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ON THE STABILITY OF THE UNIT CIRCLE WITH MINIMAL SELF-PERIMETER IN NORMED PLANES

BY

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Abstract. We prove a stability result on the minimal self-perimeter $L(B)$ of the unit disk B of a normed plane: if $L(B) = 6 + \varepsilon$ for a sufficiently small ε, then there exists an affinely regular hexagon S such that $S \subset B \subset (1 + 6\sqrt[3]{\varepsilon})S$.

1. Basic notions and introduction. Let B be a convex figure centered at the origin O of the Euclidean plane $\mathbb{R}^2$. In what follows, we identify the points of $\mathbb{R}^2$ with their position vectors. The convex figure B and its boundary ∂B are called the *unit disk* resp. *unit circle* of the *normed* (or *Minkowski*) *plane* M^2 induced by B. In the literature, B is often also called the *normalizing figure* of the normed plane M^2 (see [6, Definition 11.2]). We will use the distance function $|\cdot|$ of $\mathbb{R}^2$ as an auxiliary metric for M^2. The *Minkowskian distance function* $g_B(x)$ of M^2 is defined by

$$g_B(x) = |x|/|\widehat{x}| > 0,$$

where $x \in M^2$, $x \neq O$ and $\widehat{x} = [O, x) \cap \partial B$. Here $[O, x)$ is the *ray* with starting point O passing through x.

In a standard way (see [9]), the distance function $g_B(x)$ defines the *distance between arbitrary points* x and y of M^2 by

$$\|x - y\| = g_B(y - x). \tag{1}$$

Definition. For two distinct points a and b, the *normalizing vector* of the connecting segment ab is defined to be the point $\widehat{b - a} \in \partial B$, that is,

$$\widehat{b - a} = \overline{ab}/\|\overline{ab}\| \quad \text{with} \quad \overline{ab} = b - a. \tag{2}$$

Further on, we denote by xy the *segment* and by (xy) the *straight line* defined by the points $x \neq y$. The symbol $\triangle abc$ is used for the triangle determined by non-collinear vertices a, b and c; writing only abc, we mean the *polygonal arc* (broken line) from a to b. For more than three points, the context will clarify whether we mean a polygonal arc or an n-gon. By $\angle abc$

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we mean the *angle* with apex b, and by $\angle(\overline{mn}, \overline{qr})$ the angle between the vectors $\overline{mn} = n - m$ and $\overline{qr} = r - q$.

Let $P \subset M^2$ be a convex bounded polygon. Denote by $l(P)$ the sum of the lengths of all its sides defined via (1). Denote by $\{P\}$ the set of all convex polygons located inside a compact convex figure K. The *perimeter* of the figure $K \subset M^2$ is defined by

$$L(K) = \sup_{P \in \{P\}} l(P).$$

It is widely known (see [6, p. 110] and [15, p. 112]) that if Φ is a convex figure and $\Phi \subset K$, then

$$L(\Phi) \leq L(K). \tag{3}$$

And clearly, if $P \subset M^2$ is a convex polygon, then $L(P) = l(P)$. The perimeter $L(B)$ of the unit disk B of M^2 is called its *self-perimeter*. S. Gołąb (see [2] and also [3]) proved that

$$6 \leq L(B) \leq 8$$

and, moreover, that $L(B) = 6$ holds if B is an affinely regular hexagon, and $L(B) = 8$ holds if B is a parallelogram. E.g., Yu. G. Reshetnyak [10] and D. Laugwitz [5] reproved the result of S. Gołąb.

J. J. Schäffer [11] proved that the affinely regular hexagon is the only normalizing figure with minimal value of $L(B)$ and that the parallelogram is the only normalizing figure with the maximal value of $L(B)$.

It is natural to investigate analogous problems also in d-dimensional normed (or Minkowski) spaces, where $d \geq 3$. The most important analogues of "circumference" are the surface area measures of Holmes–Thompson (see Chapter 6 of [15]) and of Busemann (cf. Chapter 7 of that book). The case of Holmes–Thompson self-surface-area of the unit ball B is presented in [15, §6.5]; the upper bound given there is only sharp for the planar case, and non-sharp lower bounds are also given (with special results for unit balls that are zonoids or their duals). For the Busemann self-surface-area of B, discussed in [15, §7.4], the sharp upper bound is given in Theorem 7.4.1 there and attained if and only if B is a d-parallelotope; lower bounds are presented in Theorems 7.4.4 and 7.4.6.

In the case of a non-symmetric *convex distance function* (or *gauge*) on M^2 (i.e., $B \neq -B$) it is known that the oriented self-perimeters satisfy $L^{\pm}(B) \geq 6$, and that equality is possible only if B is an affinely regular hexagon (see [4], [12], and [13]). More results on the non-symmetric case can be found in [7] and [8]; see also the references given there.

2. The result. The stability of the unit disk B with respect to the value of its self-perimeter was first considered in [14]. The following stability theorem was proved there; it refers to the case when for $B = -B$ the self-perimeter is close to the *maximal* value.

Theorem A. *If for a normed plane $L(B) = 8(1-\varepsilon)$ $(0 \le \varepsilon \le 0.04)$, then there exists a parallelogram P, symmetric with respect to the origin O, such that*

$$P \subset B \subset (1+18\varepsilon)P.$$

In this paper we prove the following stability theorem related to the *minimal* value of $L(B)$, also with $B = -B$.

Theorem. *Let the self-perimeter $L(B)$ of the unit disk B of a normed plane M^2 satisfy the equality*

$$L(B) = 6 + \varepsilon \quad (0 \le \varepsilon \le 0.001). \tag{4}$$

Then there exists an affinely regular hexagon S centered at the origin O such that

$$S \subset B \subset (1 + 6\sqrt[3]{\varepsilon})S. \tag{5}$$

The authors do not know whether the dependence on ε in this theorem is best possible; this is a topic for further research.

3. Proof of the results. In the proof of our theorem we use some auxiliary statements. Without loss of generality, we consider a convex normalizing figure $B \subset M^2$ located in the Euclidean auxiliary plane $\mathbb{R}^2$. Following S. Gołąb, we inscribe an affinely regular hexagon A_6 centered at the origin O into the unit circle ∂B (see [15, §4.1]). We use the auxiliary Euclidean metric in such a way that $A_6 \subset \mathbb{R}^2$ becomes a regular hexagon $a_1a_2a_3a_4a_5a_6$ with the vertices

$$a_1(-1/2; \sqrt{3}/2), \quad a_2(1/2; \sqrt{3}/2), \quad a_3(1;0),$$
$$a_4(1/2; -\sqrt{3}/2), \quad a_5(-1/2; -\sqrt{3}/2), \quad a_6(-1;0),$$

in the Cartesian coordinate system xOy. We call A_6 the *regular unit hexagon.* For certain reasons, we designate the vertices of each polygon considered *clockwise.* We denote by $\widehat{ab}$ the arc of the unit circle ∂B between a and b, oriented clockwise, and $L(\widehat{ab})$ means the arc length of $\widehat{ab}$ with respect to the metric of M^2.

Remark 1. *If $A_6 \subset M^2$ is a regular unit hexagon inscribed in the unit circle ∂B with self-perimeter $L(B)$ satisfying* (4)*, then the lengths $\widehat{a_ka_{k+1}} \subset \partial B$ satisfy*

$$1 \le L(\widehat{a_ka_{k+1}}) \le 1 + \varepsilon/2, \quad k = 1, \dots, 6. \tag{6}$$

Proof. Evidently, $\|a_k a_{k+1}\| = 1$, $k = 1, \ldots, 6$, where $a_7 = a_1$. Due to $B = -B$ we have $L(\widehat{a_k a_{k+1}}) = L(\widehat{a_{k+3} a_{k+4}})$, $k = 1, 2, 3$, and by (4),

$$6 + \varepsilon = L(B) = 2(L(\widehat{a_6 a_1}) + L(\widehat{a_1 a_2}) + L(\widehat{a_2 a_3})).$$

Consider the convex figure A with boundary

$$\partial A = a_6 a_1 \cup \widehat{a_1 a_2} \cup a_2 a_3 \cup a_3 a_4 \cup \cup \widehat{a_4 a_5} \cup a_5 a_6.$$

The inclusions $A_6 \subset A \subset B$ and inequality (3) imply

$$6 \le 4 + 2L(\widehat{a_1 a_2}) \le 6 + \varepsilon.$$

Hence,

$$1 \le L(\widehat{a_1 a_2}) \le 1 + \varepsilon/2.$$

In an analogous way we get the same inequality for all $L(\widehat{a_k a_{k+1}})$, which completes the proof of (6).

