B(x, r))^s \tag{1.2}$$

for all $x \in F$ and $0 < r < \operatorname{diam}(F)$, for some constant $c \geq 1$. As before, if (1.2) holds for every $0 < r < r_0$, where $r_0 < \operatorname{diam}(F)$, we say that F is *locally an s-set in* (X, d, μ).

It is now easy to see that each s-set F in (X, d, μ) satisfies $\dim_H(F) = s$.

Given a space of homogeneous type (X, d, μ), in [11] there are also considered the concepts of s-sets, Hausdorff measure and Hausdorff dimension relative to a particular quasi-metric δ related to (X, d, μ). This quasi-metric was constructed by Macías and Segovia [9] in such a way that the new structure (X, δ, μ) becomes a normal space (in the sense that every δ-ball in X has μ-measure equivalent to its radius), and the topologies induced on X by d and δ coincide. This quasi-metric is defined by

$$\delta(x, y) = \inf\{\mu(B) : B \text{ is a } d\text{-ball with } x, y \in B\}$$

if $x \neq y$, and $\delta(x, y) = 0$ if $x = y$. It will also be useful to notice that in the proof of the above mentioned result of Macías and Segovia it is proved that

$$B_\delta(x, r) = \bigcup\{B : B \text{ is a } d\text{-ball with } x \in B \text{ and } \mu(B) < r\}$$

for all $x \in X$ and $r > 0$, where $B_\delta(x, r) := \{y \in X : \delta(x, y) < r\}$ denotes the ball in X relative to δ. Throughout this paper, δ will denote this quasi-metric.

Furthermore, we can consider the concepts of s-set in (X,δ), of the Hausdorff measure relative to δ and of the corresponding Hausdorff dimension. More precisely, we shall denote $G^s(F) := \lim_{\rho\to 0} G^s_\rho(F)$, where

$$G^s_\rho(F) = \inf\Big\{\sum_{i=1}^{\infty} r_i^s : F \subseteq \bigcup_i B_\delta(x_i,r_i) \text{ and } r_i \le \rho\Big\}$$

and

$$\dim_G(F) = \inf\{s>0 : G^s(F)=0\}.$$

In [11, Prop. 1.5] it is proved that $H^s(F)$ and $G^s(F)$ are equivalent, and hence $\dim_H(F) = \dim_G(F)$ for any subset F of X. In this note we explore the relationship between the concepts of s-set in (X,d,μ) and in (X,δ). This natural question completes the analysis of the concepts referring to δ and μ. It is also related to the theory of Muckenhoupt weights. The results in [2] give us a sufficient condition on a closed set F in a general space of homogeneous type (X,d,μ) for $\mu(B(x,d(x,F)))^\beta$ to become a Muckenhoupt weight for suitable values of β: that F be an s-set in (X,δ) (see [2, Thms. 1 and 9]). In this note we find a class of sets for which this condition is guaranteed if $H^s(F\cap B(x,r)) \simeq r^s$ for all $x\in F$ and $r>0$ (see Theorem 2.5 and Proposition 2.6).

The paper is organized as follows. Section 2 contains the main results. Theorem 2.1 states that under certain typical conditions, being an s-set in (X,δ) is stronger than being an s-set in (X,d,μ). A sufficient condition for every s-set in (X,d,μ) to be an s-set in (X,δ) is given in Theorem 2.5. We show that every bounded set satisfies this condition, and we give examples of unbounded sets satisfying it. In Proposition 2.6 we obtain a criterion to check the s-set condition relative to μ based on the Hausdorff measure. Section 3 is devoted to the proof of Proposition 2.6; for this we prove a lemma on covering a bounded set by balls with small measure and controlled overlap (see Lemma 3.1).

2. Main results. Let (X,d,μ) be a given space of homogeneous type, and δ the quasi-metric defined in the previous section. We shall first prove that, under a certain condition, being an s-set in (X,δ) is stronger than being an s-set in (X,d,μ).

THEOREM 2.1.

