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CONVERGENCE OF LOGARITHMIC MEANS OF QUADRATIC PARTIAL SUMS OF DOUBLE FOURIER SERIES

BY

USHANGI GOGINAVA (Tbilisi)

Abstract. We investigate some convergence and divergence properties of the logarithmic means of quadratic partial sums of double Fourier series of functions, in measure and in the L Lebesgue norm.

1. Introduction. In the literature, there is a notion of the Riesz logarithmic mean of a Fourier series. The nth mean of the Fourier series of an integrable function f is defined to be

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}, \quad l_n := \sum_{k=1}^{n}\frac{1}{k},$$

where $S_k(f)$ is the partial sum of its Fourier series. These Riesz logarithmic means with respect to the trigonometric system have been studied by many authors. We mention for instance the papers of Szász and Yabuta [13, 14]. These means with respect to the Walsh and Vilenkin systems are discussed by Simon and Gát [12, 2].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=1}^{n} q_k}\sum_{k=0}^{n-1} q_{n-k}S_k(f).$$

If $q_k = 1/k$, then we get the Nörlund logarithmic means

$$\frac{1}{l_n}\sum_{k=0}^{n-1}\frac{S_k(f)}{n-k}.$$

In this paper we call them logarithmic means. They are a kind of "reverse" Riesz logarithmic means. In [3] we proved some convergence and divergence properties of the logarithmic means of Walsh–Fourier series of continuous functions, and of functions in the Lebesgue space L. In this paper we discuss some convergence and divergence properties of logarithmic means of

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quadratic partial sums of the double Fourier series of functions in the L Lebesgue norm (see Theorems 2 and 3).

The partial sums $S_n(f)$ of the Fourier series of a function $f \in L(T)$, $T = [-\pi, \pi)$, converge in measure on T. The condition $f \in L\log L(T^2)$ provides convergence in measure on T^2 of the rectangular partial sums $S_{n,m}(f)$ of double Fourier series [15]. The first examples of functions from classes wider than $L\log L(T^2)$ with $S_{n,n}(f)$ divergent in measure on T^2 were obtained by Getsadze [7] and Konyagin [10].

In the present paper we investigate convergence in measure of logarithmic means of quadratic partial sums,

$$\frac{1}{l_n}\sum_{i=0}^{n-1}\frac{S_{i,i}(f,x,y)}{n-i},$$

of double Fourier series and prove that for any Orlicz space which is not a subspace of $L\log L(I^2)$, the set of functions such that these means converge in measure is of first Baire category (Theorem 4). From this result it follows (Corollary 1) that in classes wider than $L\log L(T^2)$ there exist functions f for which the logarithmic means $t_n(f)$ of quadratic partial sums of double Fourier series diverge in measure. Besides, it is surprising that two cases (the logarithmic means of quadratic and two-dimensional partial sums) are not different from this point of view. Namely, for instance in the case of $(C,1)$ means we have a quite different situation: it is well-known [15] that the Marcinkiewicz means $\sigma_n(f) = n^{-1}\sum_{j=1}^{n} S_{j,j}(f)$, that is, the $(C,1)$ means of quadratic partial sums of the double trigonometric Fourier series of $f \in L$, converge in L-norm and a.e. to f. Thus, as regards convergence in measure and in norm, the logarithmic means of quadratic partial sums of double Fourier series differ from the Marcinkiewicz means, and behave similarly to the usual quadratic partial sums of double Fourier series.

Some results on summability of quadratic partial sums of Walsh–Fourier series can be found in [8, 4].

2. Definitions and notation. We denote by $L_0 = L_0(T^2)$ the Lebesgue space of functions that are measurable and almost everywhere finite on $T^2 = [-\pi, \pi) \times [-\pi, \pi)$; $\operatorname{mes}(A)$ is the Lebesgue measure of the set $A \subset T^2$.