The *Hausdorff distance* $\rho(K_1; K_2)$ between convex, compact sets K_1 and K_2 is defined by

$$\rho(K_1; K_2) = \max\Big\{ \sup_{x \in K_1} \inf_{y \in K_2} |xy|, \sup_{y \in K_2} \inf_{x \in K_1} |xy| \Big\}.$$

Since $A_6 \subset B$, the Hausdorff distance between the unit disk B and its inscribed hexagon A_6 is given by

$$\rho(B; A_6) = \max_{x \in B} \min_{y \in A_6} |xy|.$$

To simplify the evaluation of $\rho(K_1; K_2)$, we use the following fact (see [6, §14, Theorem 14.1]). Note that the *support function* $h_K(u)$ of a compact convex set $K \subset \mathbb{R}^2$ is defined by $h_K(u) = \max\{\langle x, u \rangle : x \in K\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product and u is an arbitrary unit vector in the Euclidean background metric; see [6, §12].

THEOREM B. *If K_1 and K_2 are non-empty compact convex sets in $\mathbb{R}^2$ with the corresponding support functions $h_1(u)$ and $h_2(u)$, then*

$$\rho(K_1; K_2) = \max_{|u|=1} |h_2(u) - h_1(u)|. \tag{7}$$

Denote by ν one of the points on the unit circle ∂B for which the equality $\rho(\nu; A_6) = \rho(B; A_6)$ holds. To fix ideas, suppose $\nu \in \widehat{a_1 a_2}$. For the straight lines $(a_6 a_1)$ and $(a_3 a_2)$, we consider $\{q_1\} = (a_6 a_1) \cap (a_3 a_2)$. The convexity of B implies $\widehat{a_1 a_2} \subset \triangle a_1 q_1 a_2$. It is easy to see that $\rho(\nu; a_1 a_2) = \rho(B; A_6)$. We set $t = \rho(\nu; a_1 a_2)$.

REMARK 2. If $t \le 5\sqrt{\varepsilon}$ $(0 \le \varepsilon \le 0.001)$, then the inequality

$$\left(\frac{\sqrt{3}}{2} + 5\sqrt{\varepsilon} \right) : \frac{\sqrt{3}}{2} \le 1 + 2\sqrt[3]{\varepsilon}$$

implies the inclusions $A_6 \subset B \subset (1+2\sqrt[3]{\varepsilon})A_6$. Hence, to prove (5) it is sufficient to assume $S = A_6$. The case $\varepsilon = 0$ corresponds to the case $L(B) = 6$ and has already been studied in [11].

According to Remark 2 it is sufficient to consider $t > 5\sqrt{\varepsilon}$, $0 < \varepsilon \le 0.001$. However, the corresponding case analysis uses some results which are true even for $t > 4\sqrt{\varepsilon}$.

We write $c_2 = \widehat{\nu - a_1}$ and $c_3 = \widehat{a_2 - \nu}$ (cf. (2)).

PROPOSITION 1. *If* $t > 4\sqrt{\varepsilon}$ $(0 < \varepsilon \le 0.001)$, *then*

(8) $$\min\{\rho(c_2; a_2a_3); \rho(c_3; a_3a_4)\} \ge 0.9t.$$

Proof. By construction $\triangle a_1\nu a_2 \subset \triangle a_1q_1a_2$, where $\triangle a_1q_1a_2$ is a right triangle. The ray $[Oc_2)$ for $c_2 \in \widehat{a_2a_3}$ meets a_2a_3 at some point p_1, and the ray $[Oc_3)$ for $c_3 \in \widehat{a_3a_4}$ meets a_3a_4 at some point u_1.

We prove that $\rho(c_3; a_3a_4) \ge 0.9t$; the proof of $\rho(c_2; a_2a_3) \ge 0.9t$ is similar. Suppose $\rho(c_3; a_3a_4) < 0.9t$. Keeping in mind (3) and (6), we estimate the lengths of the sides of $\triangle a_1\nu a_2$ by

(9) $$1 = \|a_1a_2\| \le \|a_1\nu\| + \|\nu a_2\| \le L(\widehat{a_1a_2}) \le 1 + \varepsilon/2.$$

If νw is the height of $\triangle a_1\nu a_2$ with endpoint ν, then $|\nu w| = t$. If $\angle \nu a_1 w = \alpha$ and $\angle \nu a_2 w = \beta$, then $\angle a_3Op_1 = \alpha$ and $\angle a_3Ou_1 = \beta$. Denote by p_1p_0 and u_1u_0 the heights of $\triangle Op_1a_3$ and $\triangle Oa_3u_1$ with endpoints p_1 and u_1, respectively.

If we introduce $T = |p_1p_0|$ and $H = |u_1u_0|$, then the equalities $\angle Oa_3p_1 = \angle Oa_3u_1 = \pi/3$ imply $T(\cot\alpha + 1/\sqrt{3}) = H(\cot\beta + 1/\sqrt{3}) = 1$. If we construct a homothety $\triangle Oa_2'a_3' \approx \triangle Oa_2a_3$ so that $c_2 \in a_2'a_3'$, the ratio k of this homothety satisfies

$$k = |Op_1|/|Oc_2| \ge \sqrt{3}/(\sqrt{3} + 2t),$$

since $\rho(c_2; a_2a_3) \le \rho(B; A_6) = t$. The similarity $\triangle Op_1p_0 \sim \triangle a_1\nu w$ implies $|a_1\nu|/|Op_1| = t/T$, and hence

$$\|a_1\nu\| = \frac{|a_1\nu|}{|Oc_2|} = \frac{|a_1\nu|}{|Op_1|} \cdot \frac{|Op_1|}{|Oc_2|} \ge \frac{t}{T} \cdot \frac{\sqrt{3}}{\sqrt{3} + 2t}.$$

In a similar way, if we construct a homothety $\triangle Oa_3''a_4'' \approx \triangle Oa_3a_4$ with $c_3 \in a_3''a_4''$, since $\rho(c_3; a_3a_4) < 0.9t$ the homothety ratio is

$$|Ou_1|/|Oc_3| \ge \sqrt{3}/(\sqrt{3} + 1.8t).$$

The similarity $\triangle a_2w\nu \sim \triangle Ou_0u_1$ implies

$$\|\nu a_2\| = \frac{|\nu a_2|}{|Oc_3|} = \frac{|\nu a_2|}{|Ou_1|} \cdot \frac{|Ou_1|}{|Oc_3|} \ge \frac{t}{H} \cdot \frac{\sqrt{3}}{\sqrt{3} + 1.8t}.$$

As a consequence,

$$\|a_1\nu\| + \|\nu a_2\| \geq \sqrt{3}\,t\left(\frac{\cot\alpha + 1/\sqrt{3}}{\sqrt{3}+2t} + \frac{\cot\beta + 1/\sqrt{3}}{\sqrt{3}+1.8t}\right).$$

In $\triangle a_1\nu a_2$ we have $t(\cot\alpha + \cot\beta) = 1$, and hence

$$\begin{aligned}\|a_1\nu\| + \|\nu a_2\| &\geq t\cdot\frac{3.8\sqrt{3} + 3/t + 3.8t + 0.2\sqrt{3}\,t\cdot\cot\beta}{3.6t^2 + 3.8\sqrt{3}\,t + 3}\\ &= 1 + \frac{0.2(\sqrt{3}\cot\beta + 1)}{(\sqrt{3}+2t)(\sqrt{3}+1.8t)}\cdot t^2.\end{aligned}$$

The inclusion $\triangle a_1\nu a_2 \subset \triangle a_1 q_1 a_2$ implies $t \leq \sqrt{3}/2$ and $\cot\beta \geq 1/\sqrt{3}$. Then $\|a_1\nu\| + \|\nu a_2\| \geq 1 + 2t^2/57$. By (9) we have $t \leq \sqrt{57}\sqrt{\varepsilon}/2 < 4\sqrt{\varepsilon}$. This contradiction proves Proposition 1.

