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# A NOTE ON THE HYERS–ULAM PROBLEM

BY

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**Abstract.** Let $X, Y$ be real Banach spaces and $\varepsilon > 0$. Suppose that $f : X \to Y$ is a surjective map satisfying $|\|f(x) - f(y)\| - \|x - y\|| \le \varepsilon$ for all $x, y \in X$. Hyers and Ulam asked whether there exists an isometry $U$ and a constant $K$ such that $\|f(x) - Ux\| \le K\varepsilon$ for all $x \in X$. It is well-known that the answer to the Hyers–Ulam problem is positive and $K = 2$ is the best possible solution with assumption $f(0) = U0 = 0$. In this paper, using the idea of Figiel's theorem on nonsurjective isometries, we give a new proof of this result.

**1. Introduction.** In 1945, Hyers and Ulam [6] introduced the following notion of an approximate isometry between Banach spaces.

DEFINITION 1.1. Let $\varepsilon > 0$ and let $X$ and $Y$ be Banach spaces. A map $f : X \to Y$ is called an *$\varepsilon$-isometry* if

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \le \varepsilon \quad \text{for all } x, y \in X. \tag{1.1}$$

They asked whether for any surjective $\varepsilon$-isometry there exists an isometry which is close to this $\varepsilon$-isometry. The answer to this problem was first proved to be affirmative by Gevirtz [4] in 1983, whose proof is based on a partial result of Gruber [5]. The following sharp approximation result for this problem is due to Omladič and Šemrl [8].

THEOREM 1.2. *If $f : X \to Y$ is a surjective $\varepsilon$-isometry between Banach spaces with $f(0) = 0$, then there is a bijective linear isometry $U : X \to Y$ such that*

$$\|f(x) - Ux\| \le 2\varepsilon \quad \textit{for all } x \in X. \tag{1.2}$$

The surjectivity assumption in Theorem 1.2 cannot be omitted. We refer to the authoritative book [1] and the surveys [10, 12] for this topic and related matters.

In 1932, Mazur and Ulam [7] proved that a surjective isometry between two Banach spaces is necessarily affine. Indeed, Benyamini and Linden-

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strauss [1, p. 361] remarked that the spirit of the proof of Theorem 1.2 somewhat originates from the proof of the Mazur–Ulam theorem.

On the other hand, in 1968, Figiel [3] proved the following celebrated theorem.

Theorem 1.3 (Figiel). *Suppose that $f : X \to Y$ is an into isometry with $f(0) = 0$. Then there exists a norm one linear operator $T : \overline{\operatorname{span}}\, f(X) \to X$ such that $T \circ f = I$ (the identity) on $X$.*

It is clear that Figiel's theorem is a generalization of the result of Mazur and Ulam [7]. Hence Figiel indeed gave a different proof of the Mazur-Ulam theorem. Inspired by Figiel's theorem, nonsurjective $\varepsilon$-isometries have been studied (see [2, 9, 11]). A natural question here is whether there is a proof of Theorem 1.2 using the idea of Figiel's theorem. Our purpose in this note is to give a new proof of Theorem 1.2, which is somewhat inspired by the idea of Figiel [3].

In this paper we use standard notation. The letter $X$ will always stand for a real Banach space, and $X^*$ its dual. We denote the unit sphere of $X$ by $S(X)$ and the closed unit ball of $X$ by $B(X)$. The unique supporting functional at a smooth point $x \in X$ is denoted by $j(x)$.

**2. Main results.** We start this section with the following lemma, the idea of which is inspired by the proof of Figiel's theorem [3].

