A LIPSCHITZ FUNCTION WHICH IS $C^\infty$ ON A.E. LINE
NEED NOT BE GENERICALLY DIFFERENTIABLE

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Abstract. We construct a Lipschitz function $f$ on $X = \mathbb{R}^2$ such that, for each $0 \neq v \in X$, the function $f$ is $C^\infty$ smooth on a.e. line parallel to $v$ and $f$ is Gâteaux non-differentiable at all points of $X$ except a first category set. Consequently, the same holds if $X$ (with dim $X > 1$) is an arbitrary Banach space and “a.e.” has any usual “measure sense”. This example gives an answer to a natural question concerning the author’s recent study of linearly essentially smooth functions (which generalize essentially smooth functions of Borwein and Moors).

1. Introduction. There exist a number of results which assert that some “partial or directional smoothness property” (e.g., smoothness on some lines or directional differentiability in some directions) of a function $f$ on a Banach space $X$ implies some “global smoothness property” (e.g. Gâteaux or Fréchet differentiability at many points). For results of this sort see e.g. [JP], [S], [I], [PZ].

The present note is motivated by the special question whether a “smoothness on many lines” of a Lipschitz function $f$ on $X$ implies generic Fréchet differentiability of $f$ (where “generic” has the usual meaning “at all points except a first category set”).

A remarkable result in this direction ([S]) says that if an (a priori arbitrary) function $f$ on $X = \mathbb{R}^n$ has all partial directional derivatives at all points (in other words, $f$ is differentiable on each line parallel to a coordinate axis), then $f$ is generically Fréchet differentiable. On the other hand, if $X = \ell_2$, then (see [Pr]) there exists a Lipschitz function on $X$ which is everywhere Gâteaux differentiable (and so differentiable on all lines) but generically Fréchet non-differentiable.

A contribution to this special question is given in the article [Z2] which was motivated by the papers [BM1], [BM2] of Borwein and Moors on “essentially smooth” functions.

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For example [Z2, Theorem 5.2] reads as follows.

**Theorem A.** Let $X$ be an Asplund space and $f : X \to \mathbb{R}$ a Lipschitz function. Suppose that there exists a set $D$ which is dense in the unit sphere $S_X$ such that, for each $v \in D$, $f$ is essentially smooth on a generic line parallel to $v$. Then $f$ is generically Fréchet differentiable.

Here “$f$ is essentially smooth on the line $L$” means “the restriction of $f$ is a.e. strictly differentiable on $L$”. So each function which is $C^1$ on a line $L$ is essentially smooth on $L$. (Recall also that $X$ is Asplund if and only if $Y^*$ is separable for each separable subspace $Y \subseteq X$.)

In [Z2, Remark 1.4(iii)], it was announced that, in Theorem A, one cannot only suppose that $f$ is essentially smooth on each line from a set of lines which is dense in the space of all lines parallel to $v \in D$. (So it is not sufficient to suppose that $f$ is essentially smooth on each line from a set of lines which is dense in the space of all lines; cf. Remark 3.7).

The main aim of the present note is to construct the following much stronger example (Theorem 3.6 below), in which we obtain even generic Gâteaux non-differentiability.

**Let $X$ be a Banach space, $\dim X > 1$. Then there exists a Lipschitz function $f$ on $X$ such that, for each $v \in S_X$, $f$ is $C^\infty$ on a.e. line parallel to $v$ and $f$ is generically Gâteaux non-differentiable.**

Here “a.e. line parallel to $v$” is taken in a very strong sense (using “$\ast$-nullness”, see Definition 3.5). Note that each $\ast$-null set is clearly Lebesgue null if $X = \mathbb{R}^n$ and is Gaussian (= Aronszajn) null and also $\Gamma$-null if $X$ is separable.

We stress that our construction is “two-dimensional”; if we have an example in $\mathbb{R}^2$, then the construction in a general $X$ is rather obvious. The notion of $\ast$-nullness is not of general interest, we introduce it only to be able succinctly formulate our result in general $X$.