Corollary 1. *The angle γ between the straight lines (a_3c_2) and (a_3c_3) satisfies*

$$\sin\gamma \leq \frac{3\varepsilon}{8t}. \tag{10}$$

Proof. For $\triangle Oc_2a_3$ we write $\varphi = \angle Oa_3c_2$, and for $\triangle Oa_3c_3$ analogously $\psi = \angle Oa_3c_3$. Since a_2, c_2, a_3, c_3, a_4 lie on ∂B, we have $\pi/3 \leq \varphi, \psi \leq 2\pi/3$ and $\varphi + \psi \leq \pi$. Let T_1 and H_1 be the lengths of the heights of $\triangle Oc_2a_3$ and $\triangle Oa_3c_3$, respectively, with respect to the common base (Oa_3). Evidently, $T_1(\cot\alpha + \cot\varphi) = H_1(\cot\beta + \cot\psi) = 1 = t(\cot\alpha + \cot\beta)$. Then

$$\|a_1\nu\| + \|\nu a_2\| = \frac{|a_1\nu|}{|Oc_2|} + \frac{|\nu a_2|}{|Oc_3|} = \frac{t}{T_1} + \frac{t}{H_1} = t(\cot\varphi + \cot\psi) + 1.$$

By (9) we have $0 \leq t(\cot\varphi + \cot\psi) \leq \varepsilon/2$. Since $\gamma = \pi - (\varphi + \psi)$ and $0 \leq \gamma \leq \pi/3$, we have $0 \leq \cot\varphi - \cot(\varphi + \gamma) \leq \varepsilon/(2t)$ or $0 \leq \sin\gamma/(\sin\varphi\cdot\sin(\varphi+\gamma)) \leq \varepsilon/(2t)$. Since $\pi/3 \leq \varphi + \gamma \leq 2\pi/3$, we have $\sin\varphi \geq \sqrt{3}/2$ and $\sin(\varphi+\gamma) \geq \sqrt{3}/2$. Hence (10) follows immediately, and Corollary 1 is proved.

Proposition 2. *Let $abnm$ be a convex quadrangle. If $abnm \subset \triangle abf$, then*

$$\rho(abnm; \triangle abf) \leq \min\{|fn|; |fm|\}. \tag{11}$$

Proof. Let E denote the unit disk of the Euclidean plane $\mathbb{R}^2$. Then $f \in \{n\} + |fn|E \subset abnm + |fn|E$. Thus, by convexity, $\triangle abf \subset abnm + |fn|E$, and so $\rho(\triangle abf; abnm) \leq |fn|$. Similarly, $\rho(\triangle abf; abnm) \leq |fm|$, and Proposition 2 is proved.

Corollary 2. *If $\triangle abn \subset \triangle abm$, then $\rho(\triangle abm; \triangle abn) = |nm|$.*

Proposition 3. *Let $\triangle abc$ be a right triangle with $|ab| = 1$ and suppose $\triangle abn \subset \triangle abm \subset \triangle abc$. If the height np of $\triangle abn$ has length $|np| \geq t/\sqrt{3} > 0$,*

$\angle nbm = \mu_1 \le \mu_0$ and $\angle man = \mu_2 \le \mu_0$, then

$$|nm| \le \frac{\sqrt{3}}{t} \sin \mu_0. \tag{12}$$

Proof. For $n = m$, inequality (12) is trivial. Suppose $n \ne m$. Note that the straight line (mn) meets the side ab.

Denote by φ_1 the angle between the vectors $\overline{ca}$ and $\overline{mn}$, i.e., $\varphi_1 = \angle(\overline{ca}, \overline{mn})$. Construct the vector $\overline{cq} = \overline{mn}$. Observe that the ray $[cq)$ meets the straight line (ab), and hence $-2\pi/3 \le \varphi_1 \le \pi/3$. Denote $\varphi_2 = \angle(\overline{mn}, \overline{cb})$. For similar reasons, we have $-2\pi/3 \le \varphi_2 \le \pi/3$.

Consider $\varphi = \max\{\varphi_1; \varphi_2\}$. If $\varphi = -\pi/6$, then the vectors $\overline{mn}$ and $\overline{ab}$ are mutually orthogonal. The vectors $\overline{ca}$ and $\overline{cb}$ are symmetric with respect to the angle bisector of $\angle bca$, and hence $\varphi \ge -\pi/6$. Without loss of generality, we may assume $\varphi = \varphi_1$. With this assumption, we introduce $\{m_1\} = (bm) \cap ca$.

Considering the homothety $\triangle bmn \approx \triangle bm_1n_1$, we see that $|mn| \le |m_1n_1|$. In $\triangle bm_1n_1$, denote $\Re = \angle bm_1n_1$ and $\xi = \angle m_1n_1b$; moreover, set $\angle abn_1 = \alpha$. Then $\Re$ depends on the position of m_1 on ca, i.e., $\Re = \Re(m_1)$.

We intend to find the variation margins for $\Re(m_1)$ depending on the location of the starting point of the vector $\overline{m_1n_1}$ with fixed length $|m_1n_1|$. Observe that the constant angle $\angle(\overline{m_1n_1}, \overline{ba})$ equals $\pi - (\Re + \angle abm_1)$. Thus,

$$\Re_1 = \min_{m_1} \Re(m_1) = \Re(c) \ge \pi/6,$$

$$\Re_2 = \max_{m_1} \Re(m_1) \le \pi - \angle abm_1 = \pi - (\angle abn + \angle nbm) = \pi - (\alpha + \mu_1).$$

It follows that $\pi/6 + \mu_1 \le \Re + \mu_1 \le \pi - \alpha$, and hence $\xi = \angle m_1n_1b$ satisfies $0 < \alpha \le \xi \le 5\pi/6 - \mu_1$, where $0 \le \mu_1 < \pi/3$. Then

$$\sin \xi \ge \min\{\sin \alpha; \sin(5\pi/6 - \mu_1)\} \ge \min\{\sin \alpha; 1/2\}.$$

By hypothesis, the height np of $\triangle abn$ satisfies $|np| \ge t/\sqrt{3}$. Hence, $\sin \alpha = |np|/|bn| \ge t/\sqrt{3}$. Considering $\triangle bm_1n_1$, we get

$$|mn| \le |m_1n_1| = \sin \mu_1 \cdot \frac{|bm_1|}{\sin \xi} \le \frac{\sqrt{3}}{\min\{t; \sqrt{3}/2\}} \sin \mu_1 \le \frac{\sqrt{3}}{t} \sin \mu_0,$$

and Proposition 3 is proved.

PROPOSITION 4. *Let $\triangle abc$ and $\triangle baf$ be right triangles with $|ab| = 1$ and $c \ne f$. If $p \in bc$, $mn \subset pa$, and $\angle nfm = \mu$, then*

$$|mn| \le 2\sqrt{3} \sin \mu. \tag{13}$$

Proof. Denote by q the point on (pa) such that $fq \perp pa$. Write $s = (m + n)/2$ and $x = |sq|$. The angle function $\mu = \mu(x)$ is decreasing, and $\max \mu(x) = \mu(0)$. This means that for a fixed value of μ the quantity $|mn|$ attains its maximum either for $n = a$ or $p = m$ (we assume that $pm \subset pn \subset pa$).

If $n = a$, then for $\triangle fma$ we have $\angle afm = \mu$, $\angle maf = \pi/3+\beta$, where $0 \leq \beta \leq \pi/3$. We have $|fm| \leq |fc| = \sqrt{3}$, and moreover $\pi/3 \leq \pi/3 + \beta \leq 2\pi/3$. Then

$$(14) \qquad |mn| = \sin\mu \cdot \frac{|fm|}{\sin(\pi/3+\beta)} \leq \sin\mu \cdot \frac{\sqrt{3}}{\sin\pi/3} = 2\sin\mu.$$

If $p = m$, then for $\triangle fpa$ we write $\angle fpa = \Re$, $\Re < \pi/2$. Then $\Re$ depends on the location of p on bc, i.e., $\Re = \Re(p)$. Observe that $\Re(b) = \pi/3$ and $\Re(c) = \pi/6$. Moreover

$$\min_p \Re(p) = \min\{\Re(b); \Re(c)\} = \pi/6,$$

and hence $\sin\Re \geq 1/2$. Considering $\triangle fpn$, we can estimate the length of mn by

$$(15) \qquad |mn| = \sin\mu \cdot \frac{|fn|}{\sin\Re} \leq \sin\mu \cdot \frac{\sqrt{3}}{\sin\pi/6} = 2\sqrt{3}\sin\mu.$$

The relations (14) and (15) imply (13). Proposition 4 is proved.

According to Remark 2, we assume $t > 5\sqrt{\varepsilon}$. We remind the reader that in the proof of Proposition 1 the lengths of the sides of $\triangle a_1\nu a_2$ with respect to the metric of M^2 were estimated with the help of the polygonal arc $c_2a_3c_3$.

