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ASYMPTOTIC BEHAVIOR OF THE INVARIANT MEASURE FOR A DIFFUSION RELATED TO AN NA GROUP

BY

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Abstract. On a Lie group NA that is a split extension of a nilpotent Lie group N by a one-parameter group of automorphisms A, the heat semigroup μ_t generated by a second order subelliptic left-invariant operator $\sum_{j=0}^m Y_j + Y$ is considered. Under natural conditions there is a $\check{\mu}_t$-invariant measure m on N, i.e. $\check{\mu}_t * m = m$. Precise asymptotics of m at infinity is given for a large class of operators with $Y_0, \ldots, Y_m$ generating the Lie algebra of S.

Introduction. The aim of this paper is twofold. First, to describe the precise asymptotic behavior of the invariant measure for a diffusion process on a homogeneous Lie group N; the description is new even in the case of $N = \mathbb{R}^d$, $d > 1$. Second, to present a survey of our earlier results with simplified proofs that are needed to prove our main result.

A sequence of random variables defined recursively by

$$R_n = M_n R_{n-1} + Q_n,$$

where $(Q_n, M_n) \in \mathbb{R}^d \times \mathbb{R}^+_* = S$ is a sequence of identically distributed independent random variables with the law

$$\mathrm{P}[(Q_1, M_1) \in U] = \mu(U),$$

has attracted considerable attention during the last forty years. Of course, multiplication by scalars M_n can be replaced by other automorphisms of the group $\mathbb{R}^d$ or more generally by automorphisms of a Lie group N with $Q_n \in N$. Moreover

$$W_n = (Q_n, M_n) \cdots (Q_1, M_1) = (R_n, M_1 \cdots M_n)$$

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can be viewed as a left random walk on S, where the group multiplication in S is given by

$$(x,a)(x',a') = (x\Phi_a(x'), aa'),$$

and Φ_a, $a \in A$, is a one-parameter group of shrinking automorphisms of a nilpotent Lie group N, $S = NA$, $\dim A = 1$, by which we mean that

$$\lim_{a\to 0} \Phi_a x = e \quad \forall x \in N. \tag{0.1}$$

Each random step (M_j, Q_j) is sampled according to the law μ, a measure on $S = NA$.

We are interested in the properties of the invariant measure for the Markov chain R_n, i.e. the unique Radon measure m such that for continuous functions f with compact support we have

$$\iint f(s\cdot x)\,d\mu(s)\,dm(x) = \int f(x)\,dm(x), \quad \text{i.e.} \quad \mu * m = m,$$

or equivalently,

$$\int \mathrm{E}_x f(R_n)\,dm(x) = \int f(x)\,dm(x),$$

E being the expected value.

Let ϱ denote a norm on N, i.e. $\varrho : N \to \mathbb{R}^+$ and $\varrho(xy) \le \varrho(x) + \varrho(y)$. Assume that

$$\begin{aligned} &\text{(i)} \int_S \log a\,d\mu(xa) < 0,\\ &\text{(ii)} \int_S \log \varrho(x)\,d\mu(xa) < \infty,\\ &\text{(iii)} \int_S a^\alpha\,d\mu(a) = 1. \end{aligned} \tag{0.2}$$

Conditions (i) and (ii) imply the existence and uniqueness of a *probability* measure m on N that is invariant, $\mu * m = m$. Then the measure m is the distribution law of the random variable

$$Z = \lim_{n\to\infty} Q_1 \cdot \Phi_{a_1}(Q_2)\cdots\Phi_{a_1\ldots a_n}(Q_{n+1}).$$

This is an old result, proved and reproved by many authors [R], the idea of the proof going back to Furstenberg's famous paper [F]. More recently, Diaconis and Friedman [DF] showed a general scheme of iteration of random functions acting on a state space (in our case N) for which under conditions like (0.2)(i) and (0.2)(ii), there exists an invariant probability measure.

However, if (i) does not hold, then, as proved by Babillot, Bougerol and Élie [BBE], the conditions

$$
\begin{aligned}
&\int \log a \, d\mu(xa) = 0,\\
&\exists_{\delta>0} \int (|\log a| + |\log^+ |x| \,|^{2+\delta}) \, d\mu(xa) < \infty,\\
&\forall_{x\in N} \ \mu\{s : s\cdot x = x\} < 1,\\
&\text{the projection of } \mu \text{ on } A \text{ is not supported}\\
&\text{by a proper subgroup of } A
\end{aligned}
\tag{0.3}
$$

imply the existence of a Radon measure m_0 on N that is invariant. The measure m_0 is unique.

If $S = \mathbb{R}\cdot\mathbb{R}^+_*$, i.e. S is the "$ax+b$"-group, then for an *arbitrary* measure μ that satisfies conditions (i)–(iii) the following tail estimates have been established for the invariant measure m_α as a result of a long series of papers from Kesten to Goldie and Maller [Gu], [Go], [GM], [Gre], [K], [V]: for α as in (0.2) we have

$$
\lim_{t\to\infty} t^\alpha m_\alpha[t,\infty) = C_+, \qquad \lim_{t\to-\infty} |t|^\alpha m_\alpha(-\infty,t] = C_-, \tag{0.4}
$$

and, if $\mathrm{E}\log M = 0$, then

$$
m_0[r_1 t, r_2 t) = L(|t|)\log\frac{r_2}{r_1}, \tag{0.5}
$$

where $L(|t|)$ is a slowly varying function for $t\to\infty$ (see [BBE]). It has also been established that $C_+ + C_- > 0$. In the multi-dimensional case no precise estimates of m_0 at infinity are known. For general measures μ on S that satisfy only (0.2) or (0.3), in the case $N\neq\mathbb{R}$ we know of no results on the asymptotic behavior of the invariant measure at infinity, even if μ is absolutely continuous, smooth and compactly supported.

Before we are going to describe the setting for our results, let us recall some general facts that link the invariant measure m and the Green operator on S (cf. [BBE]). Let

$$
Uf(s) = \mathrm{E}_s \sum_{n=0}^{\infty} \phi(W_n) = \sum_{n=0}^{\infty} f(s's)\, d\mu^{*n}(s') = \int_S f(s's)\, dG(s')
$$

be the Green operator on S. Then

$$
\forall_{f\in C_c(S)} \qquad \lim_{a\to\infty} Uf(xa) = \int f(xa)\, dm(x)\, \frac{da}{a}. \tag{0.6}
$$

Let

$$
f(xa) = \phi(x)\psi(a), \qquad \int \psi(a)\,\frac{da}{a} = 1,
$$

where ϕ, ψ are continuous functions with compact support. Then

$$Uf(xa) = \int \phi(ybxa)\,dG(yb) = \int \phi(y\Phi_b(x))\psi(ba)\,dG(yb).$$

Thus, if $a \to \infty$, then $\psi(ba) \neq 0$ implies $b \to 0$ and so $\Phi_b(x) \to e$. So (0.6) says that m is a weak limit, as $b \to 0$, of the Radon measures G_b that come from the desintegration of G, $dG(yb) = dG_b(y)\,d\overline{G}(b)$. More precisely,

$$\lim_{b\to 0}\langle \phi, G_b\rangle = \langle \phi, m\rangle.$$

For the measures μ on S that are the subject of this paper, let us call them Gaussian, the fact that the invariant measure is a limit of a Green function is essential. However, we use a slightly different Green function and we do not use the facts mentioned above.

Description of the main result. Let $S = NA$ be a split extension of a nilpotent Lie group N by a one-dimensional group A of dilating automorphisms Φ_a of N (see Section 2):

$$(x, a)(x_1, a_1) = (x\Phi_a(x_1), aa_1).$$

In a number of papers (cf. e.g. [D], [DH], [DHZ], [DHU]) we considered a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y$$

on S that satisfies the Hörmander condition. Under the canonical homomorphism of S onto $A = \mathbb{R}^+_*$ the image of $\mathcal{L}$ is equal to

$$(a\partial_a)^2 - \alpha a\partial_a,$$

up to a constant. If $\alpha \geq 0$, then on N there is a positive Radon measure m_α with smooth density such that

$$\check{\mu}_t * m_\alpha = m_\alpha,$$

where μ_t is the semigroup of probability measures on S whose infinitesimal generator is $\mathcal{L}$ ([E], [R]) and for a measure μ on S we write

$$\langle \check{\mu}, f\rangle = \langle \mu, \check{f}\rangle, \quad \text{where} \quad \check{f}(s) = f(s^{-1}).$$

The invariant measure m_α is also called the Poisson kernel.

