SYMMETrY CLASSES OF TEnsORS ASSOCIATED WITH THE SEMI-DIHEDRAL GROUPS SD_{8n}

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Abstract. We discuss the existence of an orthogonal basis consisting of decomposable vectors for all symmetry classes of tensors associated with semi-dihedral groups SD_{8n}. In particular, a necessary and sufficient condition for the existence of such a basis associated with SD_{8n} and degree two characters is given.

1. Introduction. Let V be an n-dimensional complex inner product space and G be a permutation group on m elements. Let \( \chi \) be any irreducible character of G. For any \( \sigma \in G \), define the operator

\[
P_\sigma : \bigotimes_1^m V \rightarrow \bigotimes_1^m V
\]

by

\[
P_\sigma(v_1 \otimes \cdots \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}).
\]

The symmetry class of tensors associated with G and \( \chi \) is the image of the symmetry operator

\[
T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma,
\]

and it is denoted by \( V^m_\chi(G) \). We say that the tensor \( T(G, \chi)(v_1 \otimes \cdots \otimes v_m) \) is a decomposable symmetrized tensor, and we denote it by \( v_1 \ast \cdots \ast v_m \).

The inner product on \( V \) induces an inner product on \( V^m_\chi(G) \) which satisfies

\[
\langle v_1 \ast \cdots \ast v_m, u_1 \ast \cdots \ast u_m \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^m \langle v_i, u_{\sigma(i)} \rangle.
\]

Let \( \Gamma^m_n \) be the set of all sequences \( \alpha = (\alpha_1, \ldots, \alpha_m) \), with \( 1 \leq \alpha_i \leq n \). Define the action of G on \( \Gamma^m_n \) by

\[
\sigma.\alpha = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(m)}).
\]

2010 Mathematics Subject Classification: Primary 20C30; Secondary 15A69.

Key words and phrases: symmetry classes of tensors, orthogonal basis, semi-dihedral groups.

DOI: 10.4064/cm131-1-6
Let $O(\alpha) = \{\sigma.\alpha \mid \sigma \in G\}$ be the orbit of $\alpha$. We write $\alpha \sim \beta$ if $\alpha$ and $\beta$ belong to the same orbit in $\Gamma_n^m$. Let $\Delta$ be a system of distinct representatives of the orbits. We denote by $G_\alpha$ the stabilizer subgroup of $\alpha$, i.e., $G_\alpha = \{\sigma \in G \mid \sigma.\alpha = \alpha\}$. Define

$$\Omega = \{\alpha \in \Gamma_n^m \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\},$$

and put $\overline{\Delta} = \Delta \cap \Omega$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $V$, and denote by $e^*_\alpha$ the tensor $e_{\alpha_1} \ast \cdots \ast e_{\alpha_m}$. We have

$$\langle e^*_\alpha, e^*_\beta \rangle = \begin{cases} 0 & \text{if } \alpha \nott\sim \beta, \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\sigma h^{-1}) & \text{if } \alpha = h.\beta. \end{cases}$$

In particular, for $\sigma_1, \sigma_2 \in G$ and $\gamma \in \overline{\Delta}$ we obtain

$$\langle e^*_{\sigma_1.\gamma}, e^*_{\sigma_2.\gamma} \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\gamma \sigma_1^{-1}} \chi(x).$$

Moreover, $e^*_\alpha \neq 0$ if and only if $\alpha \in \Omega$.

For $\alpha \in \overline{\Delta}$, $V^*_\alpha = \langle e^*_{\sigma.\alpha} : \sigma \in G \rangle$ is called the **orbital subspace** of $V_\chi(G)$. It follows that

$$V_\chi(G) = \bigoplus_{\alpha \in \overline{\Delta}} V^*_\alpha$$

is an orthogonal direct sum. In [9] it is proved that

$$\dim V^*_\alpha = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma).$$

Thus we deduce that if $\chi$ is a linear character, then $\dim V^*_\alpha = 1$ and in this case the set

$$\{e^*_\alpha \mid \alpha \in \overline{\Delta}\}$$

is an orthogonal basis of $V_\chi(G)$.