To study the properties of $c_2a_3c_3$, we consider the following constructions. Inequality (8) implies $t_1 = \rho(c_2; a_2a_3) \geq 0.9t > 4\sqrt{\varepsilon}$ and $t_2 = \rho(c_3; a_3a_4) \geq 0.9t > 4\sqrt{\varepsilon}$. On the unit circle ∂B, we consider

$$c_4 = \widehat{c_2 - a_2}, \qquad c_5 = \widehat{a_3 - c_2}, \qquad c_6 = \widehat{c_3 - a_3}, \qquad c_7 = \widehat{a_4 - c_3}.$$

Using Proposition 1 and replacing consecutively ν by c_2, a_1 by a_2, a_2 by a_3, a_3 by a_4, c_2 by c_4, and c_3 by c_5, we get

$$(16) \qquad \rho(c_4; a_3a_4) \geq 0.9t_1 > 0.8t > t/\sqrt{3}, \qquad \rho(c_5; a_4a_5) > 0.8t.$$

For c_3, by the replacement $\nu \to c_3$ and in view of (16) we have

$$(17) \qquad \rho(c_6; a_4a_5) \geq 0.8t, \qquad \rho(c_7; a_5a_6) \geq 0.8t.$$

In what follows, it is convenient to consider the triangle $\triangle a_1\nu a_2$ together with its uniquely defined collection of triangles $\triangle a_2c_2a_3$, $\triangle a_3c_3a_4$ and the polygonal arcs $c_4a_4c_5$, $c_6a_5c_7$. Similarly, we will consider each of the triangles $\triangle a_2c_2a_3$ and $\triangle a_3c_3a_4$ together with the corresponding collection of triangles and broken lines.

We give a description of how to pass from $\triangle a_1\nu a_2$ to $\triangle a_2c_2a_3$, and from $\triangle a_1\nu a_2$ to $\triangle a_3c_3a_4$. Namely, we have the following transformations:

- the polygonal arcs: $c_2a_3c_3 \to c_4a_4c_5$ and $c_2a_3c_3 \to c_6a_5c_7$,
- the segments: $a_2c_2 \to a_3c_4$, $c_2a_3 \to c_4a_4$ and
  $a_3c_3 \to a_4c_5$, $c_3a_4 \to c_5a_5$,

- the points: $c_4 \to c_8 = (\widehat{c_4 - a_3})$, $c_5 \to c_9 = (\widehat{a_4 - c_4})$ and
  $c_6 \to c_{10} = (\widehat{c_5 - a_4})$, $c_7 \to c_{11} = (\widehat{a_5 - c_5})$,
- the angles: $\gamma_1 = \angle(\overline{a_4c_5}, \overline{c_4a_4}) \to \gamma_3 = \angle(\overline{a_5c_9}, \overline{c_8a_5})$ and
  $\gamma_2 = \angle(\overline{a_5c_7}, \overline{c_6a_5}) \to \gamma_4 = \angle(\overline{c_{11}a_6}, \overline{a_6c_{10}})$,
- and again the angles: $\angle\, c_6Oc_5 = \gamma$ and $\angle c_{10}Oc_9 = \gamma_1$.

We write $c_{12} = \widehat{c_6 - a_4}$, $c_{13} = \widehat{a_5 - c_6}$, $c_{14} = \widehat{c_7 - a_5}$ and $c_{15} = \widehat{a_6 - c_7}$. Then $\angle c_{14}Oc_{13} = \gamma_2$, and we write $\gamma_5 = \angle(\overline{c_{12}a_6}, \overline{a_6c_{13}})$ and $\gamma_6 = \angle(\overline{c_{15}a_1}, \overline{a_1c_{14}})$.

By Proposition 1, the inequalities (16), (17), and (8) imply

$$\rho(c_9; a_5a_6) \geq 0.9\rho(c_5; a_4a_5) > 0.72t > t/\sqrt{3}.$$

Similar estimates are valid for all c_k, $k = 8, 9, \dots, 15$, i.e.,

(18)
$$\min\{\rho(c_8; a_4a_5), \rho(c_{9,10,12}; a_5a_6), \rho(c_{11,13,14}; a_6a_1), \rho(c_{15}; a_1a_2)\} > 0.72t.$$

Due to inequality (10) from Corollary 1, the angles γ_k $k = 1, \dots, 6$, satisfy

$$\begin{cases} \sin\gamma_{1,2} \leq \frac{5}{12}\varepsilon/t, \\ \sin\gamma_k \leq \frac{25}{59}\varepsilon/t, \quad k = 3, 4, 5, 6. \end{cases} \tag{19}$$

Write

$$\gamma_0 = \max_{1 \leq k \leq 6}\{\gamma; \gamma_k\}; \tag{20}$$

then, evidently,

$$\sin\gamma_0 \leq \tfrac{25}{59}\varepsilon/t. \tag{21}$$

Proposition 5. *If $t = \rho(B; A_6) > 5\sqrt{\varepsilon}$ $(0 < \varepsilon \leq 0.001)$, then there exists a hexagon $B_6 = b_1b_2b_3b_4b_5b_6$ with the properties:*

(i) *$B_6 = -B_6$, i.e. B_6 is symmetric with respect to the origin O.*
(ii) *B_6 is circumscribed about B in such a way that $a_{k+1} \in b_kb_{k+1}$, $k = 1, \dots, 6$, where $b_7 = b_1$, $a_7 = a_1$.*
(iii) *The distances from b_k to the sides a_ka_{k+1} are such that*

$$\rho(b_k; a_ka_{k+1}) \geq 0.9t, \qquad k = \{1, \dots, 6\}. \tag{22}$$

(iv) *The distance from B_6 to the unit circle B satisfies*

$$\rho(B; B_6) \leq (\sqrt{3}/t + 2\sqrt{3})\sin\gamma_0, \tag{23}$$

where γ_0 is given by (20).

Proof. Denote by l_k the straight lines drawn through a_k such that

(a) l_k, $k = 1, \dots, 6$, are the supporting lines for B;
(b) $l_k \parallel l_{k+3}$, $k = 1, 2, 3$.

Write $\{b_k\} = l_k \cap l_{k+1}$, $k = 1, \dots, 6$, where $l_7 = l_1$. The convex hexagon $B_6 = b_1b_2b_3b_4b_5b_6$ just constructed, symmetric with respect to O, is inscribed in B

in accordance to (b). We have $c_3 \in \partial B$. The inclusion $B \subset B_6$ implies $c_3 \in \triangle a_3b_3a_4$. Similarly, $c_2 \in \triangle a_2b_2a_3$ and $\nu \in \triangle a_1b_1a_2$. Then (8) implies (22).

To prove (23), we simplify notations. Put $a_6 = a_0$, $a_7 = a_1$, $a_8 = a_2$, and write $\{q_k\} = (a_{k-1}a_k) \cap (a_{k+2}a_{k+1})$, $k = 1, \dots, 6$. The convexity of B implies

$$A_6 \subset B \subset B_6 \subset a_1q_1a_2q_2a_3q_3a_4q_4a_5q_5a_6q_6.$$

Consider the right triangle $\triangle a_4q_4a_5$, where $|a_4a_5| = 1$. Observe that for any $x \in \triangle a_4q_4a_5$ we have

$$\rho(x; \triangle a_3q_3a_4) = |xa_4|, \quad \rho(x; \triangle a_5q_5a_6) = |xa_5|.$$

Therefore,

$$\rho(B_6; B) = \max_{1\le k\le 6} \rho(\triangle a_kb_ka_{k+1}; B \cap \triangle a_kq_ka_{k+1}). \tag{24}$$

$1°$. *Estimating from above the distance* $\rho(\triangle a_4b_4a_5; B \cap \triangle a_4q_4a_5)$. Write $\{n_1\} = (a_4c_5) \cap (a_5c_6)$ and $\{m_1\} = (c_4a_4) \cap (c_7a_5)$. Since $c_2 \in \triangle a_2q_2a_3$ and $c_3 \in \triangle a_3q_3a_4$, we have $\angle c_6Oc_5 \subset \angle a_5Oa_4$. The points c_4, a_4, c_5, c_6, a_5 are cyclically located on the boundary ∂B of the convex figure B and the arc $\widehat{a_4a_5}$ is inside the pentagon $a_4m_1a_5c_6c_5$. By construction, $\{b_4\} = l_4 \cap l_5$, and $l_{4,5}$ are the supporting lines to B at $a_{4,5}$. Hence b_4 is inside $a_4m_1a_5n_1$, i.e., $b_4 \in a_4m_1a_5n_1 \subset a_4m_1a_5c_6c_5$. The quadrangle $a_4c_5c_6a_5$ lies in $\triangle a_4m_1a_5$, and hence by (11) we have

$$\rho(\triangle a_4m_1a_5; a_4c_5c_6a_5) \le \min\{|m_1c_5|; |m_1c_6|\} \le |m_1c_5|. \tag{25}$$

