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CLASSICAL SOLUTIONS TO THE SCALAR CONSERVATION LAW WITH DISCONTINUOUS INITIAL DATA

BY

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Abstract. Sufficient and necessary conditions for the existence and uniqueness of classical solutions to the Cauchy problem for the scalar conservation law are found in the class of discontinuous initial data and non-convex flux function. Regularity of rarefaction waves starting from discontinuous initial data and their dependence on the flux function are investigated and illustrated in a few examples.

1. Introduction. This paper deals with sufficient and necessary conditions for the existence and uniqueness of classical solutions to the Cauchy problem for the scalar conservation law

$$\begin{cases} u_t + (f(u))_x = 0 & \text{on } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \mathbb{R} \times (0, \infty)$. In contrast to most works on this topic we make no assumptions on the convexity of f or the entropy of u, but demand only that $f \in C^2(\mathbb{R})$; moreover, we admit discontinuous initial data. As a consequence, we find a new class of C^1 regular rarefaction waves starting from discontinuous initial data.

More precisely, by a *classical solution* we mean any function

$$u \in C^1(\Omega) \cap C^0(\Omega \cup (\operatorname{cont} u_0 \times (0, \infty)))$$

satisfying (1.1), $\operatorname{cont} u_0$ denoting the set of points where u_0 is continuous.

We say a function $v : \mathbb{R} \to \mathbb{R}$ belongs to the class $C^{1^\infty}(\mathbb{R})$ if for every $x \in \operatorname{cont} v$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in \operatorname{cont} v$ and $x_n \to x$ the limit (possibly infinite)

$$\lim_{n\to\infty} \frac{v(x) - v(x_n)}{x - x_n},$$

denoted by $v'(x)$, exists, the function $v' : \operatorname{cont} v \to \overline{\mathbb{R}}$ is continuous, and for every $\overline{x} \notin \operatorname{cont} v$,

$$\lim_{\operatorname{cont} v \ni x_n \to \overline{x}} |v'(x_n)| = \infty.$$

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The class of C^{1^∞} functions is discussed in Examples 3.2 and 3.3.

The main result of this work reads as follows.

THEOREM 1.1. *Suppose that* $\mathrm{cl}(\mathrm{cont}\, u_0) = \mathbb{R}$ *and* u_0 *has no removable discontinuities. Equation* (1.1) *has a unique classical solution if and only if the following conditions are satisfied:*

(1) *The superposition* $f' \circ u_0$ *is non-decreasing on* $\mathrm{cont}\, u_0$.
(2) *For every* $x \in \mathbb{R}$ *the limits*

$$\lim_{\mathrm{cont}\, u_0 \ni x_n \to x^-} u_0(x_n) \quad \text{and} \quad \lim_{\mathrm{cont}\, u_0 \ni x_n \to x^+} u_0(x_n),$$

denoted by $u_0(x^-)$ *and* $u_0(x^+)$ *respectively, both exist.*
(3) *The function* u_0 *does not jump through inflection points of* f *in the following sense: if* $0 \in f''([u_0(x^-), u_0(x^+)])$ *then* $x \in \mathrm{cont}\, u_0$.
(4) $u_0 \in C^{1^\infty}(\mathbb{R})$.
(5) *If* $f''(u_0(x)) = 0$ *for some* $x \in \mathrm{cont}\, u_0$ *then* u_0 *is differentiable at* x.

Notice that, when f is strictly convex, conditions (3) and (5) are trivially satisfied for all u_0, and condition (1) reduces to u_0 being a non-decreasing function.

In [2], Smoller discusses classical rarefaction wave for the inviscid Burgers equation ($f(u) = u^2/2$), which solves the Riemann problem with initial data

$$u_0(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

The solution is not determined on $\{(x,t) : |x| < t\}$, and cannot be C^1 regularly extended to this region. However, there exists a solution u in $C^0(\Omega)$ given by

$$u(x,t) = \begin{cases} 1 & \text{if } x \in [t, \infty), \\ x/t & \text{if } x \in (-t, t), \\ -1 & \text{if } x \in (-\infty, t], \end{cases}$$

which is a locally Lipschitz function. In Example 3.5 we present C^{1^∞}-regularization of this problem, while in Example 3.4 we discuss a C^1 rarefaction wave starting from the following discontinuous initial data

$$u_0(x) = \begin{cases} \sqrt{|x|} + 1 & \text{if } x \geq 0, \\ -(\sqrt{|x|} + 1) & \text{if } x < 0, \end{cases}$$

which satisfies (1)–(5).

Conservation laws with non-convex fluxes are discussed in [1]; however, there are several differences between LeFloch's results and those presented in this paper. Firstly, and most importantly, LeFloch discusses weak entropy solutions, while here C^1 solutions are considered. Furthermore, in this paper a solution is constructed, whereas in [1] the existence is shown by convergence methods. For technical reasons, LeFloch had to assume that the set of inflection points of the flux function is finite; we do not need this assumption.

2. Existence and uniqueness. We write $\{\Phi(x,t)\}$ for $\{(x,t) : \Phi(x,t)\}$, where Φ is a condition depending on x, t.

REMARK 2.1. Note that the ordinary differential equation

$$\dot{\mathbf{x}} = \begin{bmatrix} f'(u(\mathbf{x})) \\ 1 \end{bmatrix}$$

defines a family of characteristics, given by $\{x = f'(u_0(x_0))t + x_0\}$.