Let $L_Q = L_Q(T^2)$ be the Orlicz space [11] generated by the *Young function* Q, i.e. Q is a convex continuous even function such that $Q(0) = 0$ and

$$\lim_{u\to\infty}\frac{Q(u)}{u} = \infty, \qquad \lim_{u\to 0}\frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(T^2)} = \inf\Big\{k > 0 : \iint\limits_{T^2} Q(|f(x,y)|/k)\,dx\,dy \le 1\Big\}.$$

In particular, if $Q(u) = u\log(1+u)$ and $Q(u) = u\log^2(1+u)$, $u > 0$, then the corresponding space will be denoted by $L\log L(T^2)$ and $L\log^2 L(T^2)$, respectively.

Let $f \in L_1(T^2)$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f] := \sum_{m,n=-\infty}^{\infty} \widehat{f}(m,n)e^{imx}e^{iny},$$

where

$$\widehat{f}(m,n) = \frac{1}{4\pi^2}\iint\limits_{T^2} f(x,y)e^{-imx}e^{-iny}\,dx\,dy$$

are the Fourier coefficients of f. The rectangular partial sums of this series are defined as follows:

$$S_{M,N}(f;x,y) := \sum_{m=-M}^{M}\sum_{n=-N}^{N} \widehat{f}(m,n)e^{imx}e^{iny}.$$

The logarithmic means of quadratic partial sums of double Fourier series are defined as follows:

$$t_n(f;x,y) = \frac{1}{l_n}\sum_{i=0}^{n-1}\frac{S_{i,i}(f;x,y)}{n-i}.$$

It is evident that

$$t_n(f;x,y) = \frac{1}{\pi^2}\iint\limits_{T^2} f(s,t)F_n(x-s,y-t)\,ds\,dt,$$

where

$$F_n(t,s) = \frac{1}{l_n}\sum_{k=0}^{n-1}\frac{D_k(t)D_k(s)}{n-k}, \qquad D_n(t) := \frac{1}{2} + \sum_{j=1}^{n}\cos jt.$$

3. Main results. The following theorem is well-known (see [15]).

THEOREM 1. *Let $f \in L\log^2 L(T^2)$. Then*

$$\|S_{n,n}(f) - f\|_{L_1(T^2)} \to 0 \quad \text{as } n \to \infty.$$

Due to the inequality

$$\|t_n(f) - f\|_{L_1(T^2)} \le \frac{1}{l_n}\sum_{k=0}^{n-1}\frac{\|S_{k,k}(f) - f\|_{L_1(T^2)}}{n-k}$$

and the fact that the Nörlund logarithmic summability method is regular ([9, Ch. 3]), Theorem 1 yields

THEOREM 2. *Let $f \in L\log^2 L(T^2)$. Then*

$$\frac{1}{l_n}\sum_{k=0}^{n-1}\frac{\|S_{k,k}(f)-f\|_{L_1(T^2)}}{n-k}\to 0 \qquad \text{as } n\to\infty.$$

In this paper we investigate the sharpness of Theorem 2. Moreover, we prove

THEOREM 3. *Let $L_Q(T^2)$ be an Orlicz space such that*

$$L_Q(T^2)\nsubseteq L\log^2 L(T^2).$$

Then

(a) *we have*

$$\sup_n \|t_n\|_{L_Q(T^2)\to L_1(T^2)}=\infty;$$

(b) *there exists a function $f\in L_Q(T^2)$ such that $t_n(f)$ does not converge to f in $L_1(T^2)$-norm.*

THEOREM 4. *Let $L_Q(T^2)$ be an Orlicz space such that*

$$L_Q(T^2)\nsubseteq L\log L(T^2).$$

Then the set of functions from the Orlicz space $L_Q(T^2)$ with logarithmic means of quadratic partial sums of double Fourier series convergent in measure on T^2 is of first Baire category in $L_Q(T^2)$.

