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COMPLEXITY AND PERIODICITY

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Abstract. Let M be a finitely generated module over an Artin algebra. By considering the lengths of the modules in the minimal projective resolution of M, we obtain the Betti sequence of M. This sequence must be bounded if M is eventually periodic, but the converse fails to hold in general. We give conditions under which it holds, using techniques from Hochschild cohomology. We also provide a result which under certain conditions guarantees the existence of periodic modules. Finally, we study the case when an element in the Hochschild cohomology ring "generates" the periodicity of a module.

1. Introduction. This paper is devoted to investigating connections between periodicity and complexity for modules over Artin algebras, as was done in [9] for modules over both group rings of finite groups and commutative Noetherian local rings. More specifically, let M be a finitely generated module over an Artin algebra. By considering the lengths of the modules in the minimal projective resolution of M, we obtain the Betti sequence of M. If M is eventually periodic, i.e. if its minimal projective resolution becomes periodic from some step on, then the Betti sequence of M must be bounded. The converse, however, fails to hold in general, that is, it need not be true that M is eventually periodic even though its Betti sequence is bounded. We give conditions under which it holds, using techniques from Hochschild cohomology. In addition we provide a result which under certain conditions guarantees the existence of periodic modules.

One reason for restricting our attention to Artin algebras is that we need to make sure that every finitely generated module has a unique minimal projective resolution. Therefore, throughout this paper, we let k be a commutative Artin ring and Λ an Artin k-algebra with Jacobson radical $\mathfrak{r}$. We fix a finitely generated Λ-module M with a minimal projective resolution

$$(\mathbb{P}, d)\colon \qquad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0,$$

i.e. $\operatorname{Ker} d_i \subseteq \mathfrak{r}\, P_i$. The integers $\beta_n(M) = \ell_k(P_n)$ are called the *Betti numbers* of M, and they are all finite since a module is finitely generated over k

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whenever it is finitely generated over Λ. Moreover, these integers are well defined since any minimal projective resolution is unique up to isomorphism. Thus, we may associate the infinite sequence

$$\beta_0(M), \beta_1(M), \beta_2(M), \dots$$

to M, and this sequence is called the *Betti sequence* of M. Over a commutative Noetherian local ring it is customary to define the Betti numbers of a finitely generated module as the ranks of the modules in its minimal free resolution. However, this would not make sense in our setting since projective modules need not be free.

We say that M is *periodic* if there is an integer $p \geq 1$ such that M is isomorphic to $\Omega_\Lambda^p(M)$ (the pth syzygy in the minimal projective resolution $\mathbb{P}$), and the least such integer p is the *period* of M. Furthermore, M is *eventually periodic* if one of its syzygies (in the minimal projective resolution) is periodic. Clearly, if M has this last property, then its Betti sequence is bounded. The converse is not true in general. A counterexample was given by R. Schulz in [14, Proposition 4.1], where he considered finite-dimensional algebras of the form $k\langle x, y\rangle/(x^2, xy + qyx, y^2)$, for k a field and $q \in k$ a nonzero element.

In [9] D. Eisenbud proved that the converse does hold over group rings of finite groups, and that it also holds in the commutative Noetherian local setting when the rings considered are complete intersections. In fact, it was shown that over a hypersurface (that is, a complete intersection of codimension one) *any* minimal free resolution eventually becomes periodic. In the same paper it was therefore conjectured that over a commutative Noetherian local ring a module having bounded Betti numbers must be periodic.

However, just as in the case of Artin algebras, the conjecture fails to hold in general. An example of this was given in [11]. Here V. Gasharov and I. Peeva considered the commutative local finite-dimensional k-algebra

$$(R, \mathfrak{m}, k) = k[x_1, x_2, x_3, x_4, x_5]/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$\begin{gathered} x_1^2, \quad x_2^2, \quad x_5^2, \quad x_3x_4, \quad x_3x_5, \quad x_4x_5, \quad x_1x_4 + x_2x_4, \\ x_4^2 - x_2x_5 + x_1x_5, \quad \alpha x_1x_3 + x_2x_3, \quad x_3^2 - x_2x_5 + \alpha x_1x_5 \end{gathered}$$

for a nonzero element $\alpha \in k$ having infinite order in the multiplicative group $k \setminus \{0\}$. They constructed the free resolution

$$\cdots \xrightarrow{d_3} R^2 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R^2 \xrightarrow{d_0} M \to 0,$$

where the maps are given by the matrices

$$d_n = \begin{pmatrix} x_1 & \alpha^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix}$$

for $n \geq 0$, and $M = \operatorname{Im} d_0$. The module M has a constant Betti sequence, but is *not* eventually periodic.

We now briefly describe the contents of the three main sections of this paper. Section 2 is devoted to constructing a certain chain endomorphism on the minimal projective resolution of a module, given a finite generation hypothesis similar to the one used in [10]. This chain map eventually becomes surjective, and we use this to prove the main result: whenever the finite generation hypothesis holds, a module has bounded Betti numbers precisely when it is eventually periodic. In some cases (Theorem 2.5) we are also able to determine what the period is.

In Section 3 we develop a method for "reducing" the complexity of a module. More precisely, given a module M satisfying certain conditions (and having finite nonzero complexity), we construct a new module closely related to M and having complexity exactly 1 less than that of M. Iterating this procedure, we end up with a module having complexity 1 (i.e. having bounded Betti numbers), and this module must be periodic in view of the main result in the first section.

Finally, in Section 4 we study the case when the period of an eventually periodic module is "generated" by an element in the Hochschild cohomology ring. As with many other concepts, the inspiration comes from the group ring case, where one uses the group cohomology ring instead of the Hochschild cohomology ring.

2. Preliminary results. The existence of eventually periodic modules of infinite projective dimension—and therefore of nonzero periodic modules—is far from obvious in general. In the next section we prove a result which under certain circumstances guarantees the existence of such modules. The proof is based on the main result of this section, which for a module gives a sufficient condition under which having a bounded Betti sequence is equivalent to being periodic.

The following proposition is the key to the main results. It guarantees regular elements for graded modules, provided we go "far enough" in the grading.

PROPOSITION 2.1. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative Noetherian graded k-algebra of finite type over k (that is, each A_i is a finitely generated k-module), generated as an A_0-algebra by homogeneous elements $a_1, \dots, a_r$ of positive degrees. If $N = \bigoplus_{i=0}^{\infty} N_i$ is a finitely generated graded A-module, then there exists a homogeneous element $\eta \in A$ of positive degree such that the multiplication map*

$$N_i \xrightarrow{\eta} N_{i+|\eta|}$$

is a k-monomorphism for $i \gg 0$. Moreover, we can pick this η such that for some j, the degree of a_j divides $|\eta|$.

Proof. Consider the graded ideal $A^+ = \bigoplus_{i=1}^{\infty} A_i$ of A and the graded submodule

$$(0 :_N A^+) = \{x \in N \mid A^+ x = 0\}$$

of N. Since N is Noetherian, this submodule is finitely generated. Moreover, since it is annihilated by A^+ it is a finitely generated A_0-module ($A_0 = A/A^+$). Now A_0, being finitely generated over k, is Artinian, hence $(0 :_N A^+)$ has the descending chain condition on submodules. Therefore there exists an integer w such that $(0 :_N A^+)_i = 0$ for $i \geq w$.

Consider the graded A-submodule $N_{\geq w} = \bigoplus_{i=w}^{\infty} N_i$ of N. Since it is finitely generated, its set of associated prime ideals is finite and consists of graded ideals each of which is the annihilator of a homogeneous element (see, for example, [6, Lemma 1.5.6]); the union of these ideals is the set of zero-divisors on $N_{\geq w}$. If A^+ is contained in any of these primes, then A^+ annihilates a nonzero homogeneous element of $N_{\geq w}$, a contradiction. Therefore, by the graded version of the "prime avoidance" lemma (see, for example [6, Lemma 1.5.10]), there exists a homogeneous $N_{\geq w}$-regular element η in A^+, obviously of positive degree. Since $a_1, \dots, a_r$ generate A^+, a slight modification of the proof of [6, Lemma 1.5.10] shows that η can be chosen so that the degree of a_j divides that of η for some j. ∎

From now on, we assume that Λ is projective (or, equivalently, flat) as a k-module. We denote by Λ^{e} its enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$, and by $\mathrm{HH}^*(\Lambda)$ its Hochschild cohomology ring. Since Λ is projective as a k-module, we have

$$\mathrm{HH}^*(\Lambda) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$$

with Yoneda product as multiplication. For two Λ-modules X and Y we denote the graded k-module $\bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\Lambda}(X, Y)$ by $\mathrm{Ext}^*_{\Lambda}(X, Y)$, and this is a left and right $\mathrm{HH}^*(\Lambda)$-module via the ring homomorphisms

$$- \otimes_{\Lambda} Y\colon\ \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_{\Lambda}(Y, Y), \qquad - \otimes_{\Lambda} X\colon\ \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_{\Lambda}(X, X)$$

followed by Yoneda composition. The left and right scalar multiplications on this module are closely related as follows (see [15, Corollary 1.3]): for homogeneous elements $\eta \in \mathrm{HH}^*(\Lambda)$ and $\theta \in \mathrm{Ext}^*_{\Lambda}(X, Y)$ we have $\eta\theta = (-1)^{|\eta||\theta|}\theta\eta$, where $|\eta|$ and $|\theta|$ denote the degrees of these elements. In particular, we see that $\mathrm{Ext}^*_{\Lambda}(X, Y)$ is finitely generated as a left $\mathrm{HH}^*(\Lambda)$-module if and only if it is finitely generated as a right $\mathrm{HH}^*(\Lambda)$-module.

