COLLOQUIUM MATHEMATICUM

VOL. 104 2006 NO. 2

ON ENTROPY AND HAUSDORFF DIMENSION OF MEASURES DEFINED THROUGH A NON-HOMOGENEOUS MARKOV PROCESS

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Abstract. We study the Hausdorff dimension of measures whose weight distribution satisfies a Markov non-homogeneous property. We prove, in particular, that the Hausdorff dimensions of this kind of measures coincide with their lower Rényi dimensions (entropy). Moreover, we show that the packing dimensions equal the upper Rényi dimensions. As an application we get a continuity property of the Hausdorff dimension of the measures, when viewed as a function of the distributed weights under the ℓ^∞ norm.

1. Introduction. Let us consider the dyadic tree (even though all the results in this paper can be easily generalised to any ℓ-adic structure, $\ell \in \mathbb{N}$), let $\mathbb{K}$ be its limit (Cantor) set and denote by $(\mathcal{F}_n)_{n\in\mathbb{N}}$ the associated filtration with the usual 0-1 encoding.

We are interested in Borel measures μ on $\mathbb{K}$ constructed in the following way: Take a sequence $(p_n, q_n)_{n\in\mathbb{N}}$ of couples of real numbers satisfying $0 \le p_n, q_n \le 1$. Let $I = I_{\varepsilon_1\ldots\varepsilon_n}$ be a cylinder of the nth generation, $J = I_{\varepsilon_{n+1}}$ a cylinder of the first generation and $IJ = I_{\varepsilon_1\ldots\varepsilon_n\varepsilon_{n+1}}$ the subcylinder of I of the $(n+1)$th generation, where $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1} \in \{0,1\}$. The mass distribution of $\mu_{|I}$ will be as follows: $\mu(I_0) = p_0$, $\mu(I_1) = 1 - p_0$ and

$$\frac{\mu(IJ)}{\mu(I)} = \begin{cases} p_n\mathbf{1}_{\{\varepsilon_{n+1}=0\}} + (1-p_n)\mathbf{1}_{\{\varepsilon_{n+1}=1\}} & \text{if } \varepsilon_n = 0,\\ q_n\mathbf{1}_{\{\varepsilon_{n+1}=0\}} + (1-q_n)\mathbf{1}_{\{\varepsilon_{n+1}=1\}} & \text{if } \varepsilon_n = 1,\end{cases} \tag{1}$$

where the extreme case $\mu(I) = 0$ (and hence $\mu(IJ) = 0$) is treated in the same way by convention.

We use the notation $\dim_{\mathcal{H}}$ for the Hausdorff dimension and $\dim_{\mathcal{P}}$ for the packing dimension.

DEFINITION 1.1. If μ is a measure on $\mathbb{K}$, we will denote by $h_*(\mu)$ its *lower entropy*:

$$h_*(\mu) = \liminf_{n\to\infty} \frac{-1}{n} \sum_{I\in\mathcal{F}_n} \log\mu(I)\cdot\mu(I),$$

2000 *Mathematics Subject Classification*: 28A78, 28A80, 60J60.

Key words and phrases: Hausdorff and packing dimensions, entropy, non-homogeneous Markov processes.

by $h^*(\mu)$ its *upper entropy*:

$$h^*(\mu) = \limsup_{n\to\infty} \frac{-1}{n} \sum_{I\in\mathcal{F}_n} \log\mu(I)\cdot\mu(I),$$

by $\dim_*(\mu)$ its *lower Hausdorff dimension*:

$$\dim_*(\mu) = \inf\{\dim_{\mathcal{H}} E : E\subset\mathbb{K} \text{ and } \mu(E)>0\},$$

and by $\dim^*(\mu)$ its *upper Hausdorff dimension*:

$$\dim^*(\mu) = \inf\{\dim_{\mathcal{H}} E : E\subset\mathbb{K} \text{ and } \mu(\mathbb{K}\setminus E)=0\}.$$

In the same way we define the *lower packing dimension* of μ:

$$\mathrm{Dim}_*(\mu) = \inf\{\dim_{\mathcal{P}} E : E\subset\mathbb{K} \text{ and } \mu(E)>0\},$$

and the *upper packing dimension* of μ:

$$\mathrm{Dim}^*(\mu) = \inf\{\dim_{\mathcal{P}} E : E\subset\mathbb{K} \text{ and } \mu(\mathbb{K}\setminus E)=0\}.$$

One can show (see [Bat02], [BH02]) that

$$\dim_*(\mu)\le h_*(\mu)\le h^*(\mu)\le \mathrm{Dim}^*(\mu),$$

and there are examples of these inequalities being strict, even when the measure μ is rather “regular”.