Invariant measures for certain Markov processes on homogeneous Lie groups N have been extensively studied when N is the additive group of real numbers, or more recently when $N = \mathbb{R}^d$. The generalization from $\mathbb{R}^d$ to more general homogeneous Lie groups is straightforward as far as the formulations of the results go. The proofs, however, require different, more complicated techniques due to non-commutativity of N. The generalization, originally motivated by the study of positive harmonic functions on homogeneous spaces of negative curvature and their Martin boundaries [DHU], seems to find another justification in the fact that the processes we study are closely related to ones that have appeared in financial mathematics in which the Heisenberg group plays an important role [SS].

The behavior of m_α and its derivatives at infinity is the subject of [D], [DHZ], [DHU] and [U]. Indeed upper and lower bounds for $a^{Q+\alpha}m_\alpha(\Phi_a(x))$ as $a \to \infty$, where Q is the homogeneous dimension of N (cf. Section 1), are given in [D] and [DHU]. In the present paper, we study radial limits at ∞ and we show the following

Main Result. *For every x in the unit sphere Σ of N the limit*

$$\lim_{a\to\infty} a^{Q+\alpha}m_\alpha(\Phi_a(x)) = c(x)$$

exists and it is positive. The function $\Sigma \ni x \mapsto c(x)$ is continuous (1).

The proof relies heavily on the methods and theorems developed in [DHZ], [DHU] and [U], but many arguments have been much simplified here. The estimates for the evolution are contained in Sections 2 and 3. The Green function, that is, the density of the measure G with respect to the left invariant Haar measure on S, its continuity up to the boundary and the relation to m_α are studied in Sections 4 and 5. There an elementary argument (in the proof of (5.4)) communicated to us by Aline Bonami has simplified very much some of our original reasoning and has allowed for the proof of the theorem in its full generality.

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1. Preliminaries. Let

$$\mathcal{S} = \mathcal{N} \oplus \mathcal{A} \tag{1.1}$$

be a solvable Lie algebra that is the sum of its maximal nilpotent ideal $\mathcal{N}$ and a one-dimensional algebra $\mathcal{A} = \mathbb{R}$. We assume that there exists $H \in \mathcal{A}$ such that

(1.2) the real parts of the eigenvalues of $\mathrm{ad}_H : \mathcal{N} \to \mathcal{N}$ are positive.

This implies that multiplying H by a large constant if necessary, we may assume that the real parts of the eigenvalues of H are greater than 2.

Let N, A, S be the connected and simply connected Lie groups whose Lie algebras are $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{S}$ respectively. Then $S = NA$ is a semidirect product of N and $A = \mathbb{R}^+$.

(1) Precise assumptions on the operator are formulated in the next section (see (1.5)).

On $C_c^\infty(S)$ we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y$$

such that $Y_0, \ldots, Y_m$ generate $\mathcal{S}$ as a Lie algebra, i.e. $\mathcal{L}$ satisfies the strong Hörmander condition.

Let $\pi_A(xa) = a$ be the canonical homomorphism of S onto A. Then $\pi_A(\mathcal{L})$, up to a constant, is equal to

$$\pi_A(\mathcal{L}) = H^2 - \alpha H$$

for an $\alpha \in \mathbb{R}$. If $\alpha \geq 0$, then there is a positive Radon measure m_α, unique up to a constant, on N such that

$$\check{\mu}_t * m_\alpha = m_\alpha, \tag{1.3}$$

where μ_t is the semigroup of probability measures on S whose infinitesimal generator contains $\mathcal{L}$ ([E], [R]). For $\alpha > 0$, m_α is a bounded measure. The bounded $\mathcal{L}$-harmonic functions on S are in one-to-one correspondence with $L^\infty(N)$ via the Poisson integral

$$F(s) = \int_N f(s \cdot x)\, dm_\alpha(x),$$

where $x \mapsto s \cdot x$ denotes the action of S on $N = S/A$ (see [DH]). If $\alpha \leq 0$, then there are no bounded $\mathcal{L}$-harmonic functions and m_α is only a Radon measure. Furthermore, m_α is a smooth function and moreover, for an appropriately defined norm (see below),

$$C^{-1}(1 + |x|)^{-Q-\alpha} \leq m_\alpha(x) \leq C(1 + |x|)^{-Q-\alpha}, \tag{1.4}$$

where

$$Q = \Re \operatorname{Tr} \operatorname{ad}_H .$$

For the proof of (1.4) see [D] when $\alpha > 0$ and [DHU] when $\alpha = 0$.

Our main theorem goes a step further: instead of an upper and lower bound at infinity we prove the existence of the limit.

It follows from elementary linear algebra that $Y_0, \ldots, Y_m$ can be chosen in the way that $Y_1, \ldots, Y_m \in \mathcal{N}, Y_0 \notin \mathcal{N}$.

(1.5) We assume that $Y_1, \ldots, Y_m$ generate $\mathcal{N}$.

The general case, i.e. when $Y_0, \ldots, Y_m, Y$ generate the Lie algebra $\mathcal{S}$ is going to be the subject of the forthcoming paper [BDH].

The decomposition (1.1) is not unique, i.e. there is no canonical choice of $\mathcal{A}$. We put $A = \exp\{cY_0 : c \in \mathbb{R}\}$ and we may assume without loss of generality that the real parts of the eigenvalues of ad_{Y_0} are strictly positive.

Moreover, multiplying $\mathcal{L}$ by a constant,

$$c^2\mathcal{L} = \sum_{j=0}^{m} (cY_j)^2 + c^2 Y,$$

we see that the real parts of ad_{Y_0} may be arbitrarily large. *Clearly* $\mathcal{L}$ *and* $c^2\mathcal{L}$ *have the same harmonic functions, the semigroup for* $c^2\mathcal{L}$ *is* μ_{c^2t}, *so the boundary Radon measures are the same.* Decomposing $s \in S$ as

$$s = xa = x\exp(\log a)cY_0, \quad x \in N,\ a \in A, \tag{1.6}$$

we write

$$c^2\mathcal{L} = \mathcal{L}_{-\alpha} = (a\partial_a)^2 - \alpha(a\partial_a) + \sum_{j=1}^{m} \Phi_a(X_j)^2 + \Phi_a(X), \tag{1.7}$$

where $\Phi_a = \mathrm{Ad}_{\exp(\log a)cY_0} = e^{\mathrm{ad}_{(\log a)cY_0}}$ and $X_1, \ldots, X_m$ generate $\mathcal{N}$. We shall keep the subscript α to stress the role of the A-component of Y in (1.7). In fact, if μ_t is the semigroup generated by $\mathcal{L}_{-\alpha}$, $\alpha \geq 0$, then (0.2) or (0.3), respectively, is satisfied by $\breve{\mu}_t$.

