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ON WEAKLY LOCALLY UNIFORMLY ROTUND NORMS WHICH ARE NOT LOCALLY UNIFORMLY ROTUND

BY

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Abstract. We show that every infinite-dimensional Banach space with separable dual admits an equivalent norm which is weakly locally uniformly rotund but not locally uniformly rotund.

1. Introduction. Recall that a norm in a Banach space is called *strictly convex* (SC) if for arbitrary points x, y from the unit sphere the equality $\|x+y\| = 2$ implies that $x = y$. The norm is called *weakly locally uniformly rotund* (wLUR) if for any points x_n ($n = 1, 2, \ldots$) and x from the unit sphere the equality $\lim_{n\to\infty} \|x_n + x\| = 2$ implies the weak convergence of the sequence $(x_n)_{n=1}^{\infty}$ to x; if the convergence is strong, then the norm is called *locally uniformly rotund* (LUR). In the preceding definitions it is sufficient to require that $\lim_{n\to\infty} \|x_n\| = \|x\|$ and $\lim_{n\to\infty} \|x_n + x\| = 2\|x\|$.

It is clear that wLUR $\Rightarrow$ SC and LUR $\Rightarrow$ wLUR; it is also well-known that none of these implications reverses. Indeed, the space ℓ_∞ can be renormed in a strictly convex manner, but it does not admit an equivalent wLUR norm (cf. [Di, §4.5]). M. A. Smith [Sm, Example 2] gave an example of a wLUR norm on ℓ_2 which is not LUR; in the next section we shall present a somewhat simpler example (which is a particular case of our main result, but slightly different).

D. Yost [Yo, Theorem 2.1] showed that the implication wLUR $\Rightarrow$ SC does not reverse in the strong sense, namely, every infinite-dimensional separable Banach space admits an equivalent strictly convex norm which is not wLUR. Of course, the analogous theorem does not hold for the implication LUR $\Rightarrow$ wLUR, because of the Schur property, e.g., of the space ℓ_1. However, it is true when assuming that the dual of the underlying space is separable; this is what our main result states:

THEOREM 1. *Every infinite-dimensional Banach space with separable dual admits an equivalent* wLUR *norm which is not* LUR.

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REMARK 2. It is worth mentioning that the class of Banach spaces having a wLUR renorming coincides with the class of Banach spaces having a LUR renorming [MOT, Theorem 1.11]. However, Theorem 1 (and, all the more, Corollary 6) suggests that in a large class of Banach spaces with a wLUR renorming not every wLUR norm is automatically LUR.

2. An example of a wLUR norm which is not LUR. The norm

$$|||x||| = \|x\|_\infty + \Big(\sum_{n=1}^{\infty} 2^{-n}|x(n)|^2\Big)^{1/2} \quad \text{for } x \in c_0, \tag{1}$$

where $\|\cdot\|_\infty$ stands for the standard supremum norm, was given in [MOT, p. 1] as an example of a strictly convex norm which is not LUR. Nonetheless, we shall show that this norm is wLUR.

LEMMA 3. *Suppose that $(x_n)_{n=1}^\infty \subset c_0$ is pointwise convergent to αx, where $\alpha \in [0,\infty)$ and $x \in c_0 \setminus \{0\}$. If*

$$\lim_{n\to\infty} (\|x_n + x\|_\infty - \|x_n\|_\infty) = \|x\|_\infty$$

and the limit $\lim_{n\to\infty} \|x_n\|_\infty$ exists, then $\lim_{n\to\infty} \|x_n\|_\infty = \alpha\|x\|_\infty$.

Proof. We shall show that $(\|x_n\|_\infty)_{n=1}^\infty$ has a subsequence which is convergent to $\alpha\|x\|_\infty$.

Let

$$K = \{k\colon |x(k)| = \|x\|_\infty\};$$

by our assumptions K is a non-empty finite set. Furthermore, if n and k are positive integers such that $k \notin K$, then

$$\begin{aligned}|(x_n + x)(k)| - \|x_n\|_\infty \le |x_n(k)| + |x(k)| - \|x_n\|_\infty &\le |x(k)| \\ &\le \max\{|x(l)|\colon l \notin K\} < \|x\|_\infty.\end{aligned}$$

This means that there is a $k_0 \in K$ such that $|(x_n + x)(k_0)| = \|x_n + x\|_\infty$ for infinitely many n. Let $(n_l)_{l=1}^\infty$ be a strictly increasing sequence of positive integers such that

$$|(x_{n_l} + x)(k_0)| = \|x_{n_l} + x\|_\infty \quad \text{for } l = 1, 2, \ldots.$$