DEFINITION 2.2. Let $\overline{x}$ be a point of discontinuity of u_0. Then the *rarefaction cone* $C_{\overline{x}}$ is defined as

$$C_{\overline{x}} = \{\overline{x} + f'(u_0(\overline{x}^-))t < x < \overline{x} + f'(u_0(\overline{x}^+))t\}.$$

Analogously, the *closed rarefaction cone* $\overline{C_{\overline{x}}}$ is defined as

$$\overline{C_{\overline{x}}} = \{\overline{x} + f'(u_0(\overline{x}^-))t \le x \le \overline{x} + f'(u_0(\overline{x}^+))t\}.$$

DEFINITION 2.3. A function $P : \Omega \to \mathbb{R}$ is called a u_0*-projection* if for every $\mathbf{x} \in \Omega$ the point $(P(\mathbf{x}), 0) \in \partial\Omega$ is either a point of continuity of u_0 such that $\mathbf{x}$ lies on the characteristic containing it, or a point of discontinuity such that $\mathbf{x}$ is contained in its closed rarefaction cone.

PROPOSITION 2.4. *If* $\operatorname{cl}(\operatorname{cont} u_0) = \mathbb{R}$ *and* $f' \circ u_0$ *is non-decreasing, then there is a unique* u_0*-projection* P.

Proof. For any $(x,t) \in \Omega$ we can find real numbers a_0 and b_0 such that the characteristic containing $(a_0, 0)$ lies to the left of (x,t), and the characteristic containing $(b_0, 0)$ lies to the right of it.

Suppose we have already defined numbers $a_0 \le a_1 \le \cdots \le a_i$ and $b_i \le \cdots \le b_1 \le b_0$. From $\operatorname{cl}(\operatorname{cont} u_0) = \mathbb{R}$ we see that the ball $B\big(\frac{a_i+b_i}{2}, \frac{b_i-a_i}{6}\big)$ contains a point of continuity of u_0, say χ. Consider the characteristic containing $(\chi, 0)$. If (x,t) lies to the right of it, we define $a_{i+1} = \chi$ and $b_{i+1} = b_i$; otherwise we set $a_{i+1} = a_i$, $b_{i+1} = \chi$. The intersection of the family $([a_i, b_i])_{i\in\mathbb{N}}$ is a singleton $\{\overline{x}\}$, for $[a_i, b_i] \subset [a_{i-1}, b_{i-1}]$ and $\operatorname{diam}([a_i, b_i]) \le (2/3)^i(b_0 - a_0)$, so $\lim_{i\to\infty} \operatorname{diam}([a_i, b_i]) = 0$.

If $\overline{x} \in \operatorname{cont} u_0$ then (x,t) lies on the characteristic containing $(\overline{x}, 0)$, and therefore $P(x,t)$ is uniquely defined. Otherwise $\overline{C_{\overline{x}}}$ contains (x,t) and as the family $\{\overline{C_x}\}_{x\notin \operatorname{cont} u_0}$ is disjoint it follows that $P(x,t)$ is also uniquely defined. ∎

LEMMA 2.5. *If* $\operatorname{cl}(\operatorname{cont} u_0) = \mathbb{R}$ *and* $f' \circ u_0$ *is non-decreasing then the values of* u *and* $f' \circ u$ *are uniquely determined on, respectively,* $\Omega \setminus \bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}}$

and $\bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}}$. *Moreover*

$$\begin{cases} u(x_0,t_0) = u_0(P(x_0,t_0)) & \text{if } (x_0,t_0) \in \Omega \setminus \bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}},\\ f'(u(x_0,t_0)) = \frac{x_0 - P(x_0,t_0)}{t_0} & \text{if } (x_0,t_0) \in \bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}}. \end{cases}$$

Proof. Firstly, let $x_0 = P(\mathbf{x})$ for some $\mathbf{x} \in \Omega$, and suppose $x_0 \in \operatorname{cont} u_0$. The point $\mathbf{x}$ lies on the characteristic containing $(x_0, 0)$, so $u(\mathbf{x}) = u_0(x_0)$. The solution u is therefore determined on $\Omega \setminus \bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}}$.

Let now $x_0 = P(\mathbf{x})$ for some $\mathbf{x} \in \bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}}$. The point $\mathbf{x}$ lies between the lines $L^- = \{x = f'(u_0(x^-))t + \overline{x}\}$ and $L^+ = \{x = f'(u_0(x^+))t + \overline{x}\}$. Suppose for a moment that the solution u is given on $\Gamma = \{t = t_0\} \cap \overline{C_{\overline{x}}}$. For any $\gamma \in \Gamma$ the characteristic containing it does not intersect the lines L^- and L^+ in Ω—otherwise it would intersect a characteristic starting from some point $\tilde{x} \in \operatorname{cont} u_0$ close to $\overline{x}$. On the other hand, every characteristic starting from Γ has to intersect $\partial C_{\overline{x}} = L^- \cup L^+$. Therefore all characteristics intersect in the vertex $(\mathbb{R}^2 \setminus \Omega) \cap C_{\overline{x}} = (\overline{x}, 0)$ and $f' \circ u$ is linear with given values on $\partial\Gamma$:

$$\begin{aligned} f'(u(f'(u_0(x^-))t + \overline{x}, t)) &= f'(u_0(x^-)),\\ f'(u(f'(u_0(x^+))t + \overline{x}, t)) &= f'(u_0(x^+)). \end{aligned}$$