(1) *If F is an s-set in (X,δ) with associated measure ν and* $\operatorname{diam}(F) = \infty$*, then F is an s-set in (X,d,μ) with the same measure ν.*
(2) *If F is locally an s-set in (X,δ) with associated measure ν and $\mu(F)=0$, then F is locally an s-set in (X,d,μ) with the same measure ν.*

Proof. By hypothesis, there exist $c \geq 1$ and $r_0 > 0$ such that

$$c^{-1} r^s \leq \nu(B_\delta(x,r)) \leq c r^s$$

for all $x \in F$ and $0 < r < r_0$, where ν is a Borel measure supported in F, and $r_0 = \infty$ in case (1).

Fix $x \in F$ and $r > 0$. By definition of δ, we have

$$B(x,r) \subseteq B_\delta(x, 2\mu(B(x,r))).$$

Then

$$\nu(B(x,r)) \leq \nu(B_\delta(x, 2\mu(B(x,r)))) \leq c 2^s \mu^s(B(x,r))$$

provided that $\mu(B(x,r)) < r_0/2$. On the other hand, fix ℓ such $3K^2 \leq 2^\ell$ where K denotes the triangle constant for d. Following [9, p. 262], we shall see now that $B_\delta(x, A^{-\ell}\mu(B(x,r))) \subseteq B(x,r)$, where A is the constant from the doubling condition for μ. Indeed, for $y \in B_\delta(x, A^{-\ell}\mu(B(x,r)))$, $y \neq x$, there exists a ball $B(z,s)$ containing x and y and such that $\mu(B(z,s)) < A^{-\ell}\mu(B(x,r))$. It is easy to show that $y \in B(x, 2Ks) \subseteq B(z, 3K^2 s)$. Therefore,

$$\mu(B(x, 2Ks)) \leq \mu(B(z, 3K^2 s)) \leq A^\ell \mu(B(z,s)) < \mu(B(x,r)).$$

Consequently, $2Ks < r$. Thus $y \in B(x, 2Ks) \subseteq B(x,r)$, and the inclusion is proved. Hence

$$\nu(B(x,r)) \geq \nu(B_\delta(A^{-\ell}\mu(B(x,r)))) \geq c^{-1} A^{-\ell s} \mu^s(B(x,r)),$$

provided that $\mu(B(x,r)) < A^\ell r_0$.

Since every d-ball has finite μ-measure, (1) is proved. On the other hand, we obtain (2) if we can choose r_1 in such a way that $0 < r < r_1$ implies $\mu(B(x,r)) < \min\{r_0/2, A^\ell r_0\} = r_0/2$ for every $x \in F$. But this is possible from the hypothesis $\mu(F) = 0$. ∎

We point out that the assumption $\mu(F) = 0$ is natural in many problems related to partial differential equations, where F plays the role of the boundary of a domain in a metric measure space (X, d, μ) (see for example [6] or [5]).

To obtain a sufficient condition for every locally s-set in (X, d, μ) to be locally an s-set in (X, δ), we shall give the following definition.

Definition 2.2. Let F be a closed subset of X. We shall say that F is *consistent with* μ if there exists a positive number R such that

$$\inf_{x \in F} \mu(B(x,R)) > 0.$$

Note that if F is consistent with μ, then $\inf_{x \in F} \mu(B(x,r)) > 0$ for every $r > 0$. In fact, the inequality is trivial for $r \geq R$. On the other hand, for a fixed $0 < r < R$, for every $x \in F$ we have

$$\mu(B(x,r)) = \mu\left(x, \frac{r}{R} R\right) \geq \frac{1}{A^m} \mu(B(x,R)),$$

where m is a positive integer such that $2^m \geq R/r$ and A denotes the doubling constant for μ.

We also point out that every bounded subset of X is consistent with μ. In fact, set $R = 2K \operatorname{diam}(F)$ with K the triangle constant for d, and fix $x_0 \in F$. Hence $B(x_0, \operatorname{diam}(F)) \subseteq B(x, R)$ for every $x \in F$. Hence $\inf_{x\in F} \mu(B(x, R)) \geq \mu(B(x_0, \operatorname{diam}(F))) > 0$, since μ is doubling.