Letting $l \to \infty$ we obtain

$$(1+\alpha)|x(k_0)| = \lim_{l\to\infty} \|x_{n_l} + x\|_\infty = \lim_{l\to\infty} \|x_{n_l}\|_\infty + \|x\|_\infty,$$

which completes the proof. ■

PROPOSITION 4. *The norm given by* (1) *is wLUR.*

Proof. Fix a sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere of $(c_0, |||\cdot|||)$ and a point x from this sphere such that $\lim_{n\to\infty} |||x_n + x||| = 2$. We shall show that each subsequence of $(x_n)_{n=1}^{\infty}$ has a subsequence which is weakly convergent to x. To this end, fix a subsequence of $(x_n)_{n=1}^{\infty}$, still denoted by $(x_n)_{n=1}^{\infty}$.

Set

$$y_n = (2^{-k/2} x_n(k))_{k=1}^{\infty} \quad \text{for } n = 1, 2, \ldots \quad \text{and} \quad y = (2^{-k/2} x(k))_{k=1}^{\infty}.$$

The equality

$$\begin{aligned} 2 - |||x_n + x||| &= |||x_n||| + |||x||| - |||x_n + x||| \\ &= \|x_n\|_\infty + \|x\|_\infty - \|x_n + x\|_\infty + \|y_n\|_2 + \|y\|_2 - \|y_n + y\|_2, \end{aligned}$$

where $\|\cdot\|_2$ stands for the norm in ℓ_2, implies that

$$\lim_{n\to\infty} (\|x_n\|_\infty + \|x\|_\infty - \|x_n + x\|_\infty) = 0, \tag{2}$$

$$\lim_{n\to\infty} (\|y_n\|_2 + \|y\|_2 - \|y_n + y\|_2) = 0. \tag{3}$$

Passing to a further subsequence of $(x_n)_{n=1}^{\infty}$ (still denoted by $(x_n)_{n=1}^{\infty}$) we may assume that $\lim_{n\to\infty} \|x_n\|_\infty$ and $\lim_{n\to\infty} \|y_n\|_2$ exist. Using (3) we obtain

$$\lim_{n\to\infty} (\|y_n\|_2 + \|y\|_2)^2 = \lim_{n\to\infty} \|y_n + y\|_2^2 = \lim_{n\to\infty} (\|y_n\|_2^2 + 2(y_n|y) + \|y\|_2^2),$$

where $(\cdot|\cdot)$ stands for the real inner product. Hence

$$\lim_{n\to\infty} (y_n|y) = \lim_{n\to\infty} \|y_n\|_2 \cdot \|y\|_2 = \alpha \|y\|_2^2,$$

where $\alpha = \lim_{n\to\infty} \|y_n\|_2 / \|y\|_2$. Thus

$$\begin{aligned} \lim_{n\to\infty} \|y_n - \alpha y\|_2^2 &= \lim_{n\to\infty} (\|y_n\|_2^2 - 2\alpha(y_n|y) + \alpha^2 \|y\|_2^2) \\ &= \alpha^2 \|y\|_2^2 - 2\alpha^2 \|y\|_2^2 + \alpha^2 \|y\|_2^2 = 0, \end{aligned}$$

which means that $(y_n)_{n=1}^{\infty}$ converges (in ℓ_2) to αy. In particular, $(y_n)_{n=1}^{\infty}$ is pointwise convergent to αy, and therefore $(x_n)_{n=1}^{\infty}$ is pointwise convergent to αx. By (2) and Lemma 3, $\lim_{n\to\infty} \|x_n\|_\infty = \alpha \|x\|_\infty$. Therefore

$$\begin{aligned} 1 = \lim_{n\to\infty} |||x_n||| &= \lim_{n\to\infty} \|x_n\|_\infty + \lim_{n\to\infty} \|y_n\|_2 \\ &= \alpha \|x\|_\infty + \alpha \|y\|_2 = \alpha |||x||| = \alpha. \end{aligned}$$

Finally, $(x_n)_{n=1}^{\infty}$ is weakly convergent to x as it is bounded and converges pointwise to this point. ∎

3. The proof of the main result. Throughout this section X denotes an infinite-dimensional Banach space. We shall need a simple lemma about weak convergence (the trivial proof is omitted).