COROLLARY 1. *Let $\varphi:[0,\infty)\to[0,\infty)$ be a nondecreasing function satisfying the condition*

$$\varphi(x)=o(x\log x) \qquad \text{as } x\to\infty.$$

Then there exists a function $f\in L_1(T^2)$ such that

(a) *we have*

$$\iint_{T^2}\varphi(|f(x,y)|)\,dx\,dy<\infty;$$

(b) *the logarithmic means of the quadratic partial sums of the double Fourier series of f diverge in measure on T^2.*

4. Auxiliary results. We apply a reasoning of [1] summarized in the following proposition.

THEOREM 5. *Let $H: L_1(T^2)\to L_0(T^2)$ be a continuous linear operator which commutes with the family $\mathcal{E}$ of translations, i.e. for all $E\in\mathcal{E}$ and $f\in L_1(T^2)$, $HEf=EHf$. Let $\|f\|_{L_1(T^2)}=1$ and $\lambda>1$. Then for any $1\le r\in\mathbb{N}$ under the condition $\operatorname{mes}\{(x,y)\in T^2:|Hf|>\lambda\}\ge 1/r$ there* *exist* $E_1, \dots, E_r, E'_1, \dots, E'_r \in \mathcal{E}$ *and* $\varepsilon_i = \pm 1$, $i = 1, \dots, r$, *such that*

$$\operatorname{mes}\Big\{(x,y) \in T^2 : \Big|H\Big(\sum_{i=1}^{r} \varepsilon_i f(E_i x, E'_i y)\Big)\Big| > \lambda\Big\} \geq 1/8.$$

Lemma 1. *Let* $\{H_m\}_{m=1}^{\infty}$ *be a sequence of continuous linear operators from the Orlicz space* $L_Q(T^2)$ *into* $L_0(T^2)$. *Suppose that there exists a sequence* $\{\xi_k\}_{k=1}^{\infty}$ *of functions from the unit ball* $B_Q(0,1)$ *of* $L_Q(T^2)$, *and sequences* $\{m_k\}_{k=1}^{\infty}$ *and* $\{\nu_k\}_{k=1}^{\infty}$ *of integers increasing to infinity, such that*

$$\varepsilon_0 = \inf_k \operatorname{mes}\{(x,y) \in T^2 : |H_{m_k}\xi_k(x,y)| > \nu_k\} > 0.$$

Then the set B *of functions* $f \in L_Q(T^2)$ *for which the sequence* $\{H_m f\}$ *converges in measure to an a.e. finite function is of first Baire category in* $L_Q(T^2)$.

The proof of Lemma 1 can be found in [5].

Lemma 2. *Let* $L_\Phi(T^2)$ *be an Orlicz space and let* $\varphi : [0,\infty) \to [0,\infty)$ *be a measurable function with* $\varphi(x) = o(\Phi(x))$ *as* $x \to \infty$. *Then there exists a Young function* ω *such that* $\omega(x) = o(\Phi(x))$ *as* $x \to \infty$, *and* $\omega(x) \geq \varphi(x)$ *for* $x \geq c \geq 0$.

The proof of Lemma 2 can be found in [6].

Lemma 3. *Let*

$$\alpha_{mn} := \frac{\arccos(1/4) + 2\pi m}{2^{2n} + 1/2}, \qquad \beta_{mn} := \frac{\pi/2 + 2\pi m}{2^{2n} + 1/2}, \qquad n, m = 0, 1, \dots.$$

Then there exists a positive integer n_0 *such that*

$$F_{2^{2n}}(x,y) \geq \frac{c}{xy}, \qquad n \geq n_0,$$

whenever

$$(x,y) \in I_n := \bigcup_{l,m=1}^{2^{n-3}} \{(x,y) : \alpha_{mn} \leq x \leq \beta_{mn},\ \alpha_{ln} \leq y \leq \beta_{ln}\}.$$

Proof. We can write

$$\text{(1)} \qquad l_{2^{2n}} F_{2^{2n}}(x,y) = \sum_{k=1}^{2^{2n}} \frac{D_{2^{2n}-k}(x) D_{2^{2n}-k}(y)}{k}$$

$$= \sum_{k=1}^{2^{2n}} \frac{1}{k} \frac{\sin\,(2^{2n} + 1/2 - k)x}{2\sin\frac{x}{2}} \frac{\sin\,(2^{2n} + 1/2 - k)y}{2\sin\frac{y}{2}}$$