In view of the counterexamples provided by Schulz, Gasharov and Peeva, we need to impose some restrictions in order to be able to prove that M is eventually periodic whenever its Betti numbers are bounded. The assumption we introduce is a "local variant" of those used in [10] to develop the theory of support varieties for Artin algebras, and it enables us to use well known techniques from commutative algebra to obtain our results.

Assumption **Fg.** Given the finitely generated Λ-module M, there exists a commutative Noetherian graded subalgebra $H = \bigoplus_{i=0}^{\infty} H^i$ of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ with the property that $H^0 = \mathrm{HH}^0(\Lambda)$ (the center of Λ) and that the module $\mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ is finitely generated over H.

Now we apply Proposition 2.1 to H and to the graded H-module $\mathrm{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$. This we can do because H, being Noetherian, is generated as an algebra over H^0 by a finite set of homogeneous elements of positive degrees. From the homogeneous element granted by Proposition 2.1 we obtain a negative degree chain endomorphism $\{\xi_0, \xi_1, \dots\} : \mathbb{P} \to \mathbb{P}$ (where $\mathbb{P}$ is the minimal projective resolution of M) which eventually becomes *surjective*, that is, the map ξ_i is an epimorphism for $i \gg 0$.

Proposition 2.2. *Assume* **Fg** *holds. Then there exists an integer* $n \geq 1$ *with a map* $\xi : \Omega^n_\Lambda(M) \to M$ *and a chain map* $\{\xi_0, \xi_1, \dots\} : \mathbb{P} \to \mathbb{P}$ *over* ξ *of degree* $-n$, *such that* ξ_i *is surjective for* $i \gg 0$.

Proof. From Proposition 2.1 we see that there exists an integer w and a homogeneous element $\eta \in H^{|\eta|}$, with $|\eta| \geq 1$, such that the multiplication map

$$\mathrm{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta} \mathrm{Ext}^{i+|\eta|}_\Lambda(M, \Lambda/\mathfrak{r})$$

is injective for $i \geq w$. Moreover, since $(\mathbb{P}, d)$ is a minimal projective resolution we have $\mathrm{Im}\, d_i \subseteq \mathfrak{r}\, P_{i-1}$, implying that the differential in the complex $\mathrm{Hom}_\Lambda(\mathbb{P}, \Lambda/\mathfrak{r})$ is zero. Therefore $\mathrm{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) = \mathrm{Hom}_\Lambda(P_i, \Lambda/\mathfrak{r})$. Now consider the element η and the maps it induces. The action on $\mathrm{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r})$ is via the map $- \otimes_\Lambda M : \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_\Lambda(M, M)$, followed by Yoneda composition. Let ξ denote the image of η in $\mathrm{Ext}^*_\Lambda(M, M)$. It can be interpreted as a Λ-linear map

$$\Omega^{|\eta|}_\Lambda(M) \xrightarrow{\xi} M,$$

and so by the Comparison Theorem there exist Λ-linear maps $\{\xi_0, \xi_1, \xi_2, \dots\}$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{|\eta|+i+1} \arrow[r, "d_{|\eta|+i+1}"] \arrow[d, "\xi_{i+1}"] & P_{|\eta|+i} \arrow[r, "d_{|\eta|+i}"] \arrow[d, "\xi_i"] & \cdots \arrow[r] & P_{|\eta|} \arrow[r] \arrow[d, "\xi_0"] & \Omega^{|\eta|}_\Lambda(M) \arrow[r] \arrow[d, "\xi"] & 0 \\
\cdots \arrow[r] & P_{i+1} \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

commute. We show ξ_i is surjective for $i \geq w$.

An element $\theta \in \mathrm{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r})$ can be interpreted as a Λ-linear map $P_i \xrightarrow{\theta} \Lambda/\mathfrak{r}$ (having the property $\theta \circ d_{i+1} = 0$), and then scalar multiplication by η is given by $\theta\eta = \theta \circ \xi_i : P_{i+|\eta|} \to \Lambda/\mathfrak{r}$. Thus the map $\eta \cdot (-)$:

$\operatorname{Hom}_\Lambda(P_i, \Lambda/\mathfrak{r}) \to \operatorname{Hom}_\Lambda(P_{i+|\eta|}, \Lambda/\mathfrak{r})$ is simply given by

$$\xi_i^*\colon f \mapsto f \circ \xi_i,$$

and we know it is injective for $i \geq w$. Applying $\operatorname{Hom}_\Lambda(-, \Lambda/\mathfrak{r})$ to the exact sequence

$$P_{|\eta|+i} \xrightarrow{\xi_i} P_i \to \operatorname{Coker} \xi_i \to 0$$

therefore shows that $\operatorname{Hom}_\Lambda(\operatorname{Coker} \xi_i, \Lambda/\mathfrak{r}) = 0$ for $i \geq w$. Now if X is any finitely generated nonzero Λ-module, then $X/\mathfrak{r}X$ is nonzero by Nakayama's Lemma. This factor module is semisimple, and since every simple Λ-module occurs as a direct summand of $\Lambda/\mathfrak{r}$, there exists a nonzero map $X \to \Lambda/\mathfrak{r}$. This shows $\operatorname{Coker} \xi_i = 0$ for $i \geq w$. By taking $n = |\eta|$, we are done. ∎

We now prove the main result of this section, which gives sufficient (and necessary) conditions for a module to be eventually periodic. Having Proposition 2.2 at hand, the proof is only a formality, as it reduces to simply comparing lengths in the minimal projective resolution of a module.

THEOREM 2.3. *If* **Fg** *holds, then M has bounded Betti numbers if and only if it is an eventually periodic module.*

Proof. From the previous proposition, there exist integers $w \geq 0$ and $n \geq 1$ such that we have n sequences of surjective maps

$$\cdots \xrightarrow{\xi_{w+i+2n}} P_{w+i+2n} \xrightarrow{\xi_{w+i+n}} P_{w+i+n} \xrightarrow{\xi_{w+i}} P_{w+i}, \qquad 0 \leq i < n.$$

Considering the lengths of these modules over k, we see that we have nondecreasing sequences $\beta_{w+i}(M) \leq \beta_{w+i+n}(M) \leq \cdots$ for $0 \leq i < n$. Now if the Betti numbers of M are bounded, then these sequences must all eventually stabilize. Thus there is an integer t such that ξ_i is bijective for $i \geq t$, and diagram chasing in the commutative diagram

$$\begin{tikzcd}
P_{n+t+1} \arrow[r, "d_{n+t+1}"] \arrow[d, "\xi_{t+1}"] & P_{n+t} \arrow[r] \arrow[d, "\xi_t"] & \Omega_\Lambda^{n+t}(M) \arrow[r] & 0 \\
P_{t+1} \arrow[r, "d_{t+1}"] & P_t \arrow[r] & \Omega_\Lambda^t(M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows provides an isomorphism $\psi\colon \Omega_\Lambda^{n+t}(M) \to \Omega_\Lambda^t(M)$. ∎

From this result we obtain some insight into the structure of the sequence of Betti numbers of M. Determining how these sequences grow is a problem which has been studied for a long time in the commutative Noetherian local setting. In [1], L. Avramov asked whether the Betti sequence $(b_0, b_1, b_2, \dots)$ of a finitely generated module over such a ring is eventually nondecreasing, whereas a somewhat weaker question was asked by M. Ramras in [13]: is it true that either $(b_0, b_1, b_2, \dots)$ is eventually constant, or $\lim_{i\to\infty} b_i = \infty$?

With these questions in mind, we include the following corollary to Proposition 2.2 and Theorem 2.3. The result shows that we can split the "tail" of the sequence of Betti numbers of M into a finite number of sequences, each of which is either eventually constant or strictly increasing, depending on whether M has bounded Betti numbers or not.

Corollary 2.4. *Let the setting be as in the theorem. There exist integers $w \geq 0$ and $n \geq 1$ such that the n sequences*

$$(\beta_{w+i+jn}(M))_{j=0}^{\infty}, \qquad 0 \leq i < n,$$

are nondecreasing. In fact, these sequences are all eventually constant if the Betti numbers of M are bounded, and strictly increasing if not. In particular, if the Betti numbers of M are unbounded, then $\lim_{i\to\infty} \beta_i(M) = \infty$.