LEMMA 5. *Assume that* $(x_n)_{n=1}^\infty$ *is a bounded sequence in* X, Γ *is a set and* $\{x_\gamma^*\colon \gamma\in\Gamma\}\subset X^*$. *If* $\operatorname{span}\{x_\gamma^*\colon \gamma\in\Gamma\}$ *is dense in* X^* *and*

$$\lim_{n\to\infty} x_\gamma^*(x_n)=0 \quad \textit{for each } \gamma\in\Gamma,$$

then $(x_n)_{n=1}^\infty$ *is weakly null.*

Proof of Theorem 1. Assume that X^* is separable. According to a result of A. Pełczyński [Pe, Remark A] there exists an M-basis $(e_n,e_n^*)_{n=1}^\infty$ of X which is both bounded and shrinking. This means that

$$\sup\{\|e_n\|\cdot\|e_n^*\|\colon n=1,2\ldots\}<\infty$$

and the functionals e_n^* are linearly dense in X^*.

Without loss of generality we may assume that $\|e_n\|=1$ for $n=1,2,\ldots.$ Define a functional $\|\cdot\|_0\colon X\to[0,\infty)$ by

$$\|x\|_0=\max\bigl\{\tfrac12\|x\|,\sup_n|e_n^*(x)|\bigr\}\quad \text{for } x\in X.$$

One can easily see that $\|\cdot\|_0$ is a norm on X and by the boundedness of the M-basis $(e_n,e_n^*)_{n=1}^\infty$ this norm is equivalent to the original one.

Define a functional $|||\cdot|||\colon X\to[0,\infty)$ by

$$|||x|||^2=\|x\|_0^2+\sum_{n=1}^\infty 4^{-n}|e_n^*(x)|^2\quad \text{for } x\in X.$$

One can easily observe that $|||\cdot|||$ is an equivalent norm on X. We shall show that it is wLUR but not LUR.

To prove the first assertion, consider a sequence $(x_n)_{n=1}^\infty$ and a point x in the unit sphere of $(X,|||\cdot|||)$ such that $\lim_{n\to\infty}|||x_n+x|||=2$. Set

$$y_n=(\|x_n\|_0,2^{-1}e_1^*(x_n),2^{-2}e_2^*(x_n),\ldots)\quad \text{for } n=1,2,\ldots,$$
$$y=(\|x\|_0,2^{-1}e_1^*(x),2^{-2}e_2^*(x),\ldots).$$

We have

$$\|y_n+y\|_2^2=(\|x_n\|_0+\|x\|_0)^2+\sum_{m=1}^\infty 4^{-m}|e_m^*(x_n+x)|^2$$
$$\ge \|x_n+x\|_0^2+\sum_{m=1}^\infty 4^{-m}|e_m^*(x_n+x)|^2=|||x_n+x|||^2\xrightarrow[n\to\infty]{}4,$$

and by the local uniform rotundity of the norm in the (Hilbert) space ℓ_2, we obtain $\lim_{n\to\infty}\|y_n-y\|_2=0$. In particular,

$$\lim_{n\to\infty}e_m^*(x_n)=e_m^*(x)\quad \text{for } m=1,2,\ldots.$$

Lemma 5 and the fact that the M-basis $(e_n,e_n^*)_{n=1}^\infty$ is shrinking give the weak convergence of the sequence $(x_n)_{n=1}^\infty$ to x.

To see that the norm $|||\cdot|||$ is not LUR consider the sequence $(e_1+e_n)_{n=1}^{\infty}$ and the point e_1. One can easily verify that

$$\lim_{n\to\infty} |||e_1+e_n||| = \tfrac{1}{2}\sqrt{5} = |||e_1|||$$

and

$$\lim_{n\to\infty} |||2e_1+e_n||| = \sqrt{5},$$

while $|||e_n||| \geq 1$ for $n = 1, 2, \ldots$. ∎

COROLLARY 6. *Every Banach space which admits an equivalent* LUR *norm, in particular every separable Banach space, and has an infinite-dimensional subspace with separable dual, admits an equivalent w*LUR *norm which is not* LUR.

Proof. Suppose that Y is an infinite-dimensional subspace of X with separable dual. By Theorem 1 the space Y admits an equivalent wLUR norm which is not LUR. According to Tang's Theorem [Ta, Theorem 1.1] it extends to an equivalent wLUR norm on the whole X. Obviously, this extension fails to be LUR. ∎

REMARK 7. The statement of Tang's Theorem does not include the case of wLUR norm literally, but the theorem is also valid in this case (cf. [Ta, Remark 1.1]). Indeed, one can easily verify that the proof works without major changes.

REMARK 8. Corollary 6 implies that every Banach space which admits an equivalent LUR norm, in particular every separable Banach space, and enjoys the Schur property, has no infinite-dimensional subspace with separable dual. Of course, it is not a new result, as it is well-known that every Banach space having the Schur property is ℓ_1-saturated. However, this fact follows from Rosenthal's ℓ_1-Theorem (cf. [AK, §10.2]), so its proof is much less elementary than the one given in this paper.

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