Further note that in the case $X = \mathbb{R}^n$ the function $f$ from our example is $C^\infty$ on a.e. line in $X$, which justifies the title of the note. This is immediately seen from the canonical definition of the measure on the set of all lines in $\mathbb{R}^n$ (see [Ma, p. 53]).

Note also that the main idea of the construction is similar to that of [Po].

2. Preliminaries. In the following, if not said otherwise, $X$ will be a real Banach space. We set $S_X := \{ x \in X : \| x \| = 1 \}$. If $a, b \in X$, then $\overline{a, b}$ denotes the closed segment. By span $M$ we denote the linear span of $M \subseteq X$. The equality $X = X_1 \oplus \cdots \oplus X_n$ means that $X$ is the direct sum of non-trivial closed linear subspaces $X_1, \ldots, X_n$ and the corresponding projections $\pi_i : X \to X_i$ are continuous.
We say that a function $f : X \rightarrow \mathbb{R}$ is $C^{\infty}$ on a line $L = a + \mathbb{R}v$ if the function $h(t) := f(a + tv)$ is $C^{\infty}$ on $\mathbb{R}$. (Clearly, this definition does not depend on the choice of $a$ and $v$.)

The symbol $B(x, r)$ will denote the open ball with center $x$ and radius $r$. The word “generically” has the usual sense; it means “at all points except a first category set”.

The symbol $\mathcal{H}^k$ denotes the $k$-dimensional Hausdorff measure.

We will need the following easy well-known fact several times.

**Lemma 2.1.** Let $X$ be a Banach space, $0 \neq u \in X$, and let $X = W \oplus \text{span}\{u\}$. Then the mapping $w \in W \mapsto w + Ru \in X/\text{span}\{u\}$ is a linear homeomorphism.

In the following, $f$ is a real function defined on an open subset $G$ of $X$.

We say that $f$ has a property generically on $G$ if $f$ has this property at each point of $G$ except a first category set.

We say that $f$ is $K$-Lipschitz ($K \geq 0$) if $f$ is Lipschitz with (not necessarily least) constant $K$.

Recall the well-known easy fact that

\[ \text{(2.1) if } f \text{ is Lipschitz and } \dim X < \infty, \text{ then the Gâteaux and Fréchet derivatives of } f \text{ coincide.} \]

Recall also (see [Mo]) that $x^* \in X^*$ is called the strict derivative of $f$ at $a \in G$ if

\[ \lim_{(x, y) \to (a, a), x \neq y} \frac{f(y) - f(x) - x^*(y - x)}{\|y - x\|} = 0. \]

It is well-known and easy to see that if $f'(a)$ is the strict derivative of $f$ at $a \in X$ and $v \in X$, then

\[ \lim_{n \to \infty} \frac{f(a_n + t_n v) - f(a_n)}{t_n} = f'(a)(v) \]

whenever $a_n \to a$, $t_n \to 0^+$.  

Strict differentiability is a stronger condition than Fréchet differentiability, but (see e.g. [Z1, Theorem B, p. 476]), for an arbitrary $f$,

\[ \text{(2.3) generically, Fréchet differentiability of } f \text{ implies strict differentiability of } f. \]

The directional and one-sided directional derivatives of $f$ at $x$ in the direction $v$ are defined respectively by

\[ f'(x, v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad f'_+(x, v) := \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}. \]
We will need some well-known facts about mollification of functions. Let \( \eta : \mathbb{R}^n \to \mathbb{R} \) be the function defined as \( \eta(x) = 0 \) for \( \|x\| \geq 1 \) and \( \eta(x) = c \exp((\|x\|^2 - 1)^{-1}) \) for \( \|x\| < 1 \), where \( c \) is such that \( \int_{\mathbb{R}^n} \eta = 1 \). For \( \delta > 0 \), we define (the standard mollifier, see [4])
\[
\eta_\delta(x) = \frac{1}{\delta^n} \eta\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^n.
\]
If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), define (the standard mollifier, see [4])
\[
f_\delta(x) := \eta_\delta * f(x) = \int_{\mathbb{R}^n} \eta_\delta(x-y) f(y) \, dy = \int_{\mathbb{R}^n} \eta_\delta(y) f(x-y) \, dy, \quad x \in \mathbb{R}^n.
\]
We will need the following well-known facts.

