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A MULTIPARAMETER VARIANT OF THE SALEM–ZYGMUND CENTRAL LIMIT THEOREM ON LACUNARY TRIGONOMETRIC SERIES

BY

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Abstract. We prove the central limit theorem for the multisequence

$$\sum_{1\le n_1\le N_1}\cdots\sum_{1\le n_d\le N_d} a_{n_1,\ldots,n_d}\cos(\langle 2\pi\mathbf{m}, A_1^{n_1}\ldots A_d^{n_d}\mathbf{x}\rangle)$$

where $\mathbf{m}\in\mathbb{Z}^s$, $a_{n_1,\ldots,n_d}$ are reals, $A_1,\ldots,A_d$ are partially hyperbolic commuting $s\times s$ matrices, and $\mathbf{x}$ is a uniformly distributed random variable in $[0,1]^s$. The main tool is the S-unit theorem.

1. Introduction. In [SZ], [Z, p. 233], Salem and Zygmund proved the following theorem:

THEOREM A. *Let $\lambda_n\ge 1$ be integers with $\lambda_{n+1}/\lambda_n\ge c>1$ for $n=1,2,\ldots.$ Moreoover, let a_n,ϕ_n be reals, $\mathcal{A}_N=\big(\frac{1}{2}(a_1^2+\cdots+a_N^2)\big)^{1/2}\to\infty$,*

$$S(N,x)=\frac{1}{\mathcal{A}_N}\sum_{n=1}^{N} a_n\cos(2\pi\lambda_n x+\phi_n),$$

and

$$\max_{1\le n\le N}|a_n|/\mathcal{A}_N\to 0\quad \text{as } N\to\infty.$$

Then over any set D with $\operatorname{mes} D>0$, $S(N,x)$ tends to the Gaussian distribution with mean value 0 and dispersion 1 as $N\to\infty$.

Let A be an invertible $s\times s$ matrix with integer entries. It generates a surjective endomorphism on the s-dimensional torus $[0,1)^s$ which we will denote by the same letter A. We will also denote by A and $\mathbf{m}$ the transpose matrices $A^{(t)}$, $\mathbf{m}^{(t)}$.

DEFINITION 1. An action $\mathcal{A}$ by surjective endomorphisms $A_1,\ldots,A_d$ of $[0,1)^s$ is called *partially hyperbolic* if for all $(n_1,\ldots,n_d)\in\mathbb{Z}^d\setminus\{\mathbf{0}\}$ none of the eigenvalues of the matrix $A_1^{n_1}\cdots A_d^{n_d}$ is a root of unity.

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Examples of partially hyperbolic actions:

1. Let $\mathbb{I}$ be the $s \times s$ identity matrix, $q_1, \dots, q_d \geq 2$ pairwise coprime integers, $A_i = q_i \mathbb{I}$, $i = 1, \dots, d$.
2. Let K be an algebraic number field of degree s, $\eta_1, \dots, \eta_d$ $(d \leq s-1)$ a set of fundamental units of K, $\phi_i(x)$ the minimal polynomial of η_i, and A_i the companion matrix of $\phi_i(x)$ $(1 \leq i \leq d)$.

In this paper, we prove the following multiparameter variant of the Salem and Zygmund theorem:

Theorem. *Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, \dots, A_d$ of $[0,1)^s$, and $\mathbf{x}$ a uniformly distributed random variable on $[0,1]^s$. Let $\mathbf{m} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$, $\mathbf{N} = (N_1, \dots, N_d)$, $R(\mathbf{N}) = [1, N_1] \times \cdots \times [1, N_d]$, $N_0 = \max(N_1, \dots, N_d)$, $\mathbf{n} = (n_1, \dots, n_d)$, $a_{\mathbf{n}} \geq 0$, $\phi_{\mathbf{n}}$ be reals,*

(1.1)
$$\mathcal{A}(\mathbf{N}) = \Big(\frac{1}{2} \sum_{\mathbf{n} \in R(\mathbf{N})} a_n^2\Big)^{1/2} \to \infty, \qquad \rho(\mathbf{N}) = \max_{\mathbf{n} \in R(\mathbf{N})} a_{\mathbf{n}}/\mathcal{A}(\mathbf{N}) \xrightarrow{N_0 \to \infty} 0,$$

$$S(\mathbf{N}, \mathbf{x}) = \frac{1}{\mathcal{A}(\mathbf{N})} \sum_{\mathbf{n} \in R(\mathbf{N})} a_{\mathbf{n}} \cos(2\pi \langle \mathbf{m}, A_1^{n_1} \cdots A_d^{n_d} \mathbf{x} \rangle + \phi_{\mathbf{n}}). \tag{1.2}$$

Then over any set $D \subset [0,1]^s$ with $\operatorname{mes} D > 0$, $S(\mathbf{N}, \mathbf{x})$ tends to the Gaussian distribution with mean value 0 and dispersion 1 as $N_0 \to \infty$.

This result was announced in [Le1].

Related questions

1. *Central Limit Theorem for $\mathbb{Z}_+^d$-actions by toral endomorphisms.* In [F], [K], Fortet and Kac proved the central limit theorem (abbreviated CLT) for the sum $\sum_{n=0}^{N-1} f(q^n x)$ where $q \geq 2$ is an integer, $x \in [0,1)$ and f is a 1-periodic function. Let $(\omega_{q_1,\dots,q_d}(n))_{n \geq 1}$ be a so-called Hardy–Littlewood–Pólya sequence, consisting of the elements of the multiplicative semigroup generated by a finite set $(q_1, \dots, q_d)$ of coprime integers, arranged in increasing order. In [P], [FP], Philipp, Fukuyama and Petit obtained limit theorems for the sum $\sum_{n=0}^{N-1} f(\omega_{q_1,\dots,q_d}(n)x)$. In [Le2], we proved some limit theorems for $\sum_{n_1=0}^{N_1-1} \cdots \sum_{n_d=0}^{N_d-1} f(q_1^{n_1} \cdots q_d^{n_d} x)$ as $N_1, \dots, N_d \to \infty$, where the integers $q_1, \dots, q_d$ need not be coprime (see [Le2, Theorem 5]).

In [L], Leonov proved CLT for endomorphisms of the s-torus and Hölder continuous functions. In [Le2], we extended Leonov's result to the case of $\mathbb{Z}_+^d$-actions by endomorphisms of the s-torus and we proved the central limit theorem for the multisequence

$$\sum_{n_1=1}^{N_1} \cdots \sum_{n_d=1}^{N_d} f(A_1^{n_1} \cdots A_d^{n_d}\mathbf{x})$$

where f is a Hölder continuous function, $A_1, \dots, A_d$ are partially hyperbolic commuting integer $s \times s$ matrices, and $\mathbf{x}$ is a uniformly distributed random variable in $[0,1]^s$.

Note that mixing properties of $\mathbb{Z}^d$-actions by commuting automorphisms of the s-torus were investigated earlier by Schmidt and Ward [SW].

2. *Hardy–Littlewood–Pólya (HLP) sequence.* In [Fu], Furstenberg studied denseness properties of the HLP sequence $(\omega_{2,3}(n))_{n\geq 1}$ (see Introduction) from the ergodic point of view. He also asked the celebrated question on ergodic properties of this sequence (see e.g. [EW, p. 7]). In [P], Philipp proved the almost sure invariance principle (ASIP) for the sequence $(\cos(\omega_{q_1,\dots,q_d}(n)x))_{n\geq 1}$, and the law of the iterated logarithm (LIL) for the discrepancy of the sequence $(\omega_{q_1,\dots,q_d}(n)x)_{n\geq 1}$ (see also [BPT]). We consider the following s-dimensional variant of HLP sequence:

Let

(1.3) $\mathbf{m} \lessdot \mathbf{m}'$ if either $|\mathbf{m}| < |\mathbf{m}'|$, or $|\mathbf{m}| = |\mathbf{m}'|$ and there exists $k \in [0,s)$ with $m_1 = m_1', \dots, m_k = m_k'$ and $m_{k+1} < m_{k+1}'$, where $|\mathbf{m}| = (m_1^2 + \cdots + m_s^2)^{1/2}$.

Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, \dots, A_d$ of $[0,1)^s$. Denote $A_1^{n_1} \cdots A_d^{n_d} \lessdot A_1^{\dot{n}_1} \cdots A_d^{\dot{n}_d}$ if $(n_1, \dots, n_d) \lessdot (\dot{n}_1, \dots, \dot{n}_d)$. Let $(\Omega_n)_{n\geq 1}$ consist of the elements of the multiplicative semigroup generated by the finite set $(A_1, \dots, A_d)$, arranged in increasing order. In a forthcoming paper, we will show that the approach of [P] and [BPT] can be applied to prove ASIP for $(\cos(\Omega_n\mathbf{x}))_{n\geq 1}$ and to prove LIL for the discrepancy of $(\Omega_n\mathbf{x})_{n\geq 1}$.

In [PS], Philipp and Stout proved that if for the coefficient a_N (see Theorem A) we assume the stronger condition $a_N = O(A_N^{1-\delta})$ for some $\delta > 0$, then $S(N,x)$ obeys ASIP.

Let $(g_n)_{n\geq 1}$ consist of the elements of $\mathbb{Z}_+^d$ arranged in increasing order (see (1.3)). Let

$$\dot{\mathcal{A}}(L) = \Big(\frac{1}{2}\sum_{1\leq n\leq L} a_{g_n}^2\Big)^{1/2}, \quad \dot{S}_L = \frac{1}{\dot{\mathcal{A}}(L)} \sum_{1\leq n\leq L} a_{g_n} \cos(2\pi\langle \mathbf{m}, \Omega_{g_n}\mathbf{x}\rangle + \phi_{g_n}).$$

In a forthcoming paper, we will show that the approach of [PS] can be applied to prove ASIP for the sequence $(\dot{S}_L)_{L\geq 1}$ whenever $a_{g_L} = O(\dot{\mathcal{A}}^{1-\delta}(L))$ for some $\delta > 0$.

2. Proof of the Theorem. By the moment method, to obtain the Theorem it is sufficient to prove that

$$\lim_{N_0\to\infty}\frac{1}{\operatorname{mes} D}\int\limits_D S(\mathbf{N},\mathbf{x})^{\hbar}\,d\mathbf{x} = \begin{cases}\dfrac{\hbar!}{2^{\hbar/2}(\hbar/2)!} & \text{if } \hbar \text{ is even},\\ 0 & \text{if } \hbar \text{ is odd}.\end{cases} \tag{2.1}$$

We consider the following variant of the S-unit theorem (see [SS, Theorem 1]):

Let K be an algebraic number field of degree $s_1 \geq 1$. Write K^* for its multiplicative group of nonzero elements. We consider the equation

$$\sum_{i=1}^{h_1} P_i(\mathbf{n})\boldsymbol{\vartheta}_i^{\mathbf{n}} = 0 \tag{2.2}$$

in variables $\mathbf{n} = (n_1,\dots,n_{d_1}) \in \mathbb{Z}^{d_1}$, where the P_i are polynomials with coefficients in K, $\boldsymbol{\vartheta}_i^{\mathbf{n}} = \vartheta_{i,1}^{n_1}\cdots\vartheta_{i,d_1}^{n_{d_1}}$, and $\vartheta_{i,j} \in K^*$ $(1 \leq i \leq h_1,\ 1 \leq j \leq d_1)$.

Let U_1 be the potential number of nonzero coefficients of the polynomials $P_1,\dots,P_{h_1}$, and $U_2 = \max(d_1, U_1)$.

A solution $\mathbf{n}$ of (2.2) is called *non-degenerate* if $\sum_{i\in I} P_i(\mathbf{n})\boldsymbol{\vartheta}_i^{\mathbf{n}} \neq 0$ for every nonempty subset I of all $\{1,\dots,h_1\}$. Let G be the subgroup of $\mathbb{Z}^{d_1}$ consisting of all vectors $\mathbf{n}$ with $\boldsymbol{\vartheta}_1^{\mathbf{n}} = \cdots = \boldsymbol{\vartheta}_{h_1}^{\mathbf{n}}$.

THEOREM B ([SS]). *Suppose $G = \{\mathbf{0}\}$. Then the number $\mathfrak{U}(P_1,\dots,P_{h_1})$ of nondegenerate solutions $\mathbf{n} \in \mathbb{Z}^{d_1}$ of (2.2) satisfies the estimate*

$$\mathfrak{U}(P_1,\dots,P_{h_1}) \leq U(d_1,P) = 2^{35U_2^3} s_1^{6U_2^2}.$$

It is easy to get the following

COROLLARY. *Let $d_1 = d(h_1-1)$, $\vartheta_{h_1,j} = 1$ $(j = 1,\dots,d)$, $\vartheta_{i,j+(i-1)d} = \vartheta_j \in K^*$ and $\vartheta_{i,j+\mu d} = 1$ $(\mu \in [0, h_1-2],\ \mu \neq i-1,\ i = 1,\dots,h_1-1,\ j = 1,\dots,d)$, $\overline{\mathbf{n}} = (\mathbf{n}_1,\dots,\mathbf{n}_{h_1-1})$, $\mathbf{n}_i = (n_{i,1},\dots,n_{i,d})$ with $i = 1,\dots,h_1-1$, $P_{h_1}(\overline{\mathbf{n}}) \equiv -1$. Suppose*

$$\vartheta_1^{n_1}\cdots\vartheta_d^{n_d} = 1 \iff (n_1,\dots,n_d) = \mathbf{0}.$$

Then the number $\mathfrak{U}'(P_1,\dots,P_{h_1-1})$ of nondegenerate solutions $\overline{\mathbf{n}} \in \mathbb{Z}^{d_1}$ of the equation

$$\sum_{i=1}^{h_1-1} P_i(\overline{\mathbf{n}})\boldsymbol{\vartheta}_i^{\overline{\mathbf{n}}} = \sum_{i=1}^{h_1-1} P_i(\overline{\mathbf{n}})\vartheta_1^{n_{i,1}}\cdots\vartheta_d^{n_{i,d}} = 1$$

satisfies the estimate

$$\mathfrak{U}'(P_1,\dots,P_{h_1-1}) \leq U(d_1,P).$$

We consider a sequence of commuting $s\times s$ matrices $A_1,\dots,A_d$. By [Ga, p. 224, Corollary 2] the space $\mathbb{C}^s$ can be decomposed into a direct sum of subspaces invariant under all A_j:

$$\mathbb{C}^s = I_1 \oplus \cdots \oplus I_w,$$

such that the minimal polynomial of A_j on I_i is a power of a linear polynomial, $(x-\lambda_{i,j})^{r_j}$, over $\mathbb{C}$. According to this decomposition, the matrices $A_1,\dots,A_d$ can be simultaneously brought into the following form with square blocks along the diagonal of sizes $r_1,\dots,r_w$, $r_1+\cdots+r_w=s$:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1,1} & & 0 \\ & \ddots & \\ 0 & & \Lambda_{w,1} \end{pmatrix}, \ \dots, \ \Lambda_d = \begin{pmatrix} \Lambda_{1,d} & & 0 \\ & \ddots & \\ 0 & & \Lambda_{w,d} \end{pmatrix}$$

where all blocks $\Lambda_{i,j}$ have upper-triangular form with $\lambda_{i,j}$ on the diagonal, $1\le i\le w$, $1\le j\le d$ (see, e.g., [MM, p. 77, ref. 4.21.1]).