It is also well known (cf. [Fal97], [Bil65], [Mat95], [Fan94], [You82], [Rén70] and [Heu98]) that

$$\dim_*(\mu) = \operatorname*{ess\,inf}_{\mu} \liminf_{n\to\infty} \frac{\log\mu(I_n(x))}{-n\log 2}$$

and

$$\dim^*(\mu) = \operatorname*{ess\,sup}_{\mu} \liminf_{n\to\infty} \frac{\log\mu(I_n(x))}{-n\log 2},$$

where $I_n(x)$ is the dyadic cylinder of the nth generation containing x, $\operatorname{ess\,inf}_\mu$ is the essential infimum and $\operatorname{ess\,sup}_\mu$ is the essential supremum, taken over μ-almost all $x\in\mathbb{K}$.

Whenever μ is a shift-invariant and ergodic measure, it is well known that all limits exist and

$$\lim_{n\to\infty} \frac{\log\mu(I_n(x))}{-n\log 2} = h_*(\mu) = h^*(\mu),$$

which is the *Breiman–Shanon–McMillan formula*. This is also valid in several random settings (see for instance [Nas87], [Kah87], [KP76] and [Heu03]) and for products of Bernoulli measures (cf. [Bil65]).

In the case of measures defined by (1) we can use tools developed in [Bat96] and [Bat00] to prove they are *exact*, i.e. that $\dim_*(\mu) = \dim^*(\mu)$ or equivalently that

$$\liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{-n\log 2} = \dim_*(\mu) \quad \text{for } \mu\text{-almost all } x \in \mathbb{K}.$$

This is, for instance, the case of harmonic measure on homogeneous Cantor sets and on limit sets of a large class of iterated function systems, like the ones considered in the articles mentioned above. Nevertheless, some kind of shift-invariance is needed in replacement of the Markov condition proposed in this work. In Theorem 1.2 we prove that $\dim_*(\mu) = \dim^*(\mu)$.

In general, there is no trivial inequality between $h_*(\mu)$ and $\dim^*(\mu)$. Furthermore, it is easy to construct measures μ satisfying (1) such that $h_*(\mu) \neq h^*(\mu)$, which shows that the sequence of functions $\frac{\log \mu(I_n(x))}{-n\log 2}$ does not necessarily converge (in any space).

The proof of Theorem 1.2 implies that there is a sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\lim_{n\to\infty}\left[\frac{\log \mu(I_n(x))}{-n\log 2} - c_n\right] = 0,$$

where

$$c_n = \frac{-1}{n\log 2}\sum_{I\in\mathcal{F}_n} \log(\mu(I))\mu(I).$$

This can be seen as a Breiman–Shannon–McMillan type theorem generalised to measures defined through non-homogeneous Markov chains.

Note that the tools of [KP76] and [Kah87] can be applied to give the same results for "almost every" measure μ satisfying (1). Other results in this sense involving colouring of graphs are proposed in [Nas87].

A. Bisbas and C. Karanikas [BK94] have already partially proved the conclusions of Theorem 1.2 under some assumptions on the sequences $(p_n, q_n)_{n\in\mathbb{N}}$. In particular they proved the theorem when the sequences $(p_n, q_n)_{n\in\mathbb{N}}$ are uniformly bounded away from 0 and 1, which is the case of a perturbation of a homogeneous Markov chain. We thank A. Bisbas for informing us about that article.

Theorem 1.2. *If μ satisfies* (1) *then*

$$\dim_*(\mu) = \dim^*(\mu) = h_*(\mu) \quad \textit{and} \quad \mathrm{Dim}_*(\mu) = \mathrm{Dim}^*(\mu) = h^*(\mu).$$

Using the same type of argument we also obtain the following continuity result.

Theorem 1.3. *Let μ and μ' be measures defined by* (1) *and the respective sequences $(p_n, q_n)_{n\in\mathbb{N}}$ and $(p'_n, q'_n)_{n\in\mathbb{N}}$. Then $|\dim_*(\mu) - \dim_*(\mu')|$ and $|\mathrm{Dim}_*(\mu) - \mathrm{Dim}_*(\mu')|$ go to* 0 *as $\|(p_n, q_n)_{n\in\mathbb{N}} - (p'_n, q'_n)_{n\in\mathbb{N}}\|_\infty$ tends to* 0.

2. Lemmas and preliminary results. Let us introduce some notation: for $p \in [0,1]$ we define

$$h(p) = p\log p + (1-p)\log(1-p)$$

and if $I = I_{\varepsilon_1,\ldots,\varepsilon_{n-1}} \in \mathcal{F}_n$, we also set

$$\gamma(I,n) = \sum_{i=0,1} \log\left(\frac{\mu(II_i)}{\mu(I)}\right)\frac{\mu(II_i)}{\mu(I)}.$$

Note that $\gamma(I,n) = \mathbb{E}_I(X_n)$ in the notation of [Chu01, Section 9.1, p. 295]. We also remark that for $n \in \mathbb{N}$ and $I \in \mathcal{F}_{n-1}$, $\gamma(I,n)$ is equal to $h(p_n)$ if $\varepsilon_{n-1} = 0$ and to $h(q_n)$ if $\varepsilon_{n-1} = 1$ and therefore $|\gamma(I,n)| \le \log 2$.