Proof. The n nondecreasing sequences were given in the proof of the theorem. It is clear that if the Betti numbers of M are bounded, then the sequences are all eventually constant. In the case when there is no bound on the Betti numbers, the module M is not eventually periodic, and then ξ_i cannot be bijective for any $i \geq w$. For if ξ_t were bijective for some $t \geq w$, then since ξ_{t+1} is surjective we would (as in the proof of theorem) have an isomorphism $\Omega_{\Lambda}^{n+t}(M) \simeq \Omega_{\Lambda}^{t}(M)$. ∎

Although Theorem 2.3 provides a tool for determining whether or not a module M is eventually periodic, it does not indicate *when* the minimal projective resolution of M becomes periodic, nor what the period actually *is*, in contrast to the commutative local case. In [9, Theorem 4.1] it is shown that over a commutative local complete intersection A any minimal free resolution whose Betti sequence is bounded becomes periodic of period 2 after at most $\dim A + 1$ steps.

Question. Given the assumptions of Theorem 2.3, does there exist a computable integer s such that the minimal projective resolution of M becomes periodic after at most s steps?

As to determining the period, the degree n of the element η, which we obtain from Proposition 2.1, is of course a candidate, but all that is certain is that the (eventual) period must divide n. However, imposing moderate restrictions on the rings k and H, we get a much stronger result.

Theorem 2.5. *Suppose k contains an infinite field, and let H be generated as an algebra over H^0 by homogeneous elements $a_1, \ldots, a_r$ of positive degrees. Then if* **Fg** *holds and M has bounded Betti numbers, the eventual period of M divides $\operatorname{lcm}(|a_1|, \ldots, |a_r|)$. In particular, if $|a_i| = 1$ for all i then the eventual period of M is 1, and if $|a_i| \leq 2$ for all i then the Betti sequence of M is eventually constant.*

Proof. From Proposition 2.1 we know there exists an integer w and a homogeneous element $\eta \in H$, of positive degree, such that η is regular on the H-module $E_{\geq w} = \bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{i}(M, \Lambda/\mathfrak{r})$. Now let $\{\mathfrak{p}_i\}_{i=1}^{s}$ be the (finite) set of associated primes of $E_{\geq w}$, and denote $\operatorname{lcm}(|a_1|, \dots, |a_r|)$ by u. Then a suitable power of each a_i belongs to H^u, and so if $H^u \subseteq \mathfrak{p}_j$ for some j, then each a_i belongs to $\mathfrak{p}_j$. This implies that the ideal $H^+ = \bigoplus_{i=1}^{\infty} H^i$ is contained in $\mathfrak{p}_j$, contradicting the fact that η, which is an element of H^+, is regular on $E_{\geq w}$. Therefore $H^u \nsubseteq \mathfrak{p}_i$ for $1 \leq i \leq s$, implying $H^u \cap \mathfrak{p}_i \subset H^u$ (strict inclusion). In particular, if k' is an infinite field contained in k, then $H^u \cap \mathfrak{p}_i$ is a proper k'-subspace of H^u. Since over an infinite field no vector space can be written as a finite union of proper subspaces, we get the (strict) inclusion

$$(H^u \cap \mathfrak{p}_1) \cup \dots \cup (H^u \cap \mathfrak{p}_s) \subset H^u,$$

and so H^u must contain an element η' which is not contained in $\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_s$. This union is the set of all zero-divisors on $E_{\geq w}$, hence η' is regular on this module. In the proofs of Proposition 2.2 and Theorem 2.3 we may replace η by η', thus proving the first statement.

Suppose $|a_i| \leq 2$ for all i. Clearly, if each a_i is of degree 1 then the eventual period of M is 1, and the sequence of Betti numbers of M is eventually constant. If one of the generators is of degree 2, then the eventual period is either 1 or 2. If it is 2, then for some integer $N \geq 0$ we have $\Omega_{\Lambda}^{i}(M) \simeq \Omega_{\Lambda}^{i+2}(M)$ for $i \geq N$. By taking the alternate sum of the k-dimensions in the exact sequence

$$0 \to \Omega_{\Lambda}^{i+2}(M) \to P_{i+1} \to P_i \to \Omega_{\Lambda}^{i}(M) \to 0,$$

and recalling that this sum has to be zero, we get $\beta_i(M) = \beta_{i+1}(M)$ for $i \geq N$. Therefore the sequence of Betti numbers of M is eventually constant also in this case. ∎

The results we have proved in this section (and also the main result in the next section) depend on the assumption that $\operatorname{Ext}_{\Lambda}^{*}(M, \Lambda/\mathfrak{r})$ is finitely generated as a module over H, and therefore also as a module over $\operatorname{HH}^*(\Lambda)$. As the action of the Hochschild cohomology ring on this module factors through the rings $\operatorname{Ext}_{\Lambda}^{*}(M, M)$ and $\operatorname{Ext}_{\Lambda}^{*}(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ via ring homomorphisms, the assumption forces $\operatorname{Ext}_{\Lambda}^{*}(M, \Lambda/\mathfrak{r})$ to be finitely generated as a module over both these last rings. In which situations this happens has been studied in the commutative case.

Let $(A, \mathfrak{m}, k')$ be a commutative Noetherian local ring, and N a finitely generated A-module whose so-called *complete intersection dimension* (first defined in [3]) over A is finite (as happens for example when A is a complete intersection). Then L. Avramov and L.-C. Sun proved in [4] that the graded module $\operatorname{Ext}_{A}^{*}(N, k')$ is finitely generated over $\operatorname{Ext}_{A}^{*}(k', k')$, whereas

L. Avramov, V. Gasharov and I. Peeva proved in [3] that the module is also finitely generated over $\mathrm{Ext}^*_\Lambda(N,N)$.

QUESTION. When is $\mathrm{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ finitely generated over (one of) $\mathrm{Ext}^*_\Lambda(M,M)$ and $\mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$?

It is not difficult to see that $\mathrm{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ is finitely generated over $\mathrm{Ext}^*_\Lambda(M,M)$ whenever M is periodic: suppose $\Omega^p_\Lambda(M)=M$ and let μ denote the extension

$$0\to M\to P_{p-1}\to\cdots\to P_0\to M\to 0.$$

If $n\geq p$, say $n=qp+i$ where $0\leq i<p$, then an element of $\mathrm{Ext}^n_\Lambda(M,\Lambda/\mathfrak{r})$ can be written as $f\mu^q$ where $f\in\mathrm{Ext}^i_\Lambda(M,\Lambda/\mathfrak{r})$. Since $\mathrm{Ext}^i_\Lambda(M,\Lambda/\mathfrak{r})$ is finitely generated over k for $0\leq i<p$, the result follows.

As to the finiteness of $\mathrm{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ as an H-module, a criterion for when this always happens was given in [10, Proposition 1.4]. This result is actually much stronger, as it states that $\mathrm{Ext}^*_\Lambda(X,Y)$ is finite over H for all finite Λ-modules X and Y if and only if $\mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r},\Lambda/\mathfrak{r})$ is finite over H.

3. Reducing complexity. We can use Theorem 2.3 to construct eventually periodic modules—and therefore also periodic modules—of infinite projective dimension (the modules having finite projective dimension are not very interesting in the context of eventual periodicity). This is done by considering an (almost) arbitrary module and from it obtaining a new module whose minimal projective resolution behaves "nicer".

Let $X=\bigoplus_{n=0}^\infty X_n$ be a graded k-module of finite type. The *rate of growth* of X, denoted $\gamma(X)$, is defined as

$$\gamma(X)=\inf\{t\in\mathbb{N}_0\mid \exists a\in\mathbb{R}\text{ such that }\ell_k(X_n)\leq an^{t-1}\text{ for }n\gg 0\},$$

and it may be finite or infinite (here $\mathbb{N}_0$ denotes $\mathbb{N}\cup\{0\}$). Now consider our module M with the minimal projective resolution (P_i,d_i). The *complexity* of M, denoted $\mathrm{cx}_\Lambda M$, is defined as the rate of growth of the graded k-module $\bigoplus_{n=0}^\infty P_n$, that is,

$$\mathrm{cx}_\Lambda M=\inf\{t\in\mathbb{N}_0\mid \exists a\in\mathbb{R}\text{ such that }\beta_n(M)\leq an^{t-1}\text{ for }n\gg 0\}.$$

Thus the complexity of M indicates how the sequence of Betti numbers behaves with respect to polynomial growth. From the definition we see that M has complexity 0 if and only if it has finite projective dimension, and that it has complexity less than or equal to 1 if and only if its sequence of Betti numbers is bounded.