Hence there exists an invertible $s\times s$ matrix T such that $A_i = T^{-1}\Lambda_i T$, $1\le i\le d$,

$$A_1^{n_1}\cdots A_d^{n_d} = T^{-1}\Lambda(\mathbf{n})T, \quad \Lambda(\mathbf{n}) = \begin{pmatrix} \widetilde{\Lambda}_1(\mathbf{n}) & & 0 \\ & \ddots & \\ 0 & & \widetilde{\Lambda}_w(\mathbf{n}) \end{pmatrix}, \tag{2.3}$$

where $\mathbf{n}=(n_1,\dots,n_d)$, and $\widetilde{\Lambda}_\nu(\mathbf{n})$ is an upper-triangular matrix with $\lambda_{\nu,1}^{n_1}\cdots\lambda_{\nu,d}^{n_d}$ on the diagonal ($1\le\nu\le w$). Let $\widetilde{\Lambda}_\nu(\mathbf{n}) = (\widetilde{\lambda}^{(\nu)}_{j,j'}(\mathbf{n}))_{1\le j,j'\le r_\nu}$. Using the formula for the degree of Jordan's normal form of the matrices $\Lambda_{i,j}$ (see, e.g., [Ga, pp. 157–158]), we deduce that

$$\widetilde{\lambda}^{(\nu)}_{j,j'}(\mathbf{n}) = \lambda_{\nu,1}^{n_1}\cdots\lambda_{\nu,d}^{n_d} P^{(\nu)}_{j,j'}(\mathbf{n}) \tag{2.4}$$

for some polynomial $P^{(\nu)}_{j,j'}$. It is easy to see that

$$P^{(\nu)}_{j,j}(\mathbf{n}) = 1 \quad \text{and} \quad P^{(\nu)}_{j,j'}(\mathbf{n}) = 0 \quad \text{for } j>j'. \tag{2.5}$$

Taking into account that $\lambda_{i,1}^{n_1}\cdots\lambda_{i,d}^{n_d}$ is an eigenvalue of $A_1^{n_1}\cdots A_d^{n_d}$, we infer from Definition 1 that

$$\lambda_{i,1}^{n_1}\cdots\lambda_{i,d}^{n_d} = 1 \ \Leftrightarrow \ (n_1,\dots,n_d)=\mathbf{0}, \quad \text{with} \quad i\in[1,r]. \tag{2.6}$$

Let $\hbar\ge 1$ be an integer, $F^{(\hbar)}=\{1,\dots,\hbar\}$, $\tau_1,\dots,\tau_\hbar\in\{-1,1\}$, $f=\#F$, $F=(F(1),\dots,F(f))\subseteq F^{(\hbar)}$, $\overline{\mathbf{n}}^{(F)}=(\mathbf{n}_{F(1)},\dots,\mathbf{n}_{F(f)})$, $\overline{\mathbf{n}}=\overline{\mathbf{n}}^{(F^{(\hbar)})}=(\mathbf{n}_1,\dots,\mathbf{n}_\hbar)$, with $\mathbf{n}_i=(n_{i,1},\dots,n_{i,d})$. We will denote the transpose matrix $\mathbf{m}^{(t)}$ by the same symbol $\mathbf{m}$. Let $T\mathbf{m}=\tilde{\mathbf{m}}=(\tilde m_{1,1},\dots,\tilde m_{1,r_1},\dots,\tilde m_{w,1},\dots,\tilde m_{w,r_w})^{(t)}$,

$$C(\overline{\mathbf{n}}^{(F)}) = \sum_{\mu\in F}\tau_\mu T A_1^{n_{\mu,1}}\cdots A_d^{n_{\mu,d}}\mathbf{m}, \tag{2.7}$$

and

$$C(\overline{\mathbf{n}}^{(F)}) = (C(\overline{\mathbf{n}}^{(F)})_{1,1},\dots,C(\overline{\mathbf{n}}^{(F)})_{1,r_1},\dots,C(\overline{\mathbf{n}}^{(F)})_{w,1},\dots,C(\overline{\mathbf{n}}^{(F)})_{w,r_w})^{(t)}.$$

By (2.4) we get

$$C(\overline{\mathbf{n}}^{(F)})_{\nu,j} = \sum_{\mu\in F} \tau_\mu \lambda_{\nu,1}^{n_{\mu,1}} \cdots \lambda_{\nu,d}^{n_{\mu,d}} \sum_{j'=1}^{r_\nu} P_{j,j'}^{(\nu)}(\mathbf{n})\tilde{m}_{\nu,j'}. \tag{2.8}$$

Since $\tilde{\mathbf{m}} \neq 0$, there exist $\nu_0 \in [1, w]$, $j_0 \in [1, r_{\nu_0}]$ with

$$\tilde{m}_{\nu_0,j_0} \neq 0 \quad \text{and} \quad \tilde{m}_{\nu,j} = 0 \quad \text{for } \nu > \nu_0 \text{ and for } \nu = \nu_0,\ j > j_0. \tag{2.9}$$

From (2.5), we have

$$P_{j,j'}^{(\nu_0)}(\mathbf{n})\tilde{m}_{\nu_0,j'} = 0 \quad \text{for } j' \neq j_0. \tag{2.10}$$

By (2.5) and (2.8)–(2.10), we obtain

$$C(\overline{\mathbf{n}}^{(F)})_{\nu_0,j_0} = L(\overline{\mathbf{n}}^{(F)}, \nu_0)\tilde{m}_{\nu_0,j_0} \quad \text{with} \quad L(\overline{\mathbf{n}}^{(F)}, \nu) = \sum_{\mu\in F} \tau_\mu \lambda_{\nu,1}^{n_{\mu,1}} \cdots \lambda_{\nu,d}^{n_{\mu,d}}. \tag{2.11}$$

Hence

$$C(\overline{\mathbf{n}}^{(F)}) = \mathbf{0} \ \Rightarrow\ L(\overline{\mathbf{n}}^{(F)}, \nu_0) = 0. \tag{2.12}$$

Let

$$\delta(\mathfrak{T}) = \begin{cases} 1 & \text{if } \mathfrak{T} \text{ is true,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

$$R(\mathbf{N}, F, \nu) = \{(\mathbf{n}_{F(1)}, \dots, \mathbf{n}_{F(f)}) \in R(\mathbf{N})^f \mid \nexists F' \subsetneq F^{(\hbar)} \text{ with } L(\overline{\mathbf{n}}^{(F')}, \nu) = 0\}. \tag{2.14}$$

We denote

$$\sum_{\tau_i\in\{-1,1\},\, i=1,\dots,\hbar} \quad \text{by} \quad \sum_{\tau}, \quad \text{and} \quad \sum_{\tau_i\in\{-1,1\},\, i\in F} \quad \text{by} \quad \sum_{\tau,\, F}.$$

Lemma 1. *Let* $f = \#F$ *and*

$$\sigma = \sum_{(\mathbf{n}_{F(1)},\dots,\mathbf{n}_{F(f)})\in R(\mathbf{N},F,\nu_0)} \sum_{\tau,\, F} (2\mathcal{A}(\mathbf{N}))^{-f} a_{\mathbf{n}_{F(1)}} \cdots a_{\mathbf{n}_{F(f)}} \delta(L(\overline{\mathbf{n}}^{(F)}, \nu_0) = \gamma).$$

Then

$$\sigma = \begin{cases} 0 & \text{if } \gamma = 0,\ f = 1, \\ 1 & \text{if } \gamma = 0,\ f = 2, \\ O(\rho(\mathbf{N})) & \text{if } \gamma = 0,\ f \geq 3, \\ O(\rho(\mathbf{N})) & \text{if } \gamma \neq 0, \end{cases}$$

where $\rho(\mathbf{N}) = \max_{\mathbf{n}\in R(\mathbf{N})} |a_{\mathbf{n}}|/\mathcal{A}(\mathbf{N})$ *and the* O*-constants depend only on* $\hbar$.