Let us start with the following easy lemma.

Lemma 2.1. *For all $n,k \in \mathbb{N}$ and all $I \in \mathcal{F}_{n-1}$,*

$$\sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} = \gamma(I,n) + \sum_{i=0,1}\frac{\mu(II_i)}{\mu(I)}\sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_iK)}{\mu(II_i)}\right)\frac{\mu(II_iK)}{\mu(II_i)}. \tag{2}$$

where I_0 and I_1 are the two cylinders of the first generation. Furthermore, if we set

$$a_n^k(I) = \sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_0K)}{\mu(II_0)}\right)\frac{\mu(II_0K)}{\mu(II_0)},$$

$$b_n^k(I) = \sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_1K)}{\mu(II_1)}\right)\frac{\mu(II_1K)}{\mu(II_1)},$$

then $a_n^k(I) = a_n^k(I')$ and $b_n^k(I) = b_n^k(I')$ for all $I, I' \in \mathcal{F}_n$.

Proof. We have

$$\begin{aligned}\sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} &= \sum_{i=0,1}\sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_iK)}{\mu(I)}\right)\frac{\mu(II_iK)}{\mu(I)}\\ &= \sum_{i=0,1}\sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_iK)}{\mu(II_i)}\right)\frac{\mu(II_iK)}{\mu(I)} + \sum_{i=0,1}\log\left(\frac{\mu(II_i)}{\mu(I)}\right)\frac{\mu(II_i)}{\mu(I)}.\end{aligned} \tag{3}$$

Since we have set

$$\gamma(I,n) = \sum_{i=0,1} \log\left(\frac{\mu(II_i)}{\mu(I)}\right)\frac{\mu(II_i)}{\mu(I)},$$

the equalities (3) give

$$\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}$$
$$=\gamma(I,n)+\sum_{i=0,1}\frac{\mu(II_i)}{\mu(I)}\sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_iK)}{\mu(II_i)}\right)\frac{\mu(II_iK)}{\mu(II_i)}.$$

It is immediate that $0\le-\gamma(I,n)\le\log 2$. By the construction of the measure μ, the quantities $a_n^k(I)$ and $b_n^k(I)$ do not depend on the cylinder I but only on the cylinder's generation n, and this ends the proof. ∎

REMARK 2.2. Since the quantities $a_n^k(I)$ and $b_n^k(I)$ depend only on the generation of I and on k, we can write $a_n^k=a_n^k(I)$ and $b_n^k=b_n^k(I)$ for $I\in\mathcal{F}_n$. We also set $\varDelta_n^k=|a_n^k-b_n^k|/k$.

The following lemma is easy to prove but helps to clarify the proof.

LEMMA 2.3. *Take $\varepsilon>0$. There exists $\zeta>0$ such that for all $p,q\in[0,1]$ we have either $|h(p)-h(q)|\le\varepsilon/2$ or $|p-q|<1-\zeta$. For all $k>k_0=[4(\log 2)/\zeta\varepsilon]$ and all $\alpha>\varepsilon/2$,*

$$\frac{|h(p)-h(q)|}{k}+|p-q|\left(1-\frac{1}{k}\right)\alpha<\left(1-\frac{1}{2k}\right)\alpha,$$

and hence, for all $\alpha>0$,

$$\frac{|h(p)-h(q)|}{k}+|p-q|\left(1-\frac{1}{k}\right)\alpha<\min\left\{\varepsilon,\left(1-\frac{1}{2k}\right)\alpha\right\}.$$

The proof is elementary and therefore omitted. In the following we will denote by k_0 the positive integer defined in the previous lemma.

PROPOSITION 2.4. *Let I,I' be two cylinders of the nth generation. Then*

$$\frac{1}{k}\left|\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}-\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(I'K)}{\mu(I')}\right)\frac{\mu(I'K)}{\mu(I')}\right|<\eta(k)$$

where η is a positive function, not depending on n, such that $\eta(k)$ goes to 0 as k tends to ∞.

Proof. Take any two cylinders $I=I_{\varepsilon_1\ldots\varepsilon_n}$, $I'=I_{\varepsilon'_1\ldots\varepsilon'_n}$ of the nth generation. If $\varepsilon_n=\varepsilon'_n$ then by definition of the measure μ we get

$$\frac{1}{k}\left|\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}-\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(I'K)}{\mu(I')}\right)\frac{\mu(I'K)}{\mu(I')}\right|=0.$$