$$
\begin{aligned}
&= \frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\sum_{k=1}^{2^{2n}}\frac{\cos kx\cos ky}{k}\\
&\quad+\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\sum_{k=1}^{2^{2n}}\frac{\sin kx\sin ky}{k}\\
&\quad-\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\sum_{k=1}^{2^{2n}}\frac{\cos kx\sin ky}{k}\\
&\quad-\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\sum_{k=1}^{2^{2n}}\frac{\sin kx\cos ky}{k}\\
&= \frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\\
&\quad\times\frac{1}{2}\left\{\sum_{k=1}^{2^{2n}}\frac{\cos k(x+y)}{k}+\sum_{k=1}^{2^{2n}}\frac{\cos k(x-y)}{k}\right\}\\
&\quad+\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\\
&\quad\times\frac{1}{2}\left\{\sum_{k=1}^{2^{2n}}\frac{\cos k(x-y)}{k}-\sum_{k=1}^{2^{2n}}\frac{\cos k(x+y)}{k}\right\}\\
&\quad-\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\\
&\quad\times\frac{1}{2}\left\{\sum_{k=1}^{2^{2n}}\frac{\sin k(x+y)}{k}-\sum_{k=1}^{2^{2n}}\frac{\sin k(x-y)}{k}\right\}\\
&\quad-\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\\
&\quad\times\frac{1}{2}\left\{\sum_{k=1}^{2^{2n}}\frac{\sin k(x+y)}{k}+\sum_{k=1}^{2^{2n}}\frac{\sin k(x-y)}{k}\right\}.
\end{aligned}
$$

Since

$$\sum_{j=0}^{n}D_j(t)=\frac{\sin^2((n+1)t/2)}{2\sin^2(t/2)},$$

using Abel's transformation we obtain

$$
\begin{aligned}
\sum_{k=1}^{2^{2n}} \frac{\cos ku}{k} &= \sum_{k=1}^{2^{2n}-1} \frac{D_k(u)}{k(k+1)} + \frac{D_{2^{2n}}(u)}{2^{2n}} - \frac{1}{2} \\
&= \sum_{k=1}^{2^{2n}-2} \frac{2}{k(k+1)(k+2)} \sum_{l=0}^{k} D_l(u) + \frac{1}{2^{2n}(2^{2n}-1)} \sum_{k=0}^{2^{2n}-1} D_k(u) \\
&\quad + \frac{D_{2^{2n}}(u)}{2^{2n}} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^{2n}(2^{2n}-1)}\right) - \frac{1}{2}\,\frac{1}{2^{2n}(2^{2n}-1)} - \frac{1}{2} \\
&= \sum_{k=1}^{2^{2n}-2} \frac{2}{k(k+1)(k+2)} \frac{\sin^2\left((k+1)\frac{u}{2}\right)}{2\sin^2\frac{u}{2}} \\
&\quad + \frac{1}{2^{2n}\left(2^{2n}-1\right)} \frac{\sin^2(2^{2n}\frac{u}{2})}{2\sin^2\frac{u}{2}} + \frac{1}{2^{2n}} \frac{\sin\left(2^{2n}+1/2\right)u}{2\sin\frac{u}{2}} - \frac{3}{4}.
\end{aligned}
$$