Hence,

$$f'(u(f'(u_0(x^-))t + \overline{x} + p, t)) = f'(u_0(x^-)) + p/t$$

for any $p \in [0, f'(u_0(x^+))t - f'(u_0(x^-))t]$. By putting $x_0 = f'(u_0(x^-))t_0 + \overline{x} + p$ we obtain $p = x_0 - \overline{x} - f'(u_0(x^-))t_0$, and therefore

$$f'(u(x_0,t_0)) = \frac{x_0 - \overline{x}}{t_0}. \ \blacksquare$$

LEMMA 2.6. *If u is a solution to the scalar conservation law with initial data u_0 (either in the classical sense, or in terms of Lemma* 2.5), *and if* $\operatorname{cl}(\operatorname{cont} u_0) = \mathbb{R}$ *and* $x \in \operatorname{cont} u_0$, *then*

$$\begin{aligned} u_x(f'(u_0(x))t + x, t) &= \frac{1}{f''(u_0(x))t + \left(\lim_{\operatorname{cont} u_0 \ni x_n \to x} \frac{u_0(x) - u_0(x_n)}{x - x_n}\right)^{-1}},\\ u_t(f'(u_0(x))t + x, t) &= \frac{-f'(u_0(x))}{f''(u_0(x))t + \left(\lim_{\operatorname{cont} u_0 \ni x_n \to x} \frac{u_0(x) - u_0(x_n)}{x - x_n}\right)^{-1}}. \end{aligned}$$

Proof. Let $x_n \in \operatorname{cont} u_0$ be a sequence of points such that $x_n \to x$. The partial derivative of u in direction x at $(f'(u_0(x))t + x, t)$ is given by

$$\begin{aligned} u_x(f'(u_0(x))t+x,t) &= \lim_{n\to\infty} \frac{u(f'(u_0(x))t+x,t)-u(f'(u_0(x_n))t+x_n,t)}{[f'(u_0(x))t+x]-[f'(u_0(x_n))t+x_n]} \\ &= \lim_{n\to\infty} \frac{u_0(x)-u_0(x_n)}{(f'\circ u_0(x)-f'\circ u_0(x_n))t+(x-x_n)} \\ &= \frac{1}{f''(u_0(x))t+\left(\lim_{n\to\infty}\frac{u_0(x)-u_0(x_n)}{x-x_n}\right)^{-1}}. \end{aligned}$$

Similarly,

$$u_t(f'(u_0(x))t+x,t) = \lim_{x_n\to x} \frac{u(f'(u_0(x))t+x,t)-u(f'(u_0(x_n))t_n+x_n,t_n)}{t-t_n}$$

for $f'(u_0(x_n))t_n+x_n = f'(u_0(x))t+x$. Thus,

$$\begin{aligned} u_t(f'(u_0(x))t+x,t) &= \lim_{x_n\to x} \frac{f'(u_0(x_n))(u_0(x)-u_0(x_n))}{f'(u_0(x_n))t-f'(u_0(x))t-(x-x_n)} \\ &= \frac{-f'(u_0(x))}{f''(u_0(x))t+\left(\lim_{n\to\infty}\frac{u_0(x)-u_0(x_n)}{x-x_n}\right)^{-1}}. \ \blacksquare \end{aligned}$$

LEMMA 2.7. *If* $\mathrm{cl}(\mathrm{cont}\, u_0) = \mathbb{R}$ *and conditions* (1)–(3) *hold, then for every open interval* U *such that* $x \in U \cap \mathrm{cont}\, u_0 \Rightarrow f''(u_0(x)) < 0$ (*resp.* $f''(u_0(x)) > 0$) *the function* $u_0|_{U\cap \mathrm{cont}\, u_0}$ *is non-decreasing* (*resp. non-increasing*).

Proof. The case of $f''(u_0(x_0)) < 0$ can be handled analogously to $f''(u_0(x_0)) > 0$, so we can consider only the latter case. Suppose, to the contrary, that there exist points $a_0 < b_0$ such that $u_0(a_0) > u_0(b_0)$. The interval $[a_0, b_0]$ can then be divided by a point $\chi \in B\left(\frac{a_0+b_0}{2}, \frac{b_0-a_0}{6}\right)$ so that $\chi \in \mathrm{cont}\, u_0$ and $u_0(a_0) > u_0(\chi)$ or $u_0(\chi) > u(b_0)$. In the former case we put $a_1 := a_0$, $b_1 := \chi$, in the latter $a_1 := \chi$, $b_1 := b_0$. We continue constructing the points $\{a_i\}_{i\in\mathbb{N}}$, and $\{b_i\}_{i\in\mathbb{N}}$ with the same algorithm. Note that $u_0(a_i) > u_0(b_i)$ for every i, and $\{[a_i,b_i]\}_{i\in\mathbb{N}}$ is a descending family of sets. Let $\{\overline{x}\} = \bigcap_{i\in\mathbb{N}}[a_i, b_i]$.

If $\overline{x} \in \mathrm{cont}\, u_0$ then the continuity of f'' implies that there exists a neighborhood $V_\varepsilon = \{\tilde{u} : |\tilde{u} - u_0(\overline{x})| < \varepsilon\}$ of $u_0(\overline{x})$ such that $f''(\tilde{u}) > 0$ for all $\tilde{u} \in V_\varepsilon$, and a neighborhood $U_x \ni \overline{x}$ such that $u_0(U_{\overline{x}}) \subseteq V_\varepsilon$. For sufficiently large n we have $a_n, b_n \in U_{\overline{x}}$, which contradicts condition (1).