If $\varepsilon_n \neq \varepsilon'_n$, using Lemma 2.1 and the notation therein we obtain

$$
\begin{aligned}
(4) \quad \varDelta_{n-1}^{k+1} &= \Bigg| \frac{1}{k+1} \sum_{K\in\mathcal{F}_{k+1}} \log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} \\
&\qquad - \frac{1}{k+1} \sum_{K\in\mathcal{F}_{k+1}} \log\left(\frac{\mu(I'K)}{\mu(I')}\right)\frac{\mu(I'K)}{\mu(I')} \Bigg| \\
&= \Bigg| \frac{\gamma(I,n)-\gamma(I',n)}{k+1} + \frac{1}{k+1}\,\frac{\mu(II_0)}{\mu(I)} \sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(II_0K)}{\mu(II_0)}\right)\frac{\mu(II_0K)}{\mu(II_0)} \\
&\qquad + \frac{1}{k+1}\,\frac{\mu(II_1)}{\mu(I)} \sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(II_1K)}{\mu(II_1)}\right)\frac{\mu(II_1K)}{\mu(II_1)} \\
&\qquad - \frac{1}{k+1}\,\frac{\mu(I'I_0)}{\mu(I')} \sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(I'I_0K)}{\mu(I'I_0)}\right)\frac{\mu(I'I_0K)}{\mu(I'I_0)} \\
&\qquad - \frac{1}{k+1}\,\frac{\mu(I'I_1)}{\mu(I')} \sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(I'I_1K)}{\mu(I'I_1)}\right)\frac{\mu(I'I_1K)}{\mu(I'I_1)} \Bigg| \\
&= \Bigg| \frac{h(p_n)-h(q_n)}{k+1} + \frac{1}{k+1}\Bigg(\left(\frac{\mu(II_0)}{\mu(I)} - \frac{\mu(I'I_0)}{\mu(I')}\right)a_n^k \\
&\qquad + \left(\frac{\mu(II_1)}{\mu(I)} - \frac{\mu(I'I_1)}{\mu(I')}\right)b_n^k\Bigg)\Bigg| \\
&\leq \frac{|h(p_n)-h(q_n)|}{k+1} + \left| \frac{1}{k+1}\left(\frac{\mu(II_0)}{\mu(I)} - \frac{\mu(I'I_0)}{\mu(I')}\right)(a_n^k - b_n^k)\right|.
\end{aligned}
$$

We can rewrite (4) in the following way:

$$
\frac{|a_{n-1}^{k+1} - b_{n-1}^{k+1}|}{k+1} \leq \frac{|h(p_n)-h(q_n)|}{k+1} + |p_n - q_n|\,\frac{|a_n^k - b_n^k|}{k}\left(1 - \frac{1}{k+1}\right)
$$

and thus,

$$
(5) \qquad \varDelta_{n-1}^{k+1} \leq \frac{|h(p_n)-h(q_n)|}{k+1} + |p_n - q_n|\left(1 - \frac{1}{k+1}\right)\varDelta_n^k.
$$

Take $\varepsilon > 0$. By Lemma 2.3, for $k \geq k_0$ we have

$$
(6) \qquad \varDelta_{n-1}^{k+1} \leq \min\left\{\varepsilon\,,\left(1 - \frac{1}{2(k+1)}\right)\varDelta_n^k\right\}.
$$

We use a recursion argument to finish the proof the lemma. First observe that if for some $\ell \in \{1,\ldots,k-k_0\}$ we have

$$
(7) \qquad \varDelta_{n+\ell}^{k-\ell} < \varepsilon
$$

then we will also have

$$\Delta_{n+\ell-1}^{k-\ell+1} < \min\left\{\varepsilon, \left(1 - \frac{1}{2(k+1)}\right)\Delta_{n+\ell}^{k-\ell}\right\} \le \varepsilon$$

by (6), and therefore $\Delta_n^k < \varepsilon$.

On the other hand, if (7) does not hold for any $\ell \in \{1, \ldots, k-k_0\}$ then by (6) we get

$$\Delta_{n+\ell-1}^{k-\ell+1} \le \left(1 - \frac{1}{2(k-\ell+1)}\right)\Delta_{n+\ell}^{k-\ell}$$

and finally

$$\Delta_n^k \le \prod_{\ell=k_0}^{k+1}\left(1 - \frac{1}{2(\ell+1)}\right)\log 2, \tag{8}$$

which becomes strictly smaller than ε if k is large enough, and the proof is complete. ∎

We will also use the following two theorems of [BH02] that we include without proof for the convenience of the reader (a direct proof—without using these theorems—is possible but much longer).