From (1) we have

$$
\begin{aligned}
(2) \qquad l_{2^{2n}} F_{2^{2n}}(x,y) &= \frac{\sin\left(2^{2n}+1/2\right)x}{2\sin\frac{x}{2}} \frac{\sin\left(2^{2n}+1/2\right)y}{2\sin\frac{y}{2}} \frac{1}{2} \\
&\quad \times \sum_{k=1}^{2^{2n}-2} \frac{2}{k(k+1)(k+2)} \frac{\sin^2\left((k+1)\frac{x+y}{2}\right)}{2\sin^2\frac{x+y}{2}} \\
&\quad + \frac{\sin\left(2^{2n}+1/2\right)x}{2\sin\frac{x}{2}} \frac{\sin\left(2^{2n}+1/2\right)y}{2\sin\frac{y}{2}} \frac{1}{2} \\
&\quad \times \sum_{k=1}^{2^{2n}-2} \frac{2}{k(k+1)(k+2)} \frac{\sin^2\left((k+1)\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}} \\
&\quad + \frac{\cos\left(2^{2n}+1/2\right)x}{2\sin\frac{x}{2}} \frac{\cos\left(2^{2n}+1/2\right)y}{2\sin\frac{y}{2}} \frac{1}{2} \\
&\quad \times \sum_{k=1}^{2^{2n}-2} \frac{2}{k(k+1)(k+2)} \frac{\sin^2\left((k+1)\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}} \\
&\quad - \frac{\cos\left(2^{2n}+1/2\right)x}{2\sin\frac{x}{2}} \frac{\cos\left(2^{2n}+1/2\right)y}{2\sin\frac{y}{2}} \frac{1}{2} \\
&\quad \times \sum_{k=1}^{2^{2n}-2} \frac{2}{k(k+1)(k+2)} \frac{\sin^2\left((k+1)\frac{x+y}{2}\right)}{2\sin^2\frac{x+y}{2}} \\
&\quad + \frac{\sin\left(2^{2n}+1/2\right)x}{2\sin\frac{x}{2}} \frac{\sin\left(2^{2n}+1/2\right)y}{2\sin\frac{y}{2}} \frac{1}{2} \\
&\quad \times \frac{1}{2^{2n}(2^{2n}-1)} \frac{\sin^2(2^{2n}\frac{x+y}{2})}{2\sin^2\frac{x+y}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2}\,\frac{1}{2^{2n}}\,\frac{\sin((2^{2n}+1/2)(x+y))}{2\sin\frac{x+y}{2}}\\
&-\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{3}{2}\\
&+\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2^{2n+1}(2^{2n}-1)}\,\frac{\sin^2\left(2^{2n}\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}}\\
&+\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2^{2n+1}}\,\frac{\sin((2^{2n}+1/2)(x-y))}{2\sin\frac{x-y}{2}}\\
&+\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2^{2n+1}(2^{2n}-1)}\,\frac{\sin^2\left(2^{2n}\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}}\\
&+\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2^{2n+1}}\,\frac{\sin((2^{2n}+1/2)(x-y))}{2\sin\frac{x-y}{2}}\\
&+\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2^{2n+1}(2^{2n}-1)}\,\frac{\sin^2\left(2^{2n}\frac{x+y}{2}\right)}{2\sin^2\frac{x+y}{2}}\\
&+\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\,\frac{1}{2^{2n+1}}\,\frac{\sin((2^{2n}+1/2)(x+y))}{2\sin\frac{x+y}{2}}\\
&-\frac{\sin(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\cos(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\\
&\times\frac{1}{2}\left\{\sum_{k=1}^{2^{2n}}\frac{\sin k(x+y)}{k}-\sum_{k=1}^{2^{2n}}\frac{\sin k(x-y)}{k}\right\}\\
&-\frac{\cos(2^{2n}+1/2)x}{2\sin\frac{x}{2}}\,\frac{\sin(2^{2n}+1/2)y}{2\sin\frac{y}{2}}\\
&\times\frac{1}{2}\left\{\sum_{k=1}^{2^{2n}}\frac{\sin k(x+y)}{k}+\sum_{k=1}^{2^{2n}}\frac{\sin k(x-y)}{k}\right\}\\
:=&\sum_{j=1}^{15}R_j.
\end{aligned}
$$

Since

$$\left|\sum_{k=1}^{2^{2n}}\frac{\sin kx}{k}\right|\le c<\infty,\qquad n=1,2,\ldots,$$

and

$$|\sin x|\ge\frac{2}{\pi}|x|,\qquad -\pi/2\le x\le\pi/2, \tag{3}$$

we obtain

$$\sum_{j=5}^{15} |R_j| = O\left(\frac{1}{xy}\right). \tag{4}$$

Let $(x, y) \in I_n$. Then it is easy to show that

$$\sin(2^{2n}+1/2)x, \sin(2^{2n}+1/2)y > 1/2,$$
$$\cos(2^{2n}+1/2)x, \cos(2^{2n}+1/2)y \le 1/4.$$