A basis which consists of decomposable symmetrized tensors $e^*_\alpha$ is called an **orthogonal *-basis**. If $\chi$ is not linear, it is possible that $V_\chi(G)$ has no orthogonal *-basis. The reader can find further information about the symmetry classes of tensors in [1]–[8], [10]–[11], [13]–[15] and [17].

In this paper we discuss the existence of an orthogonal basis consisting of decomposable vectors for all symmetry classes of tensors associated with semi-dihedral groups $SD_{8n}$. 
2. Semi-dihedral groups $SD_{8n}$. The presentation for $SD_{8n}$ for $n \geq 2$ is given by

$$SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle,$$

where the embedding of $SD_{8n}$ into the symmetric group $S_{4n}$ is given by $T(a)(t) := t + 1$ and $T(b)(t) := (2n - 1)t$, where $\overline{m}$ is the remainder of $m$ divided by $4n$.

**Definition 2.1.** Define

$$C_1 := \{0, 2, 4, \ldots, 2n\},$$

$$C_2 := \{1, 3, 5, \ldots, n\} \cup \{2n + 1, 2n + 3, 2n + 5, \ldots, 3n\},$$

$$C_{\text{even}} := \{2, 4, \ldots, 2n - 2\},$$

$$C_{\text{odd}}^{\dagger} = \{1, 3, 5, \ldots, 2[n/2] - 1, 2n + 1, 2n + 3, \ldots, 2[3n/2] - 1\}.$$

We define two-dimensional representations, for each natural number $h$ and $\omega = e^{i\pi/2n}$:

$$(2.1)\quad \rho^h(a^r) = \begin{pmatrix} \omega^h r & 0 \\ 0 & \omega^{(2n-1)hr} \end{pmatrix} \quad \text{and} \quad \rho^h(ba^r) = \begin{pmatrix} 0 & \omega^{(2n-1)hr} \\ \omega^h r & 0 \end{pmatrix},$$

for each $r \in \{1, 2, \ldots, 4n\}$.

Denote $\chi_h = \text{Tr}(\rho^h)$. The non-linear irreducible complex characters of $SD_{8n}$ are the characters $\chi_h$ where $h \in C_{\text{even}}^{\dagger}$ or $h \in C_{\text{odd}}^{\dagger}$. Since the numbers of conjugacy classes of $SD_{8n}$ are different for $n$ even ($2n + 3$ classes) and $n$ odd ($2n + 6$ classes), we consider the corresponding two non-linear character tables separately.

**Table I.** The non-linear character table for $SD_{8n}$, $n$ even

<table>
<thead>
<tr>
<th>Characters</th>
<th>$[a^r], r \in C_1$</th>
<th>$[a^r], r \in C_{\text{odd}}^{\dagger}$</th>
<th>$[b]$</th>
<th>$[ba]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_h, h \in C_{\text{even}}^{\dagger}$</td>
<td>$2 \cos\left(\frac{hr\pi}{2n}\right)$</td>
<td>$2 \cos\left(\frac{hr\pi}{2n}\right)$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_h, h \in C_{\text{odd}}^{\dagger}$</td>
<td>$2 \cos\left(\frac{hr\pi}{2n}\right)$</td>
<td>$2i \sin\left(\frac{hr\pi}{2n}\right)$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

**Table II.** The non-linear character table for $SD_{8n}$, $n$ odd

<table>
<thead>
<tr>
<th>Characters</th>
<th>$[a^r], r \in C_1$</th>
<th>$[a^r], r \in C_2$</th>
<th>$[b]$</th>
<th>$[ba]$</th>
<th>$[ba^2]$</th>
<th>$[ba^3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_h, h \in C_{\text{even}}^{\dagger}$</td>
<td>$2 \cos\left(\frac{hr\pi}{2n}\right)$</td>
<td>$2 \cos\left(\frac{hr\pi}{2n}\right)$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\chi_h, h \in C_{\text{odd}}^{\dagger}$</td>
<td>$2 \cos\left(\frac{hr\pi}{2n}\right)$</td>
<td>$2i \sin\left(\frac{hr\pi}{2n}\right)$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

3. Existence of an orthogonal basis for the symmetry classes of tensors associated with $SD_{8n}$. In this section we study the existence of an orthogonal basis for the symmetry classes of tensors associated with $SD_{8n}$. As explained in the introduction, if $\chi$ is a linear character of $G$ then
the symmetry class of tensors associated with $G$ and $\chi$ has an orthogonal basis. Therefore we will concentrate on non-linear irreducible complex characters of $SD_{8n}$, i.e. the characters $\chi_h$ where $h \in C_{\text{even}}^\dagger$ or $h \in C_{\text{odd}}^\dagger$.