**Fact 2.2.** Let \( f \) be a \( K \)-Lipschitz function on \( \mathbb{R}^n \) and \( \delta > 0 \). Then
(i) \( f_\delta \in C^\infty(\mathbb{R}^n) \).
(ii) \( f_\delta \to f \) (as \( \delta \to 0^+ \)) uniformly on compact subsets of \( \mathbb{R}^n \).
(iii) \( f_\delta \) is \( K \)-Lipschitz.
(iv) If \( x \in \mathbb{R}^n, \delta > 0, \) and \( f \) equals an affine function \( \alpha \) on \( B(x,\delta) \), then \( f_\delta(x) = \alpha(x) \).

For (i) and (ii) see \[EG\,\text{Theorem 1(i),(ii), p. 123}\]; (iii) and (iv) are also well-known and almost obvious, so I omit their proof, although I have not found an explicit reference.

3. Main result

**Lemma 3.1.** Let \( K \geq 4 \) and let \( f \in C^\infty(\mathbb{R}^2) \) be a \( K \)-Lipschitz function. Let \( 0 \neq H \subset \mathbb{R}^2 \) be an open set and \( 0 < \varepsilon < 1 \). Then there exist \( \tilde{f} \in C^\infty(\mathbb{R}^2) \), \( c \in H \) and \( t > 0 \) with the following properties:
(i) \( f(x) = \tilde{f}(x) \) for each \( x \in \mathbb{R}^2 \setminus H \).
(ii) \( |f(x) - \tilde{f}(x)| < \varepsilon \) for each \( x \in \mathbb{R}^2 \).
(iii) \( \tilde{f} \) is a \((K+\varepsilon)\)-Lipschitz function.
(iv) The points \( c, c + te_1 \) and \( c - te_1 \) (where \( e_1 := (1,0) \)) belong to \( H \),

\[
\frac{\tilde{f}(c+te_1) - \tilde{f}(c)}{t} \geq 1 \quad \text{and} \quad \frac{\tilde{f}(c) - \tilde{f}(c-te_1)}{t} \leq -1.
\]

**Proof.** Choose \( c \in H \) and consider the affine function \( \alpha(x) := f(c) + f'(c)(x-c) \) for \( x \in \mathbb{R}^2 \). Since \( f \in C^1(\mathbb{R}^2) \), we can clearly choose \( r > 0 \) such that
\[
0 < r < 1, \quad B(c, r) \subset H
\]
and
\[
\text{the function } f - \alpha \text{ is } (\varepsilon/2)\text{-Lipschitz on } B(c, r).
\]
Observe that \( \|f'(c)\| \leq K \) and so \( \alpha \) is a \( K \)-Lipschitz function.
For \( x \in \mathbb{R}^2 \), set
\[
\varphi(x) := \alpha(c) - \frac{\varepsilon^2 r}{8K^2} + (K + \varepsilon/2) \|x - c\| \quad \text{and} \quad g(x) := \min(\varphi(x), \alpha(x)).
\]

We will need the following properties of the function \( g \):

(P1) \( g \) is \((K + \varepsilon/2)\)-Lipschitz.
(P2) \( g(x) = \alpha(x) \) for each \( x \in \mathbb{R}^2 \setminus B(c, r/4) \).
(P3) \(|g(x) - \alpha(x)| < \varepsilon r / K \) for each \( x \in \mathbb{R}^2 \).
(P4) There exists \( t > 0 \) such that \( c \pm te_1 \in B(c, r) \),

\[
(3.4) \quad \frac{g(c + te_1) - g(c)}{t} = K + \varepsilon/2 \quad \text{and} \quad \frac{g(c) - g(c - te_1)}{t} = -(K + \varepsilon/2).
\]

To prove these properties, first recall that \( \alpha \) is \( K \)-Lipschitz, and since \( \varphi \) is clearly \((K + \varepsilon/2)\)-Lipschitz, we obtain (P1).

