COLLOQUIUM MATHEMATICUM

VOL. 131 2013 NO. 1

A NOTE ON THE DIOPHANTINE EQUATION $P(z) = n! + m!$

BY

MACIEJ GAWRON (Kraków)

Abstract. We consider the Brocard–Ramanujan type Diophantine equation $P(z) = n! + m!$, where P is a polynomial with rational coefficients. We show that the ABC Conjecture implies that this equation has only finitely many integer solutions when $d \geq 2$ and $P(z) = a_d z^d + a_{d-3} z^{d-3} + \cdots + a_1 x + a_0$.

1. Introduction. In this paper, we consider the Diophantine equation

$$P(z) = m! + n!,$$

where $P(x) = a_d x^d + a_{d-3} x^{d-3} + \cdots + a_1 x + a_0 \in \mathbb{Q}[X]$ is any polynomial of degree $d \geq 2$ of this form. We prove that under the ABC Conjecture, this equation has only finitely many integer solutions. This is a generalization of Luca's result [7], who proved the same for the equation $P(x) = n!$ where $P \in \mathbb{Z}[X]$ is any polynomial of degree $d \geq 2$. Our result depends on the ABC Conjecture, which we recall later. If we do not assume the ABC Conjecture, the problem is still unsolved, even for the equation

$$x^2 - 1 = n!,$$

which is the well-known Brocard–Ramanujan equation [3], [8]. Berndt and Galway showed by numerical computations that this equation has only three solutions with $n \leq 10^9$ [2]. One can also consult an interesting paper of Dąbrowski [4] related to the more general Diophantine equation of the form $y^2 = x! + A$, where A is given integer, and a recent papers of Ulas [9] and Dąbrowski and Ulas [5], concerning some computational results for the equation studied by Dąbrowski.

We assume that the coefficients a_{d-1}, a_{d-2} are missing in P. Note that if $P \in \mathbb{Q}[X]$ is any polynomial of degree d, then by a standard trick, one can find integers a, b and a polynomial $Q \in \mathbb{Q}[X]$ such that $P(x) = Q(ax - b)$ and the coefficient of x^{d-1} is missing in Q. So, in general, one can assume that $a_{d-1} = 0$.

2010 *Mathematics Subject Classification*: Primary 11D85.

Key words and phrases: Brocard–Ramanujan type equation, ABC Conjecture, Diophantine equation.

DOI: 10.4064/cm131-1-5

In our paper, p is always a prime number. Moreover we will use ϑ to denote the Chebyshev function $\vartheta(x) = \sum_{p\le x} \ln p$. We will also use the fact that $\vartheta(x) \sim x$ as $x \to \infty$, which is a consequence of the prime number theorem. The integer $v_p(x)$ is the exponent of p in the factorization of the nonzero rational number x. We also use the well-known elementary estimate $\prod_{p\le n} p < 4^n$. Finally, $M, M_1, M_\epsilon, M_\theta, \ldots$ are always positive constants.

2. Preliminaries. Let us recall the ABC Conjecture.

DEFINITION 2.1. For $n \in \mathbb{Z} \setminus \{0\}$ we define the *radical* of n to be the product of all primes that divide n,

$$\operatorname{rad}(n) = \prod_{p\mid n} p.$$

CONJECTURE 2.2 (ABC Conjecture). *For each $\epsilon > 0$, there exists $M_\epsilon > 0$ such that whenever $a, b, c \in \mathbb{Z} \setminus \{0\}$ satisfy the conditions*

$$\gcd(a, b, c) = 1 \quad \text{and} \quad a + b + c = 0,$$

then

$$\max\{|a|, |b|, |c|\} \le M_\epsilon \operatorname{rad}(abc)^{1+\epsilon}.$$

We will use the following useful lemma, for triples (a, b, c) of integers with $\gcd(a, b, c) > 1$.

LEMMA 2.3. *Under the ABC Conjecture the following statement is true. For each $\epsilon > 0$, there exists $M_\epsilon > 0$ such that whenever a, b, c are positive integers with $a + b = c$, then*

$$c \le M_\epsilon (a \operatorname{rad}(bc))^{1+\epsilon}.$$

Proof. Let $d = \gcd(a, b, c)$. Then we can apply the ABC Conjecture to the triple $(a/d, b/d, c/d)$ to get

$$\frac{c}{d} \le M_\epsilon \operatorname{rad}\left(\frac{abc}{d^3}\right)^{1+\epsilon} \le M_\epsilon \left(\frac{a}{d} \operatorname{rad}(bc)\right)^{1+\epsilon},$$

and the conclusion follows. ■

Let us recall Luca's result, which we will use in our considerations.