Lemma 2.1. *Let $X$ and $Y$ be Banach spaces. Suppose that $f : X \to Y$ is an $\varepsilon$-isometry with $f(0) = 0$. Assume that $Y_1$ is a finite-dimensional subspace of $Y$ such that $Y_1 \subset f(X)$. If $z \in S(Y_1)$ is a smooth point of $B(Y_1)$, then there is a linear functional $\phi \in X^*$ with $\|\phi\| = 1$ such that*

$$|\phi(x) - j(z) \circ f(x)| \le 4\varepsilon \tag{2.1}$$

*for all $x \in X$ with $f(x) \in Y_1$.*

*Proof.* Since $z$ is a smooth point of $B(Y_1)$, by the definition of Gâteaux differentiability we easily obtain

$$\lim_{t\to\infty} (\|tz + y\| - t) = j(z)(y) \tag{2.2}$$

for all $y \in Y_1$.

Since $Y_1 \subset f(X)$, for every $t \in \mathbb{R}$ there exists $z_t \in X$ such that $f(z_t) = tz$ and so $|t| - \varepsilon \le \|z_t\| \le |t| + \varepsilon$.

Fix $n \in \mathbb{N}$. The Hahn–Banach theorem implies that there is a linear functional $\phi_n \in X^*$ with $\|\phi_n\| = 1$ such that

$$\phi_n(z_n - z_{-n}) = \|z_n - z_{-n}\|.$$

Remembering that $f$ is an $\varepsilon$-isometry we have

$$\begin{aligned}\phi_n(z_n) &= \|z_n - z_{-n}\| + \phi_n(z_{-n}) \\ &\geq \|f(z_n) - f(z_{-n})\| - \varepsilon - n - \varepsilon \\ &= \|nz - (-nz)\| - \varepsilon - n - \varepsilon = n - 2\varepsilon.\end{aligned}$$

For every $t \in [0, n]$,

$$\phi_n(z_t) = \phi_n(z_n) - \phi_n(z_n - z_t) \geq (n - 2\varepsilon) - (n - t + \varepsilon) = t - 3\varepsilon.$$

Hence

$$t - 3\varepsilon \leq \phi_n(z_t) \leq t + \varepsilon \quad \text{for all } t \in [0, n].$$

Similarly, we get

$$t - \varepsilon \leq \phi_n(z_t) \leq t + 3\varepsilon \quad \text{for all } t \in [-n, 0].$$

Note that $\|\phi_n\| = 1$ for all $n$. Alaoglu's theorem implies that the sequence $\{\phi_n\}$ has a $w^*$-cluster point $\phi \in B(Y^*)$. Then

$$t - 3\varepsilon \leq \phi(z_t) \leq t + 3\varepsilon \quad \text{for all } t \in \mathbb{R}, \tag{2.3}$$

and clearly $\|\phi\| = 1$. We will prove that $\phi$ is the desired functional.

Fix $x \in X$ such that $f(x) \in Y_1$. Now, (2.3) yields

$$t - 3\varepsilon - \phi(x) \leq \phi(z_t) - \phi(x) \leq \|z_t - x\| \leq \|tz - f(x)\| + \varepsilon.$$

Therefore,

$$\|tz - f(x)\| - t + \phi(x) \geq -4\varepsilon.$$

Letting $t \to \infty$, (2.2) yields

$$j(z) \circ f(x) - \phi(x) \leq 4\varepsilon. \tag{2.4}$$

On the other hand, we get

$$\begin{aligned}t - 3\varepsilon + \phi(x) &\leq -\phi(z_{-t}) + \phi(x) \leq \|z_{-t} - x\| \\ &\leq \|tz + f(x)\| + \varepsilon,\end{aligned}$$

which leads to

$$\|tz + f(x)\| - t - \phi(x) \geq -4\varepsilon.$$

Letting $t$ tend to $\infty$ in the inequality above, (2.2) again implies that

$$j(z) \circ f(x) - \phi(x) \geq -4\varepsilon. \tag{2.5}$$

Hence (2.1) follows from (2.4) and (2.5). ■

LEMMA 2.2. *Let $X$ be a separable Banach space, and let* $\mathrm{sm}(X)$ *denote the set of smooth points in $S(X)$. Then*

$$\|x\| = \sup_{z \in \mathrm{sm}(X)} |j(z)(x)| \quad \textit{for all } x \in X.$$