Proof. Let $\gamma \neq 0$. Bearing in mind Definition 1 and that $\lambda_{i,j}$ are algebraic integers, we can apply the Corollary. We take $h_1 = \hbar + 1$, $s_1 = s^2$, $U_2 = d_1 = d\hbar$ and $U(d_1, P) = 2^{35U_2^3} s^{12U_2^2}$. From (2.3), (2.11) and (2.14) we get

$$\sigma \leq U(d_1, P)\rho(\mathbf{N})^f.$$

Let $\gamma = 0$. We see that there are no solutions of the equation $L(\overline{\mathbf{n}}^{(F)}, \nu_0) = 0$ if $f = 1$.

Suppose $f = 2$. By (2.11), we obtain

$$- \tau_1 \tau_2 \lambda_{\nu_0,1}^{n_{F(1),1}-n_{F(2),1}} \cdots \lambda_{\nu_0,d}^{n_{F(1),d}-n_{F(2),d}} = 1. \tag{2.15}$$

Hence

$$\lambda_{\nu_0,1}^{2(n_{F(1),1}-n_{F(2),1})} \cdots \lambda_{\nu_0,d}^{2(n_{F(1),d}-n_{F(2),d})} = 1.$$

Using (2.6), we find that $n_{F(1),i} - n_{F(2),i} = 0$, $i = 1, \ldots, d$. By (2.15), we get $\tau_1 = -\tau_2$. We derive from (1.1) that $\sigma = 1$.

Suppose $f \geq 3$. We fix $\mathbf{n}_{F(f)}$. Let $n'_{F(\mu),i} = n_{F(\mu),i} - n_{F(f),i}$ $(1 \leq \mu < f)$. We see that

$$- \tau_f \sum_{\mu=1}^{f-1} \tau_\mu \lambda_{\nu_0,1}^{n'_{F(\mu),1}} \cdots \lambda_{\nu_0,d}^{n'_{F(\mu),d}} = 1. \tag{2.16}$$

Applying the Corollary, we see that the number of solutions of (2.16) is $O(\hbar)$. Let $V(\hbar)$ be the maximum of $|n'_{F(\mu),i}|$ $(\mu = 1, \ldots, f-1, i = 1, \ldots, d)$ for all solutions of (2.16). Let $W(\hbar) = [-V(\hbar), V(\hbar)]^d$. Thus

$$\begin{aligned} \sigma &\leq \sum_{\mathbf{n}'_1, \ldots, \mathbf{n}'_{f-1} \in W(\hbar)^{f-1}} \sum_{\substack{\mathbf{n}_f, \mathbf{n}_f + \mathbf{n}'_i \in R(\mathbf{N}) \\ i=1,\ldots,f-1}} \sum_{\tau} (2\mathcal{A}(\mathbf{N}))^{-f} a_{\mathbf{n}_f + \mathbf{n}'_1} \cdots a_{\mathbf{n}_f + \mathbf{n}'_{f-1}} a_{\mathbf{n}_f} \\ &\leq (2V(\hbar) + 1)^{d(f-2)} \rho(\mathbf{N})^{f-2} \sum_{\mathbf{n}'_1 \in W(\hbar)} \sum_{\mathbf{n}_f, \mathbf{n}_f + \mathbf{n}'_1 \in R(\mathbf{N})} \mathcal{A}(\mathbf{N})^{-2} a_{\mathbf{n}_f + \mathbf{n}'_1} a_{\mathbf{n}_f}. \end{aligned}$$

Bearing in mind that $|a_{\mathbf{n}_f + \mathbf{n}'_1} a_{\mathbf{n}_f}| \leq \frac{1}{2}(a^2_{\mathbf{n}_f + \mathbf{n}'_1} + a^2_{\mathbf{n}_f})$, we obtain

$$\sigma \leq (2V(\hbar) + 1)^{d(f-1)} \rho(\mathbf{N})^{f-2}.$$

Hence Lemma 1 is proved. ∎

Let $\mathcal{F}_r = (F_1, \ldots, F_r)$ be a partition of $F^{(\hbar)} = \{1, \ldots, \hbar\}$, i.e.

$$F_1 \cup \cdots \cup F_r = F^{(\hbar)}, \quad F_i \cap F_j = \emptyset, \quad i \neq j, \quad F_i(j) < F_i(k) \quad \text{for } j < k.$$

Let $(F_1, \ldots, F_{r_1}) = (F'_1, \ldots, F'_{r_2})$ if $r_1 = r_2$ and for each $i \in [1, r_1]$ there exists $k \in [1, r_1]$ such that $F_i(j) = F'_k(j)$ for all $j \in [1, f(F_i)]$. We denote by $\mathfrak{F}_\hbar$ the set of all nonequivalent partitions of $F^{(\hbar)}$.

Definition 2. Set $\mathfrak{g}(\overline{\mathbf{n}}) = 1$ if $\hbar = 2\hbar_1$ and there exists a partition $(F_1, \ldots, F_{\hbar_1}) \in \mathfrak{F}_\hbar$ with $\#F_i = 2$, $L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0$ and $\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)}$ $(i = 1, \ldots, \hbar_1)$, and set $\mathfrak{g}(\overline{\mathbf{n}}) = 0$ otherwise.

Definition 3. Set $\mathfrak{g}_1(\overline{\mathbf{n}}) = 1$ if $\mathfrak{g}(\overline{\mathbf{n}}) = 1$ and there are no $1 \leq j_1 < j_2 < j_3 \leq \hbar$ with $\mathbf{n}_{j_1} = \mathbf{n}_{j_2} = \mathbf{n}_{j_3}$, and set $\mathfrak{g}_1(\overline{\mathbf{n}}) = 0$ otherwise.

LEMMA 2. *Define*

$$\sigma_0 = \sum_{\overline{\mathbf{n}} \in R(\mathbf{N})^{\hbar}} \sum_{\tau} (2\mathcal{A}(\mathbf{N}))^{-\hbar} a_{\mathbf{n}_1} \cdots a_{\mathbf{n}_{\hbar}} \delta(C(\overline{\mathbf{n}}) = \mathbf{0}) \delta(\mathfrak{g}(\overline{\mathbf{n}}) = 0).$$

Then

$$\sigma_0 = O(\rho(\mathbf{N})).$$

Proof. Suppose $C(\overline{\mathbf{n}}) = \mathbf{0}$. By (2.12) and (2.14), there exists a partition $\mathcal{F}_r = (F_1, \dots, F_r) \in \mathfrak{F}_{\hbar}$ such that

$$L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0 \quad \text{and} \quad \overline{\mathbf{n}}^{(F_i)} \in R(\mathbf{N}, F_i, \nu_0), \qquad i = 1, \dots, r.$$

Using (2.13), we have

$$\delta(C(\overline{\mathbf{n}}) = \mathbf{0}) \leq \sum_{r=1}^{\hbar} \sum_{\mathcal{F}_r \in \mathfrak{F}_{\hbar}} \prod_{i=1}^{r} \delta(L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0) \delta(\overline{\mathbf{n}}^{(F_i)} \in R(\mathbf{N}, F_i, \nu_0)).$$