Denote by $h_1(u), h_2(u), h_3(u)$ and $h_4(u)$ the support functions for the quadrangle $a_4c_5c_6a_5$, the triangles $\triangle a_4b_4a_5$ and $\triangle a_4m_1a_5$, and $B \cap \triangle a_4q_4a_5$, respectively. It is easy to see that $a_4c_5c_6a_5 \subset \{B \cap \triangle a_4q_4a_5\} \subset \triangle a_4b_4a_5 \subset \triangle a_4m_1a_5$. Using known properties of support functions of convex figures (see [1, §4.15]), we deduce for $|u| = 1$ that

$$h_1(u) \le h_4(u) \le h_2(u) \le h_3(u).$$

Then $h_2(u) - h_4(u) \le h_3(u) - h_1(u)$. By (7) and (25) we have

$$\rho(\triangle a_4b_4a_5; B \cap \triangle a_4q_4a_5) \le |m_1c_5|. \tag{26}$$

From (16) it follows that the height of $\triangle a_4n_1a_5$ with endpoint n_1 satisfies

$$\rho(n_1; a_4a_5) \ge \rho(c_5; a_4a_5) \ge 0.8t.$$

Remember that $\gamma_1 = \angle c_5a_4m_1$ and $\gamma_2 = \angle m_1a_5c_6$ satisfy (19). Via Corollary 2 and Proposition 3, from (12) and (20) we conclude that

$$\rho(\triangle a_4m_1a_5; \triangle a_4n_1a_5) = |n_1m_1| \le \frac{\sqrt{3}}{t} \sin\gamma_0. \tag{27}$$

By construction, for A_6 in Proposition 5 we have $\angle c_6Oc_5 = \angle(\overline{a_3c_3}, \overline{c_2a_3}) = \gamma \le \gamma_0$, satisfying (10). Taking into account Proposition 4, we have

$$|n_1c_5| \le 2\sqrt{3}\sin\gamma_0. \tag{28}$$

By the triangle inequality, $|m_1c_5| \le |n_1m_1|+|n_1c_5|$. Hence, by (27) and (28) we have $|m_1c_5| \le (\sqrt{3}/t+2\sqrt{3})\sin\gamma_0$. Together with (26), the latter inequality implies

$$\rho(\triangle a_4b_4a_5; B\cap\triangle a_4q_4a_5) \le (\sqrt{3}/t+2\sqrt{3})\sin\gamma_0. \tag{29}$$

2°. *Estimating from above the distance $\rho(\triangle a_5b_5a_6; B\cap\triangle a_5q_5a_6)$ and the distance $\rho(\triangle a_6b_6a_1; B\cap\triangle a_6q_6a_1)$.* By Remark 2, we have $t > 5\sqrt{\varepsilon}$, $0<\varepsilon \le 0.001$. By Proposition 1, if $t > 4\sqrt{\varepsilon}$, then $t_1 = \rho(c_2; a_2a_3) \ge 0.9t > 4.5\sqrt{\varepsilon}$ and $t_2 = \rho(c_3; a_3a_4) > 4.5\sqrt{\varepsilon}$. In view of (18) we have

$$\begin{cases} \min(\rho(c_9; a_5a_6), \rho(c_{10}; a_5a_6)) > 0.72t > t/\sqrt{3}, \\ \min(\rho(c_{13}; a_6a_1), \rho(c_{14}; a_6a_1)) > 0.72t > t/\sqrt{3}. \end{cases}$$

Remember that the angles $\gamma_3, \gamma_4, \gamma_1 = \angle c_{10}Oc_9$ and $\gamma_5, \gamma_6, \gamma_2 = \angle c_{14}Oc_{13}$ satisfy (19) and (20). For each of the triangles $\triangle a_5q_5a_6$ and $\triangle a_6q_6a_1$ we consider constructions similar to the constructions for $\triangle a_4q_4a_5$ in the proof of 1°. Using an analogous reasoning to that from (24) to (29), we conclude that

$$\begin{cases} \rho(\triangle a_5b_5a_6; B\cap\triangle a_5q_5a_6) \le (\sqrt{3}/t+2\sqrt{3})\sin\gamma_0, \\ \rho(\triangle a_6b_6a_1; B\cap\triangle a_6q_6a_1) \le (\sqrt{3}/t+2\sqrt{3})\sin\gamma_0. \end{cases}$$

This system, together with (29) and (24), yields (23), and thus Proposition 5 is proved.

Remark 3. *The hexagon B_6 with the properties* (i) *and* (ii) *from Proposition* 5 *has at least four sides of Euclidean length not smaller than* 1/2.

Proof. We use the central symmetry $B_6 = -B_6$ and only consider the sides b_1b_2, b_2b_3, b_3b_4. By construction, for A_6 in Proposition 5 we have $A_6 \subset B_6$, $|a_ka_{k+1}| = 1$, and $\triangle a_kq_ka_{k+1}$ is a right triangle. Evidently, $\sum_{k=1}^6 |b_kb_{k+1}| \ge \sum_{k=1}^6 |a_ka_{k+1}| = 6$, and hence $|b_1b_2| + |b_2b_3| + |b_3b_4| \ge 3$. One of the sides has length at least 1. Assume $|b_1b_2| \ge 1$. If $|b_3b_4| \le 1/2$, then $|b_1b_2| + |b_2b_3| \ge 5/2$. The inclusions $b_1a_2 \subset \triangle a_1q_1a_2$ and $a_2b_2 \subset \triangle a_2q_2a_3$ imply $|b_1b_2| = |b_1a_2| + |a_2b_2| \le 2$. Therefore, $|b_2b_3| \ge 1/2$, i.e., $\min\{|b_1b_2|; |b_2b_3|\} \ge 1/2$.

Proposition 6. *If $t \ge 2\sqrt[3]{\varepsilon}$* ($0 < \varepsilon \le 0.001$), *then each side of the hexagon B_6 from Proposition* 5 *has length at least* 1/2, *i.e.,*

$$l = \min_k |b_kb_{k+1}| \ge 1/2. \tag{30}$$

Proof. Without loss of generality, assume $l = |b_1b_6| = |b_3b_4|$. By (22) we have $\min\{|a_1b_1|; |a_1b_6|\} > 0$. Consider the polygonal arc $a_1b_1b_2b_3a_4$ and observe that $|a_1b_1| + |b_3a_4| = l$.

Suppose that (30) fails, i.e., that $l < 1/2$. With $x = |a_4b_3|$ we have $|a_1b_1| = l - x > 0$. In what follows, we use the subscript "old" to denote lengths of segments and perimeters with respect to the metric generated by the 'old' normalizing figure B, for example $\|ab\| = \|ab\|_{\text{old}},\ L(B) = L_{\text{old}}(B)$. The subscript "new" indicates lengths and perimeters with respect to the new normalizing figure B_6 of M^2.

We intend to estimate the self-perimeter $L_{\text{new}}(B_6)$ from below. Following the proof of Proposition 5, we write $\{q_k\} = (a_{k-1}a_k) \cap (a_{k+2}a_{k+1})$.

By construction, $b_1 \in \triangle a_1q_1a_2$ and hence $g_1 = \widehat{b_1 - b_6} \in \widehat{a_2a_3} \subset \partial B$. The ray $[Og_1)$ meets the polygonal arc $a_2b_2a_3$ at $g_2 \in \triangle a_2q_2a_3$, and we have $|Og_2| \leq \sqrt{3}$. In view of (1), we get

$$\|b_6b_1\|_{\text{new}} = \frac{l}{|Og_2|} \geq \frac{l}{\sqrt{3}}. \tag{31}$$

We consider the points f_i satisfying the conditions

$$\begin{aligned} &f_1 \in a_1a_2, \quad |a_1f_1| = |f_1a_2|;\\ &f_2 \in a_1q_1, \quad |a_1f_2| = |f_2q_1|;\\ &f_3 \in a_2q_2, \quad |a_2f_3| = |f_3q_2|;\\ &\{f_4\} = a_3q_2 \cap (f_2a_2); \quad \{f_5\} = a_4q_3 \cap (f_3a_3);\\ &\{f_6\} = a_3a_4 \cap Oq_3; \quad \{f_7\} = a_3f_5 \cap Oq_3. \end{aligned}$$

Moreover, take $e_2 \in a_1f_2$ and $b_1'' \in a_1f_1$ such that

$$|a_1e_2| = |a_1b_1''| = |a_1b_1| = l - x < 1/2, \tag{32}$$

and $e_4 \in a_4f_5$ and $b_3'' \in a_4f_6$ such that

$$|a_4e_4| = |a_4b_3''| = |a_4b_3| = x < 1/2. \tag{33}$$

Write $g_3 = (\widehat{b_2 - b_1})_{\text{new}}$ and $g_4 = (\widehat{a_2 - e_2})_{\text{new}}$, where $g_3, g_4 \in \partial B_6$, and $\{g_3'\} = a_3e_4 \cap Og_3$ and $\{g_4'\} = a_3e_4 \cap Og_4$. We have the evident inclusions

$$\begin{aligned} &\triangle a_1b_1a_2 \subset \triangle a_1e_2a_2, \quad \triangle a_3b_3a_4 \subset \triangle a_3e_4a_4,\\ &\triangle a_3g_3O \subset \triangle a_3g_3'O \subset \triangle a_3g_4'O \subset \triangle a_3f_7O. \end{aligned}$$