The main result of this section gives the existence of a new module whose complexity is exactly 1 less than that of M. The proof uses the identity

$$\mathrm{cx}_\Lambda M=\gamma(\mathrm{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})),$$

which provides a method for computing the complexity of a module. This identity follows from the identities (see the paragraphs following [5, Definition 5.3.3])

$$\gamma(\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})) = \max\{\gamma(\operatorname{Ext}^*_\Lambda(M,S)) \mid S \text{ simple } \Lambda\text{-module}\},$$
$$\beta_n(M) = \sum_{S \,\text{simple}} \frac{\ell_k(P_S)}{\ell_k(\operatorname{Hom}_\Lambda(S,S))} \cdot \ell_k(\operatorname{Ext}^n_\Lambda(M,S)),$$

where P_S denotes the projective cover of the simple module S.

Given a homogeneous element η in $\operatorname{HH}^*(\Lambda)$ of positive degree, we can interpret it as a Λ^{e}-linear map $\eta\colon \Omega^{|\eta|}_{\Lambda^{\mathrm{e}}}(\Lambda) \to \Lambda$, where $\Omega^i_{\Lambda^{\mathrm{e}}}(\Lambda)$ denotes the ith syzygy in the minimal projective Λ^{e}-resolution of Λ. Let Q_i denote the ith module in this resolution. By taking pushout we obtain the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{|\eta|}_{\Lambda^{\mathrm{e}}}(\Lambda) \arrow[r] \arrow[d, "\eta"] & Q_{|\eta|-1} \arrow[r] \arrow[d] & \Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & K_\eta \arrow[r] & \Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda) \arrow[r] & 0
\end{tikzcd}$$

of Λ^{e}-modules, whose bottom row will be denoted by ζ_η. Since $\Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda)$ is projective as a right (and left) Λ-module, the exact sequence ζ_η splits when considered as a sequence of right (and left) Λ-modules. Applying $- \otimes_\Lambda M$ therefore gives the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{|\eta|}_{\Lambda^{\mathrm{e}}}(\Lambda) \otimes_\Lambda M \arrow[r] \arrow[d, "\eta \otimes_\Lambda M"] & Q_{|\eta|-1} \otimes_\Lambda M \arrow[r] \arrow[d] & \Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda) \otimes_\Lambda M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & K_\eta \otimes_\Lambda M \arrow[r] & \Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda) \otimes_\Lambda M \arrow[r] & 0
\end{tikzcd}$$

of left Λ-modules, whose bottom row will be denoted by $\zeta_\eta \otimes_\Lambda M$. Even though ζ_η splits when considered as a sequence of left Λ-modules, this is not necessarily the case for the new sequence. In fact, from [10, Proposition 2.2] we see that the sequence splits if and only if η annihilates $\operatorname{Ext}^*_\Lambda(M,M)$.

The module $K_\eta \otimes_\Lambda M$ is going to be the one having complexity 1 less than that of M. However, as the above shows, the element η cannot be chosen arbitrarily, for if $\zeta_\eta \otimes_\Lambda M$ splits then the complexity of $K_\eta \otimes_\Lambda M$ equals that of M. To see this, note that in a split short exact sequence the complexity of the middle term equals the maximum of the complexities of the end terms, and that the complexities of the end term modules in $\zeta_\eta \otimes_\Lambda M$ are equal since $\Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda) \otimes_\Lambda M$ is a syzygy of M (it does not matter that $\Omega^{|\eta|-1}_{\Lambda^{\mathrm{e}}}(\Lambda) \otimes_\Lambda M$ in general is not a syzygy in the *minimal* projective resolution of M, since projectively equivalent modules are of equal complexity). Thus we must pick an η not annihilating $\operatorname{Ext}^*_\Lambda(M,M)$.

Let N be any Λ-module. Applying the functor $\operatorname{Hom}_\Lambda(-,N)$ to $\zeta_\eta \otimes_\Lambda M$ gives the long exact sequence

$$\operatorname{Hom}_\Lambda(M,N) \xrightarrow{\partial_\eta} \operatorname{Ext}^{|\eta|}_\Lambda(M,N) \to \operatorname{Ext}^1_\Lambda(K_\eta \otimes_\Lambda M,N) \to \operatorname{Ext}^1_\Lambda(M,N) \xrightarrow{\partial_\eta}$$
$$\vdots$$
$$\xrightarrow{\partial_\eta} \operatorname{Ext}^{i+|\eta|-1}_\Lambda(M,N) \to \operatorname{Ext}^i_\Lambda(K_\eta \otimes_\Lambda M,N) \to \operatorname{Ext}^i_\Lambda(M,N) \xrightarrow{\partial_\eta}$$
$$\vdots$$

for Ext (we shall refer to this sequence as $\operatorname{ES}(M,N,\eta)$), where we have replaced $\operatorname{Ext}^i_\Lambda(\Omega^{|\eta|-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda M,N)$ by $\operatorname{Ext}^{i+|\eta|-1}_\Lambda(M,N)$ for $i \geq 1$ using dimension shift. By making these replacements, *the new connecting homomorphism*

$$\operatorname{Ext}^i_\Lambda(M,N) \xrightarrow{\partial_\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M,N)$$

is just multiplication by $(-1)^i\eta$, a fact which is vital for the proof of the main theorem. To see this, note that applying $\operatorname{Hom}_\Lambda(-,N)$ to the above commutative diagram gives rise to a commutative diagram of long exact sequences in Ext. Tracing the connecting homomorphism ∂_η then gives the desired result. Whenever we refer to the exact sequence $\operatorname{ES}(M,N,\eta)$, we shall drop the sign $(-1)^i$ in front of the multiplication map induced by η, as it is of no relevance.

Note that if **Fg** holds, then $\operatorname{Ext}^*_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})$ is a finitely generated H-module, regardless of the choice of η. To see this, consider the exact sequence

$$\bigoplus_{i=1}^{\infty} \operatorname{Ext}^{i+|\eta|-1}_\Lambda(M,\Lambda/\mathfrak{r}) \to \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_\Lambda(K_\eta \otimes_\Lambda M,\Lambda/\mathfrak{r}) \to \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_\Lambda(M,\Lambda/\mathfrak{r})$$

induced by $\operatorname{ES}(M,\Lambda/\mathfrak{r},\eta)$. Both the end terms are finitely generated over H, hence so is the middle term because H is Noetherian. Since in addition $\operatorname{Hom}_\Lambda(K_\eta \otimes_\Lambda M,\Lambda/\mathfrak{r})$ is finitely generated over k (which sits inside H^0), the claim follows.

In addition to giving us an important tool for computing the complexity of a module, the equality $\operatorname{cx}_\Lambda M = \gamma(\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r}))$ implies that the modules we work with have finite complexity. For if **Fg** holds, then the rate of growth of $\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ is not more than that of H, since it is a quotient of a finitely generated free H-module. Now as in the proof of Proposition 2.2, there is a finite set $\{a_1,\dots,a_r\}$ in H of homogeneous elements of positive degrees, generating H as an algebra over H^0. By the Hilbert–Serre Theorem (see [5, Proposition 5.3.1]) and [5, Proposition 5.3.2], we find that $\gamma(H)$ equals the order of the pole at $t=1$ of a certain rational function

$g(t)/\prod_{i=1}^{r}(1-t^{|a_i|})$, where $g \in \mathbb{Z}[t]$. Hence the rate of growth of H is not more than r, implying $\operatorname{cx}_{\Lambda} M \leq r$.

The aim is now to pick η such that the rate of growth of $\operatorname{Ext}^*_{\Lambda}(K_\eta \otimes_{\Lambda} M, \Lambda/\mathfrak{r})$ is 1 less than the rate of growth of $\operatorname{Ext}^*_{\Lambda}(M, \Lambda/\mathfrak{r})$. That such an element exists is a consequence of the following result.

Proposition 3.1. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative Noetherian graded k-algebra of finite type over k, generated as an A_0-algebra by homogeneous elements $a_1, \dots, a_r$ with $|a_i| = n_i > 0$. Let $N = \bigoplus_{i=0}^{\infty} N_i$ be a finitely generated graded A-module, and pick a homogeneous element $\eta \in A$ as in Proposition* 2.1. *Let $w \in \mathbb{N}$ be an integer such that $\eta\colon N_i \to N_{i+|\eta|}$ is injective for $i \geq w$, define V_{i-w} to be the cokernel of this map, and denote by V the graded k-vector space $\bigoplus_{i=0}^{\infty} V_i$. If $\gamma(N) > 0$, then $\gamma(V) = \gamma(N) - 1$.*

Proof. Consider the Poincaré series $P(N,t) = \sum_{i=0}^{\infty} \ell_k(N_i)t^i$ of N. By the Hilbert–Serre Theorem we have

$$P(N,t) = \frac{f(t)}{\prod_{i=1}^{r}(1-t^{n_i})}$$

for some $f(t) \in \mathbb{Z}[t]$, and from [5, Proposition 5.3.2] we see that $\gamma(N)$ equals the order of the pole of $P(N,t)$ at $t=1$. By assumption this integer is strictly greater than zero.