Let $f_i = \#F_i$, $\mathfrak{f}^- = \min_{1 \leq i \leq r} f_i$ and $\mathfrak{f}^+ = \max_{1 \leq i \leq r} f_i$. Moreover, define $\mathfrak{F}_{\hbar,1} = \{\mathcal{F}_r \in \mathfrak{F}_{\hbar} \mid \mathfrak{f}^- = 1\}$ and

$$\mathfrak{F}_{\hbar,2} = \{\mathcal{F}_r \in \mathfrak{F}_{\hbar} \mid \mathfrak{f}^- = \mathfrak{f}^+ = 2\}, \qquad \mathfrak{F}_{\hbar,3} = \{\mathcal{F}_r \in \mathfrak{F}_{\hbar} \mid \mathfrak{f}^+ \geq 3\}, \tag{2.17}$$

$$\sigma_{l_1,l_2,l_3} = \sum_{r=1}^{\hbar} \sum_{\mathcal{F}_r \in \mathfrak{F}_{\hbar,l_3}} \sum_{\overline{\mathbf{n}} \in R(\mathbf{N})^{\hbar}} \sum_{\tau} (2\mathcal{A}(\mathbf{N}))^{-\hbar} a_{\mathbf{n}_1} \cdots a_{\mathbf{n}_{\hbar}} \tag{2.18}$$

$$\times \prod_{i=1}^{r} \delta(L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0) \delta(\overline{\mathbf{n}}^{(F_i)} \in R(\mathbf{N}, F_i, \nu_0)) \delta(\mathfrak{g}(\overline{\mathbf{n}}) = l_1) \delta(\mathfrak{g}_1(\overline{\mathbf{n}}) = l_2).$$

Changing the order of summation, we obtain

$$\sigma_0 \leq \sum_{l_2 \in \{0,1\}} \sum_{l_3 \in \{1,2,3\}} \sigma_{0,l_2,l_3}. \tag{2.19}$$

Suppose $\mathfrak{f}^- = 1$. By (2.11), there is no solution of the equation $L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0$ with $\#F_i = 1$. Hence

$$\sum_{l_1,l_2 \in \{0,1\}} \sigma_{l_1,l_2,1} = 0. \tag{2.20}$$

Suppose $\mathfrak{f}^+ \geq 3$. We have

$$\sum_{l_1,l_2 \in \{0,1\}} \sigma_{l_1,l_2,3} \leq \sum_{r \in [1,\hbar]} \sum_{\mathcal{F}_r \in \mathfrak{F}_{\hbar,3}} \sum_{\tau} 1$$

$$\times \prod_{i \in [1,r]} \sum_{\overline{\mathbf{n}}^{(F_i)} \in R(\mathbf{N}, F_i, \nu_0)} (2\mathcal{A}(\mathbf{N}))^{-f_i} a_{\mathbf{n}_{F_i(1)}} \cdots a_{\mathbf{n}_{F_i(f_i)}} \delta(L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0).$$

Applying Lemma 1, we have

$$\sum_{l_1,l_2 \in \{0,1\}} \sigma_{l_1,l_2,3} = O(\rho(\mathbf{N})). \tag{2.21}$$

Now we consider the case $\mathfrak{f}^- = \mathfrak{f}^+ = 2$ and $l_1 = 0$. We see that $\hbar$ is even. Let $\hbar = 2\hbar_1$ and $(F_1, \ldots, F_{\hbar_1}) \in \mathfrak{F}_\hbar$. Suppose $L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0$. By (2.15) and (2.6), we get $\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)}$. Using Definition 2, we find that the inner sum in (2.18) is zero. From (2.19)–(2.21), we conclude that $\sigma_0 = O(\rho(\mathbf{N}))$. ∎

LEMMA 3. *Let*

$$\sigma^{(1)}(\mathbf{N}) = \sum_{\overline{\mathbf{n}} \in R(\mathbf{N})^\hbar} \sum_\tau (2\mathcal{A}(\mathbf{N}))^{-\hbar} a_{\mathbf{n}_1} \cdots a_{\mathbf{n}_\hbar} \delta(C(\overline{\mathbf{n}}) = \mathbf{0}) e(\tau_1 \phi_{\mathbf{n}_1} + \cdots + \tau_\hbar \phi_{\mathbf{n}_\hbar}).$$

Then

$$\sigma^{(1)}(\mathbf{N}) = \begin{cases} \dfrac{\hbar!}{2^{\hbar/2}(\hbar/2)!} + O(\rho(\mathbf{N})) & \text{if } \hbar \text{ is even,} \\ O(\rho(\mathbf{N})) & \text{if } \hbar \text{ is odd.} \end{cases}$$

Proof. Let $l_1, l_2 \in \{0, 1\}$, and define

$$\sigma_{l_1}(l_2) = \sum_{\overline{\mathbf{n}} \in R(\mathbf{N})^\hbar} \sum_\tau (2\mathcal{A}(\mathbf{N}))^{-\hbar} a_{\mathbf{n}_1} \cdots a_{\mathbf{n}_\hbar} \times e(\tau_1 \phi_{\mathbf{n}_1} + \cdots + \tau_\hbar \phi_{\mathbf{n}_\hbar}) \delta(C(\overline{\mathbf{n}}) = \mathbf{0}) \delta(\mathfrak{g}(\overline{\mathbf{n}}) = l_1) \delta(\mathfrak{g}_1(\overline{\mathbf{n}}) = l_2). \tag{2.22}$$

By (2.13),

$$\sigma^{(1)}(\mathbf{N}) = \sigma_0(0) + \sigma_0(1) + \sigma_1(0) + \sigma_1(1). \tag{2.23}$$

From Lemma 2, we have

$$|\sigma_0(0)| + |\sigma_0(1)| \leq \sigma_0 = O(\rho(\mathbf{N})).$$

Assume $\mathfrak{g}(\overline{\mathbf{n}}) = 1$. By Definition 2, $\hbar$ is even. Write $\hbar = 2\hbar_1$, and let $\mathfrak{g}_1(\overline{\mathbf{n}}) = 0$. Using (2.18), (2.20) and (2.21), we get

$$|\sigma_1(0)| \leq \sigma_{1,0,1} + \sigma_{1,0,2} + \sigma_{1,0,3} \quad \text{and} \quad \sigma_{1,0,1} + \sigma_{1,0,3} = O(\rho(\mathbf{N})). \tag{2.24}$$

Consider $\sigma_{1,0,2}$. For all $\mathcal{F}_r \in \mathfrak{F}_{\hbar,2}$ (see (2.17)), we get $r = \hbar_1$ and $\mathfrak{f}^- = \mathfrak{f}^+ = 2$. By (2.18),

$$\sigma_{1,0,2} \leq \sum_{\overline{\mathbf{n}} \in R(\mathbf{N})^{2\hbar_1}} \sum_\tau \sum_{(F_1, \ldots, F_{\hbar_1}) \in \mathfrak{F}_{2\hbar_1}} (2\mathcal{A}(\mathbf{N}))^{-2\hbar_1} a_{\mathbf{n}_1} \cdots a_{\mathbf{n}_{2\hbar_1}} \times \delta(\mathfrak{f}^- = \mathfrak{f}^+ = 2) \delta(\mathfrak{g}(\overline{\mathbf{n}}) = 1) \delta(\mathfrak{g}_1(\overline{\mathbf{n}}) = 0) \prod_{i=1}^r \delta(L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0). \tag{2.25}$$

Applying (2.15) and (2.6), we obtain $\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)}$ $(i = 1, \ldots, \hbar_1)$. Now we derive from Definition 3 that $\hbar_1 \geq 2$, and there exist $1 \leq j_1 < j_2 < j_3 \leq \hbar$ with $\mathbf{n}_{j_1} = \mathbf{n}_{j_2} = \mathbf{n}_{j_3}$. Suppose $F_i(k) = j_1$ for some $i \in [1, \hbar_1]$ and $k \in \{1, 2\}$. We see that there exist $\mu \in [1, \hbar_1] \setminus \{i\}$ and $l \in \{2, 3\}$ such that $j_l \in F_\mu$.