However, there also exist unbounded sets satisfying this condition.

Example 2.3. Consider $X = \mathbb{R}^2$ equipped with the usual distance d and the Lebesgue measure λ. Fix $a > 0$ and set $F = \{(t, 0) : t \geq a\}$. Then $\lambda(B(x, r))$ is equivalent to r^2 for every $x \in F$, so F is consistent with λ.

Recall that a quasi-metric measure space is said to be an *α-Ahlfors space* if there exists a constant $c \geq 1$ such that $c^{-1}r^\alpha \leq \mu(B(x, r)) \leq cr^\alpha$ for all $x \in X$ and $r > 0$. The most classical example of an n-Ahlfors space is the Euclidean space $\mathbb{R}^n$ equipped with the usual distance and the Lebesgue measure. So in the above example, the underlying space $(\mathbb{R}^2, d, \lambda)$ is 2-Ahlfors. Notice that if (X, d, μ) is an α-Ahlfors space, then every subset F of X is consistent with μ. In the following example we shall consider another measure μ defined on $(\mathbb{R}^2, d)$ such that $(\mathbb{R}^2, d, \mu)$ is not an Ahlfors space.

Example 2.4. Let X be $\mathbb{R}^2$ equipped with the usual distance d, and consider the measure μ defined by

$$\mu(E) = \int_E |y|^\beta \, dy$$

for a fixed $\beta > -2$. Then (X, d, μ) is a space of homogeneous type since $|x|^\beta$ is a Muckenhoupt weight (see [10] or [8]). For the set F considered in the above example, it is easy to see that $\mu(B(x, r))$ is equivalent to $r^2|x|^\beta$ for $x \in F$ and $0 < r \leq a/2$. So F is consistent with μ if and only if $\beta \geq 0$.

With this terminology, we have the following result.

Theorem 2.5.

(1) *If F is an s-set in (X, d, μ) with* $\operatorname{diam}(F) = \infty$, *then F is an s-set in (X, δ).*
(2) *If F is locally an s-set in (X, d, μ) which is consistent with μ, then F is locally an s-set in (X, δ).*

To prove the above theorem, we shall use three auxiliary results.

The first one states that, as in the case of s-sets relative to a distance, when F is an s-set relative to the measure μ, there exists essentially only one Borel measure ν satisfying the required condition. More precisely, we state the following result that we shall prove in Section 3.

PROPOSITION 2.6. *If F is (locally) an s-set in (X,d,μ) with associated measure m, then F is (locally) an s-set in (X,d,μ) with the restriction of H^s to F, where H^s denotes the s-dimensional Hausdorff measure relative to μ.*

The following statement provides a characterization of a set F consistent with a given measure: if the measure of a d-ball with center in F is sufficiently small, then so is its radius.

LEMMA 2.7. *F is consistent with μ if and only if given $r_0 > 0$, there exists C such that if $x \in F$ and $\mu(B(x,t)) \le C$, then $t < r_0$.*

Proof. Suppose first that F is consistent with μ but the above property is false. Then there exists $r_0 > 0$ such that for every natural number n we can find $x_n \in F$ and $t_n \ge r_0$ with $\mu(B(x_n,t_n)) \le 1/n$. So $\mu(B(x_n,r_0)) \le 1/n$ for every natural n, which implies that $\inf_{x\in F}\mu(B(x,r_0)) = 0$. But this is a contradiction, since F is consistent with μ.

Conversely, assume that F is not consistent with μ. Then, for every $r_0 > 0$ we have $\inf_{x\in F}\mu(B(x,r_0)) = 0$. So for every natural n there exists $x_n \in F$ such that $\mu(B(x_n,r_0)) < 1/n$. Hence, given $C > 0$ we can choose n such that $1/n \le C$ and obtain $\mu(B(x_n,r_0)) < C$, but $r_0 \not< r_0$. ∎

The last result that we shall need is a technical lemma, proved in [11].