Suppose $\overline{x} \notin \mathrm{cont}\, u_0$. Let $x_n \in \mathrm{cont}\, u_0$ be a sequence converging to $\overline{x}$ from the left. From $f''(u_0(x_n)) > 0$ it follows that $f''(u_0(\overline{x}^-)) > 0$ (see condition (3)). The function f' is therefore increasing on some open interval containing $[u_0(\overline{x}^-), u_0(\overline{x}^+)]$. The inequality $u_0(\overline{x}^-) > u_0(\overline{x}^+)$ cannot hold as $f' \circ u_0$ is non-decreasing. The inequality $u_0(\overline{x}^-) < u_0(\overline{x}^+)$ implies existence of n such that $u_0(a_n) < u_0(b_n)$, which is a contradiction. Thus we must have $u_0(\overline{x}^-) = u_0(\overline{x}^+)$, and we can assume $\overline{x} \in \mathrm{cont}\, u_0$ by putting $u_0(\overline{x}) = u_0(\overline{x}^+)$. $\blacksquare$

Proof of Theorem 1.1. First we show that conditions (1)–(5) are necessary. Suppose $\mathrm{cl}(\mathrm{cont}\, u_0) = \mathbb{R}$ and u is a classical solution to the scalar conservation law with initial data u_0. Moreover u_0 has no removable discontinuities (which implies $C_{\overline{x}} \neq \emptyset$ for every $\overline{x} \notin \mathrm{cont}\, u_0$).

CLAIM 0. *The function $x \mapsto f'(u(x,t))$ is non-decreasing on* $\mathrm{cont}\, u(\cdot,t)$ *for every $t \geq 0$.*

Suppose, to the contrary, that there exist $t_0 \geq 0$, $x_0, x_1 \in \mathrm{cont}\, u(\cdot, t_0)$ such that $x_0 < x_1$ and $f'(u(x_0,t_0)) > f'(u(x_1,t_0))$. The characteristics containing the points (x_0,t_0) and (x_1,t_0) are given by

$$\mathbf{x}_0(s) = (f'(u(x_0,t_0))(s-t_0)+x_0, s),$$
$$\mathbf{x}_1(s) = (f'(u(x_1,t_0))(s-t_0)+x_1, s).$$

Set

$$t_1 = \frac{x_1 - x_0}{f'(u(x_0,t_0)) - f'(u(x_1,t_0))} + t_0.$$

The inequalities on t_0, x_1, x_2 imply $t_1 > 0$. Thus,

$$\mathbf{x}_0(t_1) = \mathbf{x}_1(t_1) = \left(\frac{f'(u(x_0,t_0))x_1 - f'(u(x_1,t_0))x_0}{f'(u(x_0,t_0)) - f'(u(x_1,t_0))}, t_1 \right).$$

Since $u(x_0,t_0) \neq u(x_1,t_0)$ the function u cannot be defined at $\mathbf{x}_0(t_1)$.

CLAIM 1. *The limits $u_0(\overline{x}^-)$ and $u_0(\overline{x}^+)$ exist for every $\overline{x} \notin$* $\mathrm{cont}\, u_0$.

The limits $f'(u_0(\overline{x}^-))$ and $f'(u_0(\overline{x}^+))$ exist, as $f' \circ u$ is non-decreasing. Let x_n be a sequence of points in $\mathrm{cont}\, u_0$ converging to $\overline{x}$ from one side (it exists as $\mathrm{cl}(\mathrm{cont}\, u_0) = \mathbb{R}$). For every $t > 0$ the limit of $f'(u_0(x_n))t + x_n$ exists. Hence, the continuity of u implies the existence of the limit

$$\lim_{x_n \to \overline{x}} u(f'(u_0(x_n))t + x_n, t) = \lim_{x_n \to \overline{x}} u_0(x_n).$$

For given $\overline{x} \notin \mathrm{cont}\, u_0$ there is no $\theta \in [u_0(\overline{x}^-), u_0(\overline{x}^+)]$ such that $f''(\theta) = 0$.

Consider $f' \circ u$ on $\Gamma = \overline{C_{\overline{x}}} \cap \{t = t_0\}$. As $u(\cdot, t_0)$ is continuous and $u(f'(u_0(\overline{x}^-))t_0 + \overline{x}, t_0) = u_0(\overline{x}^-)$, $u(f'(u_0(\overline{x}^+))t_0 + \overline{x}, t_0) = u_0(\overline{x}^+)$, every value from $[u_0(\overline{x}^-), u_0(\overline{x}^+)]$ is in the image of Γ under u. If there existed θ in this interval such that $f''(\theta) = 0$, then also

$$\frac{d}{dx} f' \circ u(\eta, t_0) = f''(u(\eta,t)) \cdot u_x(\eta,t) = 0$$

for $\theta = u(\eta, t_0)$. However, from Lemma 2.5 we obtain $\frac{d}{dx} f' \circ u(x,t) = 1/t_0$ on Γ, which contradicts $f''(\theta) = 0$.