Theorem 2.5 ([BH02]). *Let m be a probability measure on $[0,1)^D$ equipped with the filtration of ℓ-adic cubes*, $\ell \in \mathbb{N}$. *Then*

$$\dim_*(m) \le h_*(m).$$

Moreover, the following properties are equivalent:

1. $\dim_*(m) = h_*(m)$.
2. $\dim_*(m) = \dim^*(m) = h_*(m)$.
3. *There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for m-almost every $x \in [0,1)^D$*,

$$\lim_{k\to\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \dim_*(m).$$

Theorem 2.6 ([BH02]). *We also have*

$$h^*(m) \le \mathrm{Dim}^*(m),$$

and the following properties are equivalent:

1. $\mathrm{Dim}^*(m) = h^*(m)$.
2. $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h^*(m)$.
3. *There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for m-almost every $x \in [0,1)^D$*,

$$\lim_{k\to\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \mathrm{Dim}^*(m).$$

3. Proofs of the theorems. To prove Theorem 1.2 we will use the following strong law of large numbers (cf. [HH80]).

Theorem 3.1 (Law of Large Numbers). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables uniformly bounded in $\mathcal{L}^2$ on a probability space $(\mathbb{X},\mathcal{B},P)$ and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-subalgebras of $\mathbb{B}$ such that X_n is measurable with respect to $\mathcal{F}_n$ for all $n\in\mathbb{N}$. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(X_k-\mathbb{E}(X_k\,|\,\mathcal{F}_{k-1}))=0 \qquad P\text{-almost surely.} \tag{9}$$

We point out that the assumptions on the random variables are not optimal but the result will be sufficient for our goal. The space here is $\mathbb{K}$, the filtration will be the dyadic one, and μ will take the place of the probability measure P.

Proof of Theorem 1.2. Consider the random variables X_n, $n\in\mathbb{N}$, defined on $\mathbb{K}$ by

$$X_n(x)=\log\frac{\mu(I_n(x))}{\mu(I_{n-1}(x))},$$

where, for $x\in\mathbb{K}$, we have denoted by $I_n(x)$ the unique element of $\mathcal{F}_n$ containing x. Theorem 3.1 implies that for all positive p,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}\left(\frac{1}{p}\sum_{k=1}^{p}[X_{jp+k}-\mathbb{E}(X_{jp+k}\,|\,\mathcal{F}_{jp})]\right)=0 \quad \mu\text{-almost surely.} \tag{10}$$

On the other hand, on each $I\in\mathcal{F}_n$ we have

$$\frac{1}{p}\sum_{k=1}^{p}\mathbb{E}(X_{np+k}\,|\,\mathcal{F}_{np})=\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}. \tag{11}$$

By Proposition 2.4, for every $\varepsilon>0$ there exists $p\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ and all I in $\mathcal{F}_{np}$,

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}-c_n\right|<\varepsilon, \tag{12}$$

where $c_n=p^{-1}\mathbb{E}\{\sum_{K\in\mathcal{F}_p}\log(\mu(IK)/\mu(I))\}$ is a constant depending only on n and on the chosen p but not on the cylinder I of $\mathcal{F}_n$.

It is also easy to see that the variables $(X_n)_{n\in\mathbb{N}}$ are uniformly bounded in $\mathcal{L}^2(\mu)$. We deduce, using (10) and (11), that for every $\varepsilon>0$ there exists $p\in\mathbb{N}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\begin{aligned}-\varepsilon&<\liminf_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}\left(\frac{1}{p}\sum_{k=1}^{p}X_{jp+k}-c_j\right)\\&\leq\limsup_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}\left(\frac{1}{p}\sum_{k=1}^{p}X_{jp+k}-c_j\right)<\varepsilon\end{aligned} \tag{13}$$

μ-almost everywhere on $\mathbb{K}$. This implies that

$$\liminf_{n\to\infty} \frac{-1}{n+1}\sum_{j=1}^{n} c_j - \varepsilon < \liminf_{n\to\infty} \frac{-1}{p}\,\frac{1}{n+1}\sum_{j=1}^{n}\sum_{k=1}^{p} X_{jp+k} < \liminf_{n\to\infty} \frac{-1}{n+1}\sum_{j=1}^{n} c_j + \varepsilon \tag{14}$$

and

$$\limsup_{n\to\infty} \frac{-1}{n+1}\sum_{j=1}^{n} c_j - \varepsilon < \limsup_{n\to\infty} \frac{-1}{p}\,\frac{1}{n+1}\sum_{j=1}^{n}\sum_{k=1}^{p} X_{jp+k} < \limsup_{n\to\infty} \frac{-1}{n+1}\sum_{j=1}^{n} c_j + \varepsilon \tag{15}$$

μ-almost everywhere on $\mathbb{K}$. If we set

$$\underline{c} = \liminf_{n\to\infty} \frac{-1}{(n+1)\log 2}\sum_{j=1}^{n} c_j, \qquad \overline{c} = \limsup_{n\to\infty} \frac{-1}{(n+1)\log 2}\sum_{j=1}^{n} c_j,$$

we deduce from (14) and (15) that $\dim_*(\mu) = \underline{c}$ and $\mathrm{Dim}_*(\mu) = \overline{c}$.