Hence $\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)} = \mathbf{n}_{F_\mu(1)} = \mathbf{n}_{F_\mu(2)}$. Therefore

$$\sigma_{1,0,2} \leq \sum_{i,\mu\in[1,\hbar_1],\, i\neq\mu} \;\sum_{(F_1,\dots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1}} \;\sum_{\tau} \delta(\mathfrak{f}^- = \mathfrak{f}^+ = 2)$$
$$\times \sum_{\mathbf{n}_{F_i(1)}\in R(\mathbf{N})} (2\mathcal{A}(\mathbf{N}))^{-4} a^4_{\mathbf{n}_{F_i(1)}} \prod_{l\in[1,\hbar_1],\, l\neq i,\mu} \;\sum_{\mathbf{n}_{F_l(1)}\in R(\mathbf{N})} (2\mathcal{A}(\mathbf{N}))^{-2} a^2_{\mathbf{n}_{F_l(1)}}.$$

By (2.24) and (1.1) we have

$$\sigma_1(0) = O\Big(\rho(\mathbf{N})^2 \Big(\sum_{\mathbf{n}\in\mathbf{R}(\mathbf{N})} \mathcal{A}(\mathbf{N})^2 a^2_{\mathbf{n}}\Big)^{\hbar_1-1}\Big) = O(\rho(\mathbf{N})^2). \tag{2.26}$$

Now assume $\mathfrak{g}_1(\overline{\mathbf{n}}) = 1$. We consider $\sigma_1(1)$ (see (2.22)). From Definitions 2 and 3, we have

$$\mathfrak{g}_1(\overline{\mathbf{n}}) = \sum_{(F_1,\dots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1},\, \#F_i=2,\, i=1,\dots,\hbar_1} \mathfrak{g}_1(\overline{\mathbf{n}}).$$

Thus

$$\sigma_1(1) =$$
$$\sum_{\overline{\mathbf{n}}\in R(\mathbf{N})^{\hbar}} \sum_{\tau} \sum_{(F_1,\dots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1},\, \#F_i=2,\, i=1,\dots,\hbar_1} (2\mathcal{A}(\mathbf{N}))^{-\hbar} a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_\hbar} \delta(C(\overline{\mathbf{n}}) = \mathbf{0})$$
$$\times e(\tau_1\phi_{\mathbf{n}_1} + \dots + \tau_\hbar\phi_{\mathbf{n}_\hbar})\delta(\mathfrak{g}_1(\overline{\mathbf{n}}) = 1) \prod_{i=1}^{\hbar_1} \delta(\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)})\delta(L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0).$$

Assume $L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0$. By (2.15) and (2.6), we get $\tau_{F_i(1)} = -\tau_{F_i(2)}$. Hence $e(\tau_{F_i(1)}\phi_{\mathbf{n}_{F_i(1)}} + \tau_{F_i(2)}\phi_{\mathbf{n}_{F_i(2)}}) = 1$. Thus $\sigma_1(1) = \dot{\sigma} - \ddot{\sigma}$, where

$$\dot{\sigma} = \sum_{\overline{\mathbf{n}}\in R(\mathbf{N})^{\hbar}} \sum_{\tau} \sum_{(F_1,\dots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1},\, \#F_i=2,\, i=1,\dots,\hbar_1} (2\mathcal{A}(\mathbf{N}))^{-\hbar}$$
$$\times a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_\hbar} \prod_{i=1}^{\hbar_1} \delta(\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)})\delta(\tau_{F_i(1)} = -\tau_{F_i(2)}), \tag{2.27}$$

and

$$\ddot{\sigma} = \sum_{\overline{\mathbf{n}}\in R(\mathbf{N})^{\hbar}} \sum_{\tau} \sum_{(F_1,\dots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1},\, \#F_i=2,\, i=1,\dots,\hbar_1} (2\mathcal{A}(\mathbf{N}))^{-\hbar} a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_\hbar}$$
$$\times \prod_{i=1}^{\hbar_1} \delta(\mathbf{n}_{F_i(1)} = \mathbf{n}_{F_i(2)})\delta(L(\overline{\mathbf{n}}^{(F_i)}, \nu_0) = 0)\delta(\mathfrak{g}_1(\overline{\mathbf{n}}) = 0).$$

From (2.25) and (2.26), we obtain

$$\ddot{\sigma} = O(\rho(\mathbf{N})). \tag{2.28}$$

By (2.27), we get

$$\dot{\sigma} = \sum_{(F_1,\ldots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1},\,\#F_i=2,\,i=1,\ldots,\hbar_1} \prod_{i=1}^{\hbar_1} \sum_{\mathbf{n}_{F_i(1)}\in R(\mathbf{N})} 2(2\mathcal{A}(\mathbf{N}))^{-2} a^2_{\mathbf{n}_{F_i(1)}}$$

$$= \sum_{(F_1,\ldots,F_{\hbar_1})\in\mathfrak{F}_{2\hbar_1},\,\#F_i=2,i=1,\ldots,\hbar_1} 1 = \frac{1}{\hbar_1!}\binom{2\hbar_1}{2}\binom{2\hbar_1-2}{2}\cdots\binom{2}{2} = \frac{(2\hbar_1)!}{\hbar_1!2^{\hbar_1}}.$$

Now Lemma 3 follows from (2.23), (2.24) and (2.26)–(2.28). ∎

LEMMA 4. *Let $\boldsymbol{\gamma} \neq \mathbf{0}$, and define*

$$\sigma^{(2)}(\mathbf{N},\boldsymbol{\gamma}) = \sum_{\overline{\mathbf{n}}\in R(\mathbf{N})^{\hbar}} \sum_{\tau} \mathcal{A}(\mathbf{N})^{-\hbar} a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_{\hbar}} \times \delta(C(\overline{\mathbf{n}}) = -T\boldsymbol{\gamma}) e(\tau_1\phi_{\mathbf{n}_1} + \cdots + \tau_{\hbar}\phi_{\mathbf{n}_{\hbar}}).$$

Then

$$\sigma^{(2)}(\mathbf{N},\boldsymbol{\gamma}) = O(\rho(\mathbf{N})), \tag{2.29}$$

with the O-constant depending only on $\hbar$.

Proof. Let $-T\boldsymbol{\gamma} = (\gamma_{1,1},\ldots,\gamma_{1,r_1},\ldots,\gamma_{w,1},\ldots,\gamma_{w,r_w})$ and define $\nu_1 = \max\{\nu\in[1,w] \mid (\gamma_{\nu,1},\ldots,\gamma_{\nu,r_\nu}) \neq \mathbf{0}\}$. Assume $\gamma_{\nu_1,j_1}\neq 0$, and $\gamma_{\nu_1,j}=0$ for $j>j_1$.

Suppose $(\tilde{m}_{\nu_1,1},\ldots,\tilde{m}_{\nu_1,r_{\nu_1}}) = \mathbf{0}$. By (2.8), the equation $C(\overline{\mathbf{n}})_{\nu_1,j_1} = \gamma_{\nu_1,j_1}$ has no solution, and the assertion is proved in this case.