CLAIM 2. *The function u_0 belongs to $C^{1^\infty}(\mathbb{R})$.*

By Lemma 2.6, for every sequence $x_n \in \operatorname{cont} u_0$ converging to $x \in \operatorname{cont} u_0$, we have

$$u_x(f'(u_0(x))t + x, t) = \frac{1}{f''(u_0(x))t + \left(\lim_{x_n \to x} \frac{u_0(x) - u_0(x_n)}{x - x_n}\right)^{-1}}.$$

The limit in the denominator must exist for the value to be well defined. If we denote it by $u_0'(x)$, then u_0' is a function from $\mathbb{R}$ to $\overline{\mathbb{R}}$. For every $t > 0$,

$$u_0'(x) = \frac{u_x(f'(u_0(x))t + x, t)}{1 - f''(u_0(x)) \cdot u_x(f'(u_0(x))t + x, t) \cdot t}. \tag{2.1}$$

Thus $\operatorname{cont} u_0' \supseteq D$, where

$$D = \left\{ x \in \mathbb{R} : u_x(f'(u_0(x))t + x, t) \neq \frac{1}{f''(u_0(x)) \cdot t} \right\}.$$

On every sequence x_n converging to a point $x \in \operatorname{cont} u_0 \setminus D$ the function $(u_0')^{-1}$ converges to 0. If $x \in \operatorname{cont} u_0 \setminus D$, then $f''(u_0(x)) \neq 0$, and by Lemma 2.7 there exists a neighborhood $U_\varepsilon \ni x$ on which u_0 is monotone. Therefore the derivative u_0' has a constant sign on U_ε, and so $u_0'(x_n)$ must converge to either $+\infty$ or $-\infty$. Thus, $\operatorname{cont} u_0' = \operatorname{cont} u_0$.

Let $\overline{x} \notin \operatorname{cont} u_0$. From Lemma 2.5 it follows that for every $(x,t) \in C_{\overline{x}}$,

$$\frac{d}{dx} f' \circ u(x,t) = f''(u(x,t)) \cdot u_x(x,t) = \frac{1}{t}.$$

Let now x_n be a sequence converging to $\overline{x}$ such that $x_n \in \operatorname{cont} u_0$. Then

$$f''(u(u_0(x_n)t + x_n, t)) u_x(f'(u_0(x_n)t + x_n, t) \to \frac{1}{t}.$$

On account of Lemma 2.6 we obtain

$$\lim_{n\to\infty} \frac{f''(u_0(x_n))}{f''(u_0(x_n))t + (u_0'(x_n))^{-1}} = \frac{1}{t}$$

for every t. Thus, $|u_0'(x_n)| \to \infty$.

Claim 3. *If $x \in \operatorname{cont} u_0$ and $f''(u_0(x)) = 0$ then u_0 is differentiable at x.*

If $f''(u_0(x)) = 0$ then from (2.1) we get $|u_0'(x)| = |u_x(f'(u_0(x))t + x, t)| < \infty$.

This completes the proof of the necessity of conditions (1)–(5). Now we shall show that these conditions imply the existence of a classical solution and its uniqueness in $C^0(\Omega \cup \operatorname{cont} u_0)$.

Step 4. Set

$$u(\mathbf{x}) = \begin{cases} u_0(P(\mathbf{x})) & \text{if } \mathbf{x} \in \Omega \setminus \bigcup_{x \notin \operatorname{cont} u_0} C_x, \\ (f')^{-1}\left(\dfrac{x - P(\mathbf{x})}{t}\right) & \text{if } \mathbf{x} = (x,t) \in \bigcup_{x \notin \operatorname{cont} u_0} \overline{C_x}. \end{cases} \tag{2.2}$$

The function $(f')^{-1}$ is well defined, because conditions (2) and (3) guarantee that $f' : [u_0(x^-), u_0(x^+)] \to [f'(u_0(x^-)), f'(u_0(x^+))]$ is a diffeomorphism and condition (1) implies that the u_0-projection P is well defined.

Claim 5. *We have* $u \in C^1(\bigcup_{x\notin \operatorname{cont} u_0} \overline{C_x}) \cap C^0(\Omega \setminus \bigcup_{x\notin \operatorname{cont} u_0} C_x)$.

Let $(x_0, t_0) \in \Omega\setminus\bigcup_{x\notin \operatorname{cont} u_0} \overline{C_x}$ and $(x_n, t_n) \to (x_0, t_0)$. Obviously $P(x_0, t_0) \in \operatorname{cont} u_0$, and therefore $P(x_n, t_n) \to P(x_0, t_0)$. Indeed, for sufficiently small δ, we have $B_2((x_0, t_0), \delta) \subset P^{-1}(B_1(P(x_0, t_0), \delta))$.

From now on, let $(\widetilde{x}_n, \tilde{t}_n)$ be the subsequence of (x_n, t_n) that contains all elements satisfying $(x_n, t_n) \in \Omega \setminus \bigcup_{x\notin \operatorname{cont} u_0} \overline{C_x}$, and let $(\overline{x}_n, \overline{t}_n)$ be the remaining subsequence. Thus,

$$u(\tilde{x}_n, \tilde{t}_n) = u_0(P(\tilde{x}_n, \tilde{t}_n)) \to u_0(P(x_0, t_0)) = u(x_0, t_0)$$

and

$$u(\overline{x}_n, \overline{t}_n) = (f')^{-1}\left(\frac{\overline{x}_n - P(\overline{x}_n, \overline{t}_n)}{\overline{t}_n}\right).$$

Condition (2) implies that

$$\lim_{n\to\infty} u_0(P(\overline{x}_n, \overline{t}_n)^-) = \lim_{n\to\infty} u_0(P(\overline{x}_n, \overline{t}_n)^+) = u(x_0, t_0),$$

hence

$$\lim_{n\to\infty} (\overline{x}_n - P(\overline{x}_n, \overline{t}_n)) = f'(u_0(x_0, t_0))t_0,$$

and therefore $u(\overline{x}_n, \overline{t}_n) \to u(x_0, t_0)$.