We consider $\{e_3\} = (e_2a_2) \cap (e_4a_3)$ and $\{b_2'\} = (e_4a_3) \cap b_1b_2$. On the straight line (e_2a_2), take e_5 such that $b_1''e_5 \parallel e_4a_3$. Since $\angle f_5e_4a_3 = \angle f_1b_1''e_5 < \pi/2$, we have $e_5 \in e_2a_2$. Write $\{b_1'\} = b_1a_2 \cap b_1''e_5$ and $\{e_3'\} = (e_4a_3) \cap a_2q_2$. It is important that

$$g_3 = (\widehat{b_2 - b_1})_{\text{new}} = (\widehat{b_2' - b_1'})_{\text{new}} \quad \text{and} \quad g_4 = (\widehat{a_2 - e_2})_{\text{new}} = (\widehat{a_2 - e_5})_{\text{new}}.$$

Taking into account the similarities $\triangle g_3'Og_4' \sim \triangle b_2'a_2e_3 \sim \triangle b_1'a_2e_5$ and (1), we get

$$\|b_1b_2\|_{\text{new}} = \frac{|b_1b_2|}{|Og_3|} \geq \frac{|b_1'b_2'|}{|Og_3'|} = \frac{|e_5e_3|}{|Og_4'|}. \tag{34}$$

Since $\triangle Oa_3g_4' \sim \triangle a_2e_3'e_3 \sim \triangle a_2b_1''e_5,\ \triangle a_3e_4a_4 = \triangle a_3e_3'q_2$ and $|a_1a_2| =$

$|a_2q_2| = |Oa_3| = 1$, (32)–(34) imply

$$\|b_1b_2\|_{\text{new}} \geq \frac{|b_1''e_3'|}{|Oa_3|} = 2 - |a_1b_1''| - |a_4e_4| = 2 - l.$$

In a similar way we get $\|b_2b_3\|_{\text{new}} \geq 2 - l$. From this and (31) we deduce that

$$L_{\text{new}}(B_6) \geq 2(2(2-l) + l/\sqrt{3}) = 8 - (4 - 2/\sqrt{3})l.$$

Therefore, if $l < 1/2$, then

$$L_{\text{new}}(B_6) \geq 6 + 1/\sqrt{3} > 6.57. \tag{35}$$

Now we prove that under the hypothesis of Proposition 6 the inequality (35) fails for $t \geq 2\sqrt[3]{\varepsilon}$ $(0 < \sqrt[3]{\varepsilon} \leq 0.1)$. By (23), (20) and (21),

$$\tau = \rho(B; B_6) \leq \left(\frac{\sqrt{3}}{t} + 2\sqrt{3}\right) \cdot \frac{25}{59} \cdot \frac{\varepsilon}{t} \leq \frac{25}{108} \cdot \sqrt[3]{\varepsilon}\left(\frac{\sqrt{3}}{2} + 2\sqrt{3} \cdot \sqrt[3]{\varepsilon}\right) < 0.03. \tag{36}$$

We use the formula for the Hausdorff distance which is equivalent to (7) (see, e.g., (246) in [6]) with respect to B and B_6, i.e.,

$$\rho(B; B_6) = \min\{\lambda \geq 0 : B \subset B_6 + \lambda E,\ B_6 \subset B + \lambda E\},$$

where E is the unit disk of the Euclidean plane $\mathbb{R}^2$. Then $B \subset B_6 + \tau E$ and $B_6 \subset B + \tau E$. According to our constructions, we have $(\sqrt{3}/2)E \subset A_6 \subset B_6$, and hence $E \subset (2/\sqrt{3})B_6$. Therefore,

$$B \subset B_6 + \tau \cdot \frac{2}{\sqrt{3}} B_6 = \left(1 + \frac{2}{\sqrt{3}}\tau\right) B_6.$$

Denote by $(ab)_O$ the straight line passing through the origin O which is parallel to ab, i.e., $(ab)_O \parallel ab$. The Euclidean length of the intersection of B and $(ab)_O$ satisfies

$$|B \cap (ab)_O| \leq \left(1 + \frac{2}{\sqrt{3}}\tau\right) \cdot |B_6 \cap (ab)_O|.$$

From the latter inequality and (1) it follows that for any segment ab in M^2,

$$\|ab\|_{\text{new}} \leq \left(1 + \frac{2}{\sqrt{3}}\tau\right) \cdot \|ab\|_{\text{old}},$$

and hence the self-perimeter of B_6 satisfies

$$L_{\text{new}}(B_6) \leq \left(1 + \frac{2}{\sqrt{3}}\tau\right) L_{\text{old}}(B_6) = \left(1 + \frac{2}{\sqrt{3}}\tau\right) L(B_6). \tag{37}$$

Since $(\sqrt{3}/2)E \subset A_6 \subset B$, we have $B_6 \subset (1 + (2/\sqrt{3})\tau)B$. By (37) and (3),

$$L_{\text{new}}(B_6) \leq \left(1 + \frac{2}{\sqrt{3}}\tau\right) L\left(\left(1 + \frac{2}{\sqrt{3}}\tau\right) B\right) = \left(1 + \frac{2}{\sqrt{3}}\tau\right)^2 L(B).$$

From (36) and (4) we conclude that

$$L_{\text{new}}(B_6) < \left(1 + \frac{2}{\sqrt{3}} \cdot 0.03\right)^2 \cdot 6.001 < 6.43,$$

contradicting (35). Thus, (30) is correct, and Proposition 6 is proved.

We continue with the construction of the hexagon S using the properties of B_6 stated in Propositions 5 and 6.

Recall that $c_k \in \partial B$ and $c_2 = \widehat{\nu - a_1}$, $c_3 = \widehat{a_2 - \nu}$, $c_4 = \widehat{c_2 - a_2}$, $c_5 = \widehat{a_3 - c_2}$, $c_6 = \widehat{c_3 - a_3}$, $c_7 = \widehat{a_4 - c_3}$.

Draw the straight line $l_3(O)$ through O in such a way that $l_3(O) \parallel b_2b_3 \parallel b_5b_6$. The definition of B_6 implies that $l_3(O)$ splits $\angle c_6Oc_5$. Consider the arcs $\widehat{a_4b_4a_5} \subset \partial B_6$ and $\widehat{a_1b_1a_2} \in \partial B_6$, and $\{s_4\} = l_3(O) \cap \widehat{a_4b_4a_5}$ as well as $\{s_1\} = l_3(O) \cap \widehat{a_1b_1a_2}$, where $s_4 = -s_1$.

Remark 4. It suffices to consider in detail the case $s_4 \in a_5b_4$. The case $s_4 \in b_4a_4$ is similar.

Write $\{b_3'\} = (s_4a_4) \cap (b_2b_3)$ and $\{r_4\} = (a_4b_4) \cap (Os_4)$. In view of Remark 2, it is sufficient to consider $t > 5\sqrt{\varepsilon}$ $(0 < \varepsilon \leq 0.001)$. Then (21) implies $\sin\gamma_0 \leq \frac{5}{59}\sqrt{\varepsilon} < 0.01$. Moreover, $0 \leq \gamma_0 < \pi/18$. Consider the case $\gamma_0 = 0$. Then (23) implies $B = B_6$. The polygonal arc $c_2a_3c_3$ degenerates to the segment $c_2c_3 \subset b_2b_3$. By Proposition 1 from [9] we have $\|a_1b_1\| + \|b_1a_2\| = 1$, and hence $L(B) = 6$. In [11] and [13] it was proved that in this case B is an affinely regular hexagon. Therefore, we assume $\gamma_0 \in (0; \pi/18)$.