Now suppose $(\tilde{m}_{\nu_1,1},\ldots,\tilde{m}_{\nu_1,r_{\nu_1}}) \neq \mathbf{0}$ and $\tilde{m}_{\nu_1,j_2}\neq 0$, $\tilde{m}_{\nu_1,j}=0$ for $j>j_2$. By (2.8) and (2.9), $\nu_1\le\nu_0$. Suppose $\nu_1=\nu_0$ and $j_1>j_0$. By (2.10), $P^{(\nu_0)}_{j_1,j'}(\mathbf{n})\tilde{m}_{\nu_0,j'} = 0$ for $j'\neq j_1$. By (2.8) and (2.11), $C(\overline{\mathbf{n}})_{\nu_0,j_1}=0$. Thus the equation $C(\overline{\mathbf{n}})_{\nu_0,j_1} = \gamma_{\nu_0,j_1} = \gamma_{\nu_1,j_1}\neq 0$ has no solution.

Let $\nu_1=\nu_0$ and $j_1=j_2=j_0$. From (2.3)–(2.11) we deduce that $L(\overline{\mathbf{n}},\nu_0) = \gamma_{\nu_0,j_0}/\tilde{m}_{\nu_0,j_0}\neq 0$. Analogously to the proof of Lemma 2, we obtain (2.29) from Lemma 1.

Consider the case $\nu_1<\nu_0$ or $\nu_1=\nu_0$, $j_1<j_0$. By (2.4), (2.7) and (2.11), we find that if $C(\overline{\mathbf{n}}) = -T\boldsymbol{\gamma}$, then

$$C(\overline{\mathbf{n}})_{\nu_0,j_0} = \gamma_{\nu_0,j_0} = 0 = L(\overline{\mathbf{n}},\nu_0) \quad\text{and}\quad C(\overline{\mathbf{n}})_{\nu_1,j_1} = \gamma_{\nu_1,j_1}\neq 0. \tag{2.30}$$

It is easy to see that there exists a set $F_0\subseteq F^{(\hbar)}$ with

$$C(\overline{\mathbf{n}}^{(F_0)})_{\nu_1,j_1} = \gamma_{\nu_1,j_1}, \qquad C(\overline{\mathbf{n}}^{(F')})_{\nu_1,j_1}\neq 0 \quad \forall F'\subset F_0.$$

Define

$$R^*(\mathbf{N},F) = \{\overline{\mathbf{n}}^{(F)}\in R(\mathbf{N})^f \mid C(\overline{\mathbf{n}}^{(F)})_{\nu_1,j_1} = \gamma_{\nu_1,j_1}, \nexists F'\subsetneq F \text{ with } C(\overline{\mathbf{n}}^{(F')})_{\nu_1,j_1} = 0\}.$$

Applying (2.8), (2.6) and the Corollary, we get

$$\#R^*(\mathbf{N},F_0)=O(1), \tag{2.31}$$

with the O-constant depending only on $\hbar$. If $F_0=F^{(\hbar)}$, then (2.29) easily follows from (1.1).

Now assume $F_0\neq F^{(\hbar)}$ and $\overline{\mathbf{n}}_0^{(F_0)}\in R^*(\mathbf{N},F_0)$. Let $F^{\perp}$ be the subset of $F^{(\hbar)}$ with $F\cup F^{\perp}=F^{(\hbar)}$ and $F\cap F^{\perp}=\emptyset$. We derive from (2.14) and (2.30) that there exists a partition $(F_1,\dots,F_r)$ of $F_0^{\perp}$ such that

$$L(\overline{\mathbf{n}}_0^{(F_i)},\nu_0)=\gamma^{(i)}\quad\text{and}\quad \overline{\mathbf{n}}_0^{(F_i)}\in R(\mathbf{N},F_i,\nu_0),$$

where $\gamma^{(1)}=\dots=\gamma^{(r-1)}=0$ and $\gamma^{(r)}=-L(\overline{\mathbf{n}}_0^{(F_0)},\nu_0)\tilde{m}_{\nu_0,j_0}$. Applying Lemma 1, we get

$$\sigma^{(3)}(\mathbf{N},\boldsymbol{\gamma},\overline{\mathbf{n}}_0^{(F_0)}):=\sum_{r=1}^{\hbar-1}\sum_{(F_1,\dots,F_r,F_0)\in\mathfrak{F}_\hbar}\sum_{\tau}1$$

$$\times\prod_{i=1}^{r}\sum_{\overline{\mathbf{n}}^{(F_i)}\in R(\mathbf{N},F_i,\nu_0)}\mathcal{A}(\mathbf{N})^{-f_i}a_{\mathbf{n}_{F_i(1)}}\cdots a_{\mathbf{n}_{F_i(f_i)}}\delta(L(\overline{\mathbf{n}}^{(F_i)},\nu_0)=\gamma^{(i)})=O(1).$$

By (1.1), (2.30) and (2.31), we obtain

$$|\sigma^{(2)}(\mathbf{N},\boldsymbol{\gamma})|\le\sum_{F_0\subsetneq F^{(\hbar)}}\sum_{\tau}1$$

$$\times\sum_{\overline{\mathbf{n}}_0^{(F_0)}\in R(\mathbf{N},F_0)}\mathcal{A}(\mathbf{N})^{-f_0}a_{\mathbf{n}_{0,F_0(1)}}\cdots a_{\mathbf{n}_{0,F_0(f_0)}}\sigma^{(3)}(\mathbf{N},\boldsymbol{\gamma},\overline{\mathbf{n}}_0^{(F_0)})=O(\rho(\mathbf{N})^{f_0}).$$

Thus, Lemma 4 is proved. ■

Let

$$f(\mathbf{x})=\sum_{\boldsymbol{\gamma}\in\mathbb{Z}^s}c_{\boldsymbol{\gamma}}e(2\pi\langle\boldsymbol{\gamma},\mathbf{x}\rangle)$$

be an absolutely convergent trigonometric series.

Lemma 5. *With the above notations, we have*

$$\lim_{N_0\to\infty}\int_{[0,1]^s}f(\mathbf{x})S(\mathbf{N},\mathbf{x})^{\hbar}\,d\mathbf{x}=\begin{cases}\dfrac{\hbar!}{2^{\hbar/2}(\hbar/2)!}c_{\mathbf{0}} & \text{if } \hbar \text{ is even},\\ 0 & \text{if } \hbar \text{ is odd}.\end{cases}$$

Proof. By (1.2), we get

$$f(\mathbf{x})S(\mathbf{N},\mathbf{x})^{\hbar}=\sum_{\boldsymbol{\gamma}\in\mathbb{Z}^s}c_{\boldsymbol{\gamma}}\sum_{\overline{\mathbf{n}}\in R(\mathbf{N})^{\hbar}}\sum_{\tau}\mathcal{A}(\mathbf{N})^{-\hbar}a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_\hbar}$$

$$\times e(2\pi\langle\boldsymbol{\gamma}+C(\overline{\mathbf{n}}^{(F)}),\mathbf{x}\rangle+\phi_{\mathbf{n}_1}+\dots+\phi_{\mathbf{n}_\hbar}).$$

Hence

$$\int_{[0,1]^s} f(\mathbf{x})S(\mathbf{N},\mathbf{x})^{\hbar}\,d\mathbf{x} = \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^s} c_{\boldsymbol{\gamma}} \sum_{\overline{\mathbf{n}}\in R(\mathbf{N})^{\hbar}} \sum_{\tau} \mathcal{A}(\mathbf{N})^{-\hbar} a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_{\hbar}} \delta(C(\overline{\mathbf{n}}^{(F)}) = -\boldsymbol{\gamma}) = c_{\mathbf{0}}\sigma^{(1)}(\mathbf{N}) + \sum_{\boldsymbol{\gamma}\in\mathbb{Z}^s\setminus\{\mathbf{0}\}} c_{\boldsymbol{\gamma}}\sigma^{(2)}(\mathbf{N},\boldsymbol{\gamma}).$$

Using Lemmas 3 and 4, we obtain the assertion. ■

2.1. End of the proof of the Theorem. It is sufficient to prove the Theorem for a box $D = [a_1,b_1]\times\cdots\times[a_s,b_s] \subset (0,1)^s$. Let $\chi_D(\cdot)$ be the indicator function of D, $D^- = \prod_{1\le i\le s}[a_i+\epsilon, b_i-\epsilon]$, and $D^+ = \prod_{1\le i\le s}[a_i-\epsilon, b_i+\epsilon] \subset (0,1)^s$.