Consequently, from (3) we obtain

$$\begin{aligned}
R_1 + R_2 + R_3 + R_4 &\ge R_1 + R_2 - R_3 + R_4 \\
&\ge \frac{1}{4}\frac{1}{xy}\sum_{k=1}^{2^{2n}-2}\frac{1}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x+y}{2}\right)}{2\sin^2\frac{x+y}{2}} \\
&\quad + \frac{1}{4}\frac{1}{xy}\sum_{k=1}^{2^{2n}-2}\frac{1}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}} \\
&\quad - \frac{1}{16}\left(\frac{\pi}{2}\right)^2\frac{1}{xy}\sum_{k=1}^{2^{2n}-2}\frac{1}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x+y}{2}\right)}{2\sin^2\frac{x+y}{2}} \\
&\quad - \frac{1}{16}\left(\frac{\pi}{2}\right)^2\frac{1}{xy}\sum_{k=1}^{2^{2n}-2}\frac{1}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}} \\
&\ge \frac{c}{xy}\sum_{k=1}^{2^{n-1}-1}\frac{1}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x+y}{2}\right)}{2\sin^2\frac{x+y}{2}} \\
&\quad + \frac{c}{xy}\sum_{k=1}^{2^{n-1}-1}\frac{1}{k(k+1)(k+2)}\frac{\sin^2\left((k+1)\frac{x-y}{2}\right)}{2\sin^2\frac{x-y}{2}} \\
&\ge \frac{c}{xy}\sum_{k=1}^{2^{n-1}-1}\frac{1}{k} \ge \frac{cn}{xy},
\end{aligned}$$

which completes the proof of Lemma 3. ■

COROLLARY 2. *Let $n \ge n_0$ and*

$$0 \le s, t \le \gamma_n := \frac{\pi/2 - \arccos(1/4)}{4(2^{2n}+1/2)}.$$

Then

$$F_{2^{2n}}(x-s, y-t) \ge \frac{c}{xy}$$

for

$$(x,y)\in J_n := \bigcup_{l,m=1}^{2^{n-3}} \{(x,y): \alpha_{mn}+\gamma_n \le x \le \beta_{mn}-\gamma_n,\ \alpha_{ln}+\gamma_n \le x \le \beta_{ln}-\gamma_n\}.$$

5. Proofs of the theorems

Proof of Theorem 3. (a) Let $Q(2^{4n}) \ge c2^{4n}$ for $n > n_0$. By the estimate ([11, Ch. 2])

$$\|f\|_{L_Q(T^2)} \le c(1+\|Q(|f|)\|_{L_1(T^2)})$$

we get

$$\begin{aligned}
(5)\qquad & \left\| t_{2^{2n}}\left(\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}\right)\right\|_{L_1(T^2)} \\
&\le \|t_{2^{2n}}\|_{L_Q(T^2)\to L_1(T^2)} \left\|\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}\right\|_{L_Q(T^2)} \\
&\le c\|t_{2^{2n}}\|_{L_Q(T^2)\to L_1(T^2)} \left(1+\left\|Q\left(\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}\right)\right\|_{L_1(T^2)}\right) \\
&\le c\|t_{2^{2n}}\|_{L_Q(T^2)\to L_1(T^2)} \left(1+\gamma_n^2 Q\left(\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1}{\gamma_n^2}\right)\right) \\
&\le c\|t_{2^{2n}}\|_{L_Q(T^2)\to L_1(T^2)} \frac{Q(2^{4n})}{2^{4n}}.
\end{aligned}$$

On the other hand, from Corollary 2 we obtain

$$\begin{aligned}
t_{2^{2n}}&\left(\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}; x,y\right) \\
&= \frac{1}{\pi^2}\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1}{\gamma_n^2}\int_0^{\gamma_n}\int_0^{\gamma_n} F_{2^{2n}}(x-s,y-t)\,ds\,dt \\
&\ge \frac{c}{xy}, \qquad (x,y)\in J_n.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(6)\qquad & \left\| t_{2^{2n}}\left(\left(\frac{\pi/2-\arccos(1/4)}{8}\right)^2 \frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}\right)\right\|_{L_1(T^2)} \\
&\ge c\iint_{J_n} \frac{1}{xy}\,dx\,dy \ge c\sum_{l,m=1}^{2^{n-3}} \log\left(1+\frac{c}{m}\right)\log\left(1+\frac{c}{l}\right) \ge cn^2.
\end{aligned}$$