Let $(\cdot,\cdot)_0$ be an arbitrary inner product on $\mathcal{N}$ and let $\sqrt{(X,X)_0} = \|X\|_0$. (1.2) implies that there are $p_1, p_2 > 2$ and $C > 0$ such that

$$\frac{1}{C}\min(a^{p_1}, a^{p_2})\|X\|_0 \leq \|\Phi_a(X)\|_0 \leq C\max(a^{p_1}, a^{p_2})\|X\|_0, \quad a > 0. \tag{1.8}$$

We define a "homogeneous norm" $|\cdot|$ on N. We let

$$\langle X, Y\rangle = \int_0^1 (\Phi_a(X), \Phi_a(Y))_0 \,\frac{da}{a}, \qquad \|X\| = \sqrt{\langle X, X\rangle}.$$

We put

$$|\exp X| = |X| = (\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\})^{-1}.$$

We observe that, in view of (1.6), for every $X \neq 0$,

$$a \mapsto \|\Phi_a(X)\|^2 = \int_0^a \|\Phi_b(X)\|_0^2 \,\frac{db}{b} \quad \text{is increasing},$$

$$\lim_{a\to 0} \|\Phi_a(X)\| = 0, \qquad \lim_{a\to\infty} \|\Phi_a(X)\| = \infty.$$

Therefore for every $Y \neq 0$ there is precisely one a such that

$$Y = \Phi_a(X), \qquad \|X\| = 1.$$

We put

$$|\exp Y| = |Y| = a.$$

Clearly,

$$|a\exp X a^{-1}| = a|\exp X|.$$

We introduce polar coordinates in N. For every $x \in N$ and $r \in \mathbb{R}^+$ we write $\Phi_r(x) = r \cdot \omega = |x| \cdot \sigma(x)$, where $\|\omega\| = |\omega| = 1$. Let

$$\Sigma = \{\omega \in N : |\omega| = 1\} = \{\omega \in N : \|\omega\| = 1\}.$$

Since $\|\cdot\|$ is a Euclidean sphere in $\mathcal{N}$ and $\exp : \mathcal{N} \to N$ is a diffeomorphism, Σ is a smooth compact submanifold in N. Following pages 13–15 in [FS] we see that there exists a finite measure $d\omega$ on Σ such that for the Haar measure dx on N we have

$$dx = d\omega\, r^{Q-1}\, dr.$$

Our main theorem can be written as

(1.9) Main Theorem. *For every $\sigma \in \Sigma$ the limit*

$$\lim_{r\to\infty} r^{Q+\alpha} m_\alpha(r \cdot \sigma) = c(\sigma)$$

is finite and positive. Moreover, the function $\Sigma \ni \sigma \mapsto c(\sigma)$ is continuous.

Positivity of $c(\sigma)$ follows from (1.4). The statement can also be viewed as a polar decomposition of the measure m_α at infinity. Denoting the density of m_α with respect to dx by the same letter, for $f \in C_{\mathrm{c}}(N)$ we have

$$\begin{aligned}\lim_{t\to 0} t^{-\alpha} \int_N f(tr\omega) m_\alpha(r\omega) r^{Q-1}\, dr\, d\omega \\ &= \lim_{t\to 0} \int_N f(r\omega) t^{-\alpha} m_\alpha(t^{-1}r\omega)(t^{-1}r)^{Q-1} t^{-1}\, dr\, d\omega \\ &= \lim_{t\to 0} \int_N f(r\omega) m_\alpha(t^{-1}r\omega)(t^{-1}r)^{Q+\alpha} r^{-1-\alpha}\, dr\, d\omega \\ &= \int_N f(r\omega) c(\omega) r^{-1-\alpha}\, dr\, d\omega,\end{aligned}$$

since, by (1.4), $m_\alpha(t^{-1}r\omega)(t^{-1}r)^{Q+\alpha} \le C$. Now taking $f = \mathbf{1}_{[1,\infty)}$, for $\alpha > 0$ we obtain

$$\lim_{t\to\infty} t^\alpha m_\alpha[t,\infty) = \frac{1}{\alpha} \int c(\omega)\, d\omega,$$

and for $\alpha = 0$ with $f = \mathbf{1}_{[r_1,r_2]}$ we get

$$\lim_{t\to\infty} m_0[tr_1, tr_2] = \log \frac{r_2}{r_1} \int c(\omega)\, d\omega.$$

This agrees with the estimates (0.4) and (0.5).

2. Evolution. In this section we prove basic estimates for derivatives of the evolution that will be needed later. For a multiindex $I = (I_1, \ldots, I_n)$ and a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathcal{N}$ we write

$$X^I = X_1^{I_1} \cdots X_n^{I_n} \quad \text{and} \quad |I| = \sum I_j.$$

For $k = 0, 1, \dots$, we define

$$C^k = \{f : X^I f \in C(N) \text{ for } |I| < k+1\},$$
$$C^k_\infty = \{f \in C^k : \lim_{x\to\infty} X^I f(x) \text{ exists for } |I| < k+1\}.$$

C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I|\le k} \|X^I f\|_{C(N)}.$$

For a continuous function $\sigma : [0,\infty) \to (0,\infty) = A$ let

$$L_{\sigma(t)} = \sigma(t)^{-2}\Big(\sum \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X)\Big).$$

There exists a unique family $U^\sigma(s,t)$, $0 \le s < t$, of bounded operators on $C_\infty(N) = C^0_\infty(N)$ that satisfy

$$\begin{aligned}
&U^\sigma(s,t)f = f * p^\sigma(t,s),\\
&p^\sigma(t,s) \in C^\infty, \quad \int p^\sigma(t,s;x)\,dx = 1, \quad p^\sigma(t,s) \ge 0,\\
&p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s), \quad s < r < t,\\
&\lim_{h\to 0} \|f * p^\sigma(s+h,s) - f\|_{C_\infty(N)} = 0 \quad \text{for } f \in C_\infty(N),\\
&\partial_t(f * p^\sigma(t,s)) = (L_{\sigma(t)}f) * p^\sigma(t,s) \quad \text{for } f \in C^2_\infty(N),\\
&\partial_s(f * p^\sigma(t,s)) = -L_{\sigma(s)}(f * p^\sigma(t,s)) \quad \text{for } f \in C^2_\infty(N).
\end{aligned}$$

The proof of the existence of $U^\sigma(s,t)$ follows the standard lines of, e.g., [T], once the following simple lemma is proved.

(2.1) Lemma. *Let τ be a Riemannian distance on N, $\phi \in C^\infty_c(N)$, $\phi \ge 0$, $C_T = (1+\sup_{0\le s\le T}|\sigma(s)|)^R$, where $R = 2\sup_\lambda \Re\lambda$, λ's being the eigenvalues of ad_{cY_0}. For a fixed s let $\{\mu^s_t\}$ be the semigroup with the infinitesimal generator $L_{\sigma(s)}$. There is a constant C such that for every $T > 0$, $s_1, \dots, s_n \le T$ and $t_1, \dots, t_n > 0$, for all $M > 0$,*

$$\langle \mu^{s_1}_{t_1} * \dots * \mu^{s_n}_{t_n}, e^{M(\phi*\tau)}\rangle \le e^{M\phi*\tau(0)} e^{C(M+M^2)C_T(t_1+\dots+t_n)}.$$

As a consequence we conclude that for every $k \ge 1$ there is C_k such that

$$\|U^\sigma(s,t)\|_{C^k_\infty(N)\to C^k_\infty(N)} \le C_k e^{C_k C_T(t-s)} \quad \text{for } k \ge 1,\ 0 \le s,t \le T.$$

We need some further properties of $p^\sigma(t,s)$. Let

$$A(s,t) = \int_s^t (\sigma(u)^{p_1} + \sigma(u)^{p_2})\,du.$$

There is C such that for every $s < t$ and $\beta > 0$ ((4.7) in [DHU])

$$\langle e^{\beta\tau}, p^\sigma(t,s)\rangle \le C e^{C(\beta+\beta^2)A(s,t)}. \tag{2.2}$$

There exist $C, \gamma > 0$ and $D \ge 1/2$ such that (see the proof of Theorem 4.1 in [DHU])

$$\|p^\sigma(t,s)\|_{L^\infty(N)} \le C\Big(\int_s^t \sigma(u)^\gamma\, du\Big)^{-D}. \tag{2.3}$$

For every multiindex I, there are $C, M > 0$ such that for every σ,

$$\begin{aligned}|X^I p^\sigma(t,s)\|_{L^\infty(N)} \le C &\max\{1, (t-s)^{-M}\}\\ &\times (1 + \sup_{s\le u\le t} \sigma(u) + \sup_{s\le u\le t} \sigma(u)^{-1})^M.\end{aligned} \tag{2.4}$$

For $t - s \ge 1$, (2.4) was proved in [DHZ] (Theorem 3.5); for arbitrary t, s we use a standard homogeneity argument.