If \(|x - c| \geq \varepsilon r/(4K^2)\), we obtain
\[
\alpha(x) \leq \alpha(c) + K \|x - c\| = \varphi(x) + \frac{\varepsilon^2 r}{8K^2} - \frac{\varepsilon}{2} \|x - c\|
\]
and (P2) follows since \( \varepsilon r/(4K^2) < r/4 \).

If \(|x - c| < \varepsilon r/(4K^2)\), then \(|\alpha(x) - \alpha(c)| < K(\varepsilon r/(4K^2)) \) and \(|\varphi(x) - \varphi(c)| < (K + \varepsilon/2)(\varepsilon r/(4K^2)) \). Consequently,
\[
|g(x) - \alpha(x)| \leq |\varphi(x) - \alpha(x)| \leq |\alpha(c) - \varphi(c)| + |\alpha(x) - \alpha(c)| + |\varphi(x) - \varphi(c)|
\]
\[
\leq \frac{\varepsilon^2 r}{8K^2} + K \left( \frac{\varepsilon r}{4K^2} \right) + \left( K + \frac{\varepsilon}{2} \right) \frac{\varepsilon r}{4K^2} < \varepsilon r / K,
\]
which gives (P3), since we have proved that \( g(x) = \alpha(x) \) if \(|x - c| \geq \varepsilon r/(4K^2)\).

Since \( \alpha \) and \( \varphi \) are continuous, we can clearly choose \( t > 0 \) so small that \( c \pm te_1 \in B(c, r) \), \( \varphi(c + te_1) < \alpha(c + te_1) \) and \( \varphi(c - te_1) < \alpha(c - te_1) \). Then \( g(c) = \varphi(c) \) and \( g(c + te_1) = \varphi(c + te_1) \) and so, by the definition of \( \varphi \), we clearly obtain (3.4). Thus we have proved (P4).

Now, for \( \delta > 0 \), consider the mollification \( g^\delta \) of \( g \). By Fact 2.2(i),(iii), we deduce that \( g^\delta \in C^\infty(\mathbb{R}^2) \) and \( g^\delta \) is \((K + \varepsilon/2)\)-Lipschitz.

Using (P2) and Fact 2.2(iv) we find that, if \( 0 < \delta < r/4 \), then
\[
(3.5) \quad g^\delta(x) = g(x) = \alpha(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus B(c, r/2).
\]

So, using Fact 2.2(ii) for the compact set \( B(c, r) \), we easily see that we can choose \( \delta \in (0, r/4) \) so small that
\[
(3.6) \quad |g^\delta(x) - g(x)| < \varepsilon r / K \quad \text{for each} \quad x \in \mathbb{R}^2
\]
and, using (3.4), also
\[
(3.7) \quad \frac{g^\delta(c + te_1) - g^\delta(c)}{t} \geq 2 \quad \text{and} \quad \frac{g^\delta(c) - g^\delta(c - te_1)}{t} \leq -2.
\]
By (3.6) and (P3) we obtain
\[
(3.8) \quad |g^\delta(x) - \alpha(x)| < 2\varepsilon r/K \quad \text{for each } x \in \mathbb{R}^2.
\]
Define \( \tilde{f} := f + g^\delta - \alpha \). Clearly \( \tilde{f} \in C^\infty(\mathbb{R}^2) \). We will show that \( \tilde{f} \) has also properties (i)-(iv).