Furthermore, the inequalities (13) imply that for every positive ε there is a strictly increasing sequence $(n_l)_{l\in\mathbb{N}}$ of natural numbers satisfying

$$-\varepsilon < \liminf_{l\to\infty} \frac{-1}{n_l+1}\sum_{j=1}^{n_l}\left(\frac{1}{p}\sum_{k=1}^{p} X_{jp+k}\right) - \underline{c} \le \limsup_{l\to\infty} \frac{-1}{n_l+1}\sum_{j=1}^{n_l}\left(\frac{1}{p}\sum_{k=1}^{p} X_{jp+k}\right) - \underline{c} < \varepsilon$$

for μ-almost all $x \in K$. One easily proves (using, for instance, Cantor's diagonal argument) that there exists a strictly increasing sequence $(n_l)_{l\in\mathbb{N}}$ of natural numbers such that

$$\lim_{l\to\infty} \frac{-1}{n_l \log 2}\log \mu(I_{n_l}(x)) = \dim_*(\mu) \quad \text{ for } \mu\text{-almost all } x \in \mathbb{K}.$$

Similarly, there exists a strictly increasing sequence $(\widehat{n}_l)_{l\in\mathbb{N}}$ of natural numbers such that

$$\lim_{l\to\infty} \frac{-1}{\widehat{n}_l \log 2}\log \mu(I_{\widehat{n}_l}(x)) = \mathrm{Dim}_*(\mu) \quad \text{ for } \mu\text{-almost all } x \in \mathbb{K}.$$

We use Theorems 2.5 and 2.6 to finish the proof. ■

To prove Theorem 1.3 we will use Proposition 2.4 and Lemma 3.1.

Proof of Theorem 1.3. Take $\varepsilon > 0$ and let $(p_n, q_n)_{n\in\mathbb{N}}$ and $(p'_n, q'_n)_{n\in\mathbb{N}}$ be two sequences of weights satisfying $0 < p_n, q_n, p'_n, q'_n < 1$ for all $n \in \mathbb{N}$ and

$$\|(p_n, q_n)_{n\in\mathbb{N}} - (p'_n, q'_n)_{n\in\mathbb{N}}\|_\infty < \zeta.$$

We denote by μ and μ' the measures corresponding to these two sequences of weights. We will show that

$$|\dim_*(\mu) - \dim_*(\mu')| < \varepsilon,$$

if ζ is small enough.

It follows from Proposition 2.4 that there exist a natural number p large enough and two sequences $(c_n)_{n\in\mathbb{N}}, (c'_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} - c_n\right| < \frac{\varepsilon}{4}$$

and

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu'(IK)}{\mu'(I)}\right)\frac{\mu'(IK)}{\mu'(I)} - c'_n\right| < \frac{\varepsilon}{4}$$

for all cylinders $I \in \mathcal{F}_{np}$ and all $n \in \mathbb{N}$. Since p is a fixed finite number it suffices to take ζ small in order to have

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} - \frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu'(IK)}{\mu'(I)}\right)\frac{\mu'(IK)}{\mu'(I)}\right| < \frac{\varepsilon}{2}$$

for all $I \in \mathcal{F}_{np}$ and all $n \in \mathbb{N}$. Hence,

$$-\varepsilon < \liminf_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}|c_j - c'_j| \le \limsup_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}|c_j - c'_j| < \varepsilon.$$

Now we deduce from (14) and (15) that $|\dim_*(\mu) - \dim_*(\mu')| < \varepsilon$ and $|\mathrm{Dim}_*(\mu) - \mathrm{Dim}_*(\mu')| < \varepsilon$, which completes the proof. ■

The hyphothesis on the markovian structure of the measures μ and μ' cannot be omitted as we show in the following section.

4. A counterexample. For every $\varepsilon > 0$ we construct two dyadic doubling measures μ and ν on $\mathbb{K}$ such that if

$$X_n(x) = \log\frac{\mu(I_n(x))}{\mu(I_{n-1}(x))}, \qquad Y_n(x) = \log\frac{\nu(I_n(x))}{\nu(I_{n-1}(x))}, \qquad n \in \mathbb{N},$$

then

$$\sup_{n\in\mathbb{N}}\|X_n - Y_n\|_{L^\infty} < \varepsilon \tag{16}$$

and, nevertheless, $|\dim_*(\mu) - \dim_*(\nu)| > 1/4$. A first example was proposed to us by Professor Alano Ancona; the proof provided here is of a similar nature.

The construction is carried out in two stages. We fix two Bernoulli measures satisfying (16) and we use a recurrent process to modify them in order to get the corresponding dimensions very different.

For $I \in \mathcal{F}_n$ we denote by $\widehat{I}$ the unique cylinder of the $(n-1)$th generation $\mathcal{F}_{n-1}$ containing I. Relation (16) can now be reformulated as

$$\left|\frac{\mu(I)}{\mu(\widehat{I})} : \frac{\nu(I)}{\nu(\widehat{I})} - 1\right| < \varepsilon \quad \text{for all cylinders } I \text{ of } \bigcup_{n\in\mathbb{N}} \mathcal{F}_n. \tag{17}$$

The starting point. Take $\varepsilon > 0$ and λ_0 the Lebesgue (uniform) measure (of dimension 1) on $\mathbb{K}$.