Now consider the exact sequences

$$0 \to N_i \xrightarrow{\eta} N_{i+|\eta|} \to V_{i-w} \to 0$$

for $i \geq w$. Taking k-module lengths we get $\ell_k(V_i) = \ell_k(N_{i+w+|\eta|}) - \ell_k(N_{i+w})$ for $i \geq 0$, giving

$$P(V,t) = \sum_{i=0}^{\infty} \ell_k(V_i)t^i = \sum_{i=0}^{\infty} \ell_k(N_{i+w+|\eta|})t^i - \sum_{i=0}^{\infty} \ell_k(N_{i+w})t^i.$$

Multiplying this equation by $t^{w+|\eta|}$ gives

$$\begin{aligned} t^{w+|\eta|}P(V,t) &= \sum_{i=0}^{\infty} \ell_k(N_{i+w+|\eta|})t^{i+w+|\eta|} - t^{|\eta|}\sum_{i=0}^{\infty} \ell_k(N_{i+w})t^{i+w} \\ &= (1-t^{|\eta|})P(N,t) + g(t), \end{aligned}$$

where $g(t)$ is some polynomial in $\mathbb{Z}[t]$, and therefore

$$P(V,t) = \frac{(1-t^{|\eta|})P(N,t)}{t^{w+|\eta|}} + \frac{g(t)}{t^{w+|\eta|}} = \frac{(1-t^{|\eta|})f(t)}{t^{w+|\eta|}\prod_{i=1}^{r}(1-t^{n_i})} + \frac{g(t)}{t^{w+|\eta|}}.$$

Thus the order of the pole of $P(V,t)$ at $t=1$ is 1 less than that of $P(N,t)$, showing $\gamma(V) = \gamma(N) - 1$. ∎

We now return to the setting given at the beginning of this section. With Proposition 3.1 at hand, the main theorem is merely a corollary.

THEOREM 3.2 (reducing complexity). *Assume that* **Fg** *holds and that M does not have finite projective dimension. Then there exists a homogeneous element $\eta \in H$ of positive degree such that* $\operatorname{cx}_\Lambda(K_\eta \otimes_\Lambda M) = \operatorname{cx}_\Lambda M - 1$.

Proof. We use Proposition 3.1 with $A = H$ and $N = \operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$. There is an integer $w \in \mathbb{N}$ and a homogeneous element $\eta \in H$ of positive degree such that the multiplication map $\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M, \Lambda/\mathfrak{r})$ is injective for $i \geq w$. From the long exact sequence $\operatorname{ES}(M, \Lambda/\mathfrak{r}, \eta)$ we then get short exact sequences

$$0 \to \operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M, \Lambda/\mathfrak{r}) \to \operatorname{Ext}^{i+1}_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to 0$$

for $i \geq w$. Using Proposition 3.1 we now get

$$\begin{aligned} \operatorname{cx}_\Lambda(K_\eta \otimes_\Lambda M) &= \gamma(\operatorname{Ext}^*_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})) \\ &= \gamma\Big(\bigoplus_{i=w+1}^{\infty} \operatorname{Ext}^i_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})\Big) \\ &= \gamma(\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})) - 1 = \operatorname{cx}_\Lambda M - 1. \ \blacksquare \end{aligned}$$

As a corollary we get a result which under certain conditions guarantees the existence of nonzero periodic modules. As mentioned at the beginning of the previous section, the existence of such modules is not obvious. For example, in [12], M. Ramras introduced a nonempty class of commutative Noetherian local rings called BNSI rings (short for “Betti numbers strictly increase” rings), which are rings for which every nonfree module has a strictly increasing sequence of Betti numbers. There exist a lot of finite-dimensional algebras which are BNSI rings, for example regular local rings of dimension at least two modulo any positive power of the maximal ideal. Clearly, such rings cannot have nonzero periodic modules.

COROLLARY 3.3. *Suppose that* **Fg** *holds and that M has infinite projective dimension. Then Λ has a nonzero periodic module.*

Proof. Let d denote the complexity of M (by assumption $d > 0$). If $d = 1$ then M is eventually periodic by Theorem 2.3, whereas if $d > 1$ the previous theorem provides homogeneous elements $\eta_1, \dots, \eta_{d-1} \in H$ having the property

$$\operatorname{cx}_\Lambda(K_{\eta_i} \otimes_\Lambda \cdots \otimes_\Lambda K_{\eta_1} \otimes_\Lambda M) = d - i$$

for $1 \leq i \leq d-1$. Denote the module $K_{\eta_{d-1}} \otimes_\Lambda \cdots \otimes_\Lambda K_{\eta_1} \otimes_\Lambda M$ by X. This module has complexity one, and from the discussion prior to Proposition 3.1 we see that $\operatorname{Ext}^*_\Lambda(X, \Lambda/\mathfrak{r})$ is a finitely generated H-module. Using Theorem 2.3 once more, we deduce that X is eventually periodic.

Now if we take an eventually periodic Λ-module of infinite projective dimension, one of its syzygies is a nonzero periodic module. $\blacksquare$

Remark. (i) The existence of a periodic Λ-module implies the existence of a periodic module having period 1; if M is isomorphic to $\Omega_{\Lambda}^{p}(M)$, where $p \geq 1$, then the module $\bigoplus_{i=0}^{p-1} \Omega_{\Lambda}^{i}(M)$ is periodic of period 1 (here $\Omega_{\Lambda}^{0}(M) = M$).

(ii) Suppose Λ is a BNSI ring. Then there cannot exist a nonfree Λ-module M for which **Fg** holds, i.e. $\operatorname{Ext}_{\Lambda}^{*}(M, \Lambda/\mathfrak{r})$ is not finitely generated over any commutative Noetherian graded subalgebra of $\operatorname{HH}^{*}(\Lambda)$: if such a module existed, then the corollary would imply the existence of a nonzero periodic Λ-module.

4. Generating periodicity. In this section we consider the case when the period of an eventually periodic module is "detected" by a homogeneous element in the Hochschild cohomology ring. We start by recalling the group ring case.

Assume k is an algebraically closed field and let G be a finite group. A nonzero homogeneous element $\theta \in \mathrm{H}^{|\theta|}(G,k) = \operatorname{Ext}_{kG}^{|\theta|}(k,k)$ can be interpreted as a surjective kG-homomorphism $\theta\colon \Omega_{kG}^{|\theta|}(k) \to k$; its kernel is customarily denoted by L_θ. The cohomological variety of L_θ is easily computed; from [8, Lemma 2.3] we deduce that $\mathrm{V}_G(L_\theta)$ equals $\mathrm{V}_G(\theta)$, i.e. the set of maximal ideals in the group cohomology ring $\mathrm{H}^{\cdot}(G,k)$ containing θ. Now let N be a finitely generated kG-module. If $\mathrm{V}_G(\theta) \cap \mathrm{V}_G(N) = \{0\}$, then from the above and the equality $\mathrm{V}_G(X \otimes_k Y) = \mathrm{V}_G(X) \cap \mathrm{V}_G(Y)$, which holds for all finitely generated kG-modules X and Y, we get $\mathrm{V}_G(L_\theta \otimes_k N) = \{0\}$. This is equivalent to $L_\theta \otimes_k N$ being projective, and it follows from the proof of [5, Theorem 5.10.4] that N is isomorphic to $\Omega_{kG}^{|\theta|}(N) \oplus P$, where P is a projective module. This gives

$$\Omega_{kG}^{|\theta|}(N) \simeq \Omega_{kG}^{|\theta|}(\Omega_{kG}^{|\theta|}(N) \oplus P) \simeq \Omega_{kG}^{2|\theta|}(N),$$

showing that $\Omega_{kG}^{|\theta|}(N)$ is periodic and therefore that N is isomorphic to a direct sum of a periodic module and a projective module. If N contains no nonzero projective summand we must have $N \simeq \Omega_{kG}^{|\theta|}(N)$, with the period of N dividing $|\theta|$, and in this case the element θ is said to *generate the periodicity* of N.

Returning to the setting given in the previous sections, with k a commutative Artin ring, Λ an Artin k-algebra (assumed to be projective as a k-module) and M a finitely generated Λ-module, let η be a nonzero element in $\operatorname{HH}^{*}(\Lambda)$ of positive degree. Instead of considering the kernel of the corresponding Λ^{e}-linear map $\eta\colon \Omega_{\Lambda^{\mathrm{e}}}^{|\eta|}(\Lambda) \to \Lambda$ (which is not necessarily surjective), we look at the pushout K_η and the tensor module $K_\eta \otimes_\Lambda M$, as we did in the last section.