THEOREM 2.4 (Luca [7]). *Under the ABC Conjecture, for every polynomial $P \in \mathbb{Z}[X]$ of degree $d \ge 2$, the equation*

$$P(m) = n!$$

has only finitely many integer solutions.

NOTE 2.5. We can take $P \in \mathbb{Q}[X]$ in the previous theorem as well.

We also formulate a strong conjecture about this type of Diophantine equations, which may be an object of further research.

Conjecture 2.6. *For every polynomial $P \in \mathbb{Z}[X]$ of degree $d \geq 2$ and every integer $k > 0$, the equation*

$$P(z) = x_1! + x_2! + \cdots + x_k!$$

has only finitely many integer solutions.

3. The main result. First, we prove our result for polynomials of the form $P(x) = x^d$.

Theorem 3.1. *Under the ABC Conjecture, for any integers $d \geq 2$ and $c > 0$, the equation*

$$z^d = c(m! + n!)$$

has only finitely many integer solutions.

Proof. From Theorem 2.4, we know that for fixed m the equation has only finitely many solutions. So we can assume that $n \geq m > 2c$. Let $p \in (m/2, m]$ be a prime. Then

$$d \mid v_p(z^d) = v_p(c(m! + n!)) = v_p\left(m!\left(\frac{n!}{m!} + 1\right)\right) = 1 + v_p\left(\frac{n!}{m!} + 1\right),$$

and therefore $v_p(\frac{n!}{m!} + 1) \geq d - 1 \geq 1$. Hence

$$\frac{n!}{m!} + 1 \geq \prod_{p \in (m/2, m]} p = e^{\vartheta(m) - \vartheta(m/2)}.$$

So, for sufficiently large n, m,

$$(n - m)\ln n + 1 \geq \ln\left(\frac{n!}{m!} + 1\right) \geq \vartheta(m) - \vartheta\left(\frac{m}{2}\right) \geq \frac{3}{4}m - \frac{5}{8}m = \frac{1}{8}m.$$

Therefore,

$$n - m \geq \frac{m}{9\ln n}.$$

On the other hand, from the prime gap bound [1], we know that for sufficiently large m, there exists a prime $p \in [m/2, m/2 + (m/2)^{0.525}]$. We have $2p > n$, because otherwise $v_p(n!/m! + 1) = 0$. So,

$$2m > m + 2\left(\frac{m}{2}\right)^{0.525} > 2p > n.$$

We have

$$2\left(\frac{m}{2}\right)^{0.525} \geq n - m \geq \frac{m}{9\ln n}.$$

Thus,

$$18\ln n > m^{0.475}$$

for sufficiently large m, n, which contradicts the fact that $n < 2m$. ∎

NOTE 3.2. We learned from F. Luca (personal communication) that the above equation with $c = 1$ was investigated by Erdős and Oblath [6]. Moreover, note that in essence we proved (independently of the ABC Conjecture) that there exists a constant M such that the equation $z^d = c(m! + n!)$ has no integer solutions with $\min\{m, n\} > M$.

Now we can consider the general situation.

THEOREM 3.3. *Let* $P(z) = a_d z^d + a_{d-3} z^{d-3} + a_{d-4} z^{d-4} + \cdots + a_1 z + a_0$ *be a polynomial of degree d with rational coefficients. Under the ABC Conjecture, the equation*

$$P(z) = m! + n!$$

has only finitely many integer solutions.

Proof. We multiply out by the denominators to get

$$P_1(z) = C_1(m! + n!),$$

where $P_1 \in \mathbb{Z}[X]$ and C_1 is some integer. Let $P_1(z) = b_d z^d + \text{l.o.t.}$ We multiply the above equation by b_d^{d-1} and set $w := b_d z$ to obtain

$$P_2(w) = C_2(m! + n!),$$

where P_2 is a monic polynomial with integer coefficients and C_2 is some integer. We assume that $m \leq n$. Fix some $\epsilon < 1/(3d)$. We will look at solutions satisfying the additional condition

$$m! < n!^{(d-1)/d-\epsilon}.$$

Then, for all sufficiently large w, we have

$$2w^d > P_2(w) = C(m! + n!) > n!. \tag{1}$$

Raising both sides to the power $(d-1)/d - \epsilon$, we get

$$Mw^{d-1-d\epsilon} > n!^{(d-1)/d-\epsilon} > m!,$$

where $M = 2^{(d-1)/d-\epsilon}$. Moreover, from $2w^d > n!$, we have

$$w^{\epsilon} > (n!/2)^{\epsilon/d} > 4^n \tag{2}$$

for sufficiently large n. Write $P_2(w) = w^d + R(w)$, where $\deg R \leq d-3$. Now our equation is