(2.5) Lemma. *For every $\varepsilon > 0$ and each multiindex I, if we set $X^I = X_1^{I_1} \cdots X_n^{I_n}$, then there is a constant C such that for $s < t_1 < t_2 < t$,*

$$\|X^I p^\sigma(t,s)\|_{L^\infty(N)} \le Ce^{\varepsilon A(s,t_1)} \sum_{|J|=|I|} \|X^J p^\sigma(t_2, t_1)\|_{L^\infty(N)}. \tag{2.6}$$

Proof. First we observe that for functions f, g in the Schwartz class on N we have

$$X^I(f * g) = f * X^I g, \qquad X^I(f*g) = \int_N (\mathrm{Ad}_{y^{-1}} X^I f)(xy) g(y^{-1})\, dy.$$

Moreover,

$$|\mathrm{Ad}_y X^I f(xy)| \le C(1 + \tau(y))^{C(I)} \sum_{|J|=|I|} |X^J f(xy)|.$$

Consequently, since

$$p^\sigma(t,s) = p^\sigma(t, t_2) * p^\sigma(t_2, t_1) * p^\sigma(t_1, s), \qquad \|p^\sigma(t,t_2)\|_{L^1(N)} = 1,$$

by (2.2) we have

$$\begin{aligned}\|X^I p^\sigma(t,s)\|_{L^\infty(N)} &\le \|X^I p^\sigma(t_2, s)\|_{L^\infty(N)}\\ &\le C\|(1+\tau)^{C(I)} p^\sigma(t_1, s)\|_{L^1} \sum_{|J|=|I|} \|X^J p^\sigma(t_2, t_1)\|_{L^\infty(N)}\\ &\le Ce^{\varepsilon A(s,t_1)} \sum_{|J|=|I|} \|X^J p^\sigma(t_2, t_1)\|_{L^\infty(N)}. \ \blacksquare\end{aligned}$$

(2.7) Lemma. *Let K be a compact subset of N with $e \notin K$, I a multiindex and $\varepsilon > 0$. There are C_1, C_2 such that for $s < t_1 < t_2 < t$ and $x \in K$,*

$$(2.8)\qquad |X^I p^\sigma(t,s;x)| \le C_1 e^{\varepsilon A(s,t_1)} \|p^\sigma(t_2,t_1)\|^{1/2}_{L^\infty(N)} \times e^{-C_2/A(s,t)} \sum_{|J|\le 2|I|} \|X^J p^\sigma(t_2,t_1)\|^{1/2}_{L^\infty(N)}.$$

Proof. Let $t_3 = \frac{1}{2}(t_2+t_1)$. For every $\beta>0$ we have

$$e^{\beta\tau(x)}|X^I p^\sigma(t,s;x)| \le \Big(\int e^{2\beta\tau(xy^{-1})} p^\sigma(t,t_3;xy^{-1})^2\,dy\Big)^{1/2} \times \Big(\int e^{2\beta\tau(y)}|X^I p^\sigma(t_3,s;y)|^2\,dy\Big)^{1/2}.$$

Since

$$\|X^I p^\sigma(t_3,s)\|_{L^2(e^{2\beta\tau})} \le C\|p^\sigma(t_3,s)\|_{L^1(e^{2\beta\tau})} \sum_{|J|\le 2|I|} \|X^J p^\sigma(t_3,s)\|_{L^\infty(N)},$$

we have

$$e^{\beta\tau(x)}|X^I p^\sigma(t,s;x)| \le C\|p^\sigma(t,t_3)\|^{1/2}_{L^\infty(N)} e^{C(\beta+\beta^2)A(s,t)} \times \Big(\sum_{|J|\le 2|I|} \|X^J p^\sigma(t_3,s)\|_{L^\infty(N)}\Big)^{1/2}.$$

But by (2.6),

$$\|X^J p^\sigma(t_3,s)\|_{L^\infty(N)} \le Ce^{\varepsilon A(s,t_1)} \sum_{|J'|=|J|} \|X^{J'} p^\sigma(t_3,t_1)\|_{L^\infty(N)}.$$

Hence

$$|X^I p^\sigma(t,s;x)| \le C\|p^\sigma(t,t_2)\|^{1/2}_{L^\infty(N)} e^{\varepsilon A(s,t_1)} e^{-\beta\tau(x)+C(\beta+\beta^2)A(s,t)} \times \Big(\sum_{|J|\le 2|I|} \|X^J p^\sigma(t_2,t_1)\|_{L^\infty(N)}\Big)^{1/2}.$$

Now putting $\beta = \tau(x)/2CA(s,t)$ we obtain the conclusion. ■

In view of (2.3), (2.4), we have

(2.9) Corollary. *Let K be a compact set that does not contain e, I a multiindex, $\varepsilon>0$, γ as in (2.3) and M as in (2.4). There are $C_1, C_2 > 0$ such that for $s<t_1<t_2<t$,*

$$\|X^I p^\sigma(t,s)\|_{L^\infty} \le C_1 e^{\varepsilon A(s,t_1)} \Big(\int_{t_2}^{t} \sigma(u)^\gamma\,du\Big)^{-D} \max\{1,(t_2-t_1)^{-M}\} \times (1+\sup_{t_1\le u\le t_2}\sigma(u) + \sup_{t_1\le u\le t_2}\sigma(u)^{-1})^M,$$

and for $x \in K$,

$$|X^I p^\sigma(t,s;x)| \le C_1 e^{\varepsilon A(s,t_1)} e^{-C_2/A(s,t)} \max\{1,(t_2-t_1)^{-M}\} \\ \times (1 + \sup_{t_1 \le u \le t_2} \sigma(u) + \sup_{t_1 \le u \le t_2} \sigma(u)^{-1})^M.$$

3. Evolution run by a Bessel process. Let $\mathbb{R}^+ \ni t \mapsto \sigma(t)$ denote the *Bessel process* with a parameter $\alpha \ge 0$ (see [RY]), i.e. a continuous Markov process with state space $[0,\infty)$ generated by

$$\Delta_\alpha = \partial_a^2 + \frac{\alpha+1}{a}\,\partial_a, \qquad \alpha \ge 0,$$

with $p_s(a,b)$ being the density of the transition probability with respect to the measure $a^{1+\alpha}da$, i.e.

$$P_s f(x) = \int p_s(x,y) f(y) y^{\alpha+1}\,dy.$$

We have

$$p_t(a,u) \le C t^{-1-\alpha/2} e^{-c(a-u)^2/4t}. \tag{3.1}$$

We call $\{P_t\}_{t>0}$ the *Bessel semigroup*. For $f \in L^2(u^{\alpha+1}du)$ we have

$$\lim_{t\to 0} \|P_t f - f\|_{L^2(u^{\alpha+1}du)} = 0$$

and for $f \in C_{\mathrm{c}}^\infty$,

$$\lim_{t\to 0} \left\| \frac{P_t f - f}{t} - \Delta_\alpha f \right\|_{L^2(u^{\alpha+1}du)} = 0. \tag{3.2}$$

Of course, if α is an integer ≥ 0, then Δ_α is the radial part of the Laplacean on $\mathbb{R}^{\alpha+2}$.

Let

$$\mu(a,\eta) = \int_{a-\eta}^{a+\eta} u^{\alpha+1}\,du, \qquad \chi_\eta = \mu(a,\eta)^{-1}\mathbf{1}_{[a-\eta,a+\eta]}.$$

The following facts are well known and easy to prove (cf. e.g. [RY], [DHU]):

$$\sup_{b>0} \mathrm{E}_b \chi_\eta(\sigma(t/4)) \le c t^{-1-\alpha/2} \tag{3.3}$$

and, for every $\gamma > 0$,

$$\sup_{b>0} \mathrm{E}_b \Big(\int_0^t \sigma(s)^\gamma\,ds \Big)^{-D} \le C t^{-(1+\gamma/2)D}, \tag{3.4}$$

where E_b denotes the expectation with respect to the Wiener–Bessel measure on the space of trajectories.

There are c_1, c_2 such that for every $a \geq 0$, and every $t > 0$,

$$P_a(\inf_{s\in[0,t]} \sigma(s) < a/2) \leq c_1 e^{-c_2 a^2/t}, \tag{3.5}$$

$$P_a(\sup_{s\in[0,t]} \sigma(s) > a + \lambda) \leq c_1 e^{-c_2 \lambda^2/t}. \tag{3.6}$$

There is a constant δ such that for every $a \geq 0$, and every $t > 0$,

$$P_a(\sup_{s\in[0,t]} \sigma(s) < \lambda) \leq e^{-\delta t/\lambda^2}. \tag{3.7}$$

The following basic estimate was proved in [U], but since then the proof has been essentially simplified and so we include it here.