*Proof.* Since $j(z)$ is a unique supporting functional at a smooth point $z$, we have $\|j(z)\| = 1$ and $j(z)(z) = 1$. Hence

$$\|x\| \geq \sup_{z\in \mathrm{sm}(X)} |j(z)(x)|. \tag{2.6}$$

On the other hand, since $X$ is separable, $\mathrm{sm}(X)$ is dense in the unit sphere $S(X)$ (see, for example, [1, Theorem 4.17]). Therefore, for any $\varepsilon > 0$ we can find $z_0 \in \mathrm{sm}(X)$ such that $\|x/\|x\| - z_0\| < \varepsilon$. Now

$$\begin{aligned}\left|j(z_0)\left(\frac{x}{\|x\|}\right)\right| &= \left|j(z_0)\left(\frac{x}{\|x\|} - z_0\right) + j(z_0)(z_0)\right| \\ &\geq |j(z_0)(z_0)| - \left|j(z_0)\left(\frac{x}{\|x\|} - z_0\right)\right| \\ &\geq 1 - \left\|\frac{x}{\|x\|} - z_0\right\| \geq 1 - \varepsilon.\end{aligned}$$

Since $\varepsilon$ is arbitrary, we have

$$\sup_{z\in \mathrm{sm}(X)} \left|j(z)\left(\frac{x}{\|x\|}\right)\right| \geq 1.$$

Hence

$$\sup_{z\in \mathrm{sm}(X)} |j(z)(x)| \geq \|x\|. \tag{2.7}$$

Thus the equality follows from (2.6) and (2.7). ∎

Now, we are ready to give a new proof of Theorem 1.2.

*Proof of Theorem 1.2.* Fix $m, n \in \mathbb{N}$ and $x \in X$. Let

$$Y_1 = \mathrm{span}(f(mx), f(nx)).$$

Since $f$ is surjective, Lemma 2.1 implies that for every $z \in \mathrm{sm}(Y_1)$ there is a linear functional $\phi_z \in X^*$ with $\|\phi_z\| = 1$ such that

$$|\phi_z(u) - j(z) \circ f(u)| \leq 4\varepsilon$$

for all $u \in X$ satisfying $f(u) \in Y_1$. Since $Y_1$ is finite-dimensional, Lemma 2.2 implies that

$$\begin{aligned}&\left\|\frac{nf(mx)}{m+n} - \frac{mf(nx)}{m+n}\right\| \\ &\quad = \sup_{z\in \mathrm{sm}(Y_1)} \left|j(z)\left(\frac{nf(mx)}{m+n} - \frac{mf(nx)}{m+n}\right)\right| \\ &\quad = \sup_{z\in \mathrm{sm}(Y_1)} \left|j(z)\left(\frac{nf(mx)}{m+n} - \frac{mf(nx)}{m+n}\right) - \phi_z\left(\frac{mnx}{m+n} - \frac{mnx}{m+n}\right)\right|\end{aligned}$$

$$= \sup_{z\in \mathrm{sm}(Y_1)} \left| \left( j(z)\left(\frac{nf(mx)}{m+n}\right) - \phi_z\left(\frac{mnx}{m+n}\right)\right) - \left( j(z)\left(\frac{mf(nx)}{m+n}\right) - \phi_z\left(\frac{mnx}{m+n}\right)\right)\right|$$

$$= \sup_{z\in \mathrm{sm}(Y_1)} \left| \frac{n(j(z)\circ f(mx) - \phi_z(mx))}{m+n} - \frac{m(j(z)\circ f(nx) - \phi_z(nx))}{m+n}\right|$$

$$\leq \sup_{z\in \mathrm{sm}(Y_1)} \left( \frac{n|j(z)\circ f(mx) - \phi_z(mx)|}{m+n} + \frac{m|j(z)\circ f(nx) - \phi_z(nx)|}{m+n}\right) \leq 4\varepsilon.$$