**Remark 3.1.** Let $\nu_2$ be the 2-adic valuation, that is, $\nu_2\left(\frac{2^km}{n}\right) = k$ for $m$ and $n$ odd. Then the condition $\nu_2\left(\frac{h}{2^n}\right) < 0$ means that every power of 2 that divides $h$ also divides $n$.

**Lemma 3.2.** Let $G := SD_{8n}$ and $H$ be a subgroup of $G$. Then there is a natural number $r$, $0 \leq r < 4n$, such that either $H = \langle a^r \rangle$, or $\langle a^r \rangle \leq H$ and $H \cap \langle a \rangle = \langle a^r \rangle$. In the latter case we have $|H| \geq 2|\langle a^r \rangle|$.

**Proof.** This is straightforward. ■

**Lemma 3.3.** Suppose $\chi = \chi_h$. If $r$ is defined by $G_\alpha \cap \langle a \rangle = \langle a^r \rangle$ and $l = 4n/\gcd(4n, r)$, then

$$\sum_{g \in G_\alpha} \chi(g) = \begin{cases} 2l & \text{if } rh \equiv 0 \pmod{4n}, \\ 0 & \text{if } rh \not\equiv 0 \pmod{4n}, \end{cases}$$

and for $\alpha \in \Delta$, we have $rh \equiv 0 \pmod{4n}$.

**Proof.** Since $G_\alpha$ is a subgroup of $G$, using Lemma 3.2 there is a natural number $r$, $0 \leq r < 4n$, such that either $G_\alpha = \langle a^r \rangle$ or $\langle a^r \rangle < G_\alpha$. Using Table I, we find that $\chi$ vanishes outside $\langle a \rangle$, therefore

$$\sum_{g \in G_\alpha} \chi(g) = \sum_{t=1}^l \chi(a^{tr}) = 2 \sum_{t=1}^l \cos\left(\frac{trh\pi}{2n}\right) = \begin{cases} 2l, & rh \equiv 0 \pmod{4n}, \\ 0, & rh \not\equiv 0 \pmod{4n}. \end{cases}$$

Also if $rh \not\equiv 0 \pmod{4n}$, then $\sum_{g \in G_\alpha} \chi(g) = 0$, which shows $\alpha \notin \Delta$. ■

**Lemma 3.4.** Let $1 \leq h < 2n$ and let $\nu_2$ be the 2-adic valuation. Then there exist $t_1, t_2$, $0 \leq t_1, t_2 < 4n$, such that $\cos\left(\frac{(t_1-t_2)h\pi}{2n}\right) = 0$ if and only if $\nu_2\left(\frac{h}{2^n}\right) < 0$.

**Theorem 3.5.** Let $G = SD_{8n}$ be a subgroup of $S_{4n}$, denote $\chi = \chi_h$ for $h \in C_{\text{even}}^\dagger$, and assume $d = \dim V \geq 2$. Then $V_\chi(G)$ has an orthogonal $\ast$-basis if and only if $\nu_2\left(\frac{h}{2^n}\right) < 0$.

**Proof.** It is enough to prove that for any $\alpha \in \Delta$ the orbital subspace $V_\alpha^\ast$ has an orthogonal $\ast$-basis if $\nu_2\left(\frac{h}{2^n}\right) < 0$. Let $\nu_2\left(\frac{h}{2^n}\right) < 0$ and assume $\alpha \in \Delta$. By Lemma 3.2 either $G_\alpha = \langle a^r \rangle$ or $\langle a^r \rangle < G_\alpha$. Let $l = 4n/\gcd(4n, r)$. Now we consider two cases.