(3.8) THEOREM. *Let K be a compact subset of N and $e \notin K$. For every multiindex I,*

$$\sup_{\eta<a<1,\, x\in K} \int_0^\infty \mathrm{E}_0 |X^I p^\sigma(t,0)(x)| \chi_\eta(\sigma(t))\, dt < \infty.$$

Proof. Let

$$\Omega_0 = \{\sigma : \sup_{0\leq s\leq t/2} \sigma \geq 1\}$$

and for $k = 1, 2, \ldots$ let

$$\Omega_k = \{\sigma : 2^{-k} \leq \sup_{0\leq s\leq t/2} \sigma \leq 2^{-k+1}\}.$$

We define stopping times

$$T_1^k = \inf\{t > 0 : \sigma(t) = 2^{-k-1}\}$$

and

$$T_2^k = \min\{\inf\{t > T_1^k : \sigma(t) = 3 \cdot 2^{-k-2}\}, \inf\{t > T_1^k : \sigma(t) = 2^{-k-2}\}\}.$$

Let

$$I_1 = \int_1^\infty \mathrm{E}_0 |X^I p^\sigma(t,0)(x)| \chi_\eta(\sigma(t))\, dt,$$

$$I_2 = \int_0^1 \mathrm{E}_0 |X^I p^\sigma(t,0)(x)| \chi_\eta(\sigma(t))\, dt.$$

By Corollary (2.9),

$$I_1 \leq C \sum_{k=0}^\infty 2^{Mk} \int_1^\infty \mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{\varepsilon A(0,T_1^k)}$$
$$\times \Big(\int_{T_2^k}^t \sigma(s)^\gamma\, ds \Big)^{-D} \chi_\eta(\sigma(t))\, dt$$

$$\le C\sum_{k=0}^{\infty} 2^{Mk} \int\limits_1^{\infty} \mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{\varepsilon T_1^k} \times \Big(\int\limits_{t/2}^{3t/4} \sigma(s)^{\gamma}\, ds\Big)^{-D} \chi_{\eta}(\sigma(t))\, dt.$$

Applying the Markov property twice, by (3.3) and (3.4), we have

$$\mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M_1} e^{\varepsilon T_1^k} \Big(\int\limits_{t/2}^{3t/4} \sigma(s)^{\gamma}\, ds\Big)^{-D} \chi_{\eta}(\sigma(t))$$
$$= \mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{\varepsilon T_1^k} \Big(\int\limits_{t/2}^{3t/4} \sigma(s)^{\gamma}\, ds\Big)^{-D} \mathrm{E}_{\sigma(3t/4)} \chi_{\eta}(\sigma(t/4))$$
$$\le C t^{-1-\alpha/2} \mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{\varepsilon T_1^k} \mathrm{E}_{\sigma(t/2)} \Big(\int\limits_{0}^{t/4} \sigma(s)^{\gamma}\, ds\Big)^{-D}$$
$$\le C t^{-1-\alpha/2-(1+\gamma/2)D} \mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{\varepsilon T_1^k}.$$

Thus it suffices to estimate

$$\mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{\varepsilon T_1^k} \le (\mathrm{E}_0 (T_2^k - T_1^k)^{-2M})^{1/2} (\mathrm{E}_0 \mathbf{1}_{\Omega_k} e^{2\varepsilon T_1^k})^{1/2}.$$

For $k \ge 1$, by (3.6) we have

$$\mathrm{E}_0 \mathbf{1}_{\Omega_k} e^{2\varepsilon T_1^k} \le e^{2\varepsilon t} P_0(\Omega_k) \le e^{\varepsilon t} e^{-\delta 2^{2k-2} t} \le e^{-\delta_1 2^{2k} t},$$

while

$$\mathrm{E}_0 \mathbf{1}_{\Omega_0} e^{2\varepsilon T_1^0} \le \sum_{m=1}^{[t/2]+1} e^{2\varepsilon m} P_0(\{\sigma : \sup_{0\le s\le m} \le 1/2) \le \sum_{m=1}^{[t/2]+1} e^{2(\varepsilon-\delta)m} < \infty$$

when $\varepsilon < \delta$. Finally, we estimate $\mathrm{E}_0 (T_2^k - T_1^k)^{-2M}$. Given $l > 0$, let

$$W_{l,1} = \big\{\sigma : 2^{-l} \le T_2^k - T_1^k \le 2^{-l+1},\ \sigma(T_2^k) = \tfrac{3}{2}\sigma(T_1^k)\big\},$$
$$W_{l,2} = \big\{\sigma : 2^{-l} \le T_2^k - T_1^k \le 2^{-l+1},\ \sigma(T_2^k) = \tfrac{1}{2}\sigma(T_1^k)\big\}.$$

By the strong Markov property, (3.5) and (3.6), we have

$$P_0(W_{l,1}) \le P_{\sigma(T_1^k)}\big(\big\{ \sup_{0\le s\le 2^{-l+1}} b(s) > \tfrac{3}{2} b(0)\big\}\big) \le c_1 e^{-c_2 2^{-2k+l}}$$

and

$$P_0(W_{l,2}) \le P_{\sigma(T_1^k)}\big(\big\{ \inf_{0\le s\le 2^{-l+1}} b(s) < \tfrac{1}{2} b(0)\big\}\big) \le c_1 e^{-c_2 2^{-2k+l}}.$$

Therefore,

$$\mathrm{E}_0(T_2^k - T_1^k)^{-2M} \le \sum_{l=0}^{\infty} C2^{2lM} e^{-c_2 2^{-2k+l}} \le C2^{4kM}.$$

Finally,

$$I_1 \le C \int_1^{\infty} t^{-1-\alpha/2-(1+\gamma/2)D} \Big(\sum_k 2^{3Mk} e^{-\delta_1 2^{2k} t} + 1 \Big) \le C.$$

Let $N > \alpha + 2$. By Corollary (2.9),

$$\begin{aligned} I_2 &\le C \sum_{k=0}^{\infty} 2^{Mk} \int_0^1 \mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-M} e^{-c/A(0,t)} \chi_\eta(\sigma(t))\, dt \\ &\le C_N \sum_{k=0}^{\infty} 2^{Mk} \int_0^1 (\mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-2M} \chi_\eta(\sigma(t)))^{1/2} \\ &\quad \times (\mathrm{E}_0 A(0,t)^N \chi_\eta(\sigma(t)))^{1/2}\, dt. \end{aligned}$$

Proceeding as before we have

$$\mathrm{E}_0 \mathbf{1}_{\Omega_k} (T_2^k - T_1^k)^{-2M} \chi_\eta(\sigma(t)) \le Ct^{-1-\alpha/2} 2^{4kM} e^{-\delta_1 2^{2k} t}.$$

Hence by the lemma below,

$$\begin{aligned} I_2 &\le C \sum_{k=0}^{\infty} 2^{3Mk} \int_0^1 t^{N-\alpha/2-2} e^{-\delta_1 2^{2k} t}\ dt \\ &\le C \sum_{k=0}^{\infty} 2^{3Mk-2(N-\alpha/2-1)k} \int_0^{2^{2k}} t^{N-\alpha/2-2} e^{-\delta_1 t}\, dt < \infty. \ \blacksquare \end{aligned}$$

(3.9) LEMMA. *Let $N > \alpha + 2$. There is C such that for every $t \le 1$ and $\eta, a < 1$,*

$$\mathrm{E}_0 A(0,t)^N \chi(\sigma(t)) \le Ct^{N-\alpha/2-1}.$$

Proof. By the Hölder inequality, we have

$$\begin{aligned} A(0,t)^N &\le t^{N-1} \int_0^t (\sigma(s)^{p_1} + \sigma(s)^{p_2})^N\, ds \\ &= t^{N-1} \sum_{k=0}^{N} \binom{n}{k} \int_0^t \sigma(s)^{kp_1+(N-k)p_2}\, ds. \end{aligned}$$