By (3.5) we have
\[
(3.9) \quad \tilde{f}(x) = f(x) \quad \text{for } x \in \mathbb{R}^2 \setminus B(c, r/2),
\]
which implies (i).

By (3.8) we obtain
\[
(3.10) \quad |\tilde{f}(x) - f(x)| < 2\varepsilon r/K < \varepsilon \quad \text{for each } x \in \mathbb{R}^2,
\]
so (ii) holds.

Since \( \tilde{f} := (f - \alpha) + g^\delta \), \( g^\delta \) is \((K + \varepsilon/2)\)-Lipschitz and \( f - \alpha \) is \((\varepsilon/2)\)-Lipschitz on \( B(c, r) \) (see (3.3)), we find that
\[
(3.11) \quad \tilde{f} \text{ is a } (K + \varepsilon)\text{-Lipschitz function on } B(c, r).
\]
Using (3.5) we deduce that
\[
(3.12) \quad \tilde{f} = f + (g^\delta - \alpha) \text{ is } K\text{-Lipschitz on } \mathbb{R}^2 \setminus B(c, r/2).
\]
Further, consider arbitrary \( x_1, x_2 \in \mathbb{R}^2 \) such that \( x_1 \in B(c, r/2) \) and \( x_2 \notin B(c, r) \). Then, using (3.9) and (3.10), we obtain
\[
|\tilde{f}(x_2) - \tilde{f}(x_1)| = |f(x_2) - f(x_1)| \leq |f(x_2) - f(x_1)| + |f(x_1) - \tilde{f}(x_1)| \\
\leq K|x_2 - x_1| + 2\varepsilon r/K \leq K|x_2 - x_1| + (4\varepsilon/K)|x_2 - x_1| \leq (K + \varepsilon)|x_2 - x_1|.
\]
This inequality together with (3.11) and (3.12) clearly implies (iii).

Finally, since \( \tilde{f} := (f - \alpha) + g^\delta \), (3.3), (3.7) and the fact that the points \( c, c + te_1, c - te_1 \) belong to \( B(c, r) \) easily imply (iv).

**Lemma 3.2.** Let \( M_n \subset \mathbb{R}^2, n \in \mathbb{N} \), be nowhere dense sets. Then there exists a Lipschitz function \( f \) on \( \mathbb{R}^2 \) such that

(a) \( f \) is \( C^\infty \) on each line which is contained in a set \( M_n, n \in \mathbb{N} \), and

(b) \( f \) is generically Gâteaux non-differentiable.

**Proof.** We can clearly choose a set \( D = \{d_n : n \in \mathbb{N}\} \) which is dense in \( \mathbb{R}^2 \) and \( D \cap \bigcup_{k \in \mathbb{N}} M_k = \emptyset \). For each \( n \in \mathbb{N} \), choose \( 0 < r_n < 1/n \) such that \( B(d_n, r_n) \cap \bigcup_{k=1}^n M_k = \emptyset \) and define \( B_n := B(d_n, r_n) \). Set \( \varepsilon_n := 2^{-n} \) and \( e_1 := (1, 0) \).

Now we will inductively construct sequences \( (c_n)_{n=1}^\infty \) of points in \( \mathbb{R}^2 \), \( (f_n)_{n=0}^\infty \) of \( C^\infty \) functions on \( \mathbb{R}^2 \) and \( (t_n)_{n=1}^\infty \) of positive reals such that
$f_0(x) = 0$, $x \in X$, and for each $n \in \mathbb{N}$ the following hold:

(i) $\{c_n, c_n + t_n e_1, c_n - t_n e_1\} \subset B_n$.
(ii) $\frac{f_n(c_n + t_n e_1) - f_n(c_n)}{t_n} \geq 1$ and $\frac{f_n(c_n) - f_n(c_n - t_n e_1)}{t_n} \leq -1$.
(iii) $f_n(x) = f_{n-1}(x)$ for $x \in (\mathbb{R}^2 \setminus B_n) \cup \bigcup_{k=1}^{n-1} \{c_k, c_k + t_k e_1, c_k - t_k e_1\}$.
(iv) $|f_n(x) - f_{n-1}(x)| < \varepsilon_n$ for each $x \in \mathbb{R}^2$.
(v) $f_n$ is a $(4 + \sum_{k=1}^{n} \varepsilon_k)$-Lipschitz function.