The fact that $L_Q(T^2) \not\subseteq L\log^2 L(T^2)$ is equivalent to the condition

$$\limsup_{u\to\infty} \frac{u\log^2 u}{Q(u)} = \infty.$$

Thus there exists a sequence $\{u_k : k \geq 1\}$ of positive numbers such that

$$\lim_{k\to\infty} \frac{u_k \log^2 u_k}{Q(u_k)} = \infty, \qquad u_{k+1} > u_k, \qquad k = 1, 2, \dots,$$

and an increasing sequence $\{r_k : k \geq 1\}$ of positive integers such that

$$2^{4r_k} \leq u_k < 2^{4(r_k+1)}.$$

Then

$$\frac{2^{4r_k} r_k^2}{Q(2^{4r_k})} \geq c\frac{u_k \log^2 u_k}{Q(u_k)} \to \infty \qquad \text{as } k \to \infty,$$

thus from (5) and (6) we obtain

$$\sup_n \|t_{2^{2n}}\|_{L_Q(T^2)\to L_1(T^2)} = \infty.$$

Part (b) follows immediately from (a) and the uniform boundedness principle.

This completes the proof of Theorem 3. ∎

Proof of Theorem 4. By Lemma 1 the proof of Theorem 4 will be complete if we show that there exist sequences $\{n_k : k \geq 1\}$ and $\{\nu_k : k \geq 1\}$ of integers increasing to infinity, and a sequence $\{\xi_k : k \geq 1\}$ of functions from the unit ball $B_Q(0,1)$ of $L_Q(T^2)$, such that for all k,

$$\text{(7)} \qquad \operatorname{mes}\{(x,y) \in T^2 : |t_{2^{2n_k}}(\xi_k; x, y)| > \nu_k\} \geq 1/8.$$

First, we prove that there exist $c_1, c_2 > 0$ such that

$$\text{(8)} \qquad \operatorname{mes}\left\{(x,y) \in T^2 : \left|t_{2^{2n}}\left(\frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}; x, y\right)\right| > c_1 2^{3n}\right\} > \frac{c_2 n}{2^{3n}}.$$

From Corollary 2,

$$\begin{aligned}\text{(9)} \quad \operatorname{mes}&\left\{(x,y) \in T^2 : \left|t_{2^{2n}}\left(\frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}; x, y\right)\right| > c_1 2^{3n}\right\} \\ &\geq \operatorname{mes}\left\{(x,y) \in J_n : \left|t_{2^{2n}}\left(\frac{1_{[0,\gamma_n]^2}}{\gamma_n^2}; x, y\right)\right| > c_1 2^{3n}\right\} \\ &> \operatorname{mes}\left\{(x,y) \in J_n : \frac{1}{xy} > 2^{3n}\right\}.\end{aligned}$$

Set

$$r_{n,m} := \max\left\{l : \beta_{ln} \leq \frac{1}{2^{3n}(\beta_{mn} - \gamma_n)} + \gamma_n\right\}.$$

By a simple calculation we obtain

$$r_{n,m} \sim \frac{2^n}{m}.$$

Then from (9) we have

$$\begin{aligned}\operatorname{mes}&\left\{(x,y)\in T^2 : \left|t_{2^{2n}}\left(\frac{1_{[0,\gamma_n]^2}}{\gamma_n^2};x,y\right)\right| > c_1 2^{3n}\right\}\\ &\geq c\sum_{m=1}^{2^{n-12}}\sum_{l=1}^{r_{n,m}}\operatorname{mes}(J_n) \geq \frac{c}{2^{4n}}\sum_{m=1}^{2^{n-12}} r_{n,m} \geq \frac{c}{2^{3n}}\sum_{m=1}^{2^{n-12}}\frac{1}{m} \geq \frac{c_2 n}{2^{3n}}.\end{aligned}$$

Hence (8) is proved.