Therefore, it suffices to estimate

$$\int_0^t \mathrm{E}_0 \sigma(s)^q \chi(\sigma(t))\, ds \qquad \text{for } q \ge N.$$

By the Markov property we have

$$\int_0^t \mathrm{E}_0\sigma(s)^q\chi(\sigma(t))\,ds = \int_0^t \mathrm{E}_0\sigma(s)^q\mathrm{E}_{\sigma(s)}\chi(b(t-s))\,ds$$

$$= \int_0^t\int_0^\infty u^q p_s(0,u)\,\frac{1}{\mu(a,\eta)}\int_{a-\eta}^{a+\eta} p_{t-s}(u,y)y^{\alpha+1}\,dy\,u^{\alpha+1}\,du\,ds.$$

We split the integral into two parts and by (3.1) we have

$$\int_0^t\int_0^4 u^q p_s(0,u)\frac{1}{\mu(a,\eta)}\int_{a-\eta}^{a+\eta} p_{t-s}(u,y)y^{\alpha+1}\,dy\,u^{\alpha+1}\,du\;ds$$

$$\le \frac{C}{\mu(a,\eta)}\int_0^t\int_{a-\eta}^{a+\eta}\int_0^\infty p_s(0,u)p_{t-s}(u,y)u^{\alpha+1}\,du\,y^{\alpha+1}\,dy\,ds$$

$$\le \frac{C}{\mu(a,\eta)}\int_0^t\int_{a-\eta}^{a+\eta} p_t(0,y)y^{\alpha+1}\,dy\,ds$$

$$\le \frac{C}{t^{\alpha/2}\mu(a,\eta)}\int_{a-\eta}^{a+\eta} y^{\alpha+1}\,dy \le \frac{C}{t^{\alpha/2}}$$

and

$$\int_0^t\int_4^\infty u^q p_s(0,u)\,\frac{1}{\mu(a,\eta)}\int_{a-\eta}^{a+\eta} p_{t-s}(u,y)y^{\alpha+1}\,dy\,u^{\alpha+1}\,du\,ds$$

$$\le \int_0^t\int_4^\infty u^q\,\frac{1}{s^{1+\alpha/2}}\,e^{-u^2/4s}\frac{1}{\mu(a,\eta)}$$

$$\times\int_{a-\eta}^{a+\eta}\frac{1}{(t-s)^{1+\alpha/2}}\,e^{-(u-y)^2/4(t-s)}y^{\alpha+1}\,dy\,u^{\alpha+1}\,du\,ds$$

$$\le \int_0^t\int_0^\infty u^q\frac{1}{s^{1+\alpha/2}}\,e^{-u^2/4s}\,\frac{1}{(t-s)^{1+\alpha/2}}\,e^{-u^2/16(t-s)}u^{\alpha+1}\,du\,ds$$

$$\le \int_0^t\Big(\int_0^\infty u^q e^{-u^2/2s}u^{\alpha+1}\,du\Big)^{1/2}\Big(\int_0^\infty u^q e^{-u^2/8s}u^{\alpha+1}\,du\Big)^{1/2}$$

$$\times\frac{1}{s^{1+\alpha/2}}\,\frac{1}{(t-s)^{1+\alpha/2}}\,ds$$

$$\le \int_0^t s^{(q+\alpha+1)/4-1-\alpha/2}(t-s)^{(q+\alpha+1)/4-1-\alpha/2}\,ds<\infty. \;\blacksquare$$

Let

$$L(a)=a^{-2}\Big(\sum \Phi_a(X_j)^2+\Phi_a(X)\Big),$$
$$\mathbf{L}_\alpha=a^{-2}\mathcal{L}_\alpha=\Delta_\alpha+L(a).$$

We observe that by (1.8), $L(0)=0$ is well defined. For the evolution $p^\sigma(t,s)$ described in Section 2, and $a\geq 0$, $x\in N$, let

$$T_tf(x,a)=\mathrm{E}_a\int_N f(xy^{-1},\sigma(t))p^\sigma(t,0;y)\,dy. \tag{3.10}$$

We are going to prove

(3.11) THEOREM. *$\{T_t\}_{t>0}$ is a semigroup of contractions on $L^2(dx\otimes a^{1+\alpha}da)$ whose infinitesimal generator contains $\mathbf{L}_\alpha$.*

Proof. First we observe that, if $p_t(a,b)$ is as in (3.1), we have

$$\begin{aligned}\|T_tf\|^2_{L^2(dx\otimes a^{1+\alpha}da)} &\leq \int_{\mathbb{R}^+}\mathrm{E}_a\int_N |f(xy^{-1};\sigma(t))p^\sigma(t,0;y)\,dy|^2\,dx\,a^{1+\alpha}\,da\\ &\leq \int_{\mathbb{R}^+}\mathrm{E}_a\|f(\cdot,\sigma(t))*_N p^\sigma(t,0)\|^2_{L^2(dx)}a^{1+\alpha}\,da\\ &\leq \int_{\mathbb{R}^+}\mathrm{E}_a\|f(\cdot,\sigma(t))\|^2_{L^2(dx)}a^{1+\alpha}\,da\\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}|f(x,b)|^2p_t(a,b)b^{1+\alpha}a^{1+\alpha}\,da\,db\,dx\\ &\leq \int_{\mathbb{R}^+}|f(x,b)|^2b^{1+\alpha}\,db\,dx.\end{aligned} \tag{3.12}$$

Also, by the Markov property (see e.g. [DHU, Theorem 3.2]),

$$T_sT_t=T_{s+t}.$$

Now we prove that for $f\in C_c^\infty(N)$,

$$\lim_{t\to 0}\frac{1}{t}\,[T_tf(x,a)-f(x,a)]-\mathbf{L}_\alpha f(x,a)=0$$

in $L^2(a^{1+\alpha}dxda)$, which for simplicity will be denoted L^2. For convolution of f with p^σ we adopt the notation

$$\int_N f(xy^{-1},\sigma(t))p^\sigma(t,0;dy)=f(\cdot,\sigma(t))*p^\sigma(t,0)(x).$$

We have

$$
\begin{aligned}
&\frac{1}{t}\,[T_t f(x,a) - f(x,a)] - \mathbf{L}_\alpha f(x,a)\\
&\qquad\qquad = \frac{1}{t}\,[\mathrm{E}_a f(\cdot,\sigma(t)) * p^\sigma(t,0)(x) - \mathrm{E}_a f(x,\sigma(t))] - L(a)f(x,a)\\
&\qquad\qquad\quad + \frac{1}{t}\,[\mathrm{E}_a f(x,\sigma(t)) - f(x,a)] - \Delta_\alpha f(x,a).
\end{aligned}
$$

Clearly, the second term in the above expression tends to 0 in $L^2(a^{1+\alpha}dxda)$, so we have to deal only with the first one. We write

$$
\begin{aligned}
&\mathrm{E}_a \frac{1}{t}\int_0^t (L(\sigma(s))f(\cdot,\sigma(t))) * p^\sigma(s,0)(x) - L(a)f(x,a)\\
&\qquad = \mathrm{E}_a \frac{1}{t}\int_0^t [L(\sigma(s))(f(\cdot,\sigma(t)) - f(\cdot,a))] * p^\sigma(s,0)(x)\\
&\qquad\quad + \mathrm{E}_a \frac{1}{t}\int_0^t [(L(\sigma(s)) - L(a))f(\cdot,a)] * p^\sigma(s,0)(x)\\
&\qquad\quad + \mathrm{E}_a \frac{1}{t}\int_0^t [L(a)f(\cdot,a) * p^\sigma(s,0)(x) - L(a)f(x,a)] = I_1 + I_2 + I_3.
\end{aligned}
$$