Of course, we put $\bigcup_{k=1}^{n} \{c_k, c_k + t_k e_1, c_k - t_k e_1\} := \emptyset$ (and also $\sum_{k=1}^{n} \varepsilon_k \equiv 0$ below).

We set $f_0(x) := 0$, $x \in X$. Further suppose that $m \in \mathbb{N}$ is given, $c_n, f_n, t_n$ are defined for $1 \leq n < m$, and (i)–(v) hold whenever $1 \leq n < m$.

Applying Lemma 3.1 to $K := 4 + \sum_{k=1}^{n-1} \varepsilon_k$, $f := f_{m-1}$, $H := B_m \setminus \bigcup_{k=1}^{m-1} \{c_k, c_k + t_k e_1, c_k - t_k e_1\}$ and $\varepsilon := \varepsilon_m$, we obtain a function $\tilde{f} := f_m$, $c := c_m \in H$ and $t := t_m > 0$ such that (i)–(v) clearly hold for $n = m$.

Condition (iv) shows that the series

$$f_1 + (f_2 - f_1) + (f_3 - f_2) + \cdots$$

(uniformly) converges on $\mathbb{R}^2$ and consequently the sequence $(f_n)$ converges to a function $f$. Since all $f_n$ are 5-Lipschitz by (v), so is $f$.

To prove (a), suppose that $L$ is a line in $\mathbb{R}^2$, $k \in \mathbb{N}$ and $L \subset M_k$. Since $M_k \subset \mathbb{R}^2 \setminus B_n$ for each $n \geq k$, we deduce by (iii) that $f_n(x) = f_{n-1}(x)$ for each $x \in L$ and $n \geq k$, and consequently $f(x) = f_k(x)$, $x \in L$. Since $f_k$ is $C^\infty$ on $\mathbb{R}^2$, we see that $f$ is $C^\infty$ on $L$.

To prove (b), first observe that, by (iii), for each $n > k$ and $x \in \{c_k, c_k + t_k e_1, c_k - t_k e_1\}$ we have $f_n(x) = f_{n-1}(x)$, and so $f(x) = f_k(x)$. Thus (ii) implies that, for each $k \in \mathbb{N}$,

$$f(c_k + t_k e_1) - f(c_k) \geq 1 \quad \text{and} \quad f(c_k) - f(c_k - t_k e_1) \leq -1. \quad \text{(3.13)}$$

This easily implies that

$$f \text{ is strictly differentiable at no point of } \mathbb{R}^2. \quad \text{(3.14)}$$

Indeed, suppose to the contrary that $f$ is strictly differentiable at a point $x \in \mathbb{R}^2$. Using (i), we can easily find a subsequence $(c_{n_i})$ of $(c_n)$ with $c_{n_i} \to x$. Then clearly $t_{n_i} \to 0$ and so, by (2.2) and (3.13),

$$\lim_{i \to \infty} \frac{f(c_{n_i} + t_{n_i} e_1) - f(c_{n_i})}{t_{n_i}} = f'(x)(e_1) \geq 1 \quad \text{and}$$

$$\lim_{i \to \infty} \frac{f(c_{n_i}) - f(c_{n_i} - t_{n_i} e_1)}{t_{n_i}} = f'(x)(e_1) \leq -1,$$

which is a contradiction. By (3.14), (2.3) and (2.1) we obtain (b). \(\blacksquare\)
**Proposition 3.3.** There exists a Lipschitz function $f$ on $\mathbb{R}^2$ such that

(a) for each $0 \neq v \in \mathbb{R}^2$, $f$ is $C^\infty$ on a.e. line parallel to $v$, and

(b) $f$ is generically Gâteaux non-differentiable.