**Case 1.** If $\langle a^r \rangle < G_\alpha$, then by Lemma 3.2 we obtain $|G_\alpha| \geq 2l$ where

$$\langle a^r \rangle = \langle a \rangle \cap G_\alpha = \{a^r, a^{2r}, \ldots, a^{lr} = 1\}.$$
By (1.4), \( |G_\alpha| \geq 2l \) and Lemma 3.3 we have

\[
\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) \leq \frac{2}{2l} (2l) = 2.
\]

If \( \dim V_\alpha^* = 1 \), then it is obvious that we have an orthogonal \(*\)-basis. Let us consider \( \dim V_\alpha^* = 2 \). Set \( \sigma_1 = a^j, \sigma_2 = a^i \). Then

\[
\sigma_2 G_\alpha \sigma_1^{-1} \cap \langle a \rangle = \{ a^{r+i-j}, \ldots, a^{lr+i-j} \}.
\]

Hence if \( \sigma_1 = a^j, \sigma_2 = a^i \), by (1.3), we have

\[
\langle e_{\sigma_1,\alpha}^*, e_{\sigma_2,\alpha}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\alpha \sigma_1^{-1}} \chi(x) = \frac{2}{8n} \sum_{t=1}^{l} \chi(a^{tr+i-j})
\]

\[
= \frac{4}{8n} \sum_{t=1}^{l} \cos \left( \frac{tr + i - j}{2n} \right) \pi
\]

\[
= \frac{1}{2n} \sum_{t=1}^{l} \cos \left( \frac{tr \pi + (i - j) \pi}{2n} \right)
\]

\[
= \frac{1}{2n} \sum_{t=1}^{l} \cos \left( \frac{(i - j) \pi}{2n} \right) = \frac{l}{2n} \cos \left( \frac{(i - j) \pi}{2n} \right)
\]

where the penultimate equality is due to an application of Lemma 3.3. By Lemma 3.4, there exist \( i \) and \( j \) such that

\[
\langle e_{a^i,\alpha}^*, e_{a^i,\alpha}^* \rangle = 0,
\]

which means that \( \{ e_{\sigma_1,\alpha}^*, e_{\sigma_2,\alpha}^* \} \) is an orthogonal \(*\)-basis for \( V_\alpha^* \).

**Case 2.** If \( G_\alpha = \langle a^r \rangle = \{ a^r, a^{2r}, \ldots, a^{lr} = 1 \} \), then by (1.4) and Lemma 3.3

\[
\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \frac{2}{l} (2l) = 4.
\]

For any \( \sigma_1, \sigma_2 \in G \), we have

\[
\sigma_2 G_\alpha \sigma_1^{-1} =
\begin{cases}
\{ a^{r+i-j}, a^{2r+i-j}, \ldots, a^{lr+i-j} \} & \text{if } \sigma_1 = a^j, \sigma_2 = a^i, \\
\{ a^{r+i+j(1-2n)} b, a^{2r+i+j(1-2n)} b, \ldots, a^{lr+i+j(1-2n)} b \} & \text{if } \sigma_1 = a^jb, \sigma_2 = a^i, \\
\{ a^{(1-2n)r+i-j}, a^{2r(1-2n)+i-j}, \ldots, a^{lr(1-2n)+i-j} \} & \text{if } \sigma_1 = a^jb, \sigma_2 = a^i b.
\end{cases}
\]