From the condition of the theorem we infer [11]

$$\liminf_{u\to\infty}\frac{Q(u)}{u\log u} = 0.$$

Consequently, there exists a sequence $\{n_k : k \geq 1\}$ of integers increasing to infinity such that

$$\lim_{k\to\infty} Q(2^{4n_k})2^{-4n_k}n_k^{-1} = 0, \qquad \frac{Q(2^{4n_k})}{2^{4n_k+8}} \geq 1, \quad \forall k. \tag{10}$$

From (8) we have

$$\operatorname{mes}\left\{(x,y)\in T^2 : \left|t_{2^{2n_k}}\left(\frac{1_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2};x,y\right)\right| > c_1 2^{3n_k}\right\} > \frac{c_2 n_k}{2^{3n_k}}.$$

Then by Theorem 5 there exist $E_1,\dots,E_r,E_1',\dots,E_r' \in \mathcal{E}$ and $\varepsilon_1,\dots,\varepsilon_r = \pm 1$ such that

$$\operatorname{mes}\left\{(x,y)\in T^2 : \left|\sum_{i=1}^{r}\varepsilon_i t_{2^{2n_k}}\left(\frac{1_{[0,\gamma_{n_k}]^2}}{\gamma_{n_k}^2};E_i x,E_i' y\right)\right| > 2^{3n_k}\right\} > \frac{1}{8},$$

where $r = \left[\frac{2^{3n_k}}{c_2 m_k}\right] + 1$.

Denote

$$\nu_k = \frac{2^{7n_k-1}}{rQ(2^{4n_k})}, \qquad \xi_k(x,y) = \frac{2^{4n_k-1}}{Q(2^{4n_k})}M_k(x,y),$$

where

$$M_k(x,y) = \frac{1}{r}\sum_{i=1}^{r}\varepsilon_i\frac{1_{[0,\gamma_{n_k}]^2}(E_i x,E_i' y)}{\gamma_{n_k}^2}.$$

Then (7) holds.

To finish the proof, we have to prove that $\xi_k \in B_Q(0,1)$. Since (see [16, p. 278])

$$\|M_k\|_\infty \le 2^{4n_k+8}, \qquad \|M_k\|_1 \le 1,$$
$$\|\xi_k\|_{L_Q(T^2)} \le \frac{1}{2}\Big[\iint\limits_{T^2} Q(2|\xi_k|)+1\Big],$$

and

$$\frac{Q(u)}{u} < \frac{Q(u')}{u'}, \qquad 0<u<u',$$

we have

$$\begin{aligned}\|\xi_k\|_{L_Q(T^2)} &\le \frac{1}{2}\Big[1+\iint\limits_{T^2} Q\Big(\frac{2^{4n_k}|M_k(x,y)|}{Q(2^{4n_k})}\Big)\,dx\,dy\Big]\\ &\le \frac{1}{2}\Big[1+\iint\limits_{T^2} \frac{Q(\frac{2^{4n_k}2^{4n_k+8}}{Q(2^{4m_k})})}{\frac{2^{4n_k}2^{4n_k+8}}{Q(2^{4m_k})}}\,\frac{2^{4n_k}|M_k(x,y)|}{Q(2^{4n_k})}\,dx\,dy\Big]\\ &\le \frac{1}{2}\Big[1+\iint\limits_{T^2} \frac{Q(2^{4n_k})}{2^{4n_k}}\,\frac{2^{4n_k}|M_k(x,y)|}{Q(2^{4n_k})}\,dx\,dy\Big] \le 1.\end{aligned}$$

Hence, $\xi_k \in B_Q(0,1)$, and the proof of Theorem 4 is complete. ∎

Corollary 1 follows immediately from Theorem 4 and Lemma 2.

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Ushangi Goginava
Department of Mathematics
Faculty of Exact and Natural Sciences
Iv. Javakhishvili Tbilisi State University
Chavchavadze St. 1
Tbilisi 0128, Georgia
E-mail: zazagoginava@gmail.com

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