Proposition 7. *If $t = \rho(B; A_6) > 5\sqrt{\varepsilon}$ $(0 < \varepsilon \leq 0.001)$ and $\gamma_0 > 0$, then*

$$\max\{|s_4b_4|; |b_3b_3'|\} \leq \frac{35\sqrt{3}}{2t}\sin\gamma_0. \tag{38}$$

Proof. First, we estimate $|s_4b_4|$ from above. As in Proposition 5, consider $\{n_1\} = (a_4c_5) \cap (a_5c_6)$, yielding $\angle n_1a_4b_4 < \gamma_1$ and $\angle b_4a_5n_1 < \gamma_2$. Consider $\{w_1\} = (a_4n_1) \cap a_5b_4$. If $w_1 \in a_5s_4$, then $|s_4b_4| \leq |w_1b_4|$. Since $c_5 \in n_1a_4 \subset w_1a_4$, from (16) it follows that $\rho(w_1; a_4a_5) \geq 0.8t$. Observe that $\triangle a_5a_4w_1 \subset \triangle a_5a_4b_4 \subset \triangle a_5a_4q_4$, $\angle w_1a_5b_4 = 0$, $\angle w_1a_4b_4 < \gamma_1$. Using Proposition 3, (12) and (20), we get

$$|s_4b_4| \leq |w_1b_4| \leq \frac{\sqrt{3}}{t}\sin\gamma_0. \tag{39}$$

If $s_4 \in a_5w_1$, then $|s_4b_4| \leq |s_4n_1| + |n_1b_4|$. Using Corollary 2, from (12) and (20) we deduce that

$$\rho(\triangle a_4b_4a_5; \triangle a_4n_1a_5) = |n_1b_4| \leq \frac{\sqrt{3}}{t}\sin\gamma_0. \tag{40}$$

With $\{w_2\} = (a_4w_1) \cap (Os_4)$ and $\{w_3\} = (a_5b_4) \cap (Oc_5)$ we have $w_1n_1 \subset w_2c_5 \subset (n_1a_4)$, $\angle w_2Oc_5 < \gamma$, and hence $\angle w_1On_1 < \gamma$. By (13) and (20),

$$|n_1w_1| \le 2\sqrt{3}\sin\gamma_0. \tag{41}$$

Similarly, $s_4w_1 \subset s_4w_3 \subset (a_5b_4)$, $\angle s_4Ow_1 \le \angle s_4Ow_3 < \gamma$, and hence

$$|s_4w_1| \le 2\sqrt{3}\sin\gamma_0. \tag{42}$$

Using (41) and (42), we conclude that

$$|n_1s_4| \le |n_1w_1| + |w_1s_4| \le 4\sqrt{3}\sin\gamma_0.$$

From the latter inequality and (40) we get

$$|s_4b_4| \le \sqrt{3}(1/t + 4)\sin\gamma_0. \tag{43}$$

Comparing (39) and (43), we see that the latter is more general.

By Remark 4, we assume $s_4 \in a_5b_4$ and hence $b_4 \in r_4a_4$, $b_3 \in b_2b_3'$ and $\triangle a_4b_3b_3' \approx \triangle a_4r_4s_4$. To estimate $|b_3b_3'|$ from above, we use $|s_4r_4|$.

We consider two cases:

1°. If $s_4 \in w_1b_4$, then $|r_4s_4| \le |r_4w_2|$.

2°. If $w_1 \in s_4b_4$, then

$$|r_4s_4| = |r_4w_2| + |w_2s_4|. \tag{44}$$

1°. In this case $w_2 \in \triangle a_5a_4q_4$. Then $|w_2r_4|$ (with $w_2r_4 \subset (Os_4)$) attains its maximum provided $\angle w_2a_4r_4 = \psi \le \gamma_0$, if $w_2 \in \partial\triangle a_5a_4q_4$. If $w_2 \in a_4a_5$, then $\Re = \angle r_4w_2a_4$ satisfies $\pi/3 \le \Re \le 2\pi/3$. Consequently, $\pi/3 < \Re + \psi \le \Re + \gamma_0 \le 5\pi/6$, $|w_2a_4| \le 1$, and by the law of sines

$$|r_4w_2| = \frac{|a_4w_2|}{\sin(\Re + \psi)}\sin\psi \le 2\sin\gamma_0. \tag{45}$$

If $w_2 \in a_4q_4$ and $|a_4w_2| \le 1$, then $\eta = \angle a_4w_2O$ satisfies $\pi/6 \le \eta \le \pi/3$. Then $\varphi = \angle a_4r_4w_2$ satisfies $\pi/6 - \pi/18 \le \eta - \gamma_0 \le \eta - \psi = \varphi \le \pi/3$, and hence $\sin\varphi \ge \frac{1}{3}\sin\frac{\pi}{3}$. In view of $\triangle a_4r_4w_2$ we see that

$$|r_4w_2| = \frac{|a_4w_2|}{\sin\varphi}\sin\psi \le 2\sqrt{3}\sin\gamma_0. \tag{46}$$

If $w_2 \in q_4a_5$, then again $|a_4w_2| \le 1$, $\sin\varphi \ge \sqrt{3}/6$ and hence (46) remains correct. Comparing (45) and (46), we get $|r_4s_4| \le |r_4w_2| \le 2\sqrt{3}\sin\gamma_0$.

2°. In this case it is possible that $w_2 \notin \triangle a_5a_4q_4$. Since $w_1 \in s_4b_4$, the segment w_1w_2 is in $\angle s_4Oc_5 < \gamma$, $w_1 \in \triangle a_4q_4a_5$, and $w_1w_2 \subset (a_4n_1)$. Using the same arguments as in the proof of Proposition 4, we estimate $|w_1w_2|$ in analogy with the derivation of (14) and (15). Namely, if $w_1 = a_4$ and $\sin\gamma < 0.01$, then $w_2 \in \triangle a_4q_4a_5$, and hence $|w_1w_2| \le 2\sqrt{3}\sin\gamma_0$. If $w_1 \in q_4a_5$, then $\Re = \angle a_4w_1O$ satisfies $\pi/6 \le \Re \le \pi/3$, and $\psi = \angle w_2Ow_1 \le \gamma_0 < \pi/18$. Then $\varphi = \angle w_1w_2O = \Re - \psi \ge \Re - \gamma_0$ and $\sin\varphi \ge \sin(\pi/6 - \gamma_0) \ge \sqrt{3}/6$. In

$\triangle w_1 w_2 O$ we have $|Ow_1| \le \sqrt{3}$, and hence

$$|w_1 w_2| \le \frac{|Ow_1|}{\sin\varphi}\sin\gamma \le 6\sin\gamma_0.$$

From this and (42) we conclude that

$$|s_4 w_2| \le |s_4 w_1| + |w_1 w_2| \le 10\sin\gamma_0. \tag{47}$$

We now estimate $|r_4 w_2|$ from above. Choose a_4' and a_5' on the straight lines (Oa_4) and (Oa_5) such that $a_4 \in Oa_4'$, $a_5 \in Oa_5'$ and $|Oa_4'| = |Oa_5'| = 1 + 10\sin\gamma_0$.

Construct the right triangle $\triangle a_4' q_4' a_5'$, where $a_4' q_4' \parallel a_4 q_4$ and $q_4' a_5' \parallel q_4 a_5$. Then $w_2 \in [Os_4)$, and by (47) we have $w_2 \in a_4 a_4' q_4' a_5' a_5$. By the same arguments as in 1°, it follows that $|w_2 r_4|$ attains its maximum provided that $\angle w_2 a_4 r_4 = \psi < \gamma_0$ if either $w_2 \in a_4 a_5$ or w_2 is on the polygonal arc $a_4' q_4' a_5'$, i.e., $w_2 \in \overset{\frown}{a_4' q_4' a_5'}$. If $w_2 \in a_4 a_5$, then (45) holds. If $w_2 \in \overset{\frown}{a_4' q_4' a_5'}$, then, using the same arguments as in the proof of (46), we see that

$$|w_2 r_4| \le \frac{1 + 10\sin\gamma_0}{\sin\varphi}\sin\gamma_0 \le 4\sin\gamma_0.$$

Comparing the latter inequality with (44) and (47), we get the general estimate

$$|r_4 s_4| \le 14\sin\gamma_0. \tag{48}$$

We compare $|b_3 b_3'|$ and $|r_4 s_4|$. Through $c_5 \in \triangle a_4 b_4 a_5$ we draw the straight line $l(c_5) \parallel Os_4 \parallel b_2 b_3$, and we consider $\{v_1\} = (Os_4) \cap a_4 a_5$, $\{v_2\} = (Os_4) \cap (a_4 q_4)$, $\{v_3\} = l(c_5) \cap a_4 a_5$, $\{v_4\} = l(c_5) \cap (a_4 q_4)$, and $\{v_5\} = (a_4 a_5) \cap (b_2 b_3)$. Since (Os_4) splits $\angle c_6 O c_5$, we have the inclusions $\triangle a_4 v_3 v_4 \subset \triangle a_4 v_1 v_2$ and $v_3 a_4 \subset v_1 a_4 \subset v_1 v_5$. Denote by z_1, z_2, z_3 the corresponding bases of the perpendiculars on $(a_4 a_5)$ from s_4 ,c_5, and b_3', respectively. By construction, $z_3 \in a_4 q_3$, $z_1 \in a_4 a_5$, and hence $|a_4 z_3| \le 1$ and $|a_4 z_1| \le 1$. In the right triangle $\triangle a_4 z_2 c_5$ we have $|c_5 z_2| = \rho(c_5; a_4 a_5) \ge 0.8t$, $\angle z_2 a_4 c_5 \le \pi/3$, and hence $|a_4 z_2| \ge 4t\sqrt{3}/15$. The similarity ratio between $\triangle a_4 b_3 b_3'$ and $\triangle a_4 r_4 s_4$ is

$$k = \frac{|b_3 b_3'|}{|s_4 r_4|} = \frac{|a_4 z_3|}{|a_4 z_1|} \le \frac{1}{|a_4 z_2|} \le \frac{5\sqrt{3}}{4t}.$$

From this and (48) it follows that

$$|b_3 b_3'| \le \frac{5\sqrt{3}}{4t}|s_4 r_4| \le \frac{35\sqrt{3}}{2t}\sin\gamma_0.$$

Recall that in our constructions we assume $5\sqrt{\varepsilon} < t \le \sqrt{3}/2$ $(0 < \varepsilon \le 0.001)$. The imposed restrictions and (43) imply the final inequality

$$\max\{|s_4 b_4|; |b_3 b_3'|\} \le \max\left\{\frac{\sqrt{3}(1+4t)}{t}; \frac{35\sqrt{3}}{2t}\right\}\sin\gamma_0 = \frac{35\sqrt{3}}{2t}\sin\gamma_0,$$

and Proposition 7 is proved.