Now

$$
|I_1|^2_{L^2} \le \frac{1}{t}\int_0^t \mathrm{E}_a |L(\sigma(s))(f(\cdot,\sigma(t)) - f(x,a))|^2\ a^{1+\alpha}\,dx\,da
$$

and so we have to estimate terms of the form

$$
\begin{aligned}
&\mathrm{E}_a \sigma(s)^\gamma |X^I f(x,\sigma(t)) - X^I f(x,a)|^2\\
&\qquad \le \mathrm{E}_a \sigma(s)^\gamma (|X^I f(x,\sigma(t))| + |X^I f(x,a)|)\|f\|_{C^3_\infty} |\sigma(t) - \sigma(0)|\\
&\qquad \le C\|f\|_{C^3_\infty} (\mathrm{E}_a \sigma(s)^{2\gamma})^{1/2} [(\mathrm{E}_a |X^I f(x,\sigma(t))|^2 |\sigma(t) - \sigma(0)|^2)^{1/2}\\
&\qquad\quad + |X^I f(x,a)|)(\mathrm{E}_a |\sigma(t) - \sigma(0)|^2)^{1/2}].
\end{aligned}
$$

A direct calculation involving compactness of the support of f, (3.1) and (3.6) shows that for $t \le 1$,

$$
\begin{aligned}
&|X^I f(x,a)|(\mathrm{E}_a |\sigma(t) - \sigma(0)|^2)^{1/2} \le C\sqrt{t},\\
&\mathrm{E}_a |X^I f(x,\sigma(t))|^2 |\sigma(t) - \sigma(0)|^2 \le C_1 \sqrt{t} e^{-C_2 a},\\
&\sup_{s\le t} \mathrm{E}_a \sigma(s)^{2\gamma} \le C(1+a)^{2\gamma}.
\end{aligned}
$$

Hence $\lim_{t\to 0} |I_1|^2_{L^2} = 0$. Next we have

$$
|I_2|^2_{L^2} \le \frac{1}{t}\int_0^t \mathrm{E}_a |(L(\sigma(s)) - L(a))f(x,a)|^2\ a^{1+\alpha}\,dx\,da
$$

and we have to estimate terms

$$|X^I f(x,a)|^2 \mathrm{E}_a(\sigma(t)^\gamma + \sigma(0)^\gamma)|\sigma(t)-\sigma(0)|^2.$$

Again using compactness of the support of f, we see that they do not exceed $\sqrt{t}$, which is enough to conclude $\lim_{t\to 0}|I_2|^2_{L^2}=0$. Finally,

$$I_3 = \frac{1}{t}\int_0^t \mathrm{E}_a \int_0^s L(\sigma(r))L(a)f(\cdot,a) * p^\sigma(r,0)(x),$$

and so

$$\begin{aligned}|I_3|^2_{L^2} &= \frac{1}{t}\int_0^t \mathrm{E}_a s\int_0^s |L(\sigma(r))L(a)f(\cdot,a) * p^\sigma(r,0)(x)|^2\, a^{1+\alpha}\,dx\,da\\ &\le \frac{1}{t}\int_0^t s\int_0^s \mathrm{E}_a|L(\sigma(r))L(a)f(x,a)|^2\, a^{1+\alpha}\,dx\,da.\end{aligned}$$

Inside the expected value we have terms of the form $\sigma(r)^\gamma a^\eta |X^I f(x,a)|$ and

$$\int (\mathrm{E}_a\sigma(r)^\gamma)a^\eta|X^I f(x,a)|\; a^{1+\alpha}\,dx\,da \le C.$$

Hence $\lim_{t\to 0}|I_3|_{L^2}=0$. ∎

4. The Green function. Let

$$\mathbf{L}^*_\alpha = \partial_a^2 + \frac{1+\alpha}{a}\,\partial_a + a^{-2}\Big(\sum_j \Phi_a(X_j)^2 - \Phi_a(X)\Big)$$

be the formal adjoint of $\mathbf{L}_\alpha$ on $L^2(dx\otimes a^{1+\alpha}da)$ and let $f\in C_{\mathrm{c}}(S)$. Then for T^*_t defined as in (3.10) but for $\mathbf{L}^*_\alpha$, for every $x\in N$ and $a\ge 0$ we have

$$\int_0^\infty |T^*_t f(x,a)|\,dt < \infty. \tag{4.1}$$

Indeed, $|T^*_t f(x,a)| \le \|f\|_{L^\infty}$ and for $t>1$, by (2.3), (3.1) and (3.4), we have

$$\begin{aligned}|T^*_t f(x,a)| &\le \mathrm{E}_a\|p^\sigma(t,0)\|_{L^2(N)}\|f(\cdot,\sigma(t))\|_{L^2(N)}\\ &\le (\mathrm{E}_a\|p^\sigma(t,0)\|^2_{L^2(N)})^{1/2}\Big(\int \mathrm{E}_a|f(x,\sigma(t))|^2\,dx\Big)^{1/2}\\ &\le C\mathrm{E}_a\Big(\int_0^t \sigma(s)^\gamma\,ds\Big)^{-D}\Big(\int |f(x,b)|^2 p_t(a,b) b^{1+\alpha}\,db\,dx\Big)^{1/2}\\ &\le Ct^{-1/2-\alpha/4-(1+\gamma/2)D}\|f\|_{L^2(dx\otimes a^{1+\alpha}da)}\end{aligned} \tag{4.2}$$

and $D\ge 1/2$. (4.1) defines a positive functional

$$f\mapsto \int_0^\infty T^*_t f(x,a)\,dt$$

on $C_{\mathrm{c}}(S)$. Therefore, there is a non-negative Radon measure $G^*(x,a;dy\,db)$ such that

$$\int_0^\infty T_t^* f(x,a)\,dt = \int_S G^*(x,a;dy\,db) f(y,b). \tag{4.3}$$

We are going to prove the following theorem:

(4.4) Theorem.

$$\mathbf{L}_\alpha G^*(e,0;\cdot) = 0, \tag{4.5}$$

$$G^*(e,0;y,b) = b^{-Q-\alpha} G^*(e,0;\Phi_{b^{-1}}(y),1), \tag{4.6}$$

where the measure G^* and its density with respect to $b^{1+\alpha}\,db\,dy$ are denoted by the same letter. For every multiindex I and every compact $K\subset N$ with $e\notin K$,

$$\sup_{0<b\le 1,\, y\in K} X_y^I G^*(e,0;y,b) < \infty. \tag{4.7}$$

Proof. Let $f\in C_{\mathrm{c}}^\infty(S)$. By (4.3) and Theorem 3.11, f is in the domain of the infinitesimal generator of the semigroup $\{T_t^*\}_{t>0}$ and so

$$\begin{aligned}\int_S G^*(e,0;dy\,db)\mathbf{L}_\alpha^* f(y,b) &= \int_0^\infty T_t^*\mathbf{L}_\alpha^* f(e,0)\,dt = \int_0^\infty \mathbf{L}_\alpha^* T_t^* f(e,0)\,dt \\ &= \int_0^\infty \frac{d}{dt} T_t^* f(e,0)\,dt.\end{aligned}$$

Moreover, by (4.2), the integral is absolutely convergent. Hence

$$\int_0^\infty \frac{d}{dt} T_t^* f(e,0)\,dt = \lim_{\varepsilon\to 0}(T_{\varepsilon^{-1}}^* f(e,0) - T_\varepsilon^* f(e,0)).$$

But by (2.3),

$$|T_{\varepsilon^{-1}}^* f(e,0)| \le \mathrm{E}_a\Big(\int_0^{\varepsilon^{-1}} \sigma(s)^\gamma\,ds\Big)^{-D} \le C\varepsilon^{(1+\gamma/2)D}$$

and so $\lim_{\varepsilon\to 0} T_{\varepsilon^{-1}}^* f(e,0) = 0$. Moreover, $\lim_{\varepsilon\to 0} T_\varepsilon^* f(e,0) = 0$. Hence

$$\int_S G^*(e,0;dy\,db)\mathbf{L}_\alpha^* f(y,b) = 0,$$

which implies that $G^*(e,0;\cdot)\in C^\infty(S)$ and denoting its density with respect to the measure $b^{1+\alpha}\,db\,dy$ by the same letter we have

$$\int_S \mathbf{L}_\alpha G^*(e,0;y,b) f(y,b) b^{1+\alpha}\,db\,dy = 0.$$

Let $D_r(x,a) = (\Phi_r(x), ra)$. Using directly formula (3.10) that defines T_t^* and homogeneity of the evolution, we see that

$$T_t^*(f \circ D_r) \circ D_{r^{-1}} = T_{r^2 t}^* f.$$

Hence

$$\int_0^\infty T_t^*(f \circ D_r)(e,0)\,dt = \int_0^\infty T_{r^2 t}^* f(e,0)\,dt = r^{-2} \int_0^\infty T_t^* f(e,0)\,dt$$

and so

$$\int_S f \circ D_r(y,b) G^*(e,0;y,b) b^{1+\alpha}\,db\,dy = r^{-2} \int_S f(y,b) G^*(e,0;y,b) b^{1+\alpha}\,db\,dy.$$

Changing variables we obtain

$$r^{-Q-\alpha} G^*(e,0;\Phi_{r^{-1}}(y), r^{-1}b) = G^*(e,0;y,b)$$

and (4.6) follows. Moreover, by N-left-invariance of L_α, $G^*(x,0;y,b) = G^*(e,0;x^{-1}y,b)$.