Proposition 4.1. *If the Λ-module $K_\eta \otimes_\Lambda M$ has finite projective dimension, then M is eventually periodic with period dividing $|\eta|$.*

Proof. Denote the projective dimension of $K_\eta \otimes_\Lambda M$ by d. From the long exact sequence $\mathrm{ES}(M, \Lambda/\mathfrak{r}, \eta)$ we see that scalar multiplication by η induces k-isomorphisms

$$\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M, \Lambda/\mathfrak{r})$$

for $i > d$. Now let $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ be the minimal projective resolution of M, and recall the proof of Proposition 2.2. If we denote by ξ the image of η in $\operatorname{Ext}^*_\Lambda(M, M)$, represented by a map $\Omega^{|\eta|}_\Lambda(M) \xrightarrow{\xi} M$, then in the commutative diagram (obtained from the Comparison Theorem)

$$\begin{tikzcd}
\cdots \arrow[r] & P_{|\eta|+i+1} \arrow[r, "d_{|\eta|+i+1}"] \arrow[d, "\xi_{i+1}"] & P_{|\eta|+i} \arrow[r, "d_{|\eta|+i}"] \arrow[d, "\xi_i"] & \cdots \arrow[r] & P_{|\eta|} \arrow[r] \arrow[d, "\xi_0"] & \Omega^{|\eta|}_\Lambda(M) \arrow[r] \arrow[d, "\xi"] & 0 \\
\cdots \arrow[r] & P_{i+1} \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

we see that ξ_i is surjective for $i > d$. Applying $\operatorname{Hom}_\Lambda(-, \Lambda/\mathfrak{r})$ to the exact sequences

$$0 \to \operatorname{Ker} \xi_i \to P_{|\eta|+i} \xrightarrow{\xi_i} P_i \to 0$$

(for $i > d$) then gives $\operatorname{Hom}_\Lambda(\operatorname{Ker} \xi_i, \Lambda/\mathfrak{r}) = 0$, hence $\operatorname{Ker} \xi_i = 0$. This shows ξ_i is bijective for $i > d$, and as in the proof of Theorem 2.3 we see that $\Omega^i_\Lambda(M) \simeq \Omega^{i+|\eta|}_\Lambda(M)$ for $i > d$. ∎

The proposition motivates the following definition of an element generating the periodicity of a module:

Definition 4.2. Let M be a Λ-module of infinite projective dimension. A nonzero homogeneous element $\eta \in \mathrm{HH}^*(\Lambda)$ of positive degree *generates the periodicity* of M if the Λ-module $K_\eta \otimes_\Lambda M$ has finite projective dimension.

Note that in the special case when $K_\eta \otimes_\Lambda M$ is projective, all but the end terms in the exact sequence

$$0 \to M \to K_\eta \otimes_\Lambda M \to Q_{|\eta|-2} \otimes_\Lambda M \to \cdots \to Q_0 \otimes_\Lambda M \to M \to 0$$

are projective, hence M is isomorphic to $\Omega^{|\eta|}_\Lambda(M) \oplus P$, where P is a projective Λ-module. As in the group ring case above, this implies that $\Omega^{|\eta|}_\Lambda(M) \simeq \Omega^{2|\eta|}_\Lambda(M)$, and therefore M is isomorphic to the direct sum of a periodic module and a projective module. In particular, if M contains no nonzero projective summand then it is periodic.

Also note that, whenever there exists an element η generating the periodicity of M, for every Λ-module N we see from the long exact sequence

$\mathrm{ES}(M,N,\eta)$ that multiplication

$$\mathrm{Ext}^i_\Lambda(M,N) \xrightarrow{\eta} \mathrm{Ext}^{i+|\eta|}_\Lambda(M,N)$$

is an isomorphism for $i \gg 0$. From the proof of the previous proposition we see that the converse is also true, hence the following result.

PROPOSITION 4.3. *A nonzero homogeneous element $\eta \in \mathrm{HH}^*(\Lambda)$ generates the periodicity of M if and only if scalar multiplication by η induces k-isomorphisms*

$$\mathrm{Ext}^i_\Lambda(M,N) \xrightarrow{\eta} \mathrm{Ext}^{i+|\eta|}_\Lambda(M,N)$$

for every Λ-module N and $i \gg 0$.

As an immediate corollary we obtain the following finite generation result.

COROLLARY 4.4. *If there exists an element η generating the periodicity of M, then $\mathrm{Ext}^*_\Lambda(M,N)$ is a finitely generated $\mathrm{HH}^*(\Lambda)$-module for every Λ-module N. In particular $\mathrm{Ext}^*_\Lambda(M,M)$ and $\mathrm{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})$ are finitely generated.*

Proof. The k-bases of $\mathrm{Hom}_\Lambda(M,N), \mathrm{Ext}^1_\Lambda(M,N), \dots, \mathrm{Ext}^{|\eta|+d}_\Lambda(M,N)$, where $d = \mathrm{pd}_\Lambda(K_\eta \otimes_\Lambda M)$, together form a generating set. ∎

REMARK. We actually obtain a stronger result, namely that for every Λ-module N the k-module $\mathrm{Ext}^*_\Lambda(M,N)$ is a finitely generated module over the commutative Noetherian graded subalgebra of $\mathrm{HH}^*(\Lambda)$ generated by $\mathrm{HH}^0(\Lambda)$ and η (in particular **Fg** holds). As the element η is not nilpotent, this subalgebra is actually the polynomial ring in one variable over $\mathrm{HH}^0(\Lambda)$ (the center of Λ).

It is natural to ask whether an element generating the periodicity of M always exists when M is a periodic module. The following two examples show that this is not the case. In the first example the algebra is selfinjective, whereas in the second it is commutative local.

EXAMPLE 4.5. Let k be a field, and consider the 4-dimensional algebra

$$\Lambda = k\langle x,y\rangle/(x^2, xy+qyx, y^2)$$

where $0 \neq q \in k$ is not a root of unity. Let M be any 2-dimensional k-vector space having a basis $\{u,v\}$, say. Straightforward computation shows that defining

$$xu = 0, \quad xv = 0, \quad yu = v, yv = 0$$

gives a Λ-module structure on M (this module was also studied in [14]). Define a Λ-linear map $p\colon \Lambda \to M$ by $1 \mapsto u$. This is a surjective map, and as a vector space $\operatorname{Ker} p$ has $\{x, xy\}$ as a basis. Therefore $\operatorname{Ker} p$ is contained in the radical of Λ, showing p is the projective cover of M. Define a k-linear map $f\colon M \to \operatorname{Ker} p$ by $u \mapsto x$ and $v \mapsto xy$. This is an isomorphism, and direct computation shows it is Λ-linear. Hence $M \simeq \Omega^1_\Lambda(M)$, and so M is periodic of period 1.

Suppose $0 \neq \eta \in \mathrm{HH}^{|\eta|}(\Lambda)$ is an element generating the periodicity of M. Then from Proposition 4.3 scalar multiplication by η induces isomorphisms $\operatorname{Ext}^i_\Lambda(M,M) \simeq \operatorname{Ext}^{i+|\eta|}_\Lambda(M,M)$ for $i \gg 0$. Therefore multiplication by any power of η also induces isomorphisms. However, from [7] we deduce that $\mathrm{HH}^n(\Lambda) = 0$ for $n \geq 3$, in particular η is nilpotent. Hence $\operatorname{Ext}^i_\Lambda(M,M) = 0$ for $i \gg 0$, and since for each $i > 1$ we have

$$\operatorname{Ext}^i_\Lambda(M,M) \simeq \operatorname{Ext}^1_\Lambda(\Omega^{i-1}_\Lambda(M),M) \simeq \operatorname{Ext}^1_\Lambda(M,M),$$

we get $\operatorname{Ext}^1_\Lambda(M,M) = 0$. Therefore M is projective, since it is periodic of period 1, and this is a contradiction.

Example 4.6. Let k be an algebraically closed field of characteristic different from 2, and let $k[x_1,x_2,x_3,x_4]$ be the polynomial ring in four variables over k. Denote by R the finite-dimensional local k-algebra $k[x_1,x_2,x_3,x_4]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$x_1^2, \quad x_1x_2 - x_3^2, \quad x_1x_3 - x_2x_4, \quad x_1x_4, \quad x_2^2 + x_3x_4, \quad x_2x_3, \quad x_4^2.$$

Define two R-endomorphisms $\phi, \psi\colon R^2 \to R^2$ by the matrices

$$\phi = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, \quad \psi = \begin{pmatrix} x_4 & -x_3 \\ -x_2 & x_1 \end{pmatrix},$$

and let $M = \operatorname{Im}\psi$. In [2] it is shown that $\operatorname{Im}\psi = \operatorname{Ker}\phi$ and $\operatorname{Im}\phi = \operatorname{Ker}\psi$, hence M is periodic of period 2 and has the minimal free resolution

$$\cdots \to R^2 \xrightarrow{\phi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\phi} R^2 \xrightarrow{\psi} M \to 0.$$

As a k-vector space M is 8-dimensional, and the elements

$$v_1 = \begin{pmatrix} x_4 \\ -x_2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -x_3 \\ x_1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} x_1x_3 \\ 0 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} x_1x_3 \\ x_3x_4 \end{pmatrix}, \quad v_6 = \begin{pmatrix} x_3x_4 \\ 0 \end{pmatrix}, \quad v_7 = \begin{pmatrix} -x_1x_2 \\ x_1x_3 \end{pmatrix}, \quad v_8 = \begin{pmatrix} 0 \\ x_1x_3 \end{pmatrix}$$

form a basis. Furthermore, as a graded R-module M consists of two nonzero graded components M_0 and M_1, with $\{v_1,v_2\}$ a basis for M_0 and $\{v_3,\dots,v_8\}$ a basis for M_1. The elements v_1, v_2 generate M as an R-module.