**Proof.** Choose a set $\{d_k : k \in \mathbb{N}\}$ dense in $\mathbb{R}^2$. For each $n,k \in \mathbb{N}$, set

$$B_{n,k} := B(d_k,(2^k n)^{-1}) \quad \text{and} \quad G_n := \bigcup_{k=1}^\infty B_{n,k}.$$  \ \ (3.15)

Then each $G_n$ is clearly open dense, and consequently $M_n := \mathbb{R}^2 \setminus G_n$ is nowhere dense. Applying Lemma 3.2, we obtain a Lipschitz function $f$ on $\mathbb{R}^2$ such that $f$ is generically Gâteaux non-differentiable and

$$f \text{ is } C^\infty \text{ on each line which is contained in a set } M_n, \ n \in \mathbb{N}. \ \ (3.16)$$

Fix an arbitrary $0 \neq v \in \mathbb{R}^2$. Let $W$ be the orthogonal complement of $\text{span}\{v\}$ and let $\pi$ be the orthogonal projection on $W$. Then $\pi(G_n) = \bigcup_{k=1}^\infty \pi(B_{n,k})$ and so

$$\mathcal{H}^1(\pi(G_n)) \leq \sum_{k=1}^\infty \mathcal{H}^1(\pi(B_{n,k})) = \sum_{k=1}^\infty 2(2^k n)^{-1} = \frac{2}{n}. \ \ (3.17)$$

Consequently,

$$\mathcal{H}^1\left(\bigcap_{n=1}^\infty \pi(G_n)\right) = 0. \ \ (3.17)$$

Let now $w \in W \setminus \bigcap_{n=1}^\infty \pi(G_n)$. Then there exists $n$ with $w \notin \pi(G_n)$ and so the line which contains $w$ and is parallel to $v$ is contained in $M_n$. Hence, by (3.16) and (3.17), $f$ is $C^\infty$ on a.e. line parallel to $v$. 

**Remark 3.4.** The assertion of Proposition 3.3 can easily be strengthened; namely we can consider “a.e.” with respect to any generalized Hausdorff measure $\Lambda_h$ given by a non-decreasing $h : [0, \infty) \to [0, \infty)$; see [Ma, p. 60]. Indeed, it is easy to slightly refine the proof of Proposition 3.3. Namely, it is sufficient to make two changes:

(a) to set $B_{n,k} := B(d_k,r_{n,k})$, where $r_{n,k} > 0$ and $\sum_{k=1}^\infty h(2r_{n,k}) < 1/n$;

(b) in the proof of $\Lambda_h(\bigcap_{n=1}^\infty \pi(G_n)) = 0$, to use the definition of $\Lambda_h$ (instead of the subadditivity of $\mathcal{H}^1$).

To apply Proposition 3.3 in infinite-dimensional spaces, we find it useful to introduce the following terminology.

**Definition 3.5.** Let $X$ be a Banach space with $\dim X > 1$. We say that $M \subset X$ is $*$-null if there exists $0 \neq x^* \in X^*$ such that $x^*(M) \subset \mathbb{R}$ is Lebesgue null.
Obviously, if \( X = \mathbb{R}^n \), then each \(*\)-null set in \( X \) is Lebesgue null. If \( X \) is an infinite-dimensional separable space, then each \(*\)-null set \( M \) in \( X \) is contained in an Aronszajn null (= Gauss null) set and is also \( \Gamma \)-null. This can be proved directly from definitions, but we can also use the following standard quicker argument:

Let \( x^* \) be as in Definition 3.5 and let \( h \) be a Lipschitz function on \( \mathbb{R} \) which is differentiable at no point of \( x^*(M) \) (see [BL, p. 165]). Then \( f := h \circ x^* \) is clearly a Lipschitz function on \( X \) which is Gâteaux differentiable at no point of \( M \). So our assertion follows from [BL, Theorem 6.42] and [LPT, Theorem 5.2.3].