We fix a nonnegative function $\omega(x)$, $x\in\mathbb{R}$, of class C^∞, supported inside the interval $|x|\le 1$, such that $\int_{\mathbb{R}}\omega(x)\,dx = 1$. Let $\omega_1(\mathbf{x}) = \omega(x_1)\cdots\omega(x_s)$, $\mathbf{x} = (x_1,\dots,x_s)$. The Fourier transform $\hat\omega_1(\mathbf{y}) = \int_{\mathbb{R}^s}\omega_1(\mathbf{x})e(2\pi\langle\mathbf{x},\mathbf{y}\rangle)\,d\mathbf{x}$ of $\omega_1(\mathbf{x})$ satisfies

$$\hat\omega_1(\mathbf{y}) = O((1+|\mathbf{y}|)^{-10s}). \tag{2.32}$$

Let $\omega_\epsilon(\mathbf{x}) = \epsilon^{-s}\omega_1(\epsilon^{-1}\mathbf{x})$. We consider the convolution of the indicator function $\chi_{D^\pm}(\cdot)$ with $\omega_\epsilon(\cdot)$:

$$\chi_{D^\pm,\epsilon}(\mathbf{y}) = \int_{[0,1]^s}\chi_{D^\pm}(\mathbf{x})\omega_\epsilon(\mathbf{x}-\mathbf{y})\,d\mathbf{x}. \tag{2.33}$$

Hence $\hat\chi_{D^\pm,\epsilon}(\mathbf{y}) = \hat\chi_{D^\pm}(\mathbf{y})\hat\omega_\epsilon(\mathbf{y})$. By (2.32), the Fourier series of $\chi_{D^-,\epsilon}(\mathbf{y})$ and of $\chi_{D^+,\epsilon}(\mathbf{y})$ are absolutely convergent. It is easy to verify that

$$\chi_{D^-,\epsilon}(\mathbf{y}) \le \chi_D(\mathbf{y}) \le \chi_{D^+,\epsilon}(\mathbf{y}), \tag{2.34}$$

and that

$$\prod_{i=1}^s(b_i-a_i-4\epsilon) \le \hat\chi_{D^-,\epsilon}(\mathbf{0}) \le \operatorname{mes} D, \qquad \operatorname{mes} D \le \hat\chi_{D^+,\epsilon}(\mathbf{0}) \le \prod_{i=1}^s(b_i-a_i+4\epsilon). \tag{2.35}$$

Using Lemma 5 and (2.34), we obtain

$$\begin{aligned}\frac{(2\hbar_1)!}{2^{\hbar_1}(\hbar_1)!}\hat\chi_{D^-,\epsilon}(\mathbf{0}) &= \lim_{N_0\to\infty}\int_{[0,1]^s}\chi_{D^-,\epsilon}(\mathbf{x})S(\mathbf{N},\mathbf{x})^{2\hbar_1}\,d\mathbf{x} \\ &\le \liminf_{N_0\to\infty}\int_{[0,1]^s}\chi_D(\mathbf{x})S(\mathbf{N},\mathbf{x})^{2\hbar_1}\,d\mathbf{x} \le \limsup_{N_0\to\infty}\int_{[0,1]^s}\chi_D(\mathbf{x})S(\mathbf{N},\mathbf{x})^{2\hbar_1}\,d\mathbf{x} \\ &\le \lim_{N_0\to\infty}\int_{[0,1]^s}\chi_{D^+,\epsilon}(\mathbf{x})S(\mathbf{N},\mathbf{x})^{2\hbar_1}\,d\mathbf{x} = \frac{(2\hbar_1)!}{2^{\hbar_1}(\hbar_1)!}\hat\chi_{D^+,\epsilon}(\mathbf{0}).\end{aligned}$$

Bearing in mind that the lim inf and lim sup in the middle do not depend on ϵ, from (2.35) we have

$$\frac{1}{\operatorname{mes} D} \lim_{N_0 \to \infty} \int_{[0,1]^s} \chi_D(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1} \, d\mathbf{x} = \frac{(2\hbar_1)!}{2^{\hbar_1} (\hbar_1)!}.$$

Hence (2.1) is proved for $\hbar$ even. Consider the case of $\hbar$ odd. We see that

$$\int_{[0,1]^s} \chi_D(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} = \int_{[0,1]^s} \chi_{D^{+,\epsilon}}(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} + \int_{[0,1]^s} (\chi_D(\mathbf{x}) - \chi_{D^{+,\epsilon}}(\mathbf{x})) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x}.$$

Applying the Cauchy–Schwarz inequality, we get

$$\Big(\int_{[0,1]^s} (\chi_D(\mathbf{x}) - \chi_{D^{+,\epsilon}}(\mathbf{x})) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} \Big)^2 \le \int_{[0,1]^s} (\chi_D(\mathbf{x}) - \chi_{D^{+,\epsilon}}(\mathbf{x}))^2 \, d\mathbf{x} \int_{[0,1]^s} S(\mathbf{N}, \mathbf{x})^{4\hbar_1+2} \, d\mathbf{x}$$

and

$$\Big(\int_{[0,1]^s} \chi_D(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} \Big)^2 \le 2 \Big(\int_{[0,1]^s} \chi_{D^{+,\epsilon}}(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} \Big)^2 + 2 \Big(\int_{[0,1]^s} (\chi_D(\mathbf{x}) - \chi_{D^{+,\epsilon}}(\mathbf{x})) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} \Big)^2. \tag{2.36}$$

By (2.33), we obtain $0 \le \chi_{D^{+,\epsilon}}(\mathbf{x}) \le 1$ for all $\mathbf{x}$, and

$$\int_{[0,1]^s} (\chi_D(\mathbf{x}) - \chi_{D^{+,\epsilon}}(\mathbf{x}))^2 \, d\mathbf{x} \le \prod_{i=1}^{s} (b_i - a_i + 4\epsilon) - \prod_{i=1}^{s} (b_i - a_i) = O(\epsilon). \tag{2.37}$$

Using Lemma 5, we get

$$\lim_{N_0 \to \infty} \int_{[0,1]^s} \chi_{D^{+,\epsilon}}(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} = O(\epsilon), \qquad \int_{[0,1]^s} S(\mathbf{N}, \mathbf{x})^{4\hbar_1+2} \, d\mathbf{x} = O(1).$$

From (2.36) and (2.37) we have

$$\lim_{N_0 \to \infty} \int_{[0,1]^s} \chi_D(\mathbf{x}) S(\mathbf{N}, \mathbf{x})^{2\hbar_1+1} \, d\mathbf{x} = O(\epsilon). \tag{2.38}$$

Taking into account that the left hand side of (2.38) does not depend on ϵ, we find that (2.1) is true for $\hbar$ odd. Hence, the Theorem is proved. ∎

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