LEMMA 2.8. *Given $x \in X$ and $0 < r < 2\mu(X)$, there exist $0 < a \le b < \infty$ such that*

$$B(x,a) \subseteq B_\delta(x,r) \subseteq B(x,b)$$

and

$$C_1 r \le \mu(B(x,a)) \le \mu(B(x,b)) \le C_2 r,$$

where C_1 and C_2 only depend on X.

Proof of Theorem 2.5. From Proposition 2.6, there exist $c \ge 1$ and $r_0 > 0$ such that

$$c^{-1}\mu(B(x,r))^s \le H^s(B(x,r)\cap F) \le c\mu(B(x,r))^s$$

for all $x \in F$ and $0 < r < r_0$, where $r_0 = \infty$ in case (1).

Fix $x \in F$ and $0 < r < 2\mu(X)$, and let a and b be as in Lemma 2.8. Then, if $a,b < r_0$, we have

$$H^s(B_\delta(x,r)\cap F) \le H^s(B(x,b)\cap F) \le c\mu^s(B(x,b)) \le cC_2^s r^s,$$
$$H^s(B_\delta(x,r)\cap F) \ge H^s(B(x,a)\cap F) \ge c^{-1}\mu^s(B(x,a)) \ge c^{-1}C_1^s r^s.$$

Thus (1) is proved. Moreover, (2) will be showed if we can choose $r_1 \le 2\mu(X)$ such that $r < r_1$ implies $a,b < r_0$. To do this, let C be such that if $x \in F$ and $\mu(B(x,t)) \le C$, then $t < r_0$ (see Lemma 2.7). Define $r_1 = \min\{2\mu(X), C/C_2\}$ with C_2 the constant of Lemma 2.8. Then $\mu(B(x,a))$ and $\mu(B(x,b))$ are both bounded above by C, so that $a,b < r_0$. ∎

REMARK 2.9. We point out that only in the case of a *locally* s-set F in (X,d,μ) with $\operatorname{diam}(F)=\infty$ and such that (X,d,μ) is not an Ahlfors space, we shall need to check if F is consistent with μ to conclude that F is locally an s-set in (X,δ).

In the remaining cases, being (locally) an s-set in (X,d,μ) implies being (locally) an s-set in (X,δ). Indeed, the concepts of s-set and locally s-set in (X,d,μ) coincide when F is bounded, and every bounded set is consistent with μ, just as every subset of an Ahlfors space.

3. Proof of Proposition 2.6. To prove Proposition 2.6, we shall use the following covering type lemma that we shall prove at the end of this section.

LEMMA 3.1. *Let G be a bounded subset of X. For a given $\rho>0$, there exists a finite covering $\{B(x_i,r_i): i=1,\dots,I_\rho\}$ of G by d-balls with $x_i\in G$ and $\mu(B(x_i,r_i))<\rho$. Also, each $y\in X$ belongs to at most Λ such balls, where Λ is a geometric constant which depends only on X.*

REMARK 3.2. Notice that if $\rho\le\mu(G)$, then $r_i\le\operatorname{diam}(G)$ for every i. In fact, assume that $r_i>\operatorname{diam}(G)$ for some i. Then $G\subseteq B(x_i,r_i)$, so that $\mu(G)\le\mu(B(x_i,r_i))<\rho\le\mu(G)$, which is absurd.

Proof of Proposition 2.6. By hypothesis there exist $r_0>0$, a constant $c\ge 1$ and a Borel measure m supported on F such that

$$c^{-1}\mu(B(x,r))^s\le m(B(x,r))\le c\mu(B(x,r))^s$$

for all $x\in F$ and $0<r<r_0$. Here r_0 is infinite if F is an unbounded s-set in (X,d,μ), and is finite otherwise.