Denote by μ the extension

$$0 \to M \to R^2 \xrightarrow{\phi} R^2 \xrightarrow{\psi} M \to 0$$

in $\operatorname{Ext}_R^2(M,M)$. Then for all $n \geq 0$ every element of $\operatorname{Ext}_R^{2n}(M,M)$ is of the form $f\mu^n$ for some $f \in \operatorname{Hom}_R(M,M)$. Now suppose $0 \neq \eta \in \operatorname{HH}^{|\eta|}(R)$ is an element generating the periodicity of M. Then $|\eta|$ is an even number (since M is of period 2, i.e. $\Omega_R^1(M) \not\simeq M$), and so the image of η in $\operatorname{Ext}_R^*(M,M)$ under the ring homomorphism $- \otimes_R M$ can be written as $f\mu^n$, where f is an R-endomorphism of M and $n = |\eta|/2$. Since $M = M_0 \oplus M_1$ as a graded module, we can write f as $h + g$, where h and g are homogeneous endomorphisms of degree 0 and 1, respectively (see the paragraphs following [6, Theorem 1.5.8]). Therefore the element $f\mu^n$ is the sum of two elements $h\mu^n$ and $g\mu^n$ of different internal degrees in $\operatorname{Ext}_R^{|\eta|}(M,M)$. Since the map $-\otimes_R M\colon \operatorname{HH}^{|\eta|}(R) \to \operatorname{Ext}_R^{|\eta|}(M,M)$ preserves internal grading, $h\mu^n$ and $g\mu^n$ must both lie in the image of $- \otimes_R M$.

Suppose h is nonzero. Since the degree of this map is zero, we must have $h(v_1) = c_1v_1 + c_2v_2$ and $h(v_2) = c_3v_1 + c_4v_2$ for elements $c_i \in k$. As $v_6 = x_3v_1 = -x_4v_2$ and h is R-linear, direct computation of $h(v_6)$ gives

$$c_1v_6 + c_2v_7 = c_3v_8 + c_4v_6.$$

Thus $c_2 = c_3 = 0$ and $c_1 = c_4$, and we see that h is nothing but a scalar $c \in k$ times the identity on M. Therefore the element $c\mu^n$ belongs to the image of $- \otimes_R M$, and multiplying with the inverse of c we deduce that μ^n also lies in this image. From [15, Corollary 1.3] we see that every element of $\operatorname{Im}(-\otimes_R M)$ belongs to the graded center of $\operatorname{Ext}_R^*(M,M)$, and so μ^n, which is of even degree, is a central element. We show that this is not the case.

An element of $\operatorname{Ext}_R^1(M,M)$ can be considered as an R-linear map θ: $R^2 \to M$ such that $\theta \circ \psi = 0$. Define three such elements $\theta_1, \theta_2, \theta_3$ by

$$\begin{aligned}
\theta_1 : &\quad \begin{pmatrix}1\\0\end{pmatrix} \mapsto v_6, \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto 0,\\
\theta_2 : &\quad \begin{pmatrix}1\\0\end{pmatrix} \mapsto 0, \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto v_4,\\
\theta_3 : &\quad \begin{pmatrix}1\\0\end{pmatrix} \mapsto 0, \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto v_5.
\end{aligned}$$

It is easy to check that these elements are linearly independent, and when they are multiplied on the right by any power of μ they stay linearly independent. Direct computation (lifting maps along the minimal free resolution of M) gives

$$\begin{aligned}
\mu\theta_1 &= (\theta_1/2 - \theta_2 + \theta_3/2)\mu,\\
\mu\theta_2 &= \theta_2\mu,\\
\mu\theta_3 &= (-\theta_1/2 - \theta_2 + 3\theta_3/2)\mu,
\end{aligned}$$

and so by induction we get $\mu^n\theta_1 = ([1 - n/2]\theta_1 - n\theta_2 + [n/2]\theta_3)\mu^n$ for all $n \geq 1$. This shows that μ^n is not a central element, implying h must be zero.

As a consequence, the image of η in $\mathrm{Ext}^*_R(M,M)$ is $g\mu^n$, where g is a homogeneous degree one R-endomorphism of M. In particular, since M only lives in two degrees, we must have $g^2=0$. From Proposition 4.3, the element η induces isomorphisms

$$\mathrm{Ext}^i_\Lambda(M,M) \xrightarrow{\eta} \mathrm{Ext}^{i+|\eta|}_\Lambda(M,M)$$

for $i \gg 0$, and in particular there is an $i>0$ such that $\eta \cdot g\mu^i \neq 0$ (because $g\mu^n \neq 0$ implies $g\mu^i \neq 0$, since the element μ^j is not a zero-divisor for any j). But $g\mu^n$, being the image of η, is a central element, giving $\eta \cdot g\mu^i = (g\mu^n)g\mu^i = g(g\mu^n)\mu^i = 0$, a contradiction. Therefore there does not exist a nonzero homogeneous element of positive degree in $\mathrm{HH}^*(R)$ generating the periodicity of M.

These examples show that an element generating the periodicity of an eventually periodic module does not exist in general. However, when dealing with "suitably nice" algebras, such an element always exists. Suppose M is an eventually periodic Λ-module satisfying **Fg**. Then Theorem 3.2 implies the existence of a homogeneous element $\eta \in \mathrm{HH}^*(\Lambda)$ of positive degree with the property that $K_\eta \otimes_\Lambda M$ has finite projective dimension over Λ, and so this η generates the periodicity of M.

The following example shows that it may be difficult to decide whether or not there exists an element generating the periodicity of a module, even when the algebra is commutative, local and selfinjective.

EXAMPLE 4.7. Let k be a field of characteristic different from 2, and let $k[x_1,x_2,x_3,x_4,x_5]$ be the polynomial ring in five variables over k. Denote by R the finite-dimensional local k-algebra $k[x_1,x_2,x_3,x_4,x_5]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$\begin{gathered} x_1^2, \quad x_2^2, \quad x_5^2, \quad x_3x_4, \quad x_3x_5, \quad x_4x_5, \quad x_3^2-x_4^2, \\ x_1x_3+x_2x_3, \quad x_1x_4+x_2x_4, \quad x_3^2-x_2x_5+x_1x_5. \end{gathered}$$

It is shown in [11] that R is graded selfinjective with Hilbert series $1+5t+5t^2+t^3$, and that the R-endomorphism $d\colon R^2 \to R^2$ defined by

$$d = \begin{pmatrix} x_1 & x_3+x_4 \\ 0 & x_2 \end{pmatrix}$$

satisfies $\mathrm{Im}\, d = \mathrm{Ker}\, d$. Letting $M = \mathrm{Im}\, d$, we see that M is periodic of period 1, with minimal free resolution

$$\cdots \to R^2 \xrightarrow{d} R^2 \xrightarrow{d} R^2 \xrightarrow{d} M \to 0.$$

Denote by μ the extension $0 \to M \to R^2 \xrightarrow{d} M \to 0$ in $\mathrm{Ext}^1_R(M,M)$. Then for all $n \geq 0$ every element of $\mathrm{Ext}^n_R(M,M)$ is of the form $f\mu^n$ for some endomorphism f of M.

Now suppose $0 \neq \eta \in \mathrm{HH}^{|\eta|}(R)$ is an element generating the periodicity of M. Then the image of η in $\mathrm{Ext}_R^*(M,M)$ can be written as $f\mu^{|\eta|}$, and as in the previous example this implies that $\mu^{|\eta|}$ belongs to the image of $-\otimes_R M\colon\ \mathrm{HH}^{|\eta|}(R) \to \mathrm{Ext}_R^{|\eta|}(M,M)$ (in this case M is graded and "lives" in three degrees, i.e. $M = M_0 \oplus M_1 \oplus M_2$). In particular $\mu^{|\eta|}$ belongs to the graded center of $\mathrm{Ext}_R^*(M,M)$.