Let $\phi \in C_c^\infty(N)$ and let $\widetilde{X}$ be a right-invariant vector field on N. Then, since $\widetilde{X}$ commutes with convolution on the right,

$$\begin{aligned}
\widetilde{X} G^*(\phi\chi_\eta)(x,0) &= \widetilde{X}\Big(\int G^*(e,0;x^{-1}y,b)\phi(y)\chi_\eta(b) b^{1+\alpha}\,dy\,db\Big) \\
&= \int \frac{d}{dt} G^*(e,0;x^{-1}\exp -t\widetilde{X} y, b)\Big|_{t=0} \phi(y)\chi_\eta(b) b^{1+\alpha}\,dy\,db.
\end{aligned}$$

Hence, for $x = e$, we have

$$\begin{aligned}
&\int_S \widetilde{X}^I G^*(e,0;y,b)\phi(y)\chi_\eta(b) b^{\alpha+1}\,db\,dy \\
&\qquad = (-1)^{|I|} \int_0^\infty \widetilde{X}^I T_t^*((\phi\chi_\eta))(e,0)\,dt \\
&\qquad = (-1)^{|I|} \int_0^\infty \mathrm{E}_0 \widetilde{X}^I \phi *_N p^\sigma(t,0;e)\chi_\eta(\sigma(t))\,dt \\
&\qquad = (-1)^{|I|} \int_0^\infty \mathrm{E}_0 \langle \phi, \widetilde{X}^I \breve{p}^\sigma(t,0)\rangle \chi_\eta(\sigma(t))\,dt,
\end{aligned}$$

where on the left-hand side $\widetilde{X}^I$ is applied to y. Thus, if $\phi \to \delta_x$ we have

$$\int_0^\infty \widetilde{X}^I G^*(e,0;x,b)\chi_\eta(b) b^{\alpha+1}\,db = \int_0^\infty \mathrm{E}_0 X^I p^\sigma(t,0)(x^{-1})\chi_\eta(\sigma(t))\,dt.$$

and (4.7) follows by Theorem (3.8). ∎

5. Continuity of the Green function and asymptotics of the Poisson kernel

(5.1) THEOREM. *If* $x \neq e$ *then*

$$\lim_{a\to 0} G^*(e,0;x,a) \text{ exists,} \tag{5.2}$$

it is positive, *and* $G^*(e,0;\cdot)$ *extended to* $N\setminus\{e\}\times[0,\infty)$ *by*

$$G^*(e,0;x,0)=\lim_{a\to 0} G^*(e,0;x,a)$$

is continuous on $N\setminus\{e\}\times[0,\infty)$.

Proof. Write $G^*(e,0;x,a)=G(x,a)$. We observe that by (4.5),

$$\mathcal{L}_\alpha G(x,a)=a^2\mathbf{L}_\alpha G(x,a)=0.$$

For a fixed $x\neq e$, let

$$h(a)=a\partial_a G(x,a),\qquad v(a)=\Big(-\sum_j \Phi_a(X_j)^2-\Phi_a(X)\Big)G(x,a).$$

Then

$$a\partial_a h(a)+\alpha h(a)=v(a) \tag{5.3}$$

and, by (1.8) and (4.7), there is $\beta>0$ such that $|v(a)|<a^\beta$ for $a\le 1$. Moreover, by the Harnack inequality for $\mathcal{L}_\alpha$, h is bounded. By (5.3),

$$\partial_a(a^\alpha h)=a^{\alpha-1}v(a).$$

We shall prove that

$$|h(a)|\le \frac{a^\beta}{\alpha+\beta}. \tag{5.4}$$

But (5.4) implies

$$|G(x,a)-G(x,b)|=\Big|\int_b^a u^{-1}h(u)\,du\Big|\le\int_b^a \frac{u^{\beta-1}}{\alpha+\beta}\,du=\frac{a^\beta-b^\beta}{\beta(\alpha+\beta)}$$

and from this (5.2) follows.

To prove (5.4) we take $b<a$ and we write

$$\begin{aligned}|a^\alpha h(a)-b^\alpha h(b)|&=\Big|\int_b^a \partial_u(u^\alpha h(u))\,du\Big|\le\int_b^a |u^{\alpha-1}v(u)|\,du\\&\le\int_b^a u^{\alpha+\beta-1}\,du=\frac{a^{\alpha+\beta}-b^{\alpha+\beta}}{\alpha+\beta}.\end{aligned} \tag{5.5}$$

Since h is bounded, if $\alpha>0$ we may take the limit as $b\to 0$ and so (5.4) follows. For $\alpha=0$, from (5.5) we derive the existence of

$$\lim_{a\to 0} h(a)=h(0).$$

It remains to show that $h(0) = 0$. For that we write

$$G(x,1) - G(x,a) = \int_a^1 u^{-1} h(u)\, du = \int_a^1 u^{-1}\Big(h(0) + \int_0^u \partial_s h(s)\, ds\Big) du$$
$$= -h(0)\ln(a) + \int_a^1 u^{-1} \int_0^u \partial_s h(s)\, ds\, du.$$

But by (5.5),

$$\Big|\int_a^1 u^{-1} \int_0^u \partial_s h(s)\, ds\, du\Big| \le \int_a^1 \frac{u^{\beta-1}}{\alpha+\beta}\, du = \frac{1-a^\beta}{\beta(\alpha+\beta)}.$$

Hence $h(0)\ln a$ is bounded as $a \to 0$ and so $h(0) = 0$. (5.4) follows also for $\alpha = 0$.

To prove continuity of $G^*(e, 0; \cdot)$ we take a sequence $(x_n, a_n) \to (x, 0)$ with $x \neq e$. If $a_n \neq 0$ then by (4.7),

$$|G(x_n, a_n) - G(x, a_n)| \le c\tau(x_n, x),$$

c being independent of n. If $a_n = 0$ then by (5.2) we can find b_n such that

$$|G(x_n, 0) - G(x_n, b_n)| < 1/n.$$

We also have

$$|G(x_n, b_n) - G(x, b_n)| < C\tau(x_n, x).$$

Hence

$$|G(x_n, a_n) - G(x, b_n)| < 1/n + C\tau(x_n, x).$$

In any case

$$\lim_{n\to\infty} G(x_n, a_n) = \lim_{n\to\infty} G(x, b_n) = G(x, 0). \ \blacksquare$$

Proof of the Main Theorem. We have

$$\breve{m}_\alpha(x) = G^*(e, 0; x, 1).$$

Indeed, by (4.5) and (4.6),

$$a^\alpha \mathcal{L}_\alpha(a^{-Q-\alpha} G^*(e, 0; \Phi_{a^{-1}}(x), 1)) = \mathcal{L}_{-\alpha}(a^{-Q} G^*(e, 0; \Phi_{a^{-1}}(x), 1)) = 0,$$

which implies (1.3) and so uniquely determines the measure m_α. Now again by (4.6) and the previous theorem,

$$\lim_{a\to 0} \breve{m}_\alpha(\Phi_{a^{-1}}(x)) a^{-Q-\alpha} = \lim_{a\to 0} G^*(e, 0; x, a) = G^*(e, 0; x, 0).$$

Positivity of the above limit follows from (1.4). $\blacksquare$

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