If \( \sigma_1 = a^j, \sigma_2 = a^i \), by (3.1) we have

\[
\langle e_{\sigma_1,\alpha}^*, e_{\sigma_2,\alpha}^* \rangle = \frac{l}{2n} \cos \left( \frac{(i - j) \pi}{2n} \right).
\]
If \( \sigma_1 = a^j b, \sigma_2 = a^i \), we have
\[
\langle e_{\sigma_1, \alpha}^*, e_{\sigma_2, \alpha}^* \rangle = 0,
\]
and for \( \sigma_1 = a^j b, \sigma_2 = a^i b \), we have
\[
\langle e_{\sigma_1, \alpha}^*, e_{\sigma_2, \alpha}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G \sigma_1^{-1}} \chi(x) = \frac{2}{8n} \sum_{l=1}^i \chi(a^{tr(1-2n)+i-j})
\]
\[
= \frac{4}{8n} \sum_{l=1}^i \cos \left( \frac{(tr(1-2n)+i-j)h\pi}{2n} \right)
\]
\[
= \frac{1}{2n} \sum_{l=1}^i \cos \left( \frac{trh\pi}{2n} + \frac{(i-j)h\pi}{2n} - trh\pi \right)
\]
\[
= \frac{1}{2n} \sum_{l=1}^i \cos \left( \frac{(i-j)h\pi}{2n} \right) = \frac{l}{2n} \cos \left( \frac{(i-j)h\pi}{2n} \right)
\]
where the penultimate equality uses Lemma 3.3. Therefore
\[
\langle e_{\sigma_1, \alpha}^*, e_{\sigma_2, \alpha}^* \rangle = \begin{cases} 
\frac{l}{2n} \cos \left( \frac{(i-j)h\pi}{2n} \right), & \sigma_1 = a^j, \sigma_2 = a^i, \\
0, & \sigma_1 = a^j b, \sigma_2 = a^i, \\
\frac{l}{2n} \cos \left( \frac{(i-j)h\pi}{2n} \right), & \sigma_1 = a^j b, \sigma_2 = a^i b.
\end{cases}
\]
In view of Lemma 3.4 if \( \nu_2 \left( \frac{h}{2n} \right) < 0 \), there exist \( t_1, t_2, 0 \leq t_1, t_2 < 4n \) such that \( \cos \left( \frac{(t_1-t_2)h\pi}{2n} \right) = 0 \). Put
\[
S = \{a^{t_1, \alpha}, a^{t_2, \alpha}, a^{t_1 b, \alpha}, a^{t_2 b, \alpha} \} \subseteq \Gamma_n^m.
\]
Then for every \( \alpha, \beta \in S \) and \( \alpha \neq \beta \) we have
\[
\langle e_\alpha^*, e_\beta^* \rangle = 0.
\]
But \( \dim V_\alpha^* = 4 \); hence \( \{e_\xi^* | \xi \in S\} \) is an orthogonal \(*\)-basis for \( V_\alpha^* \).

Conversely, assume that \( V_\chi(G) \) has an orthogonal basis of decomposable symmetrized tensors. Then since \( V_\chi(G) = \bigoplus_{\alpha \in \Delta} V_\alpha^* \) for all \( \alpha \in \Delta \), the orbital subspace \( V_\chi^* \) has an orthogonal basis of decomposable symmetrized tensors. Using [17, p. 642], we can choose \( \alpha \in \Gamma_n^m \) such that \( a^t \notin G_\alpha \) for \( 1 \leq t < 4n \). Thus for such \( \alpha \) we have either \( G_\alpha = \{1\} \) or \( G_\alpha = \{1, a^t b, a^{-(2n-1)t} b\} \) for some \( 1 \leq t < 4n \), since if \( G_\alpha \neq \{1\} \) and \( a^{t_1} b, a^{t_2} b \in G_\alpha \), then
\[
a^{t_1} b, a^{t_2} b = a^{t_1} b, ba^{(2n-1)t_2} = a^{t_1+(2n-1)t_2} \in G_\alpha,
\]
which shows that \( t_1 = -(2n-1)t_2 \).