$$w^d + (R(w) - Cm!) = Cn!.$$

From Theorem 2.4, we have $R(w) - Cm! = Cn! - w^d \neq 0$ for all but finitely many pairs n, w. So, we can use Lemma 2.3 for a small positive number θ to be fixed later and get

$$w^d \leq M_\theta |R(w) - Cm!|^{1+\theta} \operatorname{rad}(Cn!w^d)^{1+\theta} < M_\theta (4Cw^{d-1-d\epsilon})^{1+\theta} 4^{n(1+\theta)} w^{1+\theta}.$$

Therefore,

$$w^d < S_\theta w^{(d-1-d\epsilon+\epsilon+1)(1+\theta)}$$

for some constant S_θ. Because $d + (1 - d)\epsilon < d$, we can take θ such that

$$d > (d + (1 - d)\epsilon)(1 + \theta),$$

so w is bounded.

Now we consider solutions with the additional condition $m! > n!^{(d-2)/d+\epsilon}$. From (1) and (2), we get

$$w^\epsilon > 4^n \geq 4^m \quad \text{and} \quad n!^{1/d} < 2w.$$

We have

$$\mathrm{rad}(m! + n!) \leq 4^m \left(\frac{n!}{m!} + 1\right) \leq w^\epsilon (n!^{2/d-\epsilon} + 1) \leq 8w^{\epsilon+2-d\epsilon}.$$

Because $\deg(R) = d - 3$, for all sufficiently large w we have

$$|R(w)| < M_1 w^{d-3},$$

where M_1 is some constant which depends on P. We write our equation as $w^d + R(w) = C(m! + n!)$. If $R(w) \equiv 0$, then we use the previous theorem and we are done. Otherwise $R(w) \neq 0$ for all but finitely many w. We can use Lemma 2.3 for some small positive θ to be fixed later again and get

$$w^d \leq M_\theta |R(w)|^{1+\theta} \,\mathrm{rad}(C(m! + n!)w^d)^{1+\theta} \leq S_\theta w^{(d-3+2+(1-d)\epsilon+1)(1+\theta)}$$

for some constant S_θ. We can choose θ such that

$$d > (d + (1 - d)\epsilon)(1 + \theta),$$

therefore w is bounded.

Above, all solutions have been considered, because otherwise

$$n!^{(d-2)/d+\epsilon} \geq m! \geq n!^{(d-1)/d-\epsilon}$$

and therefore $\epsilon > 1/(2d)$, a contradiction. ■

COROLLARY 3.4 (from the proof). *Let $P(z)$ be a polynomial of degree d with rational coefficients, and $\epsilon > 0$. Under the ABC Conjecture, the equation*

$$P(z) = m! + n!$$

has only finitely many integer solutions with $m! \notin [n!^{(d-1)/d-\epsilon}, n!^{(d-1)/d+\epsilon}]$.

REFERENCES

[1] R. C. Baker, G. Harman and J. Pintz, *The difference between consecutive primes, II*, Proc. London Math. Soc. 83 (2001), 532–562.

[2] B. C. Berndt and W. F. Galway, *On the Brocard–Ramanujan Diophantine equation $n! + 1 = m^2$*, Ramanujan J. 4 (2000), 41–42.

[3] H. Brocard, *Question 1532*, Nouv. Ann. Math. 4 (1885), 291.

[4] A. Dąbrowski, *On the Diophantine equation $x! + A = y^2$*, Nieuw Arch. Wisk. 14 (1996), 321–324.

[5] A. Dąbrowski and M. Ulas, *Variations on the Brocard–Ramanujan equation*, J. Number Theory 133 (2013), 1168–1185.

[6] P. Erdős und R. Oblath, *Über diophantische Gleichungen der Form* $n! = x^p \pm y^p$ *und* $n! \pm m! = x^p$, Acta Litt. Sci. Szeged 8 (1937), 241–255.
[7] F. Luca, *The Diophantine equation* $P(x) = n!$ *and a result of M. Overholt*, Glas. Mat. Ser. III 37 (2002), 269–273.
[8] S. Ramanujan, *Question 469*, J. Indian Math. Soc. 5 (1913), 59.
[9] M. Ulas, *Some observations on the Diophantine equation* $y^2 = x! + A$ *and related results*, Bull. Austral. Math. Soc. 86 (2012), 377–388.

Maciej Gawron
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: maciej.gawron@uj.edu.pl

Received 4 September 2012;
revised 7 February 2013 (5753)