Hence

$$\left\| \frac{f(mx)}{m} - \frac{f(nx)}{n} \right\| \leq \frac{m+n}{mn} 4\varepsilon = \left(\frac{1}{n} + \frac{1}{m}\right) 4\varepsilon. \tag{2.8}$$

It follows that $\{f(nx)/n\}_{n=1}^{\infty}$ is a Cauchy sequence, and hence the limit

$$Ux := \lim_{n\to\infty} \frac{f(nx)}{n}$$

exists for every $x \in X$. Inequality (1.1) implies that $U$ is an isometry. Substituting $m = 1$ in (2.8) we get

$$\left\| f(x) - \frac{f(nx)}{n} \right\| \leq \frac{1+n}{n} 4\varepsilon.$$

Letting $n \to \infty$, we obtain

$$\|f(x) - Ux\| \leq 4\varepsilon. \tag{2.9}$$

Fix $w \in Y$. For any $n \in \mathbb{N}$ choose $x_n \in X$ such that $f(x_n) = nw$. Inequality (2.9) implies that

$$\left\| w - \frac{Ux_n}{n} \right\| = \frac{1}{n}\|f(x_n) - Ux_n\| \leq \frac{4\varepsilon}{n}.$$

Since $U$ is an isometry, its range is closed and hence contains $w$. This gives surjectivity of $U$. Moreover, because $f(0) = 0$ we have $U0 = 0$ by the definition of $U$. Now the Mazur–Ulam theorem [7] implies that $U$ is a bijective linear isometry between $X$ and $Y$.

We define $g = U^{-1} \circ f : X \to X$. Then $g$ is an $\varepsilon$-isometry with $g(0) = 0$, and (2.9) implies that $\|g(x) - x\| \leq 4\varepsilon$ for all $x \in X$. Let $x \in X$, and put $u = x - g(x)$. To prove (1.2), we only need to show that $\|u\| \leq 2\varepsilon$. In order to achieve this goal, we modify the proof of [11, Theorem 3.2]. Set

$$\alpha = \limsup_{m\to\infty} \big( \|g(x+mu)\| - \|g(x+mu) - g(x)\| \big).$$

Let $n < m$. Then

$$\begin{aligned}(2.10)\qquad &\|g(x+mu)\| - \|g(x+mu)-g(x)\| \\ &\le \left\| g(x+mu) - g(x) - \frac{n}{m}(g(x+mu)-g(x)) \right\| \\ &\quad + \left\| g(x) + \frac{n}{m}(g(x+mu)-g(x)) \right\| - \|g(x+mu)-g(x)\| \\ &= \left\| g(x) + \frac{n}{m}(g(x+mu)-g(x)) \right\| - \frac{n}{m}\|g(x+mu)-g(x)\|.\end{aligned}$$

Since $\|g(x+mu)-(x+mu)\| \le 4\varepsilon$, we have $\lim_{m\to\infty} g(x+mu)/m = u$. As $m \to \infty$, (2.10) implies that

$$\begin{aligned}\alpha &\le \|g(x)+nu\| - \|nu\| = \|x+(n-1)u\| - \|x+(n-1)u-x\| - \|u\| \\ &\le \|g(x+(n-1)u)\| - \|g(x+(n-1)u)-g(x)\| - \|u\| + 2\varepsilon.\end{aligned}$$

Letting $n \to \infty$ yields $\|u\| \le 2\varepsilon$. ∎

REMARK 2.3. The proof of improving the estimate in (2.9) from $4\varepsilon$ to $2\varepsilon$ is essentially the same as the proof of Šemrl and Väisälä in [11, Theorem 3.2]. We give some details here for completeness and the reader's convenience.

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