To prove that \( \nu_2 \left( \frac{h}{2n} \right) < 0 \) is a necessary condition for existence of an orthogonal \(*\)-basis for \( V_\chi(G) \), it is enough to consider the cases \( G_\alpha = \{1\} \).
and \( G_\alpha = \{1, a^tb, a^{-(2n-1)t}b\} \). For both, we have
\[
\|e^*_\alpha\|^2 = \frac{\chi(1)}{|G|} \sum_{g \in G_\alpha} \chi(g) \neq 0,
\]
so \( \alpha \in \overline{\Delta} \). First consider \( G_\alpha = \{1\} \). For any \( \sigma_1, \sigma_2 \in G \), we have
\[
\sigma_2 G_\alpha \sigma_1^{-1} = \begin{cases}
\{a^{i-j}\} & \text{if } \sigma_1 = a^j, \sigma_2 = a^i, \\
\{a^{i+j(1-2n)b}\} & \text{if } \sigma_1 = a^jb, \sigma_2 = a^i, \\
\{a^{(1-2n)i-j}\} & \text{if } \sigma_1 = a^jb, \sigma_2 = a^ib.
\end{cases}
\]
Therefore by (1.3) we have
\[
\langle e^*_{\sigma_1, \alpha}, e^*_{\sigma_2, \alpha} \rangle = \begin{cases}
\frac{1}{2n} \cos \left( \frac{(i-j)h\pi}{2n} \right) & \text{if } \sigma_1 = a^j, \sigma_2 = a^i, \\
0 & \text{if } \sigma_1 = a^jb, \sigma_2 = a^i, \\
\frac{1}{2n} \cos \left( \frac{(i-j)h\pi}{2n} \right) & \text{if } \sigma_1 = a^jb, \sigma_2 = a^ib.
\end{cases}
\]
Hence \( \langle e^*_{\sigma_1, \alpha}, e^*_{\sigma_2, \alpha} \rangle = 0 \) implies that there exist \( t_1 \) and \( t_2 \) such that
\[
\cos \left( \frac{(t_1-t_2)h\pi}{2n} \right) = 0,
\]
therefore by Lemma 3.4 we get \( \nu_2 \left( \frac{h}{2n} \right) < 0 \).

Now consider \( G_\alpha = \{1, a^tb, a^{-(2n-1)t}b\} \). For any \( \sigma_1, \sigma_2 \in G \), we have
\[
\sigma_2 G_\alpha \sigma_1^{-1} = \begin{cases}
\{a^{i-j}, ba^{(2n-1)(j+t)-i}, ba^{(2n-1)(j-(2n-1)t)-i}\} & \text{if } \sigma_1 = a^j, \sigma_2 = a^i, \\
\{a^{i+j(1-2n)b}, a^{j+(2n-1)t+i}, a^{j-t+i}\} & \text{if } \sigma_1 = a^jb, \sigma_2 = a^i, \\
\{a^{(1-2n)i-j}, a^{j+(2n-1)t+i}, a^{j-t+i}b\} & \text{if } \sigma_1 = a^jb, \sigma_2 = a^ib.
\end{cases}
\]
Now similar to our previous calculations in this section, we get \( \nu_2 \left( \frac{h}{2n} \right) < 0 \).

**Remark 3.6.** In the proof of the necessity part of Theorem 3.5, one can choose \( \alpha = (1, 2, \ldots, 2) \). The proof given here shows the stronger statement that the orbital subspace \( V^*_\alpha \) has an orthogonal \(*\)-basis whenever \( G_\alpha \cup \langle a \rangle = \{1\} \).

**Corollary 3.7.** Let \( G = SD_{8n} \), \( n \) odd, be a subgroup of \( S_{4n} \), denote \( \chi = \chi_h \) for \( h \in C_{\text{even}}^l \), and assume \( d = \dim V \geq 2 \). Then \( V^*_\chi(G) \) does not have an orthogonal \(*\)-basis.

**Proof.** Since \( n \) is odd we have \( \nu_2 \left( \frac{h}{2n} \right) \geq 0 \). Thus Theorem 3.5 implies \( V^*_\chi(G) \) does not have an orthogonal \(*\)-basis.

**Theorem 3.8.** Let \( G = SD_{8n} \) be a subgroup of \( S_{4n} \), denote \( \chi = \chi_h \) for \( h \in C_{\text{odd}}^l \), and assume \( d = \dim V \geq 2 \). Then \( V^*_\chi(G) \) does not have an orthogonal \(*\)-basis.
Proof. The proof is similar to the proof of Theorem 3.5. Using Table I and Table II we conclude that $\langle e^{*}_{\sigma_1, \alpha}, e^{*}_{\sigma_2, \alpha} \rangle \neq 0$ since the imaginary and real parts should both be zero; but $i \sin x$ and $\cos x$ cannot vanish simultaneously.

Acknowledgements. The authors are grateful to Professor Hjalmar Rosengren for valuable comments and for reviewing earlier drafts very carefully.

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Received 19 December 2012;
revised 17 February 2013

(5831)