An element of $\mathrm{Ext}_R^i(M,M)$ can be considered as an R-linear map $\theta\colon R^2 \to M$ with the property $\theta \circ d = 0$. In our case, $\mathrm{Ext}_R^i(M,M)$ is 12-dimensional, and the maps

$$\begin{aligned}
&\theta_1:\ e_1 \mapsto \begin{pmatrix} x_1 \\ 0 \end{pmatrix},\ e_2 \mapsto \begin{pmatrix} x_3+x_4 \\ x_2 \end{pmatrix}, && \theta_7:\ e_1 \mapsto 0,\ e_2 \mapsto \begin{pmatrix} 0 \\ x_2x_5 \end{pmatrix},\\
&\theta_2:\ e_1 \mapsto \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix},\ e_2 \mapsto 0, && \theta_8:\ e_1 \mapsto 0,\ e_2 \mapsto \begin{pmatrix} x_2x_5 \\ -x_1x_3 \end{pmatrix},\\
&\theta_3:\ e_1 \mapsto \begin{pmatrix} x_1x_5 \\ 0 \end{pmatrix},\ e_2 \mapsto 0, && \theta_9:\ e_1 \mapsto 0,\ e_2 \mapsto \begin{pmatrix} x_2x_5 \\ -x_1x_4 \end{pmatrix},\\
&\theta_4:\ e_1 \mapsto \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix},\ e_2 \mapsto 0, && \theta_{10}:\ e_1 \mapsto \begin{pmatrix} x_1x_3-x_1x_4 \\ 0 \end{pmatrix},\ e_2 \mapsto 0,\\
&\theta_5:\ e_1 \mapsto 0,\ e_2 \mapsto \begin{pmatrix} 0 \\ x_1x_2x_5 \end{pmatrix}, && \theta_{11}:\ e_1 \mapsto 0,\ e_2 \mapsto \begin{pmatrix} x_1x_3 \\ 0 \end{pmatrix},\\
&\theta_6:\ e_1 \mapsto 0,\ e_2 \mapsto \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, && \theta_{12}:\ e_1 \mapsto \begin{pmatrix} 0 \\ x_1x_4-x_1x_3 \end{pmatrix},\ e_2 \mapsto 0
\end{aligned}$$

represent a basis $\big($here $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}\big)$. Direct computation shows that $\mu\theta_j = \theta_j\mu$ for $j = 1,\dots,10$, while $\mu\theta_j = -\theta_j\mu$ for $j = 11, 12$, and therefore μ^n cannot belong to the graded center of $\mathrm{Ext}_R^*(M,M)$ when n is odd. However, the element μ^2 does belong to the graded center, hence it is possible that a homogeneous element of even degree in $\mathrm{HH}^{|\eta|}(R)$ generates the periodicity of M.

We end with a result providing a positive answer to the following natural question: if the product or sum of two homogeneous elements in the Hochschild cohomology ring generates the periodicity of a module, do the elements themselves (or one of them) generate the periodicity? As a corollary, we obtain a result analogous to [5, Corollary 5.10.6], which states that if the group cohomology ring of a finite group G is finitely generated over a subring generated by homogeneous elements $x_1,\dots,x_t$, and N is a periodic kG-module (where k is an algebraically closed field), then one of the x_i generates the periodicity of N.

Proposition 4.8. *Let $\eta_1 \in \mathrm{HH}^{|\eta_1|}(\Lambda)$ and $\eta_2 \in \mathrm{HH}^{|\eta_2|}(\Lambda)$ be nonzero homogeneous elements of positive degrees.*

(i) *If* $0 \neq \eta_1\eta_2$ *generates the periodicity of* M, *then* η_1 *and* η_2 *both generate the periodicity.*

(ii) *Suppose* Λ *is Gorenstein and* k *is an algebraically closed field. If* $|\eta_1| = |\eta_2|$, *and the sum* $\eta_1 + \eta_2$ *generates the periodicity of* M, *then either* η_1 *or* η_2 *generates the periodicity.*

Proof. (i) From Proposition 4.3 we know that multiplication

$$\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta_1\eta_2} \operatorname{Ext}^{i+|\eta_1|+|\eta_2|}_\Lambda(M, \Lambda/\mathfrak{r})$$

is a k-isomorphism for $i \gg 0$, and since $\operatorname{HH}^*(\Lambda)$ is graded commutative the element $\eta_2\eta_1$ also induces isomorphisms. Therefore the map

$$\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta_2} \operatorname{Ext}^{i+|\eta_2|}_\Lambda(M, \Lambda/\mathfrak{r})$$

is injective for $i \gg 0$, whereas the map

$$\operatorname{Ext}^{i+|\eta_1|}_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta_2} \operatorname{Ext}^{i+|\eta_1|+|\eta_2|}_\Lambda(M, \Lambda/\mathfrak{r})$$

is surjective for $i \gg 0$. It follows from the long exact sequence $\operatorname{ES}(M, \Lambda/\mathfrak{r}, \eta)$ that $\operatorname{Ext}^{i+|\eta_1|+1}_\Lambda(K_{\eta_2} \otimes_\Lambda M, \Lambda/\mathfrak{r}) = 0$ for $i \gg 0$, hence $K_{\eta_2} \otimes_\Lambda M$ has finite projective dimension. Similarly $K_{\eta_1} \otimes_\Lambda M$ has finite projective dimension.

(ii) Denote $\eta_1+\eta_2$ by η, and let H be the commutative Noetherian graded subalgebra of $\operatorname{HH}^*(\Lambda)$ generated by $\operatorname{HH}^0(\Lambda)$ and η. As mentioned in the Remark following Corollary 4.4, the graded k-vector space $\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ is a finitely generated H-module, and consequently **Fg** holds. We may therefore apply the theory of support varieties from [10].

Recall that if X is a finitely generated Λ-module, then the support variety $\operatorname{V}_H(X)$ is defined to be the variety in $\operatorname{MaxSpec} H$ defined by the annihilator of $\operatorname{Ext}^*_\Lambda(X, \Lambda/\mathfrak{r})$ in H. Denote the annihilator ideal associated to M by $\mathfrak{a}$. Since $\operatorname{cx}_\Lambda M = 1$ and $\operatorname{cx}_\Lambda(K_\eta \otimes_\Lambda M) = 0$, it follows from [10, Proposition 1.1] that the variety $\operatorname{V}_H(M) = \operatorname{V}_H(\mathfrak{a})$ is one-dimensional, whereas $\operatorname{V}_H(K_\eta \otimes_\Lambda M)$ is zero-dimensional. By [10, Proposition 3.3] the latter variety equals $\operatorname{V}_H(\eta) \cap V_H(\mathfrak{a}) = \operatorname{V}_H(\langle\eta\rangle + \mathfrak{a})$ (note that in order to apply [10, Proposition 3.3] both **Fg** and the assumption that Λ is Gorenstein are needed, as a substitute for the stronger finite generation hypothesis used in that paper).

Suppose neither of the inclusions $\sqrt{\mathfrak{a}} \subseteq \sqrt{\langle\eta_i\rangle + \mathfrak{a}}$ is strict for $i = 1, 2$. Then

$$\sqrt{\mathfrak{a}} = \sqrt{\langle\eta_1\rangle + \mathfrak{a}} = \sqrt{\langle\eta_1\rangle + \sqrt{\mathfrak{a}}} = \sqrt{\langle\eta_1\rangle + \sqrt{\langle\eta_2\rangle + \mathfrak{a}}} = \sqrt{\langle\eta_1\rangle + \langle\eta_2\rangle + \mathfrak{a}},$$

and since

$$\sqrt{\mathfrak{a}} \subseteq \sqrt{\langle\eta\rangle + \mathfrak{a}} \subseteq \sqrt{\langle\eta_1\rangle + \langle\eta_2\rangle + \mathfrak{a}}$$

we get the equality $\sqrt{\mathfrak{a}} = \sqrt{\langle\eta\rangle + \mathfrak{a}}$. Since the variety defined by any ideal equals that defined by the radical of the ideal, we deduce that $\operatorname{V}_H(M) =$

$\mathrm{V}_H(K_\eta \otimes_\Lambda M)$, which is impossible. Therefore $\sqrt{\mathfrak{a}} \subset \sqrt{\langle \eta_i \rangle + \mathfrak{a}}$ (strict inclusion) at least for one $j \in \{1,2\}$, and as k is algebraically closed and H is a finitely generated k-algebra, Hilbert's Nullstellensatz gives the strict inclusion $\mathrm{V}_H(K_{\eta_j} \otimes_\Lambda M) \subset \mathrm{V}_H(M)$. The variety $\mathrm{V}_H(K_{\eta_j} \otimes_\Lambda M)$ is then zero-dimensional, and consequently $K_{\eta_j} \otimes_\Lambda M$ has finite projective dimension. ∎

Corollary 4.9. *Suppose that Λ is Gorenstein and k is an algebraically closed field, and that* $\mathrm{HH}^*(\Lambda)$ *is generated over a subalgebra by homogeneous elements $x_1, \dots, x_t$. If M is an eventually periodic module and there exists an element in* $\mathrm{HH}^*(\Lambda)$ *generating the periodicity, then one of the x_i generates the periodicity. In particular, the period of M divides the degree of one of the x_i.*

Remark. Proposition 4.8 enables us to strengthen Theorem 2.5 in the case when Λ is Gorenstein and k is an algebraically closed field: when **Fg** holds and H is generated as an algebra over H^0 by homogeneous elements $a_1, \dots, a_r$ of positive degrees, then if M has bounded Betti numbers it is eventually periodic with period dividing one of $|a_i|$.

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