Fix $x\in F$, $0<r<r_0$ and $\varepsilon>0$. For each $\rho>0$, there exists a covering $\{B_i=B(x_i,r_i)\}$ of $B(x,r)\cap F$ by balls such that $\mu(B_i)<\rho$ and

$$\sum_{i\ge 1}\mu^s(B_i)<H^s_\rho(B(x,r)\cap F)+\varepsilon\le H^s(B(x,r)\cap F)+\varepsilon.$$

Choosing an appropriate value of ρ, we can also obtain $r_i<r_0$ for every i. In fact, take $\rho=\mu(B(x,r))/A^\ell$ with ℓ an integer such that $2^\ell\ge 3K^2$. Then, since we can assume that each $B(x_i,r_i)$ intersects $B(x,r)$, if $r_i\ge r_0$ then $B(x,r)\subseteq B(x_i,3K^2r_i)$. Hence $\mu(B(x,r))\le A^\ell\mu(B_i)<\mu(B(x,r))$, which is absurd. Thus we can assume $r_i<r_0$ for every i, and hence

$$c^{-1}\mu(B(x,r))^s\le m(B(x,r))\le\sum_i m(B_i)\le c\sum_i\mu(B_i)^s.$$

Hence, $c^{-1}\mu(B(x,r))^s<cH^s(B(x,r)\cap F)+c\varepsilon$ for every $\varepsilon>0$, which proves

$$H^s(B(x,r)\cap F)\ge c^{-2}\mu(B(x,r))^s.$$

To obtain an upper bound for $H^s(B(x,r)\cap F)$, first assume that $r < r_0/(4K^2)$ and fix $0<\rho<\mu(B(x,r)\cap F)$. From Lemma 3.1, there exists a finite covering $\{B(x_i,r_i): i=1,\ldots,I_\rho\}$ of $B(x,r)\cap F$ by d-balls with $\mu(B(x_i,r_i))<\rho$, $x_i\in F$ and $r_i\le 2Kr$. Also, each $y\in X$ belongs to at most Λ such balls, where Λ is a geometric constant which does not depend on ρ, r or x. So, we have

$$\begin{aligned}H^s_\rho(B(x,r)\cap F) &\le \sum_{i=1}^{I_\rho}\mu(B(x_i,r_i))^s \le c\sum_{i=1}^{I_\rho} m(B(x_i,r_i))\\ &\le c\Lambda m\Big(\bigcup_{i=1}^{I_\rho} B(x_i,r_i)\Big) \le c\Lambda m(B(x,4K^2r))\\ &\le c^2\Lambda\mu(B(x,4K^2r))^s = \tilde{C}\mu(B(x,r))^s\end{aligned}$$

with $\tilde{C}=c^2\Lambda A^j$, where j is a positive integer such that $2^{j-2}\ge K^2$. Taking $\rho\to 0$ we obtain the desired result for this case.

Finally, if r_0 is finite, we shall consider the case $r_0/(4K^2)\le r<r_0$. In this case, since $B(x,r)$ is bounded, there exists a finite $r_0(8K^2)^{-1}$-disperse maximal set in $B(x,r)$, say $U=\{x_1,\ldots,x_I\}$ with $I\le N^{2+\log_2 K}$. Then $B(x,r)\cap F\subseteq\bigcup_{i=1}^I B(x_i,r_0/(8K^2))$, and applying the previous case we obtain

$$H^s(B(x,r)\cap F)\le\sum_{i=1}^I H^s\bigg(B\bigg(x_i,\frac{r_0}{8K^2}\bigg)\cap F\bigg)\le \tilde{C}I\mu(B(x,2Kr))^s,$$

and the result follows from the doubling property of μ. ■

For the proof of Lemma 3.1, we shall use the next result about the behavior of the δ-diameter $\operatorname{diam}_\delta(E):=\sup\{\delta(y,w): y,w\in E\}$ of a bounded set E.

LEMMA 3.3. *Let E be a bounded subset of X. For $B=B(x,\operatorname{diam}(E))$ and $x\in E$ we have*

$$A^{-\ell}\mu(B)\le\operatorname{diam}_\delta(E)\le A\mu(B),$$

where A is the doubling constant for μ, and ℓ is a positive integer satisfying $\ell\ge\log_2(8K^3)$, with K the triangle constant for d.