Consider the Bernoulli measure ϱ_0 of weight $1/2 - \varepsilon$, i.e. such that for $I \in \mathcal{F}_n$, $n \in \mathbb{N}$,

$$\varrho_0(II_0) = (1/2-\varepsilon)\varrho(I), \qquad \varrho_0(II_1) = (1/2+\varepsilon)\varrho(I). \tag{18}$$

Put $\mu_0 = \lambda_0$ and $\nu_0 = \varrho_0$. By construction the measures λ_0 and ϱ_0 satisfy condition (16), are exact and doubling on the dyadics. Moreover, we have

$$\dim \varrho_0 = h_*(\varrho) = -\frac{1/2-\varepsilon}{\log 2}\log\left(\frac{1}{2}-\varepsilon\right) - \frac{1/2+\varepsilon}{\log 2}\log\left(\frac{1}{2}+\varepsilon\right).$$

It is clear that λ_0 and ϱ_0 are singular. Furthermore by the Shannon–MacMillan formula (cf. for instance [Zin97]),

$$\lim_{n\to\infty}\frac{\log \varrho_0(I_n(x))}{n} = h_*(\varrho_0) \qquad \varrho_0\text{-almost everywhere on } \mathbb{K}.$$

Hence, we can find $n_1 \in \mathbb{N}$ and a partition $\{F_0, F_1\}$ of $\mathcal{F}_{n_1}$ such that:

- $F_0 \cup F_1 = \mathcal{F}_{n_1}$,
- $\left|\dfrac{\log \varrho_0(I)}{n} + h_*(\varrho_0)\right| < \varepsilon$ for all $I \in F_1$,
- $\left|\dfrac{\log \lambda_0(I)}{n} + \log 2\right| < \varepsilon$ for all $I \in F_0$,
- $\displaystyle\sum_{I\in F_1} \varrho_0(I) > 1-\varepsilon$,
- $\displaystyle\sum_{I\in F_0} \lambda_0(I) > 1-\varepsilon$.

Let us also define the Bernoulli measures ϱ_1 and λ_1 on $\mathbb{K}$ by

$$\begin{aligned} \varrho_1(I_0) &= \delta, & \varrho_1(I_1) &= 1-\delta, \\ \lambda_1(I_0) &= \delta(1-\varepsilon), & \lambda_1(I_1) &= 1-\delta(1-\varepsilon), \end{aligned} \tag{19}$$

where $\delta > 0$ will be fixed later.

Going on with the construction. For $I_{i_1\ldots i_n} \subset I \in F_1$ we put

$$\begin{aligned} \mu_1(I_{i_1\ldots i_n}) &= \mu_0(I_{i_1\ldots i_{n_1}})\lambda_1(I_{i_{n_1}\ldots i_n}), \\ \nu_1(I_{i_1\ldots i_n}) &= \nu_0(I_{i_1\ldots i_{n_1}})\varrho_1(I_{i_{n_1}\ldots i_n}), \end{aligned} \tag{20}$$

and for $I_{i_1\dots i_n} \subset I \in F_0$,

$$\mu_1(I_{i_1\dots i_n}) = \mu_0(I_{i_1\dots i_n}), \quad \nu_1(I_{i_1\dots i_n}) = \nu_0(I_{i_1\dots i_n}). \tag{21}$$

We remark that for $I = I_{i_1\dots i_n}$ with $n \le n_1$ we have $\mu_1(I) = \mu_0(I)$ and $\nu_1(I) = \nu_0(I)$.

The restrictions of the measures μ_1 and ν_1 to cylinders of $\mathcal{F}_{n_1} = F_0 \cup F_1$ are Bernoulli measures of different dimensions, so they are singulars. Therefore, we can find $n_2 \in \mathbb{N}$ and a partition $\{F_{00}, F_{01}, F_{10}, F_{11}\}$ of $\mathcal{F}_{n_2}$ such that

- $I \in F_{j0} \cup F_{j1}$ if and only if there is $J \in F_j$ such that $I \subset J$, $j \in \{0,1\}$,
- $\left|\dfrac{\log \mu_1(I)}{n_2} + \log 2\right| < \varepsilon^2$ for all $I \in F_{00}$,
- $\left|\dfrac{\log \nu_1(I)}{n_2} + h_*(\varrho_1)\right| < \varepsilon^2$ for all $I \in F_{11}$,
- $\displaystyle\sum_{\substack{J\in F_{00}\\ J\subset I}} \mu_1(J) > (1-\varepsilon^2)\mu_1(I)$ and $\displaystyle\sum_{\substack{J\in F_{01}\\ J\subset I}} \nu_1(J) > (1-\varepsilon^2)\nu_1(I)$ for $I \in F_0$,
- $\displaystyle\sum_{\substack{J\in F_{10}\\ J\subset I}} \mu_1(J) > (1-\varepsilon^2)\mu_1(I)$ and $\displaystyle\sum_{\substack{J\in F_{11}\\ J\subset I}} \nu_1(J) > (1-\varepsilon^2)\nu_1(I)$ for $I \in F_1$.