Regularity of $(f')^{-1}$ implies C^1-regularity of u on $\bigcup_{x\notin \operatorname{cont} u_0} C_x$.

Claim 6. *If* $v \in C^0(\Omega \cup \operatorname{cont} u_0)$ *is constant along characteristics and* $u|_{\{t=0\}} \equiv v|_{\{t=0\}}$ *then* $u \equiv v$.

On account of Lemma 2.5 we only need to show that there is a unique $\tilde{u} \in C^0(\overline{C_{\overline{x}}} \cap \{t = t_0\})$ such that

$$\begin{cases} f'(\tilde{u}(x, t_0)) = (x - \overline{x})/t_0 \quad \text{on } \overline{C_{\overline{x}}} \cap \{t = t_0\}, \\ \tilde{u}(f'(u_0(\overline{x}^-))t + \overline{x}, t_0) = u_0(\overline{x}^-), \\ \tilde{u}(f'(u_0(\overline{x}^+))t + \overline{x}, t_0) = u_0(\overline{x}^+). \end{cases}$$

From the Darboux theorem, $[u_0(\overline{x}^-), u_0(\overline{x}^+)] \subseteq \tilde{u}\left(\overline{C_{\overline{x}}} \cap \{t = t_0\}\right)$. Suppose, to the contrary, that there exists $(\theta, t_0) \in \overline{C_{\overline{x}}} \cap \{t = t_0\}$ such that $\tilde{u}(\theta) \notin [u_0(\overline{x}^-), u_0(\overline{x}^+)]$. Then there exist $a < \theta < b$ in $\overline{C_{\overline{x}}} \cap \{t = t_0\}$ which satisfy

$$\tilde{u}(a, t_0) = \tilde{u}(b, t_0).$$

However, the equality $f'(\tilde{u}(a, t_0)) = f'(\tilde{u}(b, t_0))$ contradicts $f'(\tilde{u}(x, t_0)) = (x - \overline{x})t_0^{-1}$. Thus, $[u_0(\overline{x}^-), u_0(\overline{x}^+)] = \tilde{u}(\overline{C_{\overline{x}}} \cap \{t = t_0\})$ and hence $\tilde{u}(x, t_0) = (f')^{-1}((x - \overline{x})t_0^{-1})$.

Claim 7. *For every* $\overline{x} \notin \operatorname{cont} u_0$ *and* $(x_0, t_0) \in \partial C_{\overline{x}}$ *we have* $\nabla u(x_n, t_n) \to \nabla u(x_0, t_0)$ *whenever* $(x_n, t_n) \to (x_0, t_0)$.

Let us consider the case $(x_0,t_0)\in\{x=f'(u_0(\overline{x}^-))t+\overline{x}\}$. By (2.2), for a subsequence of (x_n,t_n) contained in $C_{\overline{x}}$, we have

$$\nabla u(x_{n_k},t_{n_k})\to\left[\frac{1}{f''(u_0(\overline{x}^-))t_0},\frac{-f'(u_0(\overline{x}^-))}{f''(u_0(\overline{x}^-))t_0}\right].$$

Let now $\widetilde{\mathbf{x}}_n=(\tilde{x}_n,\tilde{t}_n)$ and let $\overline{\mathbf{x}}_n=(\overline{x}_n,\overline{t}_n)$ be composed of only those elements that are not in $C_{\overline{x}}$. From Lemma 2.6 it follows that

$$\begin{aligned}&u_x(\tilde{x}_n,\tilde{t}_n)=[f''(u_0(P(\tilde{x}_n,\tilde{t}_n))t+u_0'(P(\tilde{x}_n,\tilde{t}_n))^{-1}]^{-1},\\&u_t(\tilde{x}_n,\tilde{t}_n)=-f'(u_0(P(\tilde{x}_n,\tilde{t}_n))[f''(u_0(P(\tilde{x}_n,\tilde{t}_n))t+u_0'(P(\tilde{x}_n,\tilde{t}_n))^{-1}]^{-1}.\end{aligned}$$

Condition (4) guarantees $u_0'(P(\tilde{x}_n,\tilde{t}_n))^{-1}\to 0$, so the desired convergence holds on $(\tilde{x}_n,\tilde{t}_n)$. On the other hand, from (2.2), we get

$$\nabla u(\overline{x}_n,\overline{t}_n)=\left[\frac{1}{f''(u(\overline{x}_n,\overline{t}_n))\overline{t}_n},-\frac{\overline{x}_n-P(\overline{x}_n,\overline{t}_n)}{f''(u(\overline{x}_n,\overline{t}_n))\overline{t}_n^2}\right],$$

and $\lim_{n\to\infty}\overline{x}_n-P(\overline{x}_n,\overline{t}_n)=f'(u_0(\overline{x}^-))t_0$.