Proof. Fix $x\in E$, and let y and w be any two points in E. Since $y,w\in B(x,2\operatorname{diam}(E))$, from the definition of δ it follows that $\delta(y,w)\le\mu(B_d(x,2\operatorname{diam}(E)))\le A\mu(B)$. Taking the supremum yields the upper bound for $\operatorname{diam}_\delta(E)$.

For the lower bound, let $y_0,w_0\in E$ be such that $\operatorname{diam}(E)<2d(y_0,w_0)$. For a given $\varepsilon>0$, let $B(x_0,r_0)$ be a ball containing y_0 and w_0 such that $\mu(B(x_0,r_0))<\delta(y_0,w_0)+\varepsilon$. We claim that $B\subseteq B(x_0,8K^3r_0)$. Assuming this is true, we have

$$\operatorname{diam}_\delta(F) \geq \delta(y_0, w_0) > \mu(B(x_0, r_0)) - \varepsilon \geq A^{-\ell}\mu(B) - \varepsilon.$$

By letting ε tend to zero we obtain the result.

It remains to prove the claim. Fix $z \in B$. Then

$$\begin{aligned} d(z, x_0) &\leq K^2[d(x, x) + d(x, w_0) + d(w_0, x_0)] \\ &< K^2[2\operatorname{diam}(E) + r_0] < K^2[4d(y_0, w_0) + r_0] \\ &< K^2[4K(d(y_0, x_0) + d(x_0, w_0)) + r_0] < 8K^3 r_0, \end{aligned}$$

and the lemma is proved. ■

Proof of Lemma 3.1. Let $\tilde{K}$ be the triangle constant for δ, and $\tilde{N}$ the constant for the finite metric dimension of (X, δ, μ). Given $\rho > 0$, let $t = \rho/(4\tilde{K}A^{\ell+1})$, with ℓ as in Lemma 3.3. Set $U = \{x_1, \dots, x_{I_t}\}$ a finite t-disperse maximal set in G with respect to the quasi-metric δ. So $\{B_\delta(x_i, t)\}$ is a covering of G. Define $B_i = B(x_i, r_i)$ with $r_i = 2\operatorname{diam}(B_\delta(x_i, t))$.

Let us first check that $\{B_i\}$ is a covering of G. In fact, if $y \in G$ then there exists i such that $y \in B_\delta(x_i, t)$. Then

$$d(x_i, y) \leq \operatorname{diam}(B_\delta(x_i, t)) < 2\operatorname{diam}(B_\delta(x_i, t))$$

so that $y \in B_i$.

To estimate the measure of each B_i, using Lemma 3.3 with $E = B_\delta(x_i, t)$ we obtain

$$\mu(B_i) \leq A\mu(B(x_i, \operatorname{diam}(B_\delta(x_i, t)))) \leq A^{\ell+1}\operatorname{diam}_\delta(B_\delta(x_i, t)) \leq A^{\ell+1} 2\tilde{K}t.$$

From the choice of t, we have $\mu(B_i) < \rho$. So it remains to prove that we can control the overlapping of these balls by a geometric constant Λ. In fact, for a fixed $y \in X$, if $y \in B(x_i, r_i)$, then $B(y, r_i) \subseteq B(x_i, 2Kr_i)$. So $\mu(B(y, r_i)) \leq A^p\rho$ with p an integer such that $2^{p-1} \geq K$, and thus

$$x_i \in B(y, r_i) \subseteq B_\delta(y, 2\mu(B(y, r_i))) \subseteq B_\delta(y, 2A^p\rho) = B_\delta(y, 8\tilde{K}A^{\ell+p+1}t).$$

Hence, the number of balls $B(x_i, r_i)$ to which y belongs is less than or equal to the cardinality of $U \cap B_\delta(y, 2^m t)$, with m a natural number such that $2^m \geq 8\tilde{K}A^{\ell+p+1}$. Since U is t-disperse with respect to δ, we find that $\Lambda \leq \tilde{N}^m$ and the lemma is proved. ■

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