If $I \in F_{00} \cup F_{10}$ and $J \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, we put

$$\mu_2(IJ) = \mu_1(I)\lambda_0(J), \quad \nu_2(IJ) = \nu_1(I)\varrho_0(J).$$

If $I \in F_{01} \cup F_{11}$ and $J \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ we put

$$\mu_2(IJ) = \mu_1(I)\lambda_1(J), \quad \nu_2(IJ) = \nu_1(I)\varrho_1(J).$$

Finally, for $I \in \mathcal{F}_n$ with $n \le n_2$, we keep the same mass distribution $\mu_2(I) = \mu_1(I)$ and $\nu_2(I) = \nu_1(I)$.

Suppose the measures μ_k, ν_k and the partition $\{F_{i_1\dots i_k} : i_1, \dots, i_k \in \{0,1\}\}$ of $\mathcal{F}_{n_k}$ are constructed. As in the two first stages, the restrictions of the measures μ_k and ν_k to each cylinder of $\mathcal{F}_{n_k}$ are supposed to be Bernoulli measures: either λ_0 and ϱ_0 or λ_1 and ϱ_1, respectively.

The measures μ_k and ν_k are mutually singular. Hence, there is $n_{k+1} > n_k$ and a partition $\{F_{i_1\dots i_{k+1}} : i_1, \dots, i_{k+1} \in \{0,1\}\}$ of $\mathcal{F}_{n_{k+1}}$ satisfying

- for any $i_1, \dots, i_k \in \{0,1\}$, $I \in F_{i_1\dots i_k 0} \cup F_{i_1\dots i_k 1}$ if and only if there is $J \in F_{i_1\dots i_k}$ such that $I \subset J$,
- $\left|\dfrac{\log \mu_k(I)}{n_{k+1}} + \log 2\right| < \varepsilon^{k+1}$ for all $I \in F_{i_1\dots i_{k-1}00}$,
- $\left|\dfrac{\log \nu_k(I)}{n_2} + h_*(\varrho_1)\right| < \varepsilon^{k+1}$ for all $I \in F_{i_1\dots i_{k-1}11}$,

- $$\sum_{\substack{J\in F_{i_1\dots i_{k-1}00}\\ J\subset I}} \mu_k(J) > (1-\varepsilon^{k+1})\mu_k(I) \text{ and}$$
  $$\sum_{\substack{J\in F_{i_1\dots i_{k-1}01}\\ J\subset I}} \nu_k(J) > (1-\varepsilon^{k+1})\nu_k(I) \text{ for all cylinders } I\in F_{i_1\dots i_{k-1}0},$$
- $$\sum_{\substack{J\in F_{i_1\dots i_{k-1}10}\\ J\subset I}} \mu_k(J) > (1-\varepsilon^{k+1})\mu_k(I) \text{ and}$$
  $$\sum_{\substack{J\in F_{i_1\dots i_{k-1}11}\\ J\subset I}} \nu_k(J) > (1-\varepsilon^{k+1})\nu_k(I) \text{ for all cylinders } I\in F_{i_1\dots i_{k-1}1}.$$

If $I\in F_{i_1\dots i_k0}$, $i_1,\dots,i_k\in\{0,1\}$, then for all $J\in\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$ we put

$$\mu_{k+1}(IJ)=\mu_k(I)\lambda_0(J),\qquad \nu_{k+1}(IJ)=\nu_k(I)\varrho_0(J).$$

If $I\in F_{i_1\dots i_k1}$, $i_1,\dots,i_k\in\{0,1\}$, then for all $J\in\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$ we put

$$\mu_{k+1}(IJ)=\mu_k(I)\lambda_1(J),\qquad \nu_{k+1}(IJ)=\nu_k(I)\varrho_1(J).$$

Properties of the measures defined. It is clear that the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ converge towards two probability measures μ and ν respectively. By the construction μ and ν are doubling on the dyadics, exact and satisfy (16).

On the other hand, clearly $\dim_*(\mu)=1$ and it is not difficult to see that $\dim_*\nu\le 1/2$ if δ is small enough, since

$$\liminf_{n\to\infty}\frac{-\log\nu(I_n(x))}{n\log 2}=\frac{h_*(\varrho_1)}{\log 2}\qquad \nu\text{-almost everywhere.}$$

Even more, the measures μ and ν satisfy the conclusion of Theorem 1.2. The counterexample is complete.

Acknowledgments. The author would like to thank the referee for having carefully read this work and for his comments on it. His remarks have been very helpful.

I would like to dedicate this paper to the memory of Martine Babillot, who encouraged and helped me throughout the writing. Martine, we miss you.

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Received 22 April 2003;
revised 10 August 2005 (4334)