Claim 8. *For every* $(x_0,t_0)\in\Omega\setminus\bigcup_{\overline{x}\notin\operatorname{cont}u_0}\overline{C_{\overline{x}}}$ *we have* $\nabla u(x_n,t_n)\to\nabla u(x_0,t_0)$ *whenever* $(x_n,t_n)\to(x_0,t_0)$.

Lemma 2.6 gives a formula for $\nabla u(x_0,t_0)$, but it does not guarantee that the denominators are non-zero. Condition (5) implies that the values $f''(u_0(P(x,t)))$ and $u_0'(P(x,t))^{-1}$ cannot both be 0. From Lemma 2.7 it follows that the values must be of the same sign. Hence,

$$\begin{aligned}&\nabla u(x,t)\\&\quad=\left[\frac{1}{f''(u_0(P(x,t)))t+u_0'(P(x,t))^{-1}},\frac{-f'(u_0(P(x,t)))}{f''(u_0(P(x,t)))t+u_0'(P(x,t))^{-1}}\right]\end{aligned}$$

is well defined for all points (x,t) such that $P(x,t)\in\operatorname{cont}u_0$. The fact that $P(\tilde{x}_n,\tilde{t}_n)\to P(x_0,t_0)$ implies $\nabla u(\tilde{x}_n,\tilde{t}_n)\to\nabla u(x_0,t_0)$.

If $|u_0'(P(x_0,t_0))|<\infty$ then $\overline{\mathbf{x}}_n$ cannot converge to (x_0,t_0). Indeed, otherwise, from condition (4), we could find a sequence of points of continuity $\check{\mathbf{x}}_n$ converging to (x_0,t_0) such that $|u_0'(\check{\mathbf{x}}_n)|\to\infty$. If $|u_0'(P(x_0,t_0))|=\infty$ then trivially $\nabla u(\overline{x}_n,\overline{t}_n)\to\nabla u(x_0,t_0)$. This completes the proof. ■

Definition 2.8. We define an operator R by

$$R(u_0)(\overline{x})=\begin{cases}\lim\limits_{x\to\overline{x}^-}u_0(x) & \text{if } \lim\limits_{x\to\overline{x}^-}u_0(x)=\lim\limits_{x\to\overline{x}^+}u_0(x),\\ u(\overline{x}) & \text{else.}\end{cases}$$

Lemma 2.9. *Suppose that* $\operatorname{cl}(\operatorname{cont}u_0)=\mathbb{R}$. *There exists a unique solution* v *to equation* (1.1) *with initial data* Ru_0 *if and only if there exists a unique solution* u *to equation* (1.1) *with initial data* u_0, *and moreover* $u\equiv v$ *on* Ω.

Proof. It has been proved in Lemma 2.5 that the solution u is uniquely determined on $\Omega \setminus \bigcup_{\overline{x}\notin \operatorname{cont} u_0} \overline{C_{\overline{x}}}$. As u must be continuous, we can determine its values on the closure of this set, that is, on $\Omega \setminus \bigcup_{\overline{x}\notin \operatorname{cont} u_0} C_{\overline{x}}$.

Let $\overline{x}$ be a point of removable discontinuity of u_0. Then $C_{\overline{x}} = \emptyset$ and $u(\mathbf{x}) = u_0(\overline{x}^+) = u_0(\overline{x}^-)$ on $\{x = f'(u_0(\overline{x}^+))t + \overline{x}\}$. The solution would not change on Ω if we put $u_0(\overline{x}) = u_0(\overline{x}^-)$. Analogously, if we change the value of Ru_0 at a point of continuity x, the region where the solution changes is $C_x = \emptyset$. ∎

COROLLARY 2.10. *Suppose that* $\operatorname{cl}(\operatorname{cont} u_0) = \mathbb{R}$ *and* f *is linear. There exists a unique classical solution to equation* (1.1) *if and only if* $Ru_0 \in C^1(\mathbb{R})$.

COROLLARY 2.11. *Suppose that* $\operatorname{cl}(\operatorname{cont} u_0) = \mathbb{R}$ *and* f *is strictly convex. There exists a unique classical solution to equation* (1.1) *if and only if* $Ru_0 \in C^{1^\infty}(\mathbb{R})$ *is non-decreasing on* $\operatorname{cont} u_0$.

3. Examples

REMARK 3.1. The inclusion $C^1(\mathbb{R}) \subset C^{1^\infty}(\mathbb{R})$ holds, but it is not true that either $C^{1^\infty}(\mathbb{R}) \subseteq C^1(\mathbb{R})$ or even $C^{1^\infty}(\mathbb{R}) \cap C^0(\mathbb{R}) \subseteq C^1(\mathbb{R})$.

EXAMPLE 3.2. Functions that are in $C^{1^\infty}(\mathbb{R})$:

1. Any function discontinuous at every point is trivially in $C^{1^\infty}(\mathbb{R})$.
2. An example of a $C^{1^\infty} \cap C^0$ function which is not in $C^1(\mathbb{R})$:

$$v(x) = (\operatorname{sgn} x) \cdot \sqrt{|x|}.$$

3. An example of a C^{1^∞} function which is continuous almost everywhere:

$$v(x) = (\operatorname{sgn} x) \cdot (\sqrt{|x|} + 1),$$
$$v(x) = \frac{1}{x}.$$

EXAMPLE 3.3. Functions that are not in $C^{1^\infty}(\mathbb{R})$:

1. Any non-constant simple function. For example,

$$v(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

2. Any function with infinite limits of both signs of the derivative near a point of discontinuity. For example

$$v(x) = |x|^{-n}.$$

EXAMPLE 3.4. Let us consider the inviscid Burgers equation with initial data given by

$$u_0(x) = \begin{cases} \sqrt{|x|} + 1 & \text{if } x \geq 0, \\ -(\sqrt{|x|} + 1) & \text{if } x < 0. \end{cases} \tag{3.1}$$

The solution, given by the closed-form formula

$$u(x,t)=\begin{cases}\sqrt{x-t+\frac{1}{2}t^2-\frac{1}{2}t\sqrt{t^2+4(x-t)}}+1 & \text{if } x>t,\\ -\sqrt{-x-t+\frac{1}{2}t^2-\frac{1}{2}t\sqrt{t^2-4(x+t)}}-1 & \text{if } -t>x,\\ x/t & \text{if } -t\le x\le t,\end{cases}$$

is presented in Figure 1.

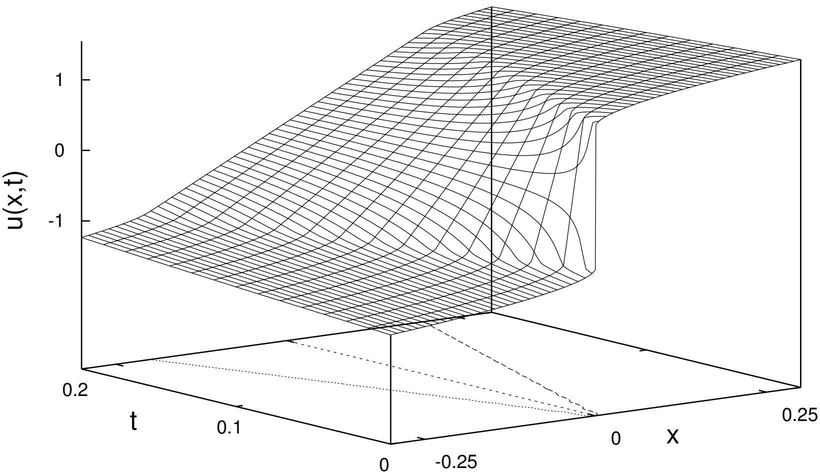

Fig. 1. Solution to the inviscid Burgers equation with initial data (3.1)

The functions f and u_0 satisfy conditions (1)–(5), so we expect u to be $C^1(\Omega)$. Indeed, we can compute

$$\frac{d}{dx}\left(\sqrt{x-t+\tfrac{1}{2}t^2-\tfrac{1}{2}t\sqrt{t^2+4(x-t)}}+1\right)=\frac{1}{2}\,\frac{1-\frac{t}{\sqrt{t^2+4(x-t)}}}{\sqrt{x-t+\frac{1}{2}t^2-\frac{1}{2}t\sqrt{t^2+4(x-t)}}}.$$

If we define $a=x-t$ and let $a\to 0$, we obtain

$$\lim_{a\to0^+}\frac{d}{dx}u(a+t,t)a=\lim_{a\to0^+}\frac{\sqrt{t^2+4a}-\sqrt{t^2}}{\sqrt{(t^2+4a)(4a+2t^2-2t\sqrt{t^2+4a})}}$$

$$=\lim_{a\to0^+}\frac{\sqrt{2t^2+4a-2t\sqrt{t^2+4a}}}{\sqrt{(t^2+4a)(4a+2t^2-2t\sqrt{t^2+4a})}}=\lim_{a\to0^+}\frac{1}{\sqrt{t^2+4a}}=\frac{1}{t}.$$

It can be similarly shown that $u_t(x,t)\to -1/t^2$ when $x\to t$.

Example 3.5. Let us consider the inviscid Burgers equation which corresponds to the Riemann problem with initial data

$$u_0(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

We can find a sequence $\{u_0^\varepsilon\}$ of C^1 regularizations that converges almost everywhere to this initial data, but no such sequence converges in $L^\infty(\mathbb{R})$. The main profit of introducing such regularizations (which is obtaining classical solutions close to a Lipschitz solution of the original problem) can also be achieved by C^{1^∞} regularizations. Let us define

$$v_0^\varepsilon(x) = \begin{cases} -1 & \text{if } x < -\varepsilon, \\ -(\sqrt{\varepsilon^2 - (\varepsilon - |x|)^2} + (1-\varepsilon)) & \text{if } x \in (-\varepsilon, 0), \\ \sqrt{\varepsilon^2 - (\varepsilon - |x|)^2} + (1-\varepsilon) & \text{if } x \in (0, \varepsilon), \\ 1 & \text{if } \varepsilon < x. \end{cases}$$

It is easy to check that v_0^ε satisfies conditions (1)–(5), and moreover $\|u_0 - v_0^\varepsilon\|_{L^\infty(\mathbb{R})} = \varepsilon$.

Let us denote the solution for initial data v_0^ε by v^ε. Then

- $u = v^\varepsilon$ on $\{|x| \in [0, t - \varepsilon t] \cup [t + \varepsilon, \infty)\}$,
- $|u - v^\varepsilon| < \varepsilon$ on $\{(1-\varepsilon)t < |x| < t + \varepsilon\}$, as $u(x,t), v(x,t) \in (1-\varepsilon, 1)$.

Thus, from Theorem 1.1 it follows that $C^1(\Omega) \ni v^\varepsilon \rightrightarrows u$.

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