Proof of the Theorem. The proof is divided into three steps and will be conducted according to the scheme $B_6 \to B_6' \to G \to S$, where B_6' and G are some special hexagons.

STEP 1. Construct the centrally symmetric hexagon $B_6' = s_4 b_5 b_6' s_1 b_2 b_3'$ (where $b_6' = -b_3'$ and $s_1 = -s_4$) which is circumscribed about A_6. Due to Corollary 2 and inequality (38), we have

$$\rho(B_6; B_6') \le \frac{35\sqrt{3}}{2t} \sin\gamma_0. \tag{49}$$

Observe that in B_6' the diagonal $s_1 s_4$ satisfies $s_1 s_4 \parallel b_2 b_3' \parallel b_5 b_6'$. Moreover, $a_2 \in s_1 b_2$, $a_4 \in s_4 b_3'$, $a_3 \in b_2 b_3 \subset b_2 b_3'$ (under the assumption that $s_4 \in a_5 b_4$). Since $l_3(O) \parallel b_2 b_3$ and $\{r_4\} = l_3(O) \cap (a_4 b_4)$, from (43) and (48) we get

$$|r_4 b_4| \le |s_4 b_4| + |s_4 r_4| \le (\sqrt{3}/t + 21) \sin\gamma_0. \tag{50}$$

Draw through the origin O the straight line $l_2(O) \parallel s_1 b_2 \parallel s_4 b_5$. Considering $\{r_3\} = l_2(O) \cap (a_3 b_3)$, it is easy to see that $|b_2 r_3| = |s_1 O| = |O s_4|$. In the constructed hexagon B_6' then $s_1 \in b_1 b_2$. Analogously to the proof of (43) and (48), but replacing $\triangle a_5 a_4 q_4$ by $\triangle a_4 a_3 q_3$, b_4 by b_3, and r_4 by r_3, we come to an inequality analogous to (50), namely

$$|b_3 r_3| \le (\sqrt{3}/t + 21) \sin\gamma_0. \tag{51}$$

STEP 2. Construct the affinely regular hexagon $G = g_1 g_2 g_3 g_4 g_5 g_6$ which is centered at O, where $g_1 = s_1$, $g_2 = b_2$, $g_3 = r_3$, $g_4 = s_4$, $g_5 = b_5$, $g_6 = -r_3$; moreover it is possible that $B_6 \not\subset G$ and $G \not\subset B_6$. According to (38) and (51),

$$|b_3' g_3| = |b_3' r_3| \le |b_3' b_3| + |b_3 r_3| \le \left(\frac{37\sqrt{3}}{2t} + 21\right) \sin\gamma_0. \tag{52}$$

Since A_6 is inscribed into B_6', we have

$$\rho(A_6 \cap G; A_6) \le |b_3' g_3|. \tag{53}$$

Without loss of generality, assume $t > 5\sqrt[3]{\varepsilon}$ ($0 < \varepsilon \le 0.001$). For completeness we conduct explicitly the reasoning analogous to Remark 2. Namely, since $(\sqrt{3}/2 + 5\sqrt[3]{\varepsilon}) \cdot 2/\sqrt{3} \le 1 + 6\sqrt[3]{\varepsilon}$ provided $\rho(B; A_6) = t \le 5\sqrt[3]{\varepsilon}$ ($0 < \varepsilon \le 0.001$), the inclusions $A_6 \subset B \subset (1 + 6\sqrt[3]{\varepsilon}) A_6$ hold, and the required hexagon S is A_6.

Since $t > 5\sqrt[3]{\varepsilon}$ and the inequalities (21) and (30) hold (in particular, $|b_2 b_3| \ge 0.5$), either $B_6' \subset G$ or $G \subset B_6'$. Then, by (52),

$$\rho(G; B_6') \le |b_3' g_3| \le \left(\frac{37\sqrt{3}}{2t} + 21\right) \sin\gamma_0.$$

Together with (49) and (23), the latter inequality yields

$$\rho(B; G) \le \rho(B; B_6) + \rho(B_6; B_6') + \rho(B_6'; G) \le (37\sqrt{3}/t + 24.5) \sin\gamma_0.$$

If $t > 5\sqrt[3]{\varepsilon}$ $(0 < \varepsilon \leq 0.001)$, then from the inequality above and (21) we get

$$\rho(B;G) \leq \left(\frac{37\sqrt{3}}{5\sqrt[3]{\varepsilon}} + 24.5\right) \cdot \frac{5}{59}\sqrt[3]{\varepsilon^2} \leq 1.3\sqrt[3]{\varepsilon}. \tag{54}$$

STEP 3. Applying (53) and (52), provided that $t > 5\sqrt[3]{\varepsilon}$, we get

$$\begin{aligned}\rho(A_6 \cap G; A_6) &\leq \left(\frac{37\sqrt{3}}{2t} + 21\right) \cdot \sin\gamma_0 \\ &\leq (3.7\sqrt{3} + 2.1)\frac{5}{59}\sqrt[3]{\varepsilon} \leq 0.73\sqrt[3]{\varepsilon}.\end{aligned} \tag{55}$$

Denote by $h_B(u)$, $h_G(u)$, $h_A(u)$ and $h_{A\cap G}(u)$ the support functions of the unit ball B on M^2, the affinely regular hexagon G, the regular unit hexagon A_6, and $A_6 \cap G$, respectively.

By Theorem B, for $|u| = 1$ the relations (7), (54) and (55) imply

$$\begin{cases} |h_B(u) - h_G(u)| \leq 1.3\sqrt[3]{\varepsilon}, \\ 0 \leq h_A(u) - h_{A\cap G}(u) \leq 0.73\sqrt[3]{\varepsilon}. \end{cases}$$

By construction, the regular hexagon A_6 is inscribed in B. Comparing the inequalities of this system, we get (for $|u| = 1$)

$$\begin{aligned} h_A(u) - 2.03\sqrt[3]{\varepsilon} \leq h_{A\cap G}(u) - 1.3\sqrt[3]{\varepsilon} &\leq h_G(u) - 1.3\sqrt[3]{\varepsilon} \\ &\leq h_B(u) \leq h_G(u) + 1.3\sqrt[3]{\varepsilon}. \end{aligned}$$

Moreover,

$$\left(1 - \frac{1.3}{h_G(u)}\sqrt[3]{\varepsilon}\right)h_G(u) \leq h_B(u) \leq \left(1 + \frac{1.3}{h_G(u)}\sqrt[3]{\varepsilon}\right)h_G(u)$$

and

$$h_G(u) \geq h_A(u) - 0.73\sqrt[3]{\varepsilon} \geq \sqrt{3}/2 - 0.73\sqrt[3]{\varepsilon}.$$

Writing

$$q = \frac{1.3}{\sqrt{3}/2 - 0.73\sqrt[3]{\varepsilon}}\sqrt[3]{\varepsilon} \geq \frac{1.3}{h_G(u)}\sqrt[3]{\varepsilon},$$

we obtain

$$(1-q)h_G(u) \leq h_B(u) \leq (1+q)h_G(u).$$

Therefore,

$$\frac{1+q}{1-q} = 1 + \frac{2.6\sqrt[3]{\varepsilon}}{\sqrt{3}/2 - 2.03\sqrt[3]{\varepsilon}} \leq 1 + \frac{5.2}{\sqrt{3} - 0.406}\sqrt[3]{\varepsilon} \leq 1 + 6\sqrt[3]{\varepsilon}.$$

Define the required hexagon by $S = (1-q)G$. The inequalities $h_S(u) \leq h_B(u) \leq (1 + 6\sqrt[3]{\varepsilon})h_S(u)$ evidently imply the inclusions (5). The Theorem is proved.

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