Note also that if \( X \) is non-separable then it is easy to see that each \(*\)-null set \( M \subset X \) is Haar null. Moreover, using [LPT, Corollary 5.6.2], it is not difficult to prove that \( M \) is \( \Gamma \)-null.

**Theorem 3.6.** Let \( X \) be a Banach space and \( \dim X \geq 2 \). Then there exists a Lipschitz function \( f \) on \( X \) such that

(i) for each \( 0 \neq v \in X \), the function \( f \) is \( C^\infty \) on \(*\)-a.e. line parallel to \( v \), and

(ii) \( f \) is generically Gâteaux non-differentiable.

(Of course, condition (i) is a short expression of the statement that there exists a \(*\)-null set \( N \) in \( X/\text{span}\{v\} \) such that \( f \) is \( C^\infty \) on each line \( L \in X/\text{span}\{v\} \setminus N \).)

**Proof.** If \( \dim X = 2 \), then the assertion clearly follows from Proposition 3.3.

So suppose \( \dim X \geq 3 \). Write \( X = P \oplus Y \) with \( \dim P = 2 \). By Proposition 3.3 choose a Lipschitz function \( g \) on \( P \) and a first category set \( A \subset P \) such that \( g \) is Gâteaux non-differentiable at all points of \( P \setminus A \) and, for each \( 0 \neq u \in P \), the function \( g \) is \( C^\infty \) on a.e. line parallel to \( u \). Let \( \pi : X \to P \) be the linear projection of \( X \) on \( P \) in the direction of \( V \). Set \( f := g \circ \pi \).

It is easy to see that \( f \) is a Lipschitz function which is Gâteaux non-differentiable at all points outside the (first category) set \( \pi^{-1}(A) \). So (ii) holds.

To prove (i), consider an arbitrary \( 0 \neq v \in X \). If \( u := \pi(v) = 0 \), then \( f \) is clearly constant on each line parallel to \( v \). So suppose \( u \neq 0 \). Then we can write \( P = \text{span}\{u\} \oplus Z \) with \( \dim Z = 1 \). Let \( \varphi : Z \to \mathbb{R} \) be a linear homeomorphism. By the choice of \( g \) and Lemma 2.1 there exists \( N \subset Z \) such that \( \varphi(N) \subset \mathbb{R} \) is Lebesgue null and the function \( h(t) := g(d + tu), \ t \in \mathbb{R}, \) is \( C^\infty \) for each \( d \in Z \setminus N \).

Observe that \( N + Y \) is \(*\)-null in \( Z + Y \). Indeed, for \( \psi := \varphi \circ (\pi|_{Z + Y}) \) we have \( 0 \neq \psi \in (Z + Y)^* \) and so \( \psi(N + Y) = \varphi(N) \) is Lebesgue null. Now let \( p \in (Z + Y) \setminus (N + Y) \). Then we can write \( p = d + y \), where \( d \in Z \setminus N \) and
Observing that $f(p + tv) = f(d + y + tv) = g(d + tu) = h(t)$ and using Lemma 2.1, we easily obtain (i).

**Remark 3.7.** Each set containing $*$-a.e. line parallel to $v$ is clearly dense in the space $X/\text{span}\{v\}$.

Consequently, the function $f$ from Theorem 3.6 is $C^\infty$ on a dense set of lines in the space $\mathcal{L}$ of all lines in $X$. Here we consider the topology on $\mathcal{L}$ in which, for a line $L = a_0 + \mathbb{R}v_0$,

$$B_L := \{\{a + \mathbb{R}v : \|a - a_0\| < \varepsilon, \|v - v_0\| < \varepsilon\} : \varepsilon > 0\}$$

is a basis of the filter of all neighbourhoods of $L$.